From hyperelliptic to superelliptic curves

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September 16, 2017

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Algebraic curves:

An **irreducible projective curve** defined over a field $k = \bar{k}$ is called the set of zeroes of the following irreducible homogenous polynomial $F(x, y, z) \in k[x, y, z]$.
We normally say: Given the curve C

$$C : \quad F(x, y, z) = 0$$

The **coordinate ring** of C is $k[C] := k[x, y, z]/(F)$. The **function field** of C is defined as

$$k(C) := \left\{ \frac{g}{h} \;\middle|\; g, h \in k[C] \text{ are forms of the same degree and } \; h \neq 0 \right\}$$

A **rational map** between two curves

$$\phi : \quad C_1 : \; F_1(x, y, z) = 0 \; \to \; C_2 : \; F_2(x, y, z) = 0$$

is a map given by

$$(x, y, z) \to (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$$

where f_1, f_2, f_3 are homogenous polynomials such that:

1. f_1, f_2, f_3 and all have the same degree.
2. There is a $P \in C_1$ such that not all $f_i(P) = 0$.
3. $F_2\left(f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)\right) = 0$

The map

$$\phi : C_1 \to C_2$$

is **regular** at $P \in C_1$ if $f_i(P) \neq 0$ for at least one i. Moreover, it is called a **morphism** if it is regular in all points $P \in C_1$ and an **isomorphism** if ϕ has an inverse

$$\phi^{-1} : C_2 \to C_1,$$

which is also a morphism.
Without any loss of generality we may assume that our curves are **non-singular**. Then,

1. Any rational map $\phi : C_1 \to C_2$ is a morphism
2. if ϕ is non-constant then ϕ is surjective.

$$\begin{tikzcd} C_1 \arrow[d, "\phi"] & - - - - - - - & k(C_1) \arrow[d, dash] \\ C_2 & & k(C_2) \end{tikzcd}$$

Moreover,

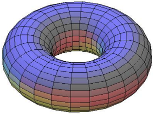
$$C_1 \cong C_2 \iff k(C_1) \cong k(C_2).$$

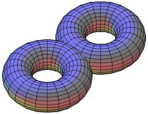
Similarly, we can define these concepts for **affine curves**.

Riemann surfaces

Riemann surfaces can be thought of as ”deformed copies” of the complex plane: locally near every point they look like patches of the complex plane.

Every algebraic curve with coefficients in $\mathbb{C}$ is a **compact Riemann surface**.

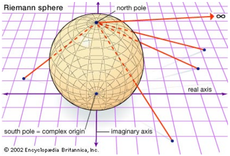




Every compact Riemann surface is a sphere with some handles attached. The number of handles is an important topological invariant called the topological **genus** of the surface.

genus of an algebraic curve = # of handles on the surface.

The most famous Riemann surface of all is the so called Riemann sphere, denoted by $\mathbb{P}^1$.



Every algebraic curve $\mathcal{X}$ is given as a covering $\mathcal{X} \mapsto \mathbb{P}^1$.

When such covering $\mathcal{X} \mapsto \mathbb{P}^1$ has degree 2, then the Riemann surface is called **hyperelliptic**.

Every hyperelliptic curve has equation $y^2 = f(x)$, for some polynomial $f(x)$.

Some examples of curves

We give some examples of some very recognizable families of curves defined over algebraically closed fields of characteristic $\neq 2$ (precise definitions will come later).

An **elliptic curve** is a curve with affine equation

$$y^2 = f(x)$$

where $f(x)$ is a degree 3 or 4 polynomial with nonzero discriminant.

An **hyperelliptic curve** is a curve with affine equation

$$y^2 = f(x)$$

where $\deg f \geq 5$ and discriminant $\Delta_f \neq 0$.

A **superelliptic curve** is a curve with affine equation

$$y^n = f(x)$$

where $n \geq 2$, $\deg f \geq 3$ and discriminant $\Delta_f \neq 0$.

My research program is to, whenever possible, extend the theory of elliptic/hyperelliptic curves to superelliptic curves (i.e. automorphisms, field of moduli versus field of definition, rational points, minimal integral models, etc).

For more details visit algcurves.org where one can find some Sage packages, a database of genus two curves, and profiles of some of my collaborators.

Automorphisms of curves

All examples above have something in common; they all have automorphisms.

Let $\mathcal{X}_g$ denote an algebraic curve of genus $g \geq 2$, defined over $\bar{k} = k$, and $K = k(\mathcal{X}_g)$.

The **automorphism group** $\mathrm{Aut}(\mathcal{X}_g)$ of $\mathcal{X}_g$ is the group of automorphisms of K defined over k. $\mathrm{Aut}(\mathcal{X}_g)$ acts on the finite set of Weierstras points of $\mathcal{X}_g$.

This action is faithful unless $\mathcal{X}_g$ is hyperelliptic, in which case its kernel is the group of order 2 containing the hyperelliptic involution of $\mathcal{X}_g$.

Thus in any case, **Aut**$(\mathcal{X}_g)$ **is a finite group.** This was first proved by **Schwartz**.

$$\begin{array}{c} \mathcal{X}_g \\ \downarrow f \\ \mathcal{X}_h \end{array}$$

The next milestone was **Hurwitz**'s seminal paper [Hur93], where he discovered what is now called the Riemann-Hurwitz formula

$$2(g-1) = 2\deg(f)\,(h-1) + \sum_{P \in \mathcal{X}_g} (e_P - 1)$$

From this he derived what is now known as the **Hurwitz bound**.

$$|\mathrm{Aut}(\mathcal{X}_g)| \;\leq\; 84\;(g-1)$$

Fix a group $G = \mathrm{Aut}(\mathcal{X}_g)$. The coverings $\mathcal{X}_g \mapsto \mathcal{X}_g/G$ for all $g \geq 2$ are studied in [MSSV02]. The space of such covers with fixed signature is a sublocus of $\mathcal{M}_g$. Studying such loci helps us determine a lattice of loci in $\mathcal{M}_g$ (cf. $g = 3, 4$).

Hyperelliptic and superelliptic curves

Let $\mathcal{X}_g$ be a genus g hyperelliptic curve with equation

$$y^2 = f(x),$$

$$\begin{tikzcd} K \arrow[d, dash, "\langle w\rangle"] \arrow[dd, bend right=60, "G"'] \\ k(x) \arrow[d, dash, "\bar{G}=G/\langle w\rangle"] \\ k \end{tikzcd}$$

where $\deg f = 2g+2$. Let $G = \mathrm{Aut}(\mathcal{X}_g)$ and $w : (x, y) \to (-x, y)$ be the hyperelliptic involution. Then, w is central in G.

The group $\bar{G} := G/\langle w\rangle$ is called the **reduced automorphism group** of $\mathcal{X}_g$.
Hence, $\bar{G} \hookrightarrow Aut(k(x)/k) \cong PGL(2, k)$ and $\bar{G}$ is finite.
Hence, $\bar{G}$ it is isomorphic to one of the following: C_n, D_n, A_4, S_4, A_5. Therefore, G is a degree 2 central extensions of $\bar{G}$.

Next, we try to generalize the above to non-hyperelliptic curves.

Let $\mathcal{X}_g$ be a curve and H be a **normal cyclic subgroup of order** n of $G = \mathrm{Aut}(\mathcal{X}_g)$ which fixes a genus 0 space $\mathcal{X}_g/H$.

The group $\bar{G} = G/H$ is called the **reduced automorphism group** of $\mathcal{X}_g$.
We call such curves **superelliptic curves**. They have affine equation

$$\begin{tikzcd} K \arrow[d, dash, "H"] \arrow[dd, bend right=60, "G"'] \\ k(x) \arrow[d, dash, "\bar{G}=G/H"] \\ k \end{tikzcd}$$

$$y^n = f(x)$$

for some polynomial $f(x)$. Then $\tau : (x, y) \to (x, \zeta y)$, where $\zeta^n = 1$, is an automorphism of $\mathcal{X}_g$.

Automorphism groups and equations for hyperelliptic curves

From [Sha03] we have the following:

