A universal pair of genus-two curves.

T. Shaska

January 6, 2017

Problem:

Let $\mathfrak{p}$ be any point in the moduli space of genus-two curves $\mathcal{M}_2$ and K its field of moduli.

Lemma

There is a genus 2 curve $\mathcal{C}$ corresponding to $\mathfrak{p}$ and defined over a quadratic extension of K.

Problem

Determine a universal equation for $\mathcal{C}$ (i.e. an equation that will work for any $\mathfrak{p} \in \mathcal{M}_2$).

More precisely,

Problem

Given the generic moduli point $\mathfrak{p} = [j_1, j_2, j_3, 1] \in \mathcal{M}_2$, find a universal equation for $\mathcal{C}$ in terms of j_1, j_2, j_3.

Preliminaries

For $\mathcal{C}$, over a field k, char $k = 0$, we can assume that its equation is given by

$$y^2 z^4 = f(x, z)$$

where $f(x, z)$ is a binary sextic defined over k given by

$$f(x, z) = a_0 x^6 + a_1 x^5 z + \cdots + a_6 z^6 = (z_1 x - x_1 z)(z_2 x - x_2 z) \dots (z_6 x - x_6 z)$$

A **covariant** I of $f(x, z)$ is a homogenous polynomial in x, z with coefficients in $k[a_0, \dots, a_6]$. The **order** of I is the degree of I as a polynomial in x, z and the **degree** of I is the degree of I as a polynomial in $k[a_0, \dots, a_6]$.

An **invariant** is a covariant of order zero. The binary form $f(x, z)$ is a covariant of order $2g + 2$ and degree 1.
For any two forms f and g the r-transvection is given by an operation called 'Überschiebung'

$$(f, g)_r =$$

Invariants and covariants via transvections

Consider the following covariants

$$\Delta = ((f,f)_4,(f,f)_4)_2\,,\ y_1 = (f,(f,f)_4)_4\,,\ y_2 = ((f,f)_4,y_1)_2\,,\ y_3 = ((f,f)_4,y_2)_2$$

The **Clebsch invariants** A, B, C, D are defined as follows

$$A = (f,f)_6,\ B = ((f,f)_4,(f,f)_4)_4\,,\ C = ((f,f)_4,\Delta)_4\,,\ D = (y_3,y_1)_2 \qquad (1)$$

see Clebsch [2] or Bolza [1, Eq. (7), (8), pg. 51] for details.
Some other invariants are

$$A_{ij} = (y_i, y_j)_2, \qquad\qquad (1 \leq i,j \leq 3)$$

Clebsch [2] showed that A_{ij} and a_{ijk} can be expressed as

$$\begin{aligned}
A_{11} &= 2\,C + \frac{1}{3}\,A\,B\,,\\
A_{22} &= A_{13} = \,D\,,\\
A_{33} &= \frac{1}{2}\,B\,D + \frac{2}{9}\,C\,(B^2 + A\,C)\,,\\
A_{23} &= \frac{1}{3}\,B\,(B^2 + A\,C) + \frac{1}{3}\,C\,(2\,C + \frac{1}{3}\,A\,B)\,,\\
A_{12} &= \frac{2}{3}\,(B^2 + A\,C)\,.
\end{aligned} \qquad (2)$$

Igusa-Clebsch invariants are

$$
\begin{aligned}
I_2 &= -120\, A\,, \\
I_4 &= -720\, A^2 + 6750\, B\,, \\
I_6 &= 8640\, A^3 - 108000\, A\, B + 202500\, C \\
I_{10} &= -62208 A^5 + 972000 A^3 B + 1620000 A^2 C - 3037500 A B^2 - 6075000 BC - 4556250 D
\end{aligned}
$$

The *Igusa functions* (i.e., $\mathrm{GL}(2,\mathbb{C})$-invariants) are defined as

$$j_1 = \frac{I_2^5}{I_{10}}, \quad j_2 = \frac{I_4 I_2^3}{I_{10}}, \quad j_3 = \frac{I_6 I_2^2}{I_{10}}$$

A **moduli point** is a projective point given by $\mathfrak{p} = [j_1, j_2, j_3, 1]$

Lemma

Two genus two curves are isomorphic over $\mathbb{C}$ if and only if they correspond to the same moduli point.

Conic

For $\mathbf{X} = [X_1 : X_2 : X_3]$ and some symmetric M the **conic** $\mathcal{C}/\mathrm{Aut}(\mathfrak{p})$ is

$$\mathcal{Q}: \quad \mathbf{X}^t \cdot M \cdot \mathbf{X} = \sum_{i,j=1}^{3} a_{ij} X_i X_j = 0$$

We want to determine M. Notice that under the operation

$$f(x) \mapsto \tilde{f}(x) = f(-x)$$

the quadrics $\mathrm{y}_i(x)$, $i = 1, 2, 3$ change according to

$$\mathrm{y}_i(x) \mapsto \tilde{\mathrm{y}}_i(x) = \mathrm{y}_i(-x).$$

Hence, they are not invariants of the sextic f.
The coefficients $a_{ij} = A_{ij}$ and are invariant under the operation $f(x) \mapsto \tilde{f}(x) = f(-x)$, and the locus $D = 0$ is equivalent to

$$D = 0 \quad \Leftrightarrow \quad (\mathrm{y}_1\mathrm{y}_3)_2 = (\mathrm{y}_2\mathrm{y}_2)_2 = 0\ . \tag{3}$$

We define R to be $1/2$ times the determinant of the three binary quadrics y_i for $i = 1, 2, 3$ with respect to the basis $x^2, x, 1$. If one extends the operation of Überschiebung by product rule [3, p. 317], then R can be re-written as

$$R = -(\mathrm{y}_1\mathrm{y}_2)_1\,(\mathrm{y}_2\mathrm{y}_3)_1\,(\mathrm{y}_3\mathrm{y}_1)_1, \tag{4}$$

It is then obvious that under the operation $f(x) \mapsto \tilde{f}(x) = f(-x)$ the determinant R changes its sign, i.e., $R(f) \mapsto R(\tilde{f}) = -R(f)$.
A straightforward calculation shows that

$$R^2 = \frac{1}{2} \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}, \tag{5}$$

Like the coefficients A_{ij}, R^2 is invariant under the operation $f(x) \mapsto \tilde{f}(x)$ and must be a polynomial in (I_2, I_4, I_6, I_{10}).

Lemma

We have the following statements:

1. *R^2 is a order 30 invariant of binary sextics expressed as a polynomial in (I_2, I_4, I_6, I_{10}) as in* [4, Eq. (17)].
2. *The locus of curves $\mathfrak{p} \in \mathcal{M}_2$ such that $V_4 \hookrightarrow Aut(\mathfrak{p})$ is a two-dimensional irreducible rational subvariety of $\mathcal{M}_2$ given by the equation $R^2 = 0$ and a birational parametrization given by the u, v-invariants as in* [4, Thm. 1].

From now on we will denote $I_{30} := R^2$.

