Heights on weighted projective spaces

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Outline

Weighted greatest common divisors

We will follow the definitions from [1]. Let $q_0, \ldots, q_n$ be positive integers. A **set of weights** is called the ordered tuple

$$\mathfrak{w} = (q_0, \ldots, q_n).$$

Denote by $r = \gcd(q_0, \ldots, q_n)$ the greatest common divisor of $q_0, \ldots, q_n$. A *weighted integer tuple* is a tuple $\mathbf{x} = (x_0, \ldots x_n) \in \mathbb{Z}^{n+1}$ such that to each coordinate x_i is assigned the weight q_i. We multiply weighted tuples by scalars $\lambda \in \mathbb{Q}$ via

$$\lambda \star (x_0, \ldots, x_n) = \left(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n \right)$$

For an ordered tuple of integers $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$, whose coordinates are not all zero. The **weighted greatest common divisor with respect to the set of weights** $\mathfrak{w}$, denoted by by $wgcd(x_0, \ldots x_n)$, is the largest integer d such that

$$d^{q_i} \mid x_i, \text{ for all } i = 0, \ldots n.$$

A tuple $\mathbf{x} = (x_0, \ldots, x_n)$ with $wgcd(\mathbf{x}) = 1$ is called **normalized**.
The **absolute weighted greatest common divisor** of a tuple $\mathbf{x} = (x_0, \ldots, x_n)$ with respect to the set of weights $\mathfrak{w} = (q_0, \ldots, q_n)$ is the largest real number d such that

$$d^{q_i} \mid x_i, \text{ for all } i = 0, \ldots n.$$

We will denote the weighted greatest common divisor by $\overline{wgcd}(x_0, \ldots x_n)$. A tuple $\mathbf{x}$ with $\overline{wgcd}(\mathbf{x}) = 1$ is called **absolutely normalized**.

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We will denote the weighted greatest common divisor by $\overline{wgcd}(x_0, \ldots x_n)$. A tuple $\mathbf{x}$ with $\overline{wgcd}(\mathbf{x}) = 1$ is called **absolutely normalized**.

Example

Consider the set of weights $\mathfrak{w} = (2, 4, 6, 10)$ *and a tuple*

$$\mathbf{x} = \left(3 \cdot 5^2, 3^2 \cdot 5^4, 3^3 \cdot 5^6, 3^5 \cdot 5^{10}\right) \in \mathbb{Z}^4.$$

Then, $wgcd(\mathbf{x}) = 5$ *and* $\overline{wgcd}(\mathbf{x}) = 5 \cdot \sqrt{3}$. *Notice that*

$$\frac{1}{5} \star \mathbf{x} = (3, 3^2, 3^3, 3^5), \qquad \frac{1}{5\sqrt{3}} \star \mathbf{x} = (1, 1, 1, 1),$$

We summarize in the following lemma.

Lemma

For any weighted integral tuple $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$*, the tuple*

$$\mathbf{y} = \frac{1}{wgcd(\mathbf{x})} \star \mathbf{x},$$

is integral and normalized. Moreover, the tuple

$$\bar{\mathbf{y}} = \frac{1}{\overline{wgcd}(\mathbf{x})} \star \mathbf{x},$$

is also integral and absolutely normalized. If $\gcd(q_0, \ldots, q_n) = 1$*, then* $wgcd(\mathbf{x}) = \overline{wgcd}(\mathbf{x})$.

Outline

Let K be a field of characteristic zero and $(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}$ a fixed tuple of positive integers called **weights**. Consider the action of $K^\star = K \setminus \{0\}$ on $\mathbb{A}^{n+1}(K)$ as follows

$$\lambda \star (x_0, \ldots, x_n) = \left(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n\right)$$

for $\lambda \in K^*$. The quotient of this action is called a **weighted projective space** and denoted by $\mathbb{WP}^n_{(q_0, \ldots, q_n)}(K)$. It is the projective variety $Proj\,(K[x_0, ..., x_n])$ associated to the graded ring $K[x_0, \ldots, x_n]$ where the variable x_i has degree q_i for $i = 0, \ldots, n$. We denote greatest common divisor of $q_0, \ldots, q_n$ by $\gcd(q_0, \ldots, q_n)$. The space $\mathbb{WP}^n_w$ is called **well-formed** if

$$\gcd(q_0, \ldots, \hat{q}_i, \ldots, q_n) = 1, \quad \text{for each } i = 0, \ldots, n.$$

While most of the papers on weighted projective spaces are on well-formed spaces, we do not assume that here. We will denote a point $\mathfrak{p} \in \mathbb{WP}^n_w(K)$ by $\mathfrak{p} = [x_0 : x_1 : \cdots : x_n]$. Let K be a number field and $\mathcal{O}_K$ its ring of integers. The group action $K^\star$ on $\mathbb{A}^{n+1}(K)$ induces a group action of $\mathcal{O}_K$ on $\mathbb{A}^{n+1}(K)$. By $Orb(\mathfrak{p})$ we denote the $\mathcal{O}_K$-orbit in $\mathbb{A}^{n+1}(\mathcal{O}_K)$ which contains $\mathfrak{p}$. For any point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}^n_w(K)$ we can assume, without loss of generality, that $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}^n_w(\mathcal{O}_K)$. The height for weighted projective spaces will be defined in the next section.