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#	G	$\overline{G}$	$\delta(G, \mathbf{C})$	δ, n, g	$\mathbf{C} = (C_1, \dots C_r)$	ϕ
1	$\mathbb{Z}_2 \otimes \mathbb{Z}_n$		$\frac{2g+2}{n} - 1$	$n < g+1$	$(n^2, n^2, 2^n, \dots, 2^n)$	
2	$\mathbb{Z}_{2n}$	$\mathbb{Z}_n$	$\frac{2g+1}{n} - 1$		$(n^2, 2n, 2^n, \dots, 2^n)$	(n, n)
3	$\mathbb{Z}_{2n}$		$\frac{2g}{n} - 1$	$n < g$	$(2n, 2n, 2^n, \dots, 2^n)$	
4	$\mathbb{Z}_2 \otimes D_n$		$\frac{g+1}{n}$		$(n^4, 2^{2n}, \dots, 2^{2n})$	
5	V_n		$\frac{g+1}{n} - \frac{1}{2}$		$(n^4, 4^n, 2^{2n}, \dots, 2^{2n})$	
6	D_{2n}	D_n	$\frac{g}{n}$		$((2n)^2, 2^{2n}, \dots, 2^{2n})$	$(2^n, 2^n, n^2)$
7	H_n		$\frac{g+1}{n} - 1$	$n < g+1$	$(4^n, 4^n, n^4, 2^{2n} \dots, 2^{2n})$	
8	U_n		$\frac{g}{n} - \frac{1}{2}$	$g \neq 2$	$(4^n, (2n)^2, 2^{2n}, \dots, 2^{2n})$	
9	G_n		$\frac{g}{n} - 1$	$n < g$	$(4^n, 4^n, (2n)^2, 2^{2n}, \dots, 2^{2n})$	
10	$\mathbb{Z}_2 \otimes A_4$		$\frac{g+1}{6}$		$(3^8, 3^8, 2^{12}, \dots, 2^{12})$	
11	$\mathbb{Z}_2 \otimes A_4$		$\frac{g-1}{6}$		$(3^8, 6^4, 2^{12}, \dots, 2^{12})$	
12	$\mathbb{Z}_2 \otimes A_4$	A_4	$\frac{g-3}{6}$	$\delta \neq 0$	$(6^4, 6^4, 2^{12}, \dots, 2^{12})$	$(2^6, 3^4, 3^4)$
13	$SL_2(3)$		$\frac{g-2}{6}$	$\delta \neq 0$	$(4^6, 3^8, 3^8, 2^{12}, \dots, 2^{12})$	
14	$SL_2(3)$		$\frac{g-4}{6}$		$(4^6, 3^8, 6^4, 2^{12}, \dots, 2^{12})$	
15	$SL_2(3)$		$\frac{g-6}{6}$	$\delta \neq 0$	$(4^6, 6^4, 6^4, 2^{12}, \dots, 2^{12})$	
16	$\mathbb{Z}_2 \otimes S_4$		$\frac{g+1}{12}$		$(3^{16}, 4^{12}, 2^{24}, \dots, 2^{24})$	
17	$\mathbb{Z}_2 \otimes S_4$		$\frac{g-3}{12}$		$(6^8, 4^{12}, 2^{24}, \dots, 2^{24})$	
18	$GL_2(3)$		$\frac{g-2}{12}$		$(3^{16}, 8^6, 2^{24}, \dots, 2^{24})$	
19	$GL_2(3)$	S_4	$\frac{g-6}{12}$		$(6^8, 8^6, 2^{24}, \dots, 2^{24})$	$(2^{12}, 3^8, 4^6)$
20	W_2		$\frac{g-5}{12}$		$(4^{12}, 4^{12}, 3^{16}, 2^{24}, \dots, 2^{24})$	
21	W_2		$\frac{g-9}{12}$		$(4^{12}, 4^{12}, 6^8, 2^{24}, \dots, 2^{24})$	
22	W_3		$\frac{g-8}{12}$		$(4^{12}, 3^{16}, 8^6, 2^{24}, \dots, 2^{24})$	
23	W_3		$\frac{g-12}{12}$		$(4^{12}, 6^8, 8^6, 2^{24}, \dots, 2^{24})$	
24	$\mathbb{Z}_2 \otimes A_5$		$\frac{g+1}{30}$		$(3^{40}, 5^{24}, 2^{60}, \dots, 2^{60})$	
25	$\mathbb{Z}_2 \otimes A_5$		$\frac{g-5}{30}$		$(3^{40}, 10^{12}, 2^{60}, \dots, 2^{60})$	
26	$\mathbb{Z}_2 \otimes A_5$		$\frac{g-15}{30}$		$(6^{20}, 10^{12}, 2^{60}, \dots, 2^{60})$	
27	$\mathbb{Z}_2 \otimes A_5$	A_5	$\frac{g-9}{30}$		$(6^{20}, 5^{24}, 2^{60}, \dots, 2^{60})$	$(2^{30}, 3^{20}, 5^{12})$
28	$SL_2(5)$		$\frac{g-14}{30}$		$(4^{30}, 3^{40}, 5^{24}, 2^{60}, \dots, 2^{60})$	
29	$SL_2(5)$		$\frac{g-20}{30}$		$(4^{30}, 3^{40}, 10^{12}, 2^{60}, \dots, 2^{60})$	
30	$SL_2(5)$		$\frac{g-24}{30}$		$(4^{30}, 6^{20}, 5^{24}, 2^{60}, \dots, 2^{60})$	
31	$SL_2(5)$		$\frac{g-30}{30}$		$(4^{30}, 6^{20}, 10^{12}, 2^{60}, \dots, 2^{60})$	

#	$y^2 = f(x)$
1	$x^{2g+2} + a_1 x^{n(t-1)} + \dots + a_\delta x^n + 1, \quad t = \frac{2g+2}{n}$
2	$x^{2g+1} + a_1 x^{n(t-1)} + \dots + a_\delta x^n + 1, \quad t = \frac{2g+1}{n}$
3	$x(x^{nt} + a_1 x^{n(t-1)} + \dots + a_\delta x^n + 1), \quad t = \frac{2g}{n}$
4	$F(x) := \prod_{i=1}^t (x^{2n} + \lambda_i x^n + 1), \quad t = \frac{g+1}{n}$
5	$(x^n - 1) \cdot F(x)$
6	$x \cdot F(x)$
7	$(x^{2n} - 1) \cdot F(x)$
8	$x(x^n - 1) \cdot F(x)$
9	$x(x^{2n} - 1) \cdot F(x)$
10	$G(x) := \prod_{i=1}^\delta \left(x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1\right)$
11	$(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
12	$(x^8 + 14x^4 + 1) \cdot G(x)$
13	$x(x^4 - 1) \cdot G(x)$
14	$x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
15	$x(x^4 - 1)(x^8 + 14x^4 + 1) \cdot G(x)$
16	$M(x)$
17	$S(x) \cdot M(x)$
18	$T(x) \cdot M(x)$
19	$S(x) \cdot T(x) \cdot M(x)$
20	$R(x) \cdot M(x)$
21	$R(x) \cdot S(x) \cdot M(x)$
22	$R(x) \cdot T(x) \cdot M(x)$
23	$R(x) \cdot S(x) \cdot T(x) \cdot M(x)$
24	$\Lambda(x)$
25	$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \Lambda(x)$
26	$(x\ (x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
27	$\psi \cdot \Lambda(x)$
28	$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot (x\ (x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
29	$(x\ (x^{10} + 11x^5 - 1)) \cdot \psi \cdot \Lambda(x)$
30	$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \psi \cdot \Lambda(x)$
31	$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot (x\ (x^{10} + 11x^5 - 1)) \cdot \psi \cdot \Lambda(x)$

Automorphism groups and equations for superelliptic curves

Theorem (Sanjeewa-Sh)

For any superelliptic curve $\mathcal{X}_g$ of genus $g \geq 2$ defined over a field k, char $k = p \neq 2$, the automorphism group $Aut(\mathcal{X}_g)$ and the equation of $\mathcal{X}_g$ are given below:

Theorem 3.2. *Let $\mathcal{X}_g$ be a genus $g \geq 2$ irreducible cyclic curve defined over an algebraically closed field k of characteristic char $(k) = p$, $G = Aut(\mathcal{X}_g)$, and $\bar{G}$ its reduced automorphism group. If $|G| > 1$ then is G is one of the following:*

(1) $\bar{G} \cong C_m$: Then, $G \cong C_{mn}$ or $\langle r, s \,|\, r^n = 1, s^m = 1, srs^{-1} = r^l \rangle$, $(l, n) = 1$ and $l^m \equiv 1$ (mod n).

(2) If $\bar{G} \cong D_{2m}$ then $G \cong D_{2m} \times C_n$ or

$$G'_4 = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^2 = 1, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^l \rangle$$
$$G'_7 = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = r^{\frac{n}{2}}, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^l \rangle$$

where $(l, n) = 1$ and $l^2 \equiv 1$ (mod n) or

$$G_4 = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^2 = 1, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$
$$G_5 = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = 1, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$
$$G_6 = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^2 = 1, (st)^m = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$
$$G_7 = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = r^{\frac{n}{2}}, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$
$$G_8 = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = 1, (st)^m = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$
$$G_9 = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = r^{\frac{n}{2}}, (st)^m = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$

where $(l, n) = 1$ and $l^2 \equiv 1$ (mod n), $(k, n) = 1$ and $k^2 \equiv 1$ (mod n).

(3) If $\bar{G} \cong A_4$ and $p \neq 2, 3$ then $G \cong A_4 \times C_n$ or

$$G'_{10} = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^3 = 1, (st)^3 = 1, srs^{-1} = r, trt^{-1} = r^l \rangle$$
$$G'_{12} = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^3 = r^{\frac{n}{3}}, (st)^3 = r^{\frac{n}{3}}, srs^{-1} = r, trt^{-1} = r^l \rangle$$

where $(l, n) = 1$ and $l^3 \equiv 1$ (mod n) or

$$\langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^3 = r^{\frac{n}{2}}, (st)^5 = r^{\frac{n}{2}}, srs^{-1} = r, trt^{-1} = r \rangle, \text{ or}$$
$$G_{10} = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^3 = 1, (st)^3 = 1, srs^{-1} = r, trt^{-1} = r^k \rangle$$
$$G_{13} = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^3 = 1, (st)^3 = 1, srs^{-1} = r, trt^{-1} = r^k \rangle$$

where $(k, n) = 1$ and $k^3 \equiv 1$ (mod n).

(4) If $\bar{G} \cong S_4$ and $p \neq 2, 3$ then $G \cong S_4 \times C_n$ or

$$G_{16} = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^3 = 1, (st)^4 = 1, srs^{-1} = r^l, trt^{-1} = r \rangle$$
$$G_{18} = \langle r, s, t \,|\, r^n = 1, s^2 = 1, t^3 = 1, (st)^4 = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r \rangle$$
$$G_{20} = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^3 = 1, (st)^4 = 1, srs^{-1} = r^l, trt^{-1} = r \rangle$$
$$G_{22} = \langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^3 = 1, (st)^4 = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r \rangle$$

where $(l, n) = 1$ and $l^2 \equiv 1$ (mod n).