Cubic

Similarly, there is also a cubic curve given by the equation

$$\mathcal{T}: \quad \sum_{1\le i,j,k\le 3} a_{ijk}\, X_i X_j X_k = 0\,,$$

where a_{ijk} are of order zero, invariant under $f(x) \mapsto \tilde{f}(x)$, and given by

$$a_{ijk} = (f, \mathrm{y}_i)_2\, (f, \mathrm{y}_j)_2\, (f, \mathrm{y}_k)_2\ . \tag{6}$$

The coefficients a_{ijk} are given explicitly as follows:

$$\begin{aligned}
36\, a_{111} &= 8(A^2C - 6BC + 9D),\\
36\, a_{112} &= 4(2B^3 + 4ABC + 12C^2 + 3AD),\\
36\, a_{113} &= 36\, a_{122} = 4(AB^3 + 4/3\, A^2BC + 4B^2C + 6AC^2 + 3BD),\\
36\, a_{123} &= 2(2B^4 + 4AB^2C + 4/3\, A^2C^2 + 4BC^2 + 3ABD + 12CD),\\
36\, a_{133} &= 2\left(AB^4 + 4/3\, A^2B^2C + 16/3\, B^3C + 26/3\, ABC^2 + 8C^3 + 3B^2D + 2ACD\right),\\
36\, a_{222} &= 4(3B^4 + 6AB^2C + 8/3\, A^2C^2 + 2BC^2 - 3CD),\\
36\, a_{223} &= 2(-2/3\, B^3C - 4/3\, ABC^2 - 4C^3 + 9B^2D + 8ACD),\\
36\, a_{233} &= 2(B^5 + 2AB^3C + 8/9\, A^2BC^2 + 2/3\, B^2C^2 - BCD + 9D^2),\\
36\, a_{333} &= -2B^4C - 4AB^2C^2 - 16/9\, A^2C^3 - 4/3\, BC^3 + 9B^3D + 12ABCD + 20C^2D.
\end{aligned} \tag{7}$$

Mestre's method

The intersection of the conic $\mathcal{Q}$ with the cubic $\mathcal{T}$ consists of six points which are the zeroes of a polynomial $f(t)$. Hence, the affine equation of the curve corresponding to $\mathfrak{p}$ is given by $y^2 = f(t)$.

Lemma

There exists a model of $\mathcal{C}$ defined over k if $\mathcal{Q}(k) \neq \emptyset$.

If $\mathcal{Q}$ has a rational point over k, then this leads to a parametrization

$$(h_1(t), h_2(t), h_3(t))$$

Substitute X_1, X_2, X_3 by $h_1(t), h_2(t), h_3(t)$ in the cubic $\mathcal{T}$ and we get the degree 6 polynomial $f(t)$. However, if the conic has no rational point or

$$R^2 = \det M = I_{30} = 0$$

the method obviously fails. This locus is parametrized by dihedral invariants u and v. In this case the equation of the curve is given in [5, Lemma 4] and [6, Thm. 3]

A universal curve

We start with $\mathfrak{p} = [j_1, j_2, j_3, 1]$. The plane conic $\mathcal{Q}$ is

$$A_{11}\,X_1^2 + A_{22}\,X_2^2 + A_{33}\,X_3^2 + 2A_{12}\,X_1X_2 + 2A_{13}\,X_1X_3 + 2A_{23}\,X_2X_3 = 0\ . \qquad (8)$$

Over the field extension $\mathbb{Q}[d]$, where d is given by

$$d^2 = -2\,A_{22}\,R^2 = -A_{22}\ \det M, \qquad (9)$$

we can re-express A_{11} in terms of d and the other coefficients as

$$A_{11} = -\frac{d^2}{(A_{22}A_{33} - A_{23}^2)\,A_{22}} + \frac{A_{12}^2A_{33} - 2A_{12}A_{13}A_{23} + A_{13}^2A_{22}}{A_{22}A_{33} - A_{23}^2}\ . \qquad (10)$$

A rational point $[X_1^{(0)} : X_2^{(0)} : X_3^{(0)}]$ of the conic $\mathcal{Q}$ over $\mathbb{Q}[d, A, B, C, D]$.

$$\begin{aligned} X_1^{(0)} &= A_{22}(A_{22}A_{33} - A_{23}^2)\ , \\ X_2^{(0)} &= \mp A_{23}d - A_{22}\left(A_{12}A_{33} - A_{13}A_{23}\right)\ , \\ X_3^{(0)} &= A_{22}\ \left(\pm d + A_{12}A_{23} - A_{13}A_{22}\right)\ . \end{aligned} \qquad (11)$$

We substitute

$$X_3^{(0)} X_2 = X_2^{(0)} X_3 + t \left(X_3^{(0)} X_1 - X_1^{(0)} X_3 \right) \tag{12}$$

into Equation (8). One of the roots of this quadratic, since it must be satisfied if $[X_1 : X_2 : X_3] = [X_1^{(0)} : X_2^{(0)} : X_3^{(0)}]$. The second root is given by

$$\begin{aligned}
X_1 =& A_{22}^3 (A_{22}A_{33} - A_{23}^2)^2\, t^2 + 2A_{12}A_{22}^2 (A_{22}A_{33} - A_{23}^2)\, t + A_{22}(A_{22}A_{33} - A_{23}^2) \\
& \left(A_{12}^2 A_{22}A_{33} - 2A_{12}A_{13}A_{22}A_{23} + A_{13}^2 A_{22}^2 \pm 2(A_{12}A_{23} - A_{13}A_{22})\, d + d^2 \right), \\
X_2 =& A_{22}^2 (A_{23}^2 - A_{22}A_{33})(A_{12}A_{22}A_{33} - A_{13}A_{22}A_{23} \pm A_{23} d)\, t^2 + 2A_{22}(A_{22}A_{33} - A_{23}^2) \\
& (-A_{12}^2 A_{22}A_{33} + A_{12}A_{13}A_{22}A_{23} \mp A_{13}A_{22} d + d^2)\, t \\
& + (A_{12}A_{22}A_{33} - A_{13}A_{22}A_{23} \pm A_{23} d)\, (-A_{12}^2 A_{22}A_{33} + 2A_{12}A_{13}A_{22}A_{23} - A_{13}^2 A_{22}^2 + d^2), \\
X_3 =& A_{22}^3 (A_{22}A_{33} - A_{23}^2)(A_{12}A_{23} - A_{13}A_{22} \pm d)\, t^2 + 2A_{12}A_{22}^2 (A_{22}A_{33} - A_{23}^2)(A_{12}A_{23} - A_{13}A_{22} \\
& - A_{22}(A_{12}A_{23} - A_{13}A_{22} \pm d)\, (-A_{12}^2 A_{22}A_{33} + 2A_{12}A_{13}A_{22}A_{23} - A_{13}^2 A_{22}^2 + d^2).
\end{aligned}$$

Using $A_{13} = A_{22}$ and $d^2 = -2\,A_{22}\,R^2$ the point $[X_1 : X_2 : X_3]$ is easily shown to be equivalent to

$$\begin{aligned}
X_1 &= A_{22}^2(A_{22}A_{33} - A_{23}^2)^2\,t^2 + 2A_{12}A_{22}(A_{22}A_{33} - A_{23}^2)\,t \\
&- A_{11}A_{22}^2A_{33} + A_{11}A_{22}A_{23}^2 + 2A_{12}^2A_{22}A_{33} - 4A_{12}A_{22}^2A_{23} \\
&\pm 2\,(A_{12}A_{23} - A_{22})\,d, \\
X_2 &= -A_{22}(A_{12}A_{22}A_{33} - A_{22}^2A_{23} \pm A_{23}d)\,t^2 \\
&- 2A_{22}(A_{11}A_{22}A_{33} - A_{11}A_{23}^2 + A_{12}A_{22}A_{23} - A_{22}^3 \pm A_{22}d)\,t \\
&- A_{11}(A_{12}A_{22}A_{33} - A_{22}^2A_{23} \pm A_{23}d), \\
X_3 &= A_{22}^2(A_{12}A_{23} - A_{22}^2 \pm d)\,t^2 + 2A_{12}A_{22}(A_{12}A_{23} - A_{22}^2 \pm d)\,t \\
&+ A_{11}A_{22}(A_{12}A_{23} - A_{22} \pm d).
\end{aligned} \tag{13}$$

Equations (13) give for any $t \in \mathbb{Q}$ a rational parametrization of the conic $\mathcal{Q}$ over $\mathbb{Q}[d, A, B, C, D]$.