(5) If $\bar{G} \cong A_5$ and $p \neq 2, 5$ then $G \cong A_5 \times C_n$ or

$$\langle r, s, t \,|\, r^n = 1, s^2 = r^{\frac{n}{2}}, t^3 = r^{\frac{n}{2}}, (st)^5 = r^{\frac{n}{2}}, srs^{-1} = r, trt^{-1} = r \rangle$$

(6) If $\bar{G} \cong U$ then $G \cong U \times C_n$ or

$$\langle r, s_1, s_2, \ldots, s_t | r^n = s_1^p = s_2^p = \ldots = s_t^p = 1, s_i s_j = s_j s_i, s_i r s_i^{-1} = r^l, 1 \leq i, j \leq t \rangle$$

where $(l, n) = 1$ and $l^p \equiv 1$ (mod n).

(7) If $\bar{G} \cong K_m$ then $G \cong < r, s_1, \ldots, s_t, v | r^n = s_1^p = \ldots = s_t^p = v^m = 1, s_i s_j = s_j s_i, v r v^{-1} = r, s_i r s_i^{-1} = r^l, s_i v s_i^{-1} = v^k, 1 \leq i, j \leq t >$ where $(l, n) = 1$ and $l^p \equiv 1$ (mod n), $(k, m) = 1$ and $k^p \equiv 1$ (mod m) or

$$G_{35} = \langle r, s_1, \ldots, s_t | r^{nm} = s_1^p = \ldots = s_t^p = 1, s_i s_j = s_j s_i, s_i r s_i^{-1} = r^l, 1 \leq i, j \leq t \rangle$$

where $(l, nm) = 1$ and $l^p \equiv 1($ mod $nm)$.

(8) If $\bar{G} \cong PSL_2(q)$ then $G \cong PSL_2(q) \times C_n$ or $SL_2(3)$.

(9) If $\bar{G} \cong PGL(2, q)$ then $G \cong PGL(2, q) \times C_n$.

Equations for superelliptic curves

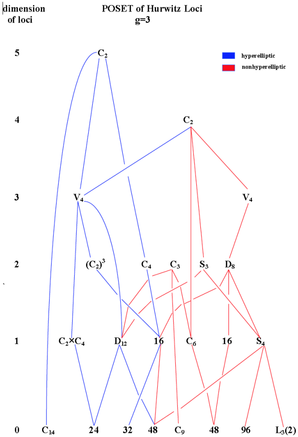
#	G	$\delta(G, C)$	δ, n, g	$\mathcal{C} = (C_1, \dots, C_r)$
1	$(p, m) = 1$	$\frac{2(g+n-1)}{m(n-1)} - 1$	$n < g + 1$	$(m, m, n, \dots, n)$
2	C_m	$\frac{2g+n-1}{m(n-1)} - 1$		$(m, mn, n, \dots, n)$
3		$\frac{2g}{m(n-1)} - 1$	$n < g$	$(mn, mn, n, \dots, n)$
4	$(p, m) = 1$	$\frac{g+n-1}{m(n-1)}$		$(2, 2, m, n, \dots, n)$
5		$\frac{2g+m+2n-nm-2}{2m(n-1)}$		$(2n, 2, m, n, \dots, n)$
6	D_{2m}	$\frac{g}{m(n-1)}$		$(2, 2, mn, n, \dots, n)$
7		$\frac{g+m+n-mn-1}{m(n-1)}$	$n < g + 1$	$(2n, 2n, m, n, \dots, n)$
8		$\frac{2g+m-mn}{2m(n-1)}$	$g \neq 2$	$(2n, 2, mn, n, \dots, n)$
9		$\frac{g+m-mn}{m(n-1)}$	$n < g$	$(2n, 2n, mn, n, \dots, n)$
10		$\frac{n+g-1}{6(n-1)}$		$(2, 3, 3, n, \dots, n)$
11	A_4	$\frac{g-n+1}{6(n-1)}$		$(2, 3n, 3, n, \dots, n)$
12		$\frac{g-3n+3}{6(n-1)}$		$(2, 3n, 3n, n, \dots, n)$
13		$\frac{g-2n+2}{6(n-1)}$	$\delta \neq 0$	$(2n, 3, 3, n, \dots, n)$
14		$\frac{g-4n+4}{6(n-1)}$		$(2n, 3n, 3, n, \dots, n)$
15		$\frac{g-6n+6}{6(n-1)}$	$\delta \neq 0$	$(2n, 3n, 3n, n, \dots, n)$
16		$\frac{g+n-1}{12(n-1)}$		$(2, 3, 4, n, \dots, n)$
17		$\frac{g-3n+3}{12(n-1)}$		$(2, 3n, 4, n, \dots, n)$
18		$\frac{g-2n+2}{12(n-1)}$		$(2, 3, 4n, n, \dots, n)$
19		$\frac{g-6n+6}{12(n-1)}$		$(2, 3n, 4n, n, \dots, n)$
20	S_4	$\frac{g-5n+5}{12(n-1)}$		$(2n, 3, 4, n, \dots, n)$
21		$\frac{g-9n+9}{12(n-1)}$		$(2n, 3n, 4, n, \dots, n)$
22		$\frac{g-8n+8}{12(n-1)}$		$(2n, 3, 4n, n, \dots, n)$
23		$\frac{g-12n+12}{12(n-1)}$		$(2n, 3n, 4n, n, \dots, n)$
24		$\frac{g+n-1}{30(n-1)}$		$(2, 3, 5, n, \dots, n)$
25		$\frac{g-5n+5}{30(n-1)}$		$(2, 3, 5n, n, \dots, n)$
26		$\frac{g-15n+15}{30(n-1)}$		$(2, 3n, 5n, n, \dots, n)$
27		$\frac{g-9n+9}{30(n-1)}$		$(2, 3n, 5, n, \dots, n)$
28	A_5	$\frac{g-14n+14}{30(n-1)}$		$(2n, 3, 5, n, \dots, n)$
29		$\frac{g-20n+20}{30(n-1)}$		$(2n, 3, 5n, n, \dots, n)$
30		$\frac{g-24n+24}{30(n-1)}$		$(2n, 3n, 5, n, \dots, n)$
31		$\frac{g-30n+30}{30(n-1)}$		$(2n, 3n, 5n, n, \dots, n)$
32		$\frac{2g+2n-2}{p^t(n-1)} - 2$		$(p^t, n, \dots, n)$
33	U	$\frac{2g+np^t-p^t}{p^t(n-1)} - 2$	$(n, p) = 1, n \mid p^t - 1$	$(np^t, n, \dots, n)$
34		$\frac{2(g+n-1)}{mp^t(n-1)} - 1$	$(m, p) = 1, m \mid p^t - 1$	$(mp^t, m, n, \dots, n)$
35		$\frac{2g+2n+p^t-np^t-2}{mp^t(n-1)} - 1$	$(m, p) = 1, m \mid p^t - 1$	$(mp^t, nm, n, \dots, n)$
36	K_m	$\frac{2g+np^t-p^t}{mp^t(n-1)} - 1$	$(nm, p) = 1, nm \mid p^t - 1$	$(nmp^t, m, n, \dots, n)$
37		$\frac{2g}{mp^t(n-1)} - 1$	$(nm, p) = 1, nm \mid p^t - 1$	$(nmp^t, nm, n, \dots, n)$
38		$\frac{2(g+n-1)}{m(n-1)} - 1$	$\left(\frac{q-1}{2}, p\right) = 1$	$(\alpha, \beta, n, \dots, n)$
39	$PSL_2(q)$	$\frac{2g+q(q-1)-n(q+1)(q-2)-2}{m(n-1)} - 1$	$\left(\frac{q-1}{2}, p\right) = 1$	$(\alpha, n\beta, n, \dots, n)$
40		$\frac{2g+nq(q-1)+q-q^2}{m(n-1)} - 1$	$\left(\frac{n(q-1)}{2}, p\right) = 1$	$(n\alpha, \beta, n, \dots, n)$
41		$\frac{2g}{m(n-1)} - 1$	$\left(\frac{n(q-1)}{2}, p\right) = 1$	$(n\alpha, n\beta, n, \dots, n)$
42		$\frac{2(g+n-1)}{m(n-1)} - 1$	$(q-1, p) = 1$	$(2\alpha, 2\beta, n, \dots, n)$
43	$PGL_2(q)$	$\frac{2g+q(q-1)-n(q+1)(q-2)-2}{m(n-1)} - 1$	$(q-1, p) = 1$	$(2\alpha, 2n\beta, n, \dots, n)$
44		$\frac{2g+nq(q-1)+q-q^2}{m(n-1)} - 1$	$(n(p-1), p) = 1$	$(2n\alpha, 2\beta, n, \dots, n)$
45		$\frac{2g}{m(n-1)} - 1$	$(n(q-1), p) = 1$	$(2n\alpha, 2n\beta, n, \dots, n)$

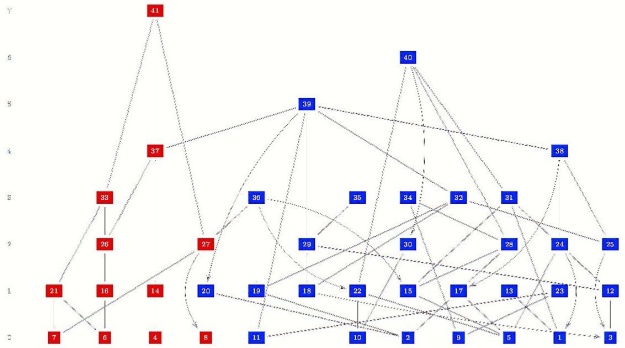
#	$\bar{G}$	$y^n = f(x)$
1		$x^{m\delta} + a_1 x^{m(\delta-1)} + \cdots + a_\delta x^m + 1$
2	C_m	$x^{m\delta} + a_1 x^{m(\delta-1)} + \cdots + a_\delta x^m + 1$
3		$x(x^{m\delta} + a_1 x^{m(\delta-1)} + \cdots + a_\delta x^m + 1)$
4		$F(x) := \prod_{i=1}^{\delta}(x^{2m} + \lambda_i x^m + 1)$
5		$(x^m - 1) \cdot F(x)$
6		$x \cdot F(x)$
7	D_{2m}	$(x^{2m} - 1) \cdot F(x)$
8		$x(x^m - 1) \cdot F(x)$
9		$x(x^{2m} - 1) \cdot F(x)$
10		$G(x) := \prod_{i=1}^{\delta}(x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1)$
11		$(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
12	A_4	$(x^8 + 14x^4 + 1) \cdot G(x)$
13		$x(x^4 - 1) \cdot G(x)$
14		$x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
15		$x(x^4 - 1)(x^8 + 14x^4 + 1) \cdot G(x)$
16		$M(x)$
17		$S(x) \cdot M(x)$
18		$T(x) \cdot M(x)$
19		$S(x) \cdot T(x) \cdot M(x)$
20	S_4	$R(x) \cdot M(x)$
21		$R(x) \cdot S(x) \cdot M(x)$
22		$R(x) \cdot T(x) \cdot M(x)$
23		$R(x) \cdot S(x) \cdot T(x).M(x)$
24		$\Lambda(x)$
25		$(x(x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
26		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
27		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \Lambda(x)$
28	A_5	$Q(x) \cdot \Lambda(x)$
29		$x(x^{10} + 11x^5 - 1).\psi(x) \cdot \Lambda(x)$
30		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \psi(x) \cdot \Lambda(x)$
31		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \psi(x) \cdot \Lambda(x)$
32	U	$B(x)$
33		$B(x)$
34		$\Theta(x)$
35	K_m	$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
36		$\Theta(x)$
37		$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
38		$\Delta(x)$
39	$PSL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
40		$(x^q - x) \cdot \Delta(x)$
41		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
42		$\Omega(x)$
43	$PGL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Omega(x)$
44		$(x^q - x) \cdot \Omega(x)$
45		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Omega(x)$

TABLE 4. The equations of the curves related to the cases in Table 2

Inclusion among the loci

Inclusion of the loci in $\mathcal{M}_g$ for genus 3 and 4:





The majority of curves are superelliptic

In [BSZ15] we focus on $g = 4$. Red and yellow entries denote the superelliptic curves (hyperelliptic and non-hyperelliptic respectively). From 41 total cases, only 13 are non-superelliptic.

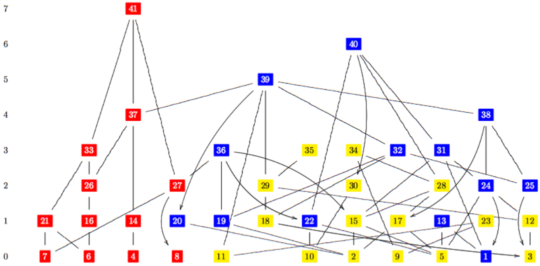


Table 2. Equations of genus 4 superelliptic curves

#	dim	aut	equation
23	1	(5,1)	$y^5 = x(x-1)(x-\lambda)$
9	0	(15,1)	$y^5 = x^3 - 1$
11	0	(10,2)	$y^5 = x(x^2-1)$
34	3	(3,1)	$y^3 = x(x-1)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$
28	2	(6,2)	$y^3 = (x^2-1)(x^2-\alpha_1)(x^2-\alpha_2)$
15	1	(18,3)	$y^3 = x^6 + \lambda x^3 + 1$
17	1	(12,5)	$y^3 = (x^2-1)(x^4 - \lambda x^2 + 1)$
2	0	(72,42)	$y^3 = x(x^4-1)$
5	0	(36,12)	$y^3 = x^6 - 1$
35	3	(3,1)	$y^3 = (x^2-2)(x^4 + bx^2 + cx + d)$
29	2	(6,1)	$y^3 - 1 = x\left(x^5 + (b-2)x^3 + x^3c - (2b+1/2)x - 2c\right)$
12	1	(36,10)	$y^3 - 1 = x^6 + \lambda x^3 + 1$
18	1	(12,4)	$y^3 - 1 = (x^2-1)(x^2-\alpha_1)(x^2-\alpha_2)$
3	0	(72,40)	$y^3 - 1 = x^6 - 1$
22	1	(6,2)	$y^6 = x(x-1)(x-\alpha)$
30	2	(4,1)	$y^4 = x^2(x-1)(x-\alpha_1)(x-\alpha_2)$
10	0	(12,2)	$y^4 = x^2(x^3-1)$
41	7	(2,1)	$y^2 = f(x),\ \deg f = 9, 10$
37	4	(4,2)	$y^2 = x^{10} + a_1x^8 + a_2x^6 + a_3x^4 + a_4x^2 + 1$
33	3	(4,1)	$y^2 = x(x^8 + a_1x^6 + a_2x^4 + a_3x^2 + 1)$
26	2	(8,3)	$y^2 = x(x^4 + \lambda_1x^2 + 1)(x^4 + \lambda_2x^2 + 1)$
27	2	(6,2)	$y^2 = x^9 + a_1x^6 + a_2x^3 + 1$
4	0	(40,8)	$y^2 = x^{10} - 1$
6	0	(32,19)	$y^2 = x(x^8-1)$
7	0	(24,3)	$y^2 = x(x^4-1)(x^4 + 2i\sqrt{3}x^2 + 1)$
8	0	(18,2)	$y^2 = x^9 + 1$
21	1	(8,4)	$y^2 = x(x^4-1)(x^4 + \lambda x^2 + 1)$
14	1	(20,4)	$y^2 = x^{10} + \lambda x^5 + 1$
16	1	(16,7)	$y^2 = x(x^8 + \lambda x^4 + 1)$

The easiest case, as always, $g = 2$. Let $\mathfrak{p} \in \mathcal{M}_2$. Find an equation for the curve.

Mestre (83) provided an algorithm, which worked for $\mathrm{Aut}(\mathcal{X}_2) \cong C_2$. In [Sha02] equations for cases $|\mathrm{Aut}(\mathcal{X}_2)| > 2$ were determined.

Theorem (Malmendier-Sh, 2016)

For every point $\mathfrak{p} \in \mathcal{M}_2$ such that $\mathfrak{p} \in \mathcal{M}_2(K)$, for some number field K, there is a pair of genus-two curves $\mathcal{C}^\pm$ given by

$$\mathcal{C}^\pm : \quad y^2 = \sum_{i=0}^{6} a_{6-i}^\pm \, x^i \, ,$$

corresponding to $\mathfrak{p}$, such that $a_i^\pm \in K(d)$, $i = 0, \ldots, 6$ as given explicitly in Equation (45) of [MS16]. *Moreover, $K(d)$ is the minimal field of definition of $\mathfrak{p}$.*

Here d is given in terms of $\mathfrak{p}$. In particular, if $|\mathrm{Aut}(\mathfrak{p})| > 2$, then $d \in K$.

Question: Can the above approach be generalized to all superelliptic curves?

There is no theoretical reason why it shouldn't, at least for hyperelliptic curves. However, difficulties arise with invariants of binary forms of higher degree.

Superelliptic curves with extra automorphisms

From the previous tables, when the curve has an extra automorphism, then it has equation

$$y^n = x^{\delta(s+1)} + a_s x^{\delta s} + a_{s-1} x^{\delta(s-1)} + \cdots + a_2 x^{\delta\cdot 2} + a_1 x^{\delta} + 1$$

Dihedral invariants, as defined in [GS05] are

$$\mathfrak{u}_i = a_1^{s+1-i} a_i + a_s^{s+1-i} a_{s+1-i}, \qquad i = 0, \ldots, s$$

Theorem ([BT14])

Let $\mathcal{X}_g$ and $\mathfrak{u}_1, \ldots, \mathfrak{u}_g$ be as above. Then,
i) $K = \mathbb{Q}(\mathfrak{u}_1, \ldots, \mathfrak{u}_s)$ is a quadratic extension of the field of moduli F of $\mathcal{X}_g$ such that $K = F(\sqrt{\Delta_\mathfrak{u}})$, where $\Delta_\mathfrak{u} = 2^{s+1}\mathfrak{u}_1^2 - 2^{s+3}\mathfrak{u}_s^{s+1}$.
iii) The equation of $\mathcal{X}$ over K is

$$y^n = A\,x^{\delta(s+1)} + A\,x^{\delta s} + \sum_{i=1}^{s-1} A\,\frac{2^{s+1} A\mathfrak{u}_{s+1-i} - 2^{s+1-i}{\mathfrak{u}_s}^i \mathfrak{u}_i}{2^{s+1}A^2 - \mathfrak{u}_s^{s+1}} \cdot x^{\delta\cdot i} + 1 \tag{1}$$

where $2^{s+1}A^2 - 2^{s+1}\mathfrak{u}_1 A + \mathfrak{u}_s^{s+1} = 0.$

Hence, a minimal field of definition is at most a degree 2 extension of the field of moduli.