Similarly, associated to the coefficients (a_{ijk}) in Equation (7) is a plane cubic curve $\mathcal{T}$ in the variables $[X_1 : X_2 : X_3] \in \mathbb{P}^3$ given by

$$\begin{aligned} & a_{111}\, X_1^3 + a_{222}\, X_2^3 + a_{333}\, X_3^3 + 6\, a_{123}\, X_1 X_2 X_3 \\ + & 3\, a_{112}\, X_1^2 X_2 + 3\, a_{113}\, X_1^2 X_3 + 3\, a_{122}\, X_1 X_2^2 + 3\, a_{223}\, X_2^2 X_3 \\ + & 3\, a_{133}\, X_1 X_3^2 + 3\, a_{233}\, X_2 X_3^2 = 0\ . \end{aligned} \tag{14}$$

Substituting the rational parametrization of the conic $\mathcal{Q}$ from Equations (13) into the cubic $\mathcal{T}$ in Equation (14), one obtains the ramificattion locus of sextic curve. The ramification locus is equivalent to

$$0 = \sum_{i=0}^{6} \underbrace{18^{-\lfloor \frac{i+1}{2} \rfloor}\, \kappa_i\, \left(\delta_i\, (54D)^{\lfloor \frac{i+1}{2} \rfloor} \pm 54 \cdot 3^{\lfloor \frac{(i-3)^2}{2} \rfloor - 3 \lfloor \frac{(i-3)^2}{5} \rfloor}\, \epsilon_i\, (54D)^{\lfloor \frac{i}{2} \rfloor}\, d \right)}_{=:\, a^{\pm}_{6-i}}\, t^i\ , \tag{15}$$

where δ_i, ϵ_i are irreducible polynomials in $\mathbb{Z}[A, B, C, D]$ and $\kappa_i = 1, 12, 15B$, 360, 15, 12, 1 for $i = 0, \ldots, 6$ such that $a^{\pm}_{6-i} \in \mathbb{Q}[d, A, B, C, D]$.

(A, B, C, D) are given as polynomial in terms of the invariants (I_2, I_4, I_6, I_{10}). Thus, we can express all coefficients of the sextic as polynomials in $\mathbb{Q}[d, I_2, I_4, I_6, I_{10}]$, and we have

$$d^2 = \frac{I_{30}^2}{2^{11}3^{27}5^{30}}(9I_2^5+700I_2^3I_4-3600I_2^2I_6-12400I_2I_4^2+48000I_4I_6+10800000I_{10}). \tag{16}$$

Notice that d^2 has two significant factors: one is I_{30}^2 which correspond exactly to the locus of the curves with extra involutions, and the other one is the Clebsch invariant D. Next we have our main result:

Theorem

For every point $\mathfrak{p} \in \mathcal{M}_2$ such that $\mathfrak{p} \in \mathcal{M}_2(k)$, for some number field K, there is a pair of genus-two curves $\mathcal{C}^{\pm}$ given by

$$\mathcal{C}^{\pm} : \quad y^2 = \sum_{i=0}^{6} a_{6-i}^{\pm} x^i \ ,$$

corresponding to $\mathfrak{p}$, such that $a_i^{\pm} \in K(d)$, $i = 0, \dots, 6$ as given explicitly in Equation (45).

Corollary

Let j_1, j_2, j_3 be transcendentals. There exists a pair of genus-two curves $\mathcal{C}^{\pm}$ defined over $\mathbb{Q}(j_1, j_2, j_3)[d]$ such that

$$j_1(\mathcal{C}^{\pm}) = j_1, \quad j_2(\mathcal{C}^{\pm}) = j_2, \quad j_3(\mathcal{C}^{\pm}) = j_3,$$

where d^2 is given in terms of (j_1, j_2, j_3) in Equation (44).

Computing expressions for $a_i^{\pm} \in K[d] = \mathbb{Q}(j_1, j_2, j_3)[d]$ is straightforward.

Corollary

The following are true:

1. *If $|Aut(\mathfrak{p})| > 2$, then the curve of defined over the field of moduli.*
2. *If the Clebsch discriminant $D = 0$, then the curve of defined over the field of moduli.*

The universal equation

5. Appendix

In Equations (45) we will display the polynomials δ_i, ϵ_i for $i = 0, \ldots, 6$ that determine the pair of genus-two curves $\mathcal{C}^{\pm}$ g

$$y^2 = \sum_{i=0}^{6} a_{6-i}^{\pm} x^i = \sum_{i=0}^{6} 18^{-\lfloor \frac{i+1}{2} \rfloor} \kappa_i \left(\delta_i \, (54D)^{\lfloor \frac{i+1}{2} \rfloor} \pm 54 \cdot 3^{\lfloor \frac{(i-3)^2}{2} \rfloor - 3 \lfloor \frac{(i-3)^2}{5} \rfloor} \, \epsilon_i \, (54D)^{\lfloor \frac{i}{2} \rfloor} \, d \right) x^i \tag{43}$$

where (A, B, C, D) are the Clebsch invariants and $\kappa_i = 1, 12, 15B, 360, 15, 12, 1$ for $i = 0, \ldots, 6$. The Clebsch invariants ar polynomial in terms of (I_2, I_4, I_6, I_{10}) in Equations (17). The square d^2 is given in terms of (j_1, j_2, j_3) by

$$\begin{aligned} d^2 = \frac{I_2^{20}}{2^{22} 3^{36} 5^{30} j_1^9} \Big(& j_2^4 j_1^3 - 12\, j_2^3 j_3 j_1^3 + 54\, j_2^2 j_3^2 j_1^3 - 108\, j_2 j_3^3 j_1^3 + 81\, j_3^4 j_1^3 + 78\, j_2^5 j_1^2 - 1332\, j_2^4 j_3 j_1^2 + 8910\, j_2^3 j_3^2 j_1^2 - 29376\, j_2^2 j_3^3 j_1^2 + 47952\, j_2 j_3^4 j_1^2 - 311 \\ & - 159\, j_2^6 j_1 + 1728\, j_2^5 j_3 j_1 - 6048\, j_2^4 j_3^2 j_1 + 6912\, j_2^3 j_3^3 j_1 + 80\, j_2^7 - 384\, j_2^6 j_3 - 972\, j_2^2 j_1^4 + 5832\, j_2 j_3 j_1^4 - 8748\, j_3^2 j_1^4 - 77436\, j_2^3 j_1^3 + 870912\, j_2^2 j_3 j_1^3 \\ & - 3090960\, j_2 j_3^2 j_1^3 + 3499200\, j_3^3 j_1^3 + 592272\, j_2^4 j_1^2 - 4743360\, j_2^3 j_3 j_1^2 + 9331200\, j_2^2 j_3^2 j_1^2 - 41472\, j_2^5 j_1 + 236196\, j_1^5 + 19245600\, j_2 j_1^4 - 104976000\, j_3 j_1^4 \\ & - 507384000\, j_2^2 j_1^3 + 2099520000\, j_2 j_3 j_1^3 + 125971200000\, j_1^4 \Big) \left(9\, j_1^2 + 700\, j_2 j_1 - 3600\, j_3 j_1 - 12400\, j_2^2 + 48000\, j_2 j_3 + 10800000\, j_1 \right). \end{aligned} \tag{44}$$