Field of moduli versus field of definition

Theorem (Hidalgo-Sh)

Let $\mathcal{X}$ be a superelliptic curve of genus $g \geq 2$ with superelliptic group $H \cong C_n$. If the reduced group of automorphisms $\overline{Aut}(\mathcal{X}) = Aut(\mathcal{X})/H$ is different from trivial or cyclic, then $\mathcal{X}$ is definable over its field of moduli.

Next we display all genus $g \leq 10$ superelliptic curves which are defined over its field of moduli.

TABLE 1. Genus 3 curves No. 1 and 2 are the only one whose field of moduli is not necessarily a field of definition

Nr.	$\overline{G}$	G	n	m	sig.	δ	Equation $y^n = f(x)$
1	$\{I\}$	C_2	2	1	2^8	5	$x\left(x^6 + \sum_{i=1}^5 a_i x^i + 1\right)$
2	C_2	V_4	2	2	2^6	3	$x^8 + a_1x^2 + a_2x^4 + a_3x^6 + 1$
3	C_2	C_4	2	2	$2^3, 4^2$	2	$x\left(x^6 + a_1x^2 + a_2x^4 + 1\right)$
4	C_2	C_6	3	2	$2, 3^2, 6$	1	$x^4 + a_1x^2 + 1$
5	V_4	$V_4 \times C_4$	4	2	$2^3, 4$	1	$x^4 + a_1x^2 + 1$

TABLE 2. Genus 4 curves No. 1, 3 and 5 are the only ones whose field of moduli is not necessarily a field of definition

Nr.	$\overline{G}$	G	n	m	sig.	δ	Equation $y^n = f(x)$
1	C_m	C_2	2	1	2^{10}	7	$x\left(x^8 + \sum_{i=1}^7 a_i x^i + 1\right)$
2		V_4	2	2	2^7	4	$x^{10} + \sum_{i=1}^4 a_i x^{2i} + 1$
3		C_4	2	2	$2^4, 4^2$	3	$x(x^8 + a_3x^6 + a_2x^4 + a_1x^2 + 1)$
4		C_6	2	3	$2^3, 3, 6$	2	$x^9 + a_1x^3 + a_2x^6 + 1$
5		C_3	3	1	3^6	3	$x(x^4 + a_1x + a_2x^2 + a_3x^3 + 1)$
6		$C_2 \times C_3$	3	2	$2^2, 3^3$	2	$x^6 + a_2x^4 + a_1x^2 + 1$
7	D_{2m}	$D_6 \times C_3$	3	3	$2^2, 3^2$	1	$x^6 + a_1x^3 + 1$
8		$V_4 \times C_3$	3	2	$2^2, 3, 6$	1	$(x^2 - 1)(x^4 + a_1x^2 + 1)$
9		$V_4 \times C_3$	3	2	$2^2, 3, 6$	1	$x(x^4 + a_1x^2 + 1)$

Nr.	$\overline{G}$	G	n	m	sig.	δ	Equation $y^n = f(x)$
				Genus 5			
1	C_m	V_4	2	2	2^8	5	$x^{12}+\sum_{i=1}^5 a_i x^{2i}+1$
2		$C_3\times C_2$	2	3	$2^4, 3^2$	3	$x^{12}+\sum_{i=1}^3 a_i x^{3i}+1$
3		$C_2\times C_4$	2	4	$2^3, 4^2$	2	$x^{12}+a_2x^8+a_1x^4+1$
4		C_{22}	2	11	2, 11, 22	0	$x^{11}+1$
5		C_{22}	11	2	2, 22, 22	0	x^2+1
6		C_2	2	1	2^{12}	9	$x(x^{10}+\sum_{i=1}^9 a_i x^i+1)$
7		C_4	2	2	$2^5, 4^2$	4	$x(x^{10}+\sum_{i=1}^4 a_i x^{2i}+1)$
8	D_{2m}		2	2	2^6	3	$\prod_{i=1}^3(x^4+a_ix^2+1)$
9			2	3	$2^4, 3$	2	$(x^6+a_1x^3+1)(x^6+a_2x^3+1)$
10			2	6	$2^3, 6$	1	$x^{12}+a_1x^6+1$
11			2	4	$2^2, 4^2$	1	$(x^4-1)(x^8+a_1x^4+1)$
12			2	12	2, 4, 12	0	$x^{12}-1$
13			2	5	2^3, 10	1	$x(x^{10}+a_1x^5+1)$
14			2	2	$2^3, 4^2$	2	$(x^4-1)(x^4+a_1x^2+1)(x^4+a_2x^2+1)$
15			2	3	2, 3, 4^2	1	$(x^6-1)(x^6+a_1x^3+1)$
16			2	2	$2^3, 4^2$	2	$x(x^2-1)(x^4+a_1x^2+1)(x^4+a_2x^2+1)$
17			2	10	2, 4, 20	0	$x(x^{10}-1)$
18	A_4		2		$2^2, 3^2$	1	$f_1(x)$
19	S_4		2	0	3, 4^2	0	$x^{12}-33x^8-33x^4+1$
20	A_5		2		2,3,10	0	$x(x^{10}+11x^5-1)$
				Genus 6			
1	C_m	V_4	2	2	2^9	6	$x^{14}+\sum_{i=1}^6 a_i x^{2i}+1$
2		C_{26}	2	13	2, 13, 26	0	$x^{13}+1$
3		C_{21}	3	7	3, 7, 21	0	x^7+1
4		C_{20}	4	5	4, 5, 20	0	x^5+1
5		C_{10}	5	2	2, 5, 5, 10	1	$x^4+a_1x^2+1$
6		C_{20}	5	4	4, 5, 20	0	x^4+1
7		C_{21}	7	3	3, 7 , 21	0	x^3+1
8		C_{26}	13	2	2, 13, 26	0	x^2+1
9		C_2	2	1	2^{14}	11	$x(x^{12}+\sum_{i=1}^{11} a_i x^i+1)$
10		C_4	2	2	$2^6, 4^2$	5	$x(x^{12}+\sum_{i=1}^5 a_i x^{2i}+1)$
11		C_6	2	3	$2^3, 3^2, 6^2$	3	$x(x^{12}+\sum_{i=1}^3 a_i x^{3i}+1)$
12		C_8	2	4	$2^3, 8^2$	2	$x(x^{12}+\sum_{i=1}^2 a_i x^{4i}+1)$
13		C_3	3	1	3^8	5	$x^6+\sum_{i=1}^5 a_i x^i+1$
14		C_6	3	2	$3^3, 6^2$	2	$x^6+a_2x^4+a_1x^2+1$
15		C_4	4	1	4^6	3	$x^4+\sum_{i=1}^3 a_i x^i+1$
16		C_5	5	1	5^5	2	$x^3+a_1x+a_2x^2+1$
17	D_{2m}	$D_{14}\times C_2$	2	7	$2^3, 7$	1	$x^{14}+a_1x^7+1)$
18		G_5	2	2	2^5, 4	3	$(x^2-1)\prod_{i=1}^3(x^4+a_ix^2+1)$
19		G_5	2	14	2, 4, 14	0	$x^{14}-1$
20		$D_{10}\times C_2$	5	5	2, 5, 10	0	x^5-1
21		D_8	2	2	$2^5, 4$	3	$x\cdot\prod_{i=1}^3(x^4+a_ix^2+1)$
22		$D_6\times C_2$	2	3	2^4, 6	2	$x\cdot\prod_{i=1}^2(x^4+a_ix^2+1)$
23		D_{24}	2	6	2^3, 12	1	$x(x^{12}+a_1x^6+1)$
24		$D_6\times C_3$	3	3	$2^2, 3, 9$	1	$x(x^6+a_1x^3+1)$
25		D_{16}	4	2	2^2, 4, 8	1	$x(x^4+a_1x^2+1)$
26		G_8	2	4	2^2, 4, 8	1	$x(x^4-1)(x^8+a_1x^4+1)$
27		G_8	2	12	2, 4, 24	0	$x(x^{12}-1)$
28		$V_4\times C_3$	3	2	2, 3, 6^2	1	$x(x^2-1)(x^4+a_1x^2+1)$
29		$D_{12}\times C_3$	3	6	2, 6, 18	0	$x(x^6-1)$
30		G_8	4	4	2, 8, 16	0	$x(x^4-1)$
31		$D_6\times C_5$	5	3	2, 10, 15	0	$x(x^3-1)$
32		$V_4\times C_7$	7	2	2, 14^2	0	$x(x^2-1)$
33		G_9	2	2	$2^2, 4^3$	2	$x(x^4-1)\cdot\prod_{i=1}^2(x^4+a_ix^2+1)$
34		G_9	2	3	$2, 4^2, 6$	1	$x(x^6-1)(x^6+a_1x^3+1)$
35	S_4	G_{18}	4	0	2, 3, 16	0	$x(x^4-1)$
36		G_{19}	2	0	2, 6, 8	0	$x(x^4-1)(x^8+14x^4+1)$

Table 3. (Cont.)

Nr.	$\overline{G}$	G	n	m	sig.	δ	Equation $y^n = f(x)$
				Genus 7			
1	C_m	V_4	2	2	2^{10}	7	$x^{16}+\sum_{i=1}^7 a_i x^{2i}+1$
2		$C_2\times C_4$	2	4	$2^4, 4^2$	3	$x^{16}+\sum_{i=1}^3 a_i x^{4i}+1$
3		${C_3}^2$	3	3	3^5	2	$x^9+a_2x^6+a_1x^3+1$
4		C_6	2	3	2^5, 3, 6	4	$x^{15}+\sum_{i=1}^4 a_1 x^{3i}+1$
5		C_{10}	2	5	2^3, 5, 10	2	$x^{15}+a_1x^5+a_2x^{10}+1$
6		C_{30}	2	15	2, 15, 30	0	$x^{15}+1$
7		C_6	3	2	2, 3^4, 6	3	$x^8+a_3x^6+a_2x^4+a_1x^2+1$
8		C_{12}	3	4	3^2, 4, 12	1	$x^8+a_1x^4+1$
9		C_{24}	3	8	3, 8, 24	0	x^8+1
10		C_{30}	15	2	2, 15, 30	0	x^2+1
11		C_2	2	1	2^{16}	13	$x(x^{14}+\sum_{i=1}^{13} a_i x^i+1)$
12		C_4	2	2	$2^7, 4^2$	6	$x(x^{14}+\sum_{i=1}^6 a_i x^{2i}+1)$
13		C_3	3	1	3^9	6	$x^7+\sum_{i=1}^6 a_i x^i+1$
14	D_{2m}	$V_4\times C_2$	2	2	2^7	4	$\prod_{i=1}^4(x^4+a_ix^2+1)$
15		$D_8\times C_2$	2	4	2^4, 4	2	$(x^8+a_1x^4+1)(x^8+a_2x^4+1)$
16		$D_{16}\times C_2$	2	8	2^3, 8	1	$x^{16}+a_1x^8+1$
17	D_{2m}	G_5	2	16	2, 4, 16	0	$x^{16}-1$
18		$D_6\times C_3$	3	3	2, 3^2, 6	1	$(x^3-1)(x^6+a_1x^3+1)$
19		$D_{18}\times C_3$	3	9	2, 6, 9	0	x^9-1
20		$D_{14}\times C_2$	2	7	2^3, 14	1	$x(x^{14}+a_1x^7+1)$
21		G_7	2	2	2^4, 4^2	3	$(x^4-1)\prod_{i=1}^3(x^4+a_ix^2+1)$
22		G_7	2	4	2, 4^3	1	$(x^8-1)(x^8+a_1x^4+1)$
23		G_8	2	2	2^4, 4^2	3	$x(x^2-1)\prod_{i=1}^3(x^4+a_ix^2+1)$
24		G_8	2	14	2, 4, 28	0	$x(x^{14}-1)$
25		$D_{14}\times C_3$	3	7	2, 6, 21	0	$x(x^7-1)$
26		G_8	8	2	$2, 16^2$	0	$x(x^2-1)$
27	A_4	K	2	0	2^2, 3, 6	1	$(x^4+2\sqrt{-3}x^2+1)\,f_1(x)$
				Genus 8			
1	C_m	V_4	2	2	2^{11}	8	$x^{18}+\sum_{i=1}^8 a_i x^{2i}+1$
2		$C_2\times C_3$	2	3	$2^6, 3^2$	5	$x^{18}+\sum_{i=1}^5 a_i x^{3i}+1$
3		$C_2\times C_6$	2	6	$2^3, 6^2$	2	$x^{18}+a_1x^6+a_2x^{12}+1$
4		C_{34}	2	17	2, 17, 34	0	$x^{17}+1$
5		C_{34}	17	2	2, 17 , 34	0	x^2+1
6		C_2	2	1	2^{18}	15	$x(x^{16}+\sum_{i=1}^1 5a_i x^i+1)$
7		C_4	2	2	$2^8, 4^2$	7	$x(x^{16}+\sum_{i=1}^7 a_i x^{2i}+1)$
8		C_8	2	4	$2^4, 8^2$	3	$x(x^{16}+a_1x^4+a_2x^8+a_3x^{12}+1)$
9	D_{2m}	$D_6\times C_2$	2	3	2^5, 3	3	$\prod_{i=1}^3(x^6+a_ix^3+1)$
10		$D_{18}\times C_2$	2	9	2^3, 9	1	$x^{18}+a_1x^9+1$
11		G_5	2	2	2^6, 4	4	$(x^2-1)\prod_{i=1}^4(x^4+a_ix^2+1)$
12		G_5	2	6	2^2, 4, 6	1	$(x^6-1)(x^{12}+a_1x^6+1)$
13		G_5	2	18	2, 4, 18	0	$x^{18}-1$
14		D_8	2	2	2^6, 4	4	$x\prod_{i=1}^4(x^4+a_ix^2+1)$
15		D_{16}	2	4	2^4, 8	2	$x(x^8+a_1x^4+1)(x^8+a_2x^4+1)$
16		D_{32}	2	8	2^3, 16	1	$x(x^{16}+a_1x^8+1)$
17		G_9	2	3	2^2, 3, 4^2	2	$(x^6-1)(x^6+a_1x^3+1)(x^6+a_2x^3+1)$
18		G_8	2	16	2, 4, 32	0	$x(x^{16}-1)$
19		G_9	2	2	$2^3, 4^3$	3	$x\prod_{i+1}^3(x^6+a_ix^3+1)$
20		G_9	2	4	2, 4^2, 8	1	$x(x^8-1)(x^8+a_1x^4+1)$
21	A_4	K	2	0	2, 3^2, 4	1	$x(x^4-1)\,f_1(x)$
22	S_4	G_{22}	2	0	3, 4, 8	0	$x(x^4-1)(x^{12}-33x^8-33x^4+1)$

TABLE 3. (Cont.)

Nr.	$\overline{G}$	G	n	m	sig.	δ	Equation $y^n = f(x)$
					Genus 9		
1	C_m	V_4	2	2	2^{12}	9	$x^{20} + \sum_{i=1}^{9} a_i x^{2i} + 1$
2		$C_2 \times C_4$	2	4	$2^5, 4^2$	4	$x^{20} + \sum_{i=1}^{4} a_i x^{4i} + 1$
3		$C_2 \times C_5$	2	5	$2^4, 5^2$	3	$x^{20} + a_1 x^5 + a_2 x^{10} + a_3 x^{15} + 1$
4		$C_2 \times C_4$	4	2	$2^2, 4^4$	3	$x^8 + a_1 x^2 + a_2 x^4 + a_3 x^6 + 1$
5		C_{38}	2	19	2, 19, 38	0	$x^{19} + 1$
6		C_6	3	2	2, 3^5, 6	4	$x^{10} + a_1 x^2 + a_2 x^4 + a_3 x^6 + a_4 x^8 + 1$
7		C_{15}	3	5	3^2, 5, 15	1	$x^{10} + a_1 x^5 + 1$
8		C_{30}	3	10	3, 10^2	0	$x^{10} + 1$
9		C_{28}	4	7	4, 7^2	0	$x^7 + 1$
10		C_{14}	7	2	$2, 7^2, 14$	1	$x^4 + a_1 x^2 + 1$
11		C_{28}	7	4	4^2, 7	0	$x^4 + 1$
12		C_{30}	10	3	3^2, 10	0	$x^3 + 1$
13		C_{38}	19	2	2^2, 19	0	$x^2 + 1$
14		C_2	2	1	2^{20}	17	$x(x^{18} + \sum_{i=1}^{17} a_i x^i + 1)$
15		C_4	2	2	$2^9, 4^2$	8	$x(x^{18} + \sum_{i=1}^{8} a_i x^{2i} + 1)$
16		C_6	2	3	$2^6, 6^2$	5	$x(x^{18} + \sum_{i=1}^{5} a_i x^{3i} + 1)$
17		C_{12}	2	6	$2^3, 12^2$	2	$x(x^{18} + a_1 x^6 + a_2 x^{12} + 1)$
18		C_3	3	1	3^{11}	8	$x^9 + \sum_{i=1}^{8} a_i x^i + 1$
19		C_9	3	3	$3^3, 9^2$	2	$x^9 + a_2 x^6 + a_1 x^3 + 1$
20		C_4	4	1	4^8	5	$x^6 + \sum_{i=1}^{5} a_i x^i + 1$
21		C_8	4	2	$4^3, 8^2$	2	$x^6 + a_2 x^4 + a_1 x^2 + 1$
22		C_7	7	1	7^5	2	$x^3 + a_1 x + a_2 x^2 + 1$
23	D_{2m}	$V_4 \times C_2$	2	2	2^8	5	$\prod_{i=1}^{5} (x^4 + a_i x^2 + 1))$
24		$D_{10} \times C_2$	2	5	2^4, 5	2	$(x^{10} + a_1 x^5 + 1)(x^{10} + a_2 x^5 + 1)$
25		$D_{20} \times C_2$	2	10	2^3, 10	1	$x^{20} + a_1 x^{10} + 1$
26		$V_4 \times C_4$	4	2	$2^3, 4^2$	2	$(x^4 + a_1 x^2 + 1)(x^4 + a_2 x^2 + 1)$
27		$D_8 \times C_4$	4	4	$2^2, 4^2$	1	$x^8 + a_1 x^4 + 1$
28		G_5	2	4	$2^3, 4^2$	2	$(x^4 - 1)(x^8 + a_1 x^4 + 1)(x^8 + a_2 x^4 + 1)$
29		G_5	2	20	2, 4, 20	0	$x^{20} - 1$
30		G_5	4	8	2, 8^2	0	$x^8 - 1$
31		$D_6 \times C_2$	2	3	2^5, 6	3	$x \prod_{i=1}^{3} (x^6 + a_i x^3 + 1)$
32		$D_{18} \times C_2$	2	9	2^3, 18	1	$x(x^{18} + a_1 x^9 + 1)$
33		$D_6 \times C_4$	4	3	2^2, 4, 12	1	$x(x^6 + a_1 x^3 + 1)$
34		G_7	2	2	$2^5, 4^2$	4	$(x^4 - 1) \prod_{i=1}^{4} (x^4 + a_i x^2 + 1)$
35		G_9	2	5	2, 4^2, 5	1	$(x^{10} - 1)(x^{10} + a_1 x^5 + 1)$
36		G_7	4	2	2, 4, 8^2	1	$(x^4 - 1)(x^4 + a_1 x^2 + 1)$
37		G_8	2	2	$2^5, 4^2$	4	$x(x^2 - 1) \prod_{i=1}^{4} (x^4 + a_i x^2 + 1)$
38		G_8	2	6	2^2, 4, 12	1	$x(x^6 - 1)(x^{12} + a_1 x^6 + 1)$
39		G_8	2	18	2, 4, 36	0	$x(x^{18} - 1)$
40		$D_6 \times C_3$	3	3	2, 3, 6, 9	1	$x(x^3 - 1)(x^6 + a_1 x^3 + 1)$
41		$D_{18} \times C_3$	3	9	2, 6, 27	0	$x(x^9 - 1)$
42		G_8	4	2	2, 4, 8^2	1	$x(x^2 - 1)(x^4 + a_1 x^2 + 1)$
43		G_8	4	6	2, 8, 24	0	$x(x^6 - 1)$
44		$D_6 \times C_7$	7	3	2, 14, 21	0	$x(x^3 - 1)$
45		G_8	10	2	2, 20^2	0	$x(x^2 - 1)$
46		G_9	2	3	$2^2, 4^2, 6$	2	$x(x^6 - 1)(x^6 + a_1 x^3 + 1)(x^6 + a_2 x^3 + 1)$
47	A_4	K	2	0	$2^2, 6^2$	1	$(x^8 + 14x^4 + 1)\, f_1(x)$
48	S_4	G_{17}	4	0	2, 4, 12	0	$x^8 + 14x^4 + 1$
49		G_{21}	2	0	4^2, 6	0	$(x^8 + 14x^4 + 1)(x^{12} - 33x^8 - 33x^4 + 1)$
50	A_5		2		2, 5, 6	0	$x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1$