The irreducible polynomials δ_i, ϵ_i in $\mathbb{Z}[A, B, C, D]$ for $i = 0, \ldots, 6$ are given by

$$\begin{aligned} \delta_6 = & -2048\, A^5 B^3 C^5 - 9216\, A^4 B^5 C^4 - 16128\, A^3 B^7 C^3 - 13824\, A^2 B^9 C^2 - 5832\, A B^{11} C - 972\, B^{13} - 9216\, A^4 B^2 C^6 - 36096\, A^3 B^4 C^5 \\ & - 50112\, A^2 B^6 C^4 - 29808\, A B^8 C^3 - 6480\, B^{10} C^2 - 6912\, A^4 B C^5 D - 35712\, A^3 B^3 C^4 D - 13824\, C^7 A^3 B - 55728\, A^2 B^5 C^3 D - 48384\, C^6 A^2 B^3 \\ & - 34992\, A B^7 C^2 D - 49248\, C^5 A B^5 - 7776\, B^9 C D - 15552\, C^4 B^7 + 25920\, A^3 B^2 C^3 D^2 - 10368\, A^3 C^6 D + 54432\, A^2 B^4 C^2 D^2 - 90720\, A^2 B^2 C^5 D \\ & - 6912\, C^8 A^2 + 37908\, A B^6 C D^2 - 114048\, A B^4 C^4 D - 25920\, C^7 A B^2 + 8748\, B^8 D^2 - 38880\, B^6 C^3 D - 15552\, C^6 B^4 + 136080\, A^2 B C^4 D^2 \\ & + 208008\, A B^3 C^3 D^2 - 108864\, A B C^6 D + 79704\, B^5 C^2 D^2 - 77760\, B^3 C^5 D - 5184\, C^8 B + 7776\, A^2 C^3 D^3 + 46656\, A B^2 C^2 D^3 + 139968\, A C^5 D^2 \\ & + 34992\, B^4 C D^3 + 116640\, B^2 C^4 D^2 - 62208\, C^7 D - 52488\, A B C D^4 - 19683\, B^3 D^4 + 23328\, B C^3 D^3 - 139968\, C^2 D^4, \\ \epsilon_6 = & -128\, A^3 B^2 C^3 - 288\, A^2 B^4 C^2 - 216\, A B^6 C - 54\, B^8 - 384\, C^4 A^2 B - 576\, C^3 A B^3 - 216\, B^5 C^2 - 48\, A^2 C^3 D + 108\, A B^2 C^2 D \\ & - 288\, C^5 A + 108\, B^4 C D - 216\, B^2 C^4 + 324\, A B C D^2 + 243\, B^3 D^2 + 288\, B C^3 D + 864\, C^2 D^2, \\ \delta_5 = & \; 1024\, A^5 B^4 C^4 + 3072\, A^4 B^6 C^3 + 3456\, A^3 B^8 C^2 + 1728\, A^2 B^{10} C + 324\, A B^{12} + 8064\, A^4 B^3 C^5 + 20352\, A^3 B^5 C^4 + 18648\, A^2 B^7 C^3 \\ & + 7236\, A B^9 C^2 + 972\, C B^{11} - 6912\, A^4 B^2 C^4 D - 25056\, A^3 B^4 C^3 D + 20736\, C^6 A^3 B^2 - 32832\, A^2 B^6 C^2 D + 38880\, C^5 A^2 B^4 - 18630\, A B^8 C D \\ & + 23544\, C^4 A B^6 - 3888\, B^{10} D + 4536\, C^3 B^8 - 5184\, A^3 B^3 C^2 D^2 - 17712\, A^3 B C^5 D - 7776\, A^2 B^5 C D^2 - 56376\, A^2 B^3 C^4 D + 21600\, C^7 A^2 B \end{aligned} \tag{45}$$