Nr.	$\overline{G}$	G	n	m	sig.	δ	Equation $y^n = f(x)$
					Genus 10		
1	C_m	V_4	2	2	2^{13}	10	$x^{22} + \sum_{i=1}^{10} a_i x^{2i} + 1$
2		$C_2 \times C_3$	3	2	$2^2, 3^6$	5	$x^{12} + \sum_{i=1}^{5} a_i x^{2i} + 1$
3		C_3^2	3	3	3^6	3	$x^{12} + a_1 x^3 + a_2 x^6 + a_3 x^9 + 1$
4		$C_3 \times C_4$	3	4	$3^3, 4^2$	2	$x^{12} + a_1 x^4 + a_2 x^8 + 1$
5		$C_2 \times C_6$	6	2	$2^2, 6^3$	2	$x^6 + a_1 x^2 + a_2 x^4 + 1$
6		C_6	2	3	2^7, 3, 6	6	$x^{21} + \sum_{i=1}^{6} a_i x^{3i} + 1$
7		C_{14}	2	7	2^3, 7, 14	2	$x^{21} + a_1 x^7 + a_2 x^{14} + 1$
8		C_{42}	2	21	2, 4, 21	0	$x^{21} + 1$
9		C_{33}	3	11	3, 11^2	0	$x^{11} + 1$
10		C_{10}	5	2	2, 5^3, 10	2	$x^6 + a_2 x^4 + a_1 x^2 + 1$
11		C_{15}	5	3	3, 5^2, 15	1	$x^6 + a_1 x^3 + 1$
12		C_{30}	5	6	5, 6^2	0	$x^6 + 1$
13		C_{30}	6	5	5^2, 6	0	$x^5 + 1$
14		C_{33}	11	3	3^2, 11	0	$x^3 + 1$
15		C_{42}	21	2	2, 21, 42	0	$x^2 + 1$
16		C_2	2	1	2^{22}	19	$x(x^{20} + \sum_{i=1}^{19} a_i x^i + 1)$
17		C_4	2	2	$2^{10}, 4^2$	9	$x(x^{20} + \sum_{i=1}^{9} a_i x^{2i} + 1)$
18		C_8	2	4	$2^5, 8^2$	4	$x(x^{20} + a_1 x^4 + a_2 x^8 + a_3 x^{12} + a_4 x^{16} + 1)$
19		C_{10}	2	5	$2^4, 10^2$	3	$x(x^{20} + a_1 x^5 + a_2 x^{10} + a_3 x^{15} + 1)$
20		C_3	3	1	3^{12}	9	$x^{10} + \sum_{i=1}^{9} a_i x^i + 1$
21		C_6	3	2	$3^5, 6^2$	4	$x^{10} + a_1 x^2 + a_2 x^4 + a_3 x^6 + a_4 x^8 + 1$
22		C_5	5	1	5^7	4	$x^5 + \sum_{i=1}^{4} a_i x^i + 1$
23		C_6	6	1	6^6	3	$x^4 + a_1 x + a_2 x^2 a_3 x^3 + 1$
24	D_{2m}	$D_{22} \times C_2$	2	11	2^3, 11	1	$x^{22} + a_1 x^{11} + 1$
25		$V_4 \times C_3$	3	2	$2^3, 3^3$	3	$\prod_{i=1}^{3} (x^4 + a_i x^2 + 1)$
26		$D_6 \times C_3$	3	3	$2^2, 3^3$,	2	$(x^6 + a_1 x^3 + 1)(x^6 + a_2 x^3 + 1)$
27		$D_{12} \times C_3$	3	6	$2^2, 3, 6$	1	$(x^{12} + a_1 x^6 + 1$
28		$D_6 \times C_6$	6	3	$2^2, 3, 6$	1	$x^6 + a_1 x^3 + 1$
29		G_5	2	2	2^7, 4	5	$(x^2 - 1) \prod_{i=1}^{5} (x^4 + a_i x^2 + 1)$
30		G_5	2	22	2, 4, 22	0	$x^{22} - 1$
31		$D_8 \times C_3$	3	4	2, 3, 4, 6	1	$(x^4 - 1)(x^8 + a_1 x^4 + 1)$
32		$D_{24} \times C_3$	3	12	2, 6, 12	0	$x^{12} - 1$
33		G_5	6	2	2^2, 6, 12	1	$(x^2 - 1)(x^4 + a_1 x^2 + 1)$
34		G_5	6	6	2, 6, 12	0	$x^6 - 1$
35		D_8	2	2	2^7, 4	5	$x \prod_{i=1}^{5} (x^4 + a_i x^2 + 1)$
36		$D_{10} \times C_2$	2	5	2^4, 10	2	$x(x^{10} + a_1 x^5 + 1)(x^{10} + a_2 x^5 + 1)$
37		D_{40}	2	10	2^3, 20	1	$x(x^{20} + a_1 x^{10} + 1)$
38		$D_{10} \times C_3$	3	5	2^2, 3, 15	1	$x(x^{10} + a_1 x^5 + 1)$
39		D_{24}	6	2	2^2, 6, 12	1	$x(x^4 + a_1 x^2 + 1)$
40		$V_4 \times C_3$	3	2	2, 3^2, 6^2	2	$(x^2 - 1)(x^4 + a_1 x^2 + 1)(x^4 + a_2 x^2 + 1)$
41		$D_6 \times C_3$	3	3	$3^2, 6^2$	1	$(x^6 - 1)(x^6 + a_1 x^3 + 1)$
42		G_8	2	4	2^3, 4, 8	2	$x(x^4 - 1)(x^8 + a_1 x^4 + 1)(x^8 + a_2 x^4 + 1)$
43		G_8	2	20	2, 4, 40	0	$x(x^{20} - 1)$
44		$V_4 \times C_3$	3	2	$2, 3^2, 6^2$	2	$x(x^2 - 1)(x^4 + a_1 x^2 + 1)(x^4 + a_2 x^2 + 1)$
45		$D_{20} \times C_3$	3	10	2, 6, 30	0	$x(x^{10} - 1)$
46		$D_{10} \times C_5$	5	5	2, 10, 25	0	$x(x^5 - 1)$
47		G_8	6	4	2, 12, 24	0	$x(x^4 - 1)$
48		$V_4 \times C_{11}$	11	2	2, 22^2	0	$x(x^2 - 1)$
49		G_9	2	2	$2^4, 4^3$	4	$x(x^4 - 1) \prod_{i=1}^{4} (x^4 + a_i x^2 + 1)$
50		G_9	2	5	$2, 4^2, 10$	1	$x(x^{10} - 1)(x^{10} + a_1 x^5 + 1)$
51	A_4		3	0	2, 3^3	1	$f_1(x)$
52			2	0	2, 3, 4, 6	1	$x(x^4 - 1)(x^4 + 2\sqrt{-3}\, x^2 + 1)\, f_1(x)$
53	S_4	G_{18}	6	0	2, 3, 24	0	$x(x^4 - 1)$
54		$S_4 \times C_3$	3	0	3, 4, 6	0	$x^{12} - 33x^8 - 33x^4 + 1$
55	A_5	$A_5 \times C_3$	3	0	2, 3, 15	0	$x(x^{10} + 11x^5 - 1)$

It's all about the best curves

All equations of curves over $\mathbb{Q}$ have very large coefficients. This leads to the natural question:

Can we find a twist of the curve with smallest coefficients?

This is done via reduction theory of binary forms.

Example

Let $\mathcal{X}$ be a genus 2 curve with equation

$$y^2 = 7\,t^6 - \left(78 + 16\sqrt{5}\right) t^5 + \left(72\sqrt{5} + 617\right) t^4 - \left(320\sqrt{5} + 2148\right) t^3 + \left(4961 + 456\sqrt{5}\right) t^2 - \left(5214 + 672\sqrt{5}\right) t + 3167$$

Then, the algorithm in [MS16] gives

$$y^2 = 359785557t^6 + 4935433518t^5 + 29692428795t^4 + 98737979076t^3 + 193917220155t^2 + 210507034158t + 100220296853$$

Can we get a "better" equation? Can we get "the best" equation?

With a reduction algorithm which will explain later we get

$$y^2 = t^6 + 2t^4 + t^2 + 3$$

Binary forms

There is one important correspondence in all of this:

superelliptic curves $y^n z^{d-n} = f(x,z)$ —— degree d binary forms $f(x,z)$.

Let $k = \bar{k}$ be a characteristic 0 field, $k[x,z]$ be the polynomial ring in two variables, and let V_d denote the $(d+1)$-dimensional subspace of $k[x,z]$ consisting of homogeneous polynomials.

$$f(x,z) = a_0 x^d + a_1 x^{d-1} z + \cdots + a_d z^d \tag{2}$$

of degree d. Elements in V_d are called **binary forms of degree** d.

Let $GL_2(k)$ act as a group of automorphisms on $k[x,z]$ as follows:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k), \text{ then } \quad M \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} ax+bz \\ cx+dz \end{pmatrix}. \tag{3}$$

This action of $GL_2(k)$ leaves V_d invariant and acts irreducibly on V_d.

Given $f(x,z)$ a binary form we denote with $Orb(f)$ its $GL_2(K)$-orbit in V_d.
• Two binary forms f and f' of the same degree d are called **equivalent** or $GL_2(k)$-conjugate if there is an $M \in GL_2(k)$ such that $f' = f^M$.
Problem: Given a binary form $f(x,y)$ over $\mathcal{O}_K$ we determine its integral model with minimal height $\mathrm{H}(f)$.

Minimal height of forms

Let K be a number field and $\mathcal{O}_K$ its ring of integers.

Let $f(x, z)$ be a binary form and $Orb(f)$ its $\mathrm{GL}_2(K)$-orbit in V_d.

Remark

There are only finitely many $f' \in Orb(f)$ such that $H(f') \leq H(f)$.

Define the **minimal height of the binary form** $f(x, z)$ as follows

$$\tilde{H}(f) := \min \left\{ H(f') | f' \in Orb(f), \; H(f') \leq H(f) \right\}$$

From Northcot's theorem there are only finitely many orbitz for a given binary form with height c_0. Define the **minimal absolute height of the binary form** to be the minimal height throughout all the orbitz.

Problem: Given a binary form $f(x, y)$ over $\mathcal{O}_K$ we determine its integral model with minimal height $\mathrm{H}(f)$.

Julia quadratic and Julia invariant

Let $f(x,z) \in \mathbb{R}[x,z]$ be a degree n binary form given as follows

$$f(x,z) = a_0 x^n + a_1 x^{n-1} z + \cdots + a_n z^n$$

and suppose that $a_0 \neq 0$. Let the real roots of $f(x,z)$ be α_i, for $1 \leq i \leq r$ and the pair of complex roots β_j, $\bar{\beta}_j$ for $1 \leq j \leq s$, where $r + 2s = n$. We associate to f the two quadratic forms $T_r(x,z)$ and $S_s(x,z)$ respectively given by the formulas

$$T_r(x,z) = \sum_{i=1}^{r} t_i^2 (x - \alpha_i z)^2, \quad \textit{and} \quad S_s(x,z) = \sum_{j=1}^{s} 2u_j^2 (x - \beta_j z)(x - \bar{\beta}_j z), \tag{4}$$

where t_i, u_j are to be determined.

Proposition

$Q_f = T_r + S_s$ is a positive definite quadratic form with discriminant $\mathfrak{D}_f$

$$\mathfrak{D}_f = \Delta(T_r) + \Delta(S_s) - 8 \sum_{i,j} t_i^2 u_j^2 \left((a_i - a_j)^2 + b_j^2 \right)$$

We define the θ_0 of a binary form as follows

$$\theta_0(f) = \frac{a_0^2 \cdot |\mathfrak{D}_f|^{n/2}}{\prod_{i=1}^{r} t_i^2 \prod_{j=1}^{s} u_j^4}.$$

We pick $t_1, \ldots, t_r, u_1, \ldots, u_s$ such that θ_0 obtains a minimum.

Reduction of higher degree binary forms

Proposition (Julia 1917)

$\theta_0 : \mathbb{R}^{r+s} \to \mathbb{R}$ *obtains a minimum at a unique point.*

Denote $(\bar{t}_1, \dots, \bar{t}_r, \bar{u}_1, \dots, \bar{u}_s)$ this unique point.

The quadratic $\mathcal{J}_f := Q_f(\bar{t}_1, \dots, \bar{t}_r, \bar{u}_1, \dots, \bar{u}_s)(x, z)$ is called the **Julia's quadratic** of f and $\theta_f := \theta_0(\bar{t}_1, \dots, \bar{t}_r, \bar{u}_1, \dots, \bar{u}_s)$ is called the **Julia invariant**.

Theorem (Julia 1917)

i) θ_f *is an* $SL_2(\mathbb{C})$ *invariant*
ii) $\mathcal{J}_f(x, z) \in \mathbb{R}[x, z]$ *is a positive definite quadratic.*

Define the zero map for a binary form as

$$\begin{aligned} \zeta : V_{n,\mathbb{R}} &\longrightarrow V^+_{2,\mathbb{R}} \longrightarrow \mathcal{H}_2 \\ f &\longrightarrow \mathcal{J}_f \longrightarrow \xi(\mathcal{J}_f) \end{aligned}$$

A binary form $f \in \mathbb{R}[x, z]$ is said to be a **reduced binary form** if $\zeta(f) \in \mathcal{F}$.

Algorithm: Finding the minimum absolute height

The following algorithm finds the form with minimal absolute height; [SB15]

Input: A degree n binary form $f(x, y) \in V_{n,\mathcal{O}_K}$
Output: A binary form $F \in V_{n,\mathcal{O}_K}$ which is $\mathrm{GL}_2(\bar{K})$-equivalent to f and has minimal absolute height.

STEP 1: Find the reduced form $f :=$ **red** (f) and the Julia quadratic J associated to it using reduction theory.
STEP 2: Compute the discriminant $\mathfrak{D}_f$ of the quadratic form J.

STEP 3: Let $L := K(\mathfrak{D}_f)$

STEP 4: Determine all quadratics $\{J_1, \dots, J_r\}$ equivalent to J over L and let $M_1, \dots, M_r \in \mathrm{GL}_2(L)$ be the matrices such that $J = J_i^{M_i}$, for $i = 1, \dots, r$.

STEP 5: Compute the set of forms

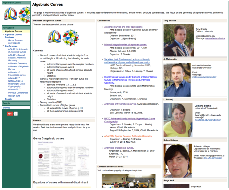
$$f_1 := f^{M_1}, \dots, f_r := f^{M_r}.$$

STEP 6: For each $i = 1, \dots, r$, find the minimal of **red** (f_i)

Computing with superelliptic curves

```
Info(t^6-t^4-t^2-1)

Initial equation of the curve
t^6 - t^4 - t^2 - 1

Clebsch invariants [A, B, C, D] are:
[-28/15, 1288/1875, 40144/140625, 6722048/791015625]

Igusa-Clebsch invariants as in Magma  [I2, I4, I6, I10] are:
[3584, 544768, 573833216, 129922760704]

Igusa invariants  [J_2, J_4, J_6, J_10] are:
[224, 2128, 140096, 123904]

The moduli point for this curve  p=(J2, i1, i2, i3)
(-1, 171/28, -23787/2744, 29403/275365888)

The Automorphism group is isomorphic to the group with GapId
[4, 2]

The invariants  u and v are:
[-1, 0]

The rational model of this curve via Mestre's algorithm is:
Mestre's approach does not find an equation in this case

The minimal field of definition is:  K=Q[d]
0

The universal curve  over K is:

-11480452289971705958247681268959080770459825322061725696/201847884758859965397491918748240635750335059128701686859130859375*t^6 + 6664
The moduli point matches that of f
(-1, 171/28, -23787/2744, 29403/275365888)
```

A database for superelliptic curves



The main website of the project is at algcurves.org. It contains:

- A Sage package for genus 2 curves
- A Sage package for genus 3 hyperelliptic curves
- The genus 2 database with over 1 million curves
  - A python dictionary with integral binary sextics with minimum absolute height H $\leq$ 10.
  - A python dictionary with decomposable integral binary sextics $f(x^2, z^2)$ with minimum absolute H $\leq$ 101
  - A python dictionary with integral binary sextics with moduli height $\mathfrak{H} \leq 20$

Coming conferences:

- Algebraic Curves and their Applications, AMS Meeting, Orlando, September, 2017.

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