$$
\begin{aligned}
&- 2916\,AB^7D^2 - 53784\,AB^5C^3D + 24624\,C^6AB^3 - 16092\,B^7C^2D + 6480\,C^5B^5 - 3888\,A^3C^4D^2 - 46656\,A^2B^2C^3D^2 - 11664\,A^2C^6D \\
&- 51030\,AB^4C^2D^2 - 40824\,AB^2C^5D + 7776\,C^8A - 13608\,B^6CD^2 - 23328\,B^4C^4D + 2592\,C^7B^2 + 48600\,A^2BC^2D^3 + 83835\,AB^3CD^3 \\
&- 78732\,ABC^4D^2 + 34992\,B^5D^3 - 49572\,B^3C^3D^2 - 11664\,BC^6D + 6561\,AB^2D^4 + 96228\,AC^3D^3 + 97686\,B^2C^2D^3 - 52488\,C^5D^2 \\
&+ 41553\,BCD^4 - 78732\,D^5, \\
\epsilon_5 &= 128\,A^3B^4C^3 + 288\,A^2B^6C^2 + 216\,AB^8C + 54\,B^{10} + 576\,A^2B^3C^4 + 864\,AB^5C^3 + 324\,B^7C^2 + 1296\,A^2B^2C^3D + 2052\,AB^4C^2D \\
&+ 864\,C^5AB^2 + 810\,B^6CD + 648\,C^4B^4 - 3456\,A^2BC^2D^2 - 5508\,AB^3CD^2 + 3240\,ABC^4D - 2187\,B^5D^2 + 2592\,B^3C^3D + 432\,C^6B \\
&- 4860\,AC^3D^2 - 4050\,B^2C^2D^2 + 1944\,C^5D - 243\,BCD^3 + 8748\,D^4, \\
\delta_4 &= 2048\,A^6B^4C^5 + 7168\,A^5B^6C^4 + 9984\,A^4B^8C^3 + 6912\,A^3B^{10}C^2 + 2376\,A^2B^{12}C + 324\,AB^{14} + 15360\,A^5B^3C^6 + 40704\,A^4B^5C^5 \\
&+ 38208\,A^3B^7C^4 + 13392\,A^2B^9C^3 - 648\,B^{13}C - 20736\,A^5B^2C^5D - 93312\,A^4B^4C^4D + 41472\,A^4C^7B^2 - 167184\,A^3B^6C^3D \\
&+ 66816\,A^3C^6B^4 - 149040\,A^2B^8C^2D + 16416\,A^2C^5B^6 - 66096\,AB^{10}CD - 18144\,AB^8C^4 - 11664\,B^{12}D - 7776\,B^{10}C^3 - 5184\,A^4B^3C^3D^2 \\
&- 51840\,A^4BC^6D - 12960\,A^3B^5C^2D^2 - 194400\,A^3B^3C^5D + 48384\,A^3BC^8 - 10692\,A^2B^7CD^2 - 272160\,A^2B^5C^4D + 8640\,A^2B^3C^7 \\
&- 2916\,AB^9D^2 - 168480\,AB^7C^3D - 62208\,AB^5C^6 - 38880\,B^9C^2D - 31104\,B^7C^5 - 10368\,A^4C^5D^2 - 81648\,A^3B^2C^4D^2 - 31104\,A^3C^7D \\
&- 141912\,A^2B^4C^3D^2 - 93312\,A^2B^2C^6D + 20736\,C^9A^2 - 88128\,AB^6C^2D^2 - 93312\,AB^4C^5D - 57024\,C^8AB^2 - 17496\,B^8CD^2 - 31104\,B^6C^4D \\
&- 51840\,C^7B^4 + 132192\,A^3BC^3D^3 + 369360\,A^2B^3C^2D^3 - 139968\,A^2BC^5D^2 + 344088\,AB^5CD^3 - 221616\,AB^3C^4D^2 + 104976\,B^7D^3 \\
&- 81648\,B^5C^3D^2 - 31104\,C^9B + 256608\,A^2C^4D^3 + 6561\,AB^4D^4 + 501552\,AB^2C^3D^3 - 139968\,AC^6D^2 + 268272\,B^4C^2D^3 - 139968\,B^2C^5D^2 \\
&+ 52488\,ABC^2D^4 + 91854\,B^3CD^4 - 69984\,BC^4D^3 - 209952\,ACD^5 - 236196\,B^2D^5, \\
\epsilon_4 &= -128\,A^4B^3C^3 - 288\,A^3B^5C^2 - 216\,A^2B^7C - 54\,AB^9 + 576\,A^2B^4C^3 + 864\,AB^6C^2 + 324\,B^8C - 1584\,A^3BC^3D - 3924\,A^2B^3C^2D \\
&+ 864\,A^2BC^5 - 3348\,AB^5CD + 2376\,AB^3C^4 - 972\,B^7D + 1296\,B^5C^3 + 324\,A^2B^2CD^2 - 2160\,A^2C^4D + 243\,AB^4D^2 - 2808\,AB^2C^3D \\
&+ 864\,C^6A - 1080\,B^4C^2D + 1296\,C^5B^2 + 972\,ABC^2D^2 + 486\,B^3CD^2 + 1296\,BC^4D + 3888\,ACD^3 + 4374\,B^2D^3, \\
\delta_3 &= -512\,A^6B^2C^6 - 3456\,A^5B^4C^5 - 8864\,A^4B^6C^4 - 11464\,A^3B^8C^3 - 8028\,A^2B^{10}C^2 - 2916\,AB^{12}C - 432\,B^{14} + 384\,A^5B^3C^4D \\
&- 1536\,A^5BC^7 + 864\,A^4B^5C^3D - 11904\,A^4B^3C^6 + 648\,A^3B^7C^2D - 28320\,A^3B^5C^5 + 162\,A^2B^9CD - 29976\,A^2B^7C^4 - 14832\,AB^9C^3 \\
&- 2808\,B^{11}C^2 - 192\,A^5C^6D + 3024\,A^4B^2C^5D - 1152\,A^4C^8 + 6120\,A^3B^4C^4D - 16416\,A^3B^2C^7 + 3768\,A^2B^6C^3D - 35856\,A^2B^4C^6 \\
&+ 720\,AB^8C^2D - 27936\,AB^6C^5 - 7344\,B^8C^4 + 2592\,A^4BC^4D^2 + 15120\,A^3B^3C^3D^2 + 10224\,A^3BC^6D + 25434\,A^2B^5C^2D^2 + 12312\,A^2B^3C^5D \\
&- 12960\,A^2BC^8 + 16848\,AB^7CD^2 + 3024\,AB^5C^4D - 22464\,AB^3C^7 + 3888\,B^9D^2 - 360\,B^7C^3D - 9504\,B^5C^6 - 972\,A^3B^2C^2D^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 6048\, A^3C^5D^2 - 729\, A^2B^4CD^3 + 44388\, A^2B^2C^4D^2 + 7776\, A^2C^7D + 58968\, AB^4C^3D^2 - 2592\, AB^2C^6D - 5184\, AC^9 + 21600\, B^6C^2D^2 \\
&- 5184\, B^4C^5D - 5184\, B^2C^8 - 14580\, A^2BC^3D^3 - 15552\, AB^3C^2D^3 + 34992\, ABC^5D^2 - 3888\, B^5CD^3 + 29160\, B^3C^4D^2 - 7776\, BC^7D \\
&- 3888\, A^2C^2D^4 - 16767\, AB^2CD^4 - 29160\, AC^4D^3 - 8748\, B^4D^4 - 20412\, B^2C^3D^3 + 11664\, C^6D^2 - 24786\, BC^2D^4 + 17496\, CD^5, \\
\epsilon_3 &= -512\, A^5B^3C^4 - 1536\, A^4B^5C^3 - 1728\, A^3B^7C^2 - 864\, A^2B^9C - 162\, AB^{11} - 2304\, A^4C^5B^2 - 4416\, A^3C^4B^4 - 2160\, A^2C^3B^6 + 324\, AC^2B^8 \\
&+ 324\, CB^{10} - 1728\, A^4BC^4D - 8064\, A^3B^3C^3D - 3456\, A^3BC^6 - 12636\, A^2B^5C^2D - 1728\, A^2B^3C^5 - 8262\, AB^7CD + 3240\, AB^5C^4 - 1944\, B^9D \\
&+ 1944\, B^7C^3 + 1296\, A^3B^2C^2D^2 - 2592\, A^3C^5D + 1944\, A^2B^4CD^2 - 14904\, A^2B^2C^4D - 1728\, C^7A^2 + 729\, AB^6D^2 - 18792\, AB^4C^3D \\
&+ 3888\, C^6AB^2 - 6804\, B^6C^2D + 3888\, C^5B^4 + 12636\, A^2BC^3D^2 + 17982\, AB^3C^2D^2 - 9720\, ABC^5D + 6318\, B^5CD^2 - 7776\, B^3C^4D + 2592\, BC^7 \\
&+ 3888\, A^2C^2D^3 + 15309\, AB^2CD^3 + 17496\, AC^4D^2 + 8748\, B^4D^3 + 14580\, B^2C^3D^2 - 3888\, C^6D + 16038\, BC^2D^3 - 17496\, CD^4, \\
\delta_2 &= 6144\, A^7B^4C^5 + 23552\, A^6B^6C^4 + 36096\, A^5B^8C^3 + 27648\, A^4B^{10}C^2 + 10584\, A^3B^{12}C + 1620\, A^2B^{14} + 39936\, A^6C^6B^3 + 80640\, A^5B^5C^5 \\
&- 17856\, A^4B^7C^4 - 168432\, A^3B^9C^3 - 168048\, A^2B^{11}C^2 - 68688\, AB^{13}C - 10368\, B^{15} - 34560\, A^6B^2C^5D - 114048\, A^5B^4C^4D + 96768\, A^5C^7B^2 \\
&- 140400\, A^4B^6C^3D - 76032\, A^4B^4C^6 - 76464\, A^3B^8C^2D - 733536\, A^3B^6C^5 - 15552\, A^2B^{10}CD - 981504\, A^2B^8C^4 - 515808\, AB^{10}C^3 \\
&- 97200\, B^{12}C^2 - 36288\, A^5B^3C^3D^2 - 93312\, A^5BC^6D - 80352\, A^4B^5C^2D^2 + 75168\, A^4B^3C^5D + 103680\, A^4C^8B - 59292\, A^3B^7CD^2 \\
&+ 803520\, A^3B^5C^4D - 675648\, A^3B^3C^7 - 14580\, A^2B^9D^2 + 1143072\, A^2B^7C^3D - 1829952\, A^2B^5C^6 + 632448\, AB^9C^2D - 1425600\, AB^7C^5 \\
&+ 124416\, B^{11}CD - 357696\, B^9C^4 - 20736\, A^5C^5D^2 - 423792\, A^4B^2C^4D^2 - 62208\, A^4C^7D - 740664\, A^3B^4C^3D^2 + 1041984\, A^3B^2C^6D \\
&+ 41472\, A^3C^9 - 434808\, A^2B^6C^2D^2 + 2799360\, A^2B^4C^5D - 1104192\, A^2B^2C^8 - 81648\, AB^8CD^2 + 2317248\, AB^6C^4D - 1710720\, AB^4C^7 \\
&+ 622080\, B^8C^3D - 642816\, B^6C^6 + 256608\, A^4BC^3D^3 + 412128\, A^3B^3C^2D^3 - 855360\, A^3BC^5D^2 + 163296\, A^2B^5CD^3 - 746496\, A^2B^3C^4D^2 \\
&+ 1399680\, A^2BC^7D + 128304\, AB^5C^3D^2 + 2426112\, AB^3C^6D - 746496\, ABC^9 + 159408\, B^7C^2D^2 + 1026432\, B^5C^5D - 559872\, B^3C^8 \\
&+ 52488\, A^3B^2CD^4 + 419904\, A^3C^4D^3 + 32805\, A^2B^4D^4 - 1539648\, A^2B^2C^3D^3 - 699840\, A^2C^6D^2 - 2939328\, AB^4C^2D^3 + 489888\, AB^2C^5D^2 \\
&+ 559872\, AC^8D - 1119744\, B^6CD^3 + 629856\, B^4C^4D^2 + 559872\, B^2C^7D - 186624\, C^{10} + 1364688\, A^2BC^2D^4 + 1758348\, AB^3CD^4 \\
&- 4199040\, ABC^4D^3 + 629856\, B^5D^4 - 3359232\, B^3C^3D^3 + 839808\, BC^6D^2 - 419904\, A^2CD^5 + 2519424\, AC^3D^4 + 1810836\, B^2C^2D^4 \\
&- 1679616\, C^5D^3 + 2519424\, BCD^5 - 1889568\, D^6, \\
\epsilon_2 &= 128\, A^5B^3C^3 + 288\, A^4B^5C^2 + 216\, A^3B^7C + 54\, A^2B^9 + 1152\, A^4C^4B^2 + 2112\, A^3C^3B^4 + 1224\, A^2C^2B^6 + 216\, ACB^8 - 3024\, A^4BC^3D \\
&- 5868\, A^3B^3C^2D + 2592\, A^3C^5B - 3564\, A^2B^5CD + 1944\, A^2C^4B^3 - 648\, AB^7D - 864\, AC^3B^5 - 648\, B^7C^2 - 324\, A^3B^2CD^2 - 4320\, A^3C^4D \\
&- 243\, A^2B^4D^2 + 11664\, A^2B^2C^3D + 1728\, A^2C^6 + 24624\, AB^4C^2D - 2592\, AC^5B^2 + 9936\, B^6CD - 2592\, B^4C^4 - 13608\, A^2BC^2D^2
\end{aligned}
$$

$$
\begin{aligned}
&- 20412\,AB^3CD^2 + 34992\,ABC^4D - 7776\,B^5D^2 + 27216\,B^3C^3D - 2592\,BC^6 + 7776\,A^2CD^3 + 2916\,AB^2D^3 - 23328\,AC^3D^2 - 20412\,B^2C^2D^2 \\
&+ 15552\,C^5D - 29160\,BCD^3 + 34992\,D^4, \\
\delta_1 &= -1024\,A^7B^6C^4 - 3072\,A^6B^8C^3 - 3456\,A^5B^{10}C^2 - 1728\,A^4B^{12}C - 324\,A^3B^{14} + 3072\,A^7C^6B^3 + 31104\,A^6B^5C^5 + 120384\,A^5B^7C^4 \\
&+ 209496\,A^4B^9C^3 + 181764\,A^3B^{11}C^2 + 77436\,A^2B^{13}C + 12960\,B^{15}A + 41472\,A^6B^4C^4D + 82944\,A^6C^7B^2 + 130464\,A^5B^6C^3D \\
&+ 705024\,A^5C^6B^4 + 153360\,A^4B^8C^2D + 2112480\,A^4C^5B^6 + 79866\,A^3B^{10}CD + 2975832\,A^3B^8C^4 + 15552\,A^2B^{12}D + 2151144\,A^2B^{10}C^3 \\
&+ 772416\,AB^{12}C^2 + 108864\,B^{14}C + 38016\,A^6BC^6D + 5184\,A^5B^5C^2D^2 + 352080\,A^5B^3C^5D + 228096\,A^5C^8B + 7776\,A^4B^7CD^2 \\
&+ 456840\,A^4B^5C^4D + 2178144\,A^4C^7B^3 + 2916\,A^3B^9D^2 - 246456\,A^3B^7C^3D + 5903280\,A^3B^5C^6 - 771876\,A^2B^9C^2D + 6724944\,A^2B^7C^5 \\
&- 474336\,AB^{11}CD + 3446064\,AB^9C^4 - 93312\,B^{13}D + 657072\,B^{11}C^3 - 260496\,A^5B^2C^4D^2 + 82944\,A^5C^7D - 972000\,A^4B^4C^3D^2 \\
&+ 382320\,A^4B^2C^6D + 165888\,A^4C^9 - 1404054\,A^3B^6C^2D^2 - 1235736\,A^3B^4C^5D + 2744928\,A^3C^8B^2 - 896184\,A^2B^8CD^2 - 3742848\,A^2B^6C^4D \\
&+ 6757344\,A^2B^4C^7 - 209952\,AB^{10}D^2 - 2978856\,AB^8C^3D + 5632416\,AB^6C^6 - 751680\,B^{10}C^2D + 1531872\,B^8C^5 - 229392\,A^4B^3C^2D^3 \\
&- 1492992\,A^4BC^5D^2 - 359397\,A^3B^5CD^3 - 5794092\,A^3B^3C^4D^2 - 497664\,A^3BC^7D - 139968\,A^2B^7D^3 - 7986924\,A^2B^5C^3D^2 \\
&- 4210704\,A^2B^3C^6D + 1866240\,A^2BC^9 - 4537296\,AB^7C^2D^2 - 5668704\,AB^5C^5D + 3779136\,AB^3C^8 - 886464\,B^9CD^2 - 2072304\,B^7C^4D \\
&+ 1726272\,B^5C^7 - 139968\,A^4C^4D^3 - 6561\,A^3B^4D^4 - 708588\,A^3B^2C^3D^3 - 1819584\,A^3C^6D^2 + 1010394\,A^2B^4C^2D^3 - 8275608\,A^2B^2C^5D^2 \\
&- 419904\,A^2C^8D + 2239488\,AB^6CD^3 - 9710280\,AB^4C^4D^2 - 2729376\,AB^2C^7D + 653184\,C^{10}A + 839808\,B^8D^3 - 3289248\,B^6C^3D^2 \\
&- 1959552\,B^4C^6D + 839808\,C^9B^2 + 1399680\,A^3BC^2D^4 + 2464749\,A^2B^3CD^4 + 1994544\,A^2BC^4D^3 + 1102248\,AB^5D^4 + 7269588\,AB^3C^3D^3 \\
&- 4304016\,ABC^6D^2 + 4199040\,B^5C^2D^3 - 3464208\,B^3C^5D^2 - 419904\,BC^8D + 314928\,A^2B^2D^5 + 3569184\,A^2C^3D^4 + 5616216\,AB^2C^2D^4 \\
&+ 3044304\,AC^5D^3 + 1784592\,B^4CD^4 + 5196312\,B^2C^4D^3 - 1889568\,C^7D^2 - 472392\,ABCD^5 - 1889568\,B^3D^5 + 1495908\,BC^3D^4 \\
&- 1889568\,AD^6 - 2834352\,C^2D^5, \\
\epsilon_1 &= -1024\,A^6B^4C^4 - 3200\,A^5B^6C^3 - 3744\,A^4B^8C^2 - 1944\,A^3B^{10}C - 378\,A^2B^{12} + 1536\,A^5C^5B^3 + 22848\,A^4C^4B^5 + 58944\,A^3B^7C^3 \\
&+ 61668\,B^9C^2A^2 + 29160\,AB^{11}C + 5184\,B^{13} + 10368\,A^5B^2C^4D + 40176\,A^4B^4C^3D + 20736\,A^4C^6B^2 + 56484\,A^3B^6C^2D + 124704\,A^3C^5B^4 \\
&+ 34506\,A^2B^8CD + 211464\,A^2C^4B^6 + 7776\,AB^{10}D + 141264\,AB^8C^3 + 33048\,B^{10}C^2 + 9504\,A^4B^3C^2D^2 + 57024\,A^4BC^5D + 14580\,A^3B^5CD^2 \\
&+ 220968\,A^3B^3C^4D + 38016\,A^3C^7B + 5589\,A^2B^7D^2 + 291600\,A^2B^5C^3D + 183600\,A^2C^6B^3 + 160056\,AB^7C^2D + 215136\,AC^5B^5 + 31104\,B^9CD \\
&+ 73872\,B^7C^4 + 5184\,A^4C^4D^2 - 32076\,A^3B^2C^3D^2 + 62208\,A^3C^6D - 199746\,A^2B^4C^2D^2 + 274104\,A^2B^2C^5D + 20736\,C^8A^2 - 223236\,AB^6CD^2 \\
&+ 291600\,AB^4C^4D + 98496\,C^7AB^2 - 69984\,B^8D^2 + 91368\,B^6C^3D + 64800\,C^6B^4 - 108864\,A^3BC^2D^3 - 190755\,A^2B^3CD^3 - 244944\,A^2BC^4D^2
\end{aligned}
$$

$$
\begin{aligned}
&- 81648\, AB^5 D^3 - 612360\, AB^3 C^3 D^2 + 139968\, ABC^6 D - 312012\, B^5 C^2 D^2 + 93312\, B^3 C^5 D + 15552\, C^8 B - 17496\, A^2 B^2 D^4 - 256608\, A^2 C^3 D^3 \\
&- 352836\, AB^2 C^2 D^3 - 221616\, AC^5 D^2 - 93312\, B^4 C D^3 - 332424\, B^2 C^4 D^2 + 69984\, C^7 D + 157464\, ABCD^4 \\
&+ 209952\, B^3 D^4 - 8748\, BC^3 D^3 + 209952\, AD^5 + 314928\, C^2 D^4, \\
\delta_0 &= 10240\, A^8 B^6 C^5 + 39936\, A^7 B^8 C^4 + 62208\, A^6 B^{10} C^3 + 48384\, A^5 B^{12} C^2 + 18792\, A^4 B^{14} C + 2916\, A^3 B^{16} + 98304\, A^8 B^3 C^7 + 580608\, A^7 B^5 C^6 \\
&+ 1208064\, A^6 B^7 C^5 + 1082688\, A^5 B^9 C^4 + 267984\, A^4 B^{11} C^3 - 220320\, A^3 B^{13} C^2 - 162648\, A^2 B^{15} C - 31104\, AB^{17} + 172800\, A^7 B^4 C^5 D \\
&+ 442368\, A^7 C^8 B^2 + 620928\, A^6 B^6 C^4 D + 1110528\, A^6 C^7 B^4 + 895536\, A^5 B^8 C^3 D - 2439936\, A^5 C^6 B^6 + 649296\, A^4 B^{10} C^2 D - 11284704\, A^4 C^5 B^8 \\
&+ 237168\, A^3 B^{12} CD - 15591456\, A^3 C^4 B^{10} + 34992\, A^2 B^{14} D - 10362816\, A^2 C^3 B^{12} - 3386448\, AB^{14} C^2 - 435456\, B^{16} C + 331776\, A^7 BC^7 D \\
&- 67392\, A^6 B^5 C^3 D^2 + 4648320\, A^6 B^3 C^6 D + 663552\, A^6 C^9 B - 147744\, A^5 B^7 C^2 D^2 + 14486688\, A^5 B^5 C^5 D - 3103488\, A^5 C^8 B^3 \\
&- 107892\, A^4 B^9 CD^2 + 21505824\, A^4 B^7 C^4 D - 25468992\, A^4 C^7 B^5 - 26244\, A^3 B^{11} D^2 + 18424800\, A^3 B^9 C^3 D - 52265088\, A^3 C^6 B^7 \\
&+ 9572256\, A^2 B^{11} C^2 D - 47708352\, A^2 C^5 B^9 + 2846016\, AB^{13} CD - 20497536\, AC^4 B^{11} + 373248\, B^{15} D - 3382560\, B^{13} C^3 - 2519424\, A^6 B^2 C^5 D^2 \\
&+ 497664\, A^6 C^8 D - 7235568\, A^5 B^4 C^4 D^2 + 12845952\, A^5 B^2 C^7 D + 331776\, A^5 C^{10} - 5445144\, A^4 B^6 C^3 D^2 + 31290624\, A^4 B^4 C^6 D \\
&- 10886400\, A^4 C^9 B^2 + 1279152\, A^3 B^8 C^2 D^2 + 31648320\, A^3 B^6 C^5 D - 50258880\, A^3 C^8 B^4 + 2886840\, A^2 B^{10} CD^2 + 17184960\, A^2 B^8 C^4 D \\
&- 74442240\, A^2 C^7 B^6 + 839808\, AB^{12} D^2 + 5520960\, AB^{10} C^3 D - 45567360\, AC^6 B^8 + 886464\, B^{12} C^2 D - 9984384\, B^{10} C^5 - 925344\, A^5 B^3 C^3 D^3 \\
&- 6718464\, A^5 BC^6 D^2 - 1924560\, A^4 B^5 C^2 D^3 + 23001408\, A^4 B^3 C^5 D^2 + 15676416\, A^4 BC^8 D - 1347192\, A^3 B^7 CD^3 + 116301744\, A^3 B^5 C^4 D^2 \\
&+ 9859968\, A^3 B^3 C^7 D - 10948608\, A^3 C^{10} B - 314928\, A^2 B^9 D^3 + 148785984\, A^2 B^7 C^3 D^2 - 20435328\, A^2 B^5 C^6 D - 40217472\, A^2 C^9 B^3 \\
&+ 76667472\, AB^9 C^2 D^2 - 20808576\, AB^7 C^5 D - 42177024\, AC^8 B^5 + 13996800\, B^{11} CD^2 - 4945536\, B^9 C^4 D - 13561344\, B^7 C^7 - 1119744\, A^5 C^5 D^3 \\
&+ 104976\, A^4 B^4 CD^4 - 36181728\, A^4 B^2 C^4 D^3 - 3359232\, A^4 C^7 D^2 + 59049\, A^3 B^6 D^4 - 96799536\, A^3 B^4 C^3 D^3 + 145006848\, A^3 B^2 C^6 D^2 \\
&+ 8584704\, A^3 C^9 D - 104941008\, A^2 B^6 C^2 D^3 + 380642976\, A^2 B^4 C^5 D^2 - 40310784\, A^2 B^2 C^8 D - 3732480\, C^{11} A^2 - 52488000\, AB^8 CD^3 \\
&+ 311148864\, AB^6 C^4 D^2 - 67184640\, AB^4 C^7 D - 12877056\, C^{10} AB^2 - 10077696\, B^{10} D^3 + 81671328\, B^8 C^3 D^2 - 23887872\, B^6 C^6 D \\
&- 7838208\, B^4 C^9 + 11757312\, A^4 BC^3 D^4 + 11354904\, A^3 B^3 C^2 D^4 - 115893504\, A^3 BC^5 D^3 - 6337926\, A^2 B^5 CD^4 - 257191200\, A^2 B^3 C^4 D^3 \\
&+ 193155840\, A^2 BC^7 D^2 - 5668704\, AB^7 D^4 - 203443488\, AB^5 C^3 D^3 + 353139264\, AB^3 C^6 D^2 - 47029248\, ABC^9 D - 55007424\, B^7 C^2 D^3 \\
&+ 151585344\, B^5 C^5 D^2 - 33592320\, B^3 C^8 D - 1119744\, C^{11} B + 1259712\, A^3 B^2 CD^5 + 15116544\, A^3 C^4 D^4 + 708588\, A^2 B^4 D^5 \\
&- 151795296\, A^2 B^2 C^3 D^4 - 116733312\, A^2 C^6 D^3 - 251233812\, AB^4 C^2 D^4 - 159983424\, AB^2 C^5 D^3 + 85660416\, AC^8 D^2 - 89439552\, B^6 CD^4 \\
&- 71803584\, B^4 C^4 D^3 + 95738112\, B^2 C^7 D^2 - 13436928\, C^{10} D + 117153216\, A^2 BC^2 D^5 + 178564176\, AB^3 CD^5 - 319336992\, ABC^4 D^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 68024448\, B^5 D^5 - 268791048\, B^3 C^3 D^4 + 42830208\, BC^6 D^3 - 15116544\, A^2 C D^6 + 11337408\, AB^2 D^6 + 249422976\, AC^3 D^5 \\
&+ 229582512\, B^2 C^2 D^5 - 120932352\, C^5 D^4 + 158723712\, BCD^6 - 136048896\, D^7, \\
\epsilon_0 &= 128\, A^6 B^5 C^3 + 288\, A^5 B^7 C^2 + 216\, A^4 B^9 C + 54\, A^3 B^{11} + 6144\, A^6 B^2 C^5 + 22272\, A^5 B^4 C^4 + 28992\, A^4 B^6 C^3 + 16272\, A^3 B^8 C^2 + 3348\, A^2 B^{10} C \\
&+ 48\, A^5 B^3 C^3 D + 18432\, A^5 C^6 B + 468\, A^4 B^5 C^2 D + 6048\, A^4 C^5 B^3 + 756\, A^3 B^7 CD - 108936\, A^3 C^4 B^5 + 324\, A^2 B^9 D - 178992\, A^2 C^3 B^7 \\
&- 103896\, AC^2 B^9 - 20736\, CB^{11} + 2304\, A^5 C^5 D - 324\, A^4 B^4 CD^2 + 73872\, A^4 B^2 C^4 D + 13824\, A^4 C^7 - 243\, A^3 B^6 D^2 + 237096\, A^3 B^4 C^3 D \\
&- 151200\, A^3 C^6 B^2 + 292392\, A^2 B^6 C^2 D - 460080\, A^2 C^5 B^4 + 157680\, AB^8 CD - 400896\, AC^4 B^6 + 31104\, B^{10} D - 110160\, C^3 B^8 - 51840\, A^4 BC^3 D^2 \\
&- 88020\, A^3 B^3 C^2 D^2 + 235008\, A^3 BC^5 D - 36450\, A^2 B^5 CD^2 + 648000\, A^2 B^3 C^4 D - 228096\, A^2 C^7 B + 589680\, AB^5 C^3 D - 432864\, AC^6 B^3 \\
&+ 174096\, B^7 C^2 D - 191808\, B^5 C^5 - 72576\, A^3 C^4 D^2 - 1458\, A^2 B^4 D^3 + 326592\, A^2 B^2 C^3 D^2 + 212544\, A^2 C^6 D + 615276\, AB^4 C^2 D^2 \\
&+ 427680\, AB^2 C^5 D - 93312\, AC^8 + 238464\, B^6 CD^2 + 225504\, B^4 C^4 D - 108864\, B^2 C^7 - 489888\, A^2 BC^2 D^3 - 833976\, AB^3 CD^3 + 828144\, ABC^4 D^2 \\
&- 326592\, B^5 D^3 + 659016\, B^3 C^3 D^2 - 15552\, BC^6 D + 93312\, A^2 CD^4 - 979776\, AC^3 D^3 - 1032264\, B^2 C^2 D^3 + 373248\, C^5 D^2 - 559872\, BCD^4 \\
&+ 839808\, D^5.
\end{aligned}
$$

An implementation in Sage

```
f=t^6-5*t^4+2*t^2-1; Info(f)
```

```
Initial equation of the curve
t^6 - 5*t^4 + 2*t^2 - 1


Clebsch invariants [A, B, C, D] are:
[-10/3, 1894/1125, -1534/1875, -220634944/56953125]


Igusa-Clebsch invariants as in Magma  [I2, I4, I6, I10] are:
[6400, 861184, 493191168, 5223821082624]


Igusa invariants  [J_2, J_4, J_6, J_10] are:
[400, 3364, 120408, 4981824]


The moduli point for this curve  p=(J2, i1, i2, i3)
(-1, 7569/2500, -3322269/125000, 18915363/80000000000)


The Automorphism group is isomorphic to the group with GapId
[4, 2]


The invariants  u and v are:
[10, 133]
```

$3966289681441136517376316469376919845960722877641746919813283 84/1984245446333497003843504558062704745680093765258789062 5*$

$t^6 -$

$30573250572833626471373749387359269206167271106404793194643456/2645660595111329338458006077416939660906791687011718 75*$

$t^5 +$

$98336693442317455924012012802141722568261366005870627284582 4/352754746014843911794400810322258621454238891601562 5*$

$t^4 -$

$84341674113646761568160038456190944145520379601772471123968/2351698306765626078629338735481724143028259277343 75*$

$t^3 +$

$40631080612905040344276724217405253790052119418617588613 12/156779887117708405241955915698781609535217285156 25*$

$t^2 - 20819767415096954598228565325389778874548957158366511104/20903984949027787365594122093170881271362304687 5*$

$t + 73776744988999047290470748077270101837883013580455936/4645329988672841636798693798482418060302734375$

```
The universal curve  over K is:


3966289681441136517376316469376919845960722877641746919813283 84/1984245446333497
The moduli point matches that of f
(-1, 7569/2500, -3322269/125000, 18915363/80000000000)
```

[1] Oskar Bolza, *On binary sextics with linear transformations into themselves*, Amer. J. Math. **10** (1887), no. 1, 47–70. MR1505464

[2] A. Clebsch and P. Gordan, *Theorie der Abelschen Functionen*, Physica-Verlag, WÃ$\frac{1}{4}$rzburg, 1967. Thesaurus Mathematicae, (Neudrucke wichtiger mathematischer Werke), Band 7. MR0485106

[3] Jean-François Mestre, *Construction de courbes de genre* 2 *Ã partir de leurs modules*, Effective methods in algebraic geometry (Castiglioncello, 1990), 1991, pp. 313–334. MR1106431

[4] Tanush Shaska and Helmut Voelklein, *Elliptic subfields and automorphisms of genus 2 function fields*, Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), 2004, pp. 703–723. MR2037120

[5] Tony Shaska, *Genus 2 curves with* $(3,3)$*-split Jacobian and large automorphism group*, Algorithmic number theory (Sydney, 2002), 2002, pp. 205–218. MR2041085

[6] ______, *Genus two curves with many elliptic subcovers*, Comm. Algebra **44** (2016), no. 10, 4450–4466. MR3508311