For the rest of this section we assume $K = \mathbb{Q}$. For the tuple $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ we define the **weighted greatest common divisor** with respect to the absolute value $|\cdot|_v$, denoted by $wgcd_v(\mathbf{x})$,

$$wgcd_v(\mathbf{x}) := \prod_{\substack{d^{q_i} \, | \, x_i \\ d \in \mathbb{Z}}} |d|_v$$

as the product of all divisors $d \in \mathbb{Z}$ such that for all $i = 0, \ldots, n$, we have $d^i \mid x_i$. We will call a point $\mathfrak{p} \in \mathbb{WP}_{\mathfrak{w}}^n(\mathbb{Q})$ **normalized** if $wgcd(\mathfrak{p}) = 1$.

Definition
We will call a point $\mathfrak{p} \in \mathbb{WP}_{\mathfrak{w}}^n(\mathbb{Q})$ a **normalized point** if the weighted greatest common divisor of its coordinates is 1.

Lemma
Let $\mathfrak{w} = (q_0, \ldots, q_n)$ be a set of weights and $d = \gcd(q_0, \ldots, q_n)$. For any point $\mathfrak{p} \in \mathbb{WP}_{\mathfrak{w}}^n(\mathbb{Q})$, the point

$$\mathfrak{q} = \frac{1}{wgcd(\mathfrak{p})} \star \mathfrak{p}$$

is normalized. Moreover, this normalization is unique up to a multiplication by a d-root of unity.

Proof: Let $\mathfrak{p} = [x_0 : \ldots, x_n] \in \mathbb{WP}_w^n(\mathbb{Q})$ and $\mathfrak{p}_1 = [\alpha_0 : \cdots : \alpha_n]$ and $\mathfrak{p}_2 = [\beta_0 : \cdots : \beta_n]$ two different normalizations of $\mathfrak{p}$. Then exists non-zero $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that

$$\mathfrak{p} = \lambda_1 \star \mathfrak{p}_1 = \lambda_2 \star \mathfrak{p}_2,$$

or in other words

$$(x_0, \ldots, x_n) = \left(\lambda_1^{q_0} \alpha_0, \ldots, \lambda_1^{q_i} \alpha_i, \ldots \right) = \left(\lambda_2^{q_0} \beta_0, \ldots, \lambda_2^{q_i} \beta_i, \ldots \right).$$

Thus,

$$(\alpha_0, \ldots, \alpha_i, \ldots, \alpha_n) = \left(r^{q_0} \beta_0, \ldots, r^{q_i} \beta_i, \ldots, r^{q_n} \beta_n \right).$$

for $r = \frac{\lambda_2}{\lambda_1} \in K$. Thus, $r^{q_i} = 1$ for all $i = 0, \ldots, n$. Therefore, $r^d = 1$. This completes the proof.

□

Thus we have the following:

Corollary
For any point $\mathfrak{p} = [x_0 : \cdots : \mathbf{x}_n] \in \mathbb{WP}_w^n(\mathbb{Q})$, if the greatest common divisors of non-zero coordinates is 1, then the normalization of $\mathfrak{p}$ is unique.

Here is an example which illustrates the Lemma.

Example

Let $\mathfrak{p} = [x_0, x_1, x_2, x_3] \in \mathbb{WP}^3_{(2,4,6,10)}(\mathbb{Q})$ be a normalized point. Hence,

$$wgcd(x_0, x_1, x_2, x_3) = 1.$$

Since $d = \gcd(2, 4, 6, 10) = 2$, then we can take r such that $r^2 = 1$. Hence, $r = \pm 1$. Therefore, the point

$$(-1) \star \mathfrak{p} = [-x_0 : x_1 : -x_2 : -x_3]$$

is also be normalized.

However, if $\mathfrak{p} = [x_0, x_1, x_2, x_3] \in \mathbb{WP}^3_{(1,2,3,5)}(\mathbb{Q})$ is normalized then it is unique, unless some of the coordinates are zero. For example the points $[0, 1, 0, 0]$ and $[0, -1, 0, 0]$ are equivalent and both normalized.

Next we give two examples, which were the main motivation behind this note.

Example (Weighted projective space of binary sextics)

*The ring of invariants of binary sextics is generated by the **basic arithmetic invariants**, or as they sometimes called, **Igusa invariants** (J_2, J_4, J_6, J_{10}) as defined in [2]. Two genus 2 curves $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic if and only if there exists $\lambda \in K^*$ such that*

$$J_{2i}(\mathcal{X}) = \lambda^{2i} J_{2i}(\mathcal{X}'), \quad \text{for} \quad i = 1, 2, 3, 5.$$

We take the set of weights $\mathfrak{w} = (2, 4, 6, 10)$ and considered the weighted projective space $\mathbb{WP}_{(2,4,6,10)}(\mathbb{Q})$. Thus, the invariants of a sextic define a point in a weighted projective space $[J_2 : J_4 : J_6 : J_{10}] \in \mathbb{WP}_{\mathfrak{w}}(\mathbb{Q})$ and every genus 2 curve correspond to a point in $\mathbb{WP}_{\mathfrak{w}}^3(\mathbb{Q}) \setminus \{J_{10} \neq 0\}$. There is a bijection between

$$\phi : \quad \mathbb{WP}_{(2,4,6,10)}^3(\mathbb{Q}) \setminus \{J_{10} \neq 0\} \to \mathcal{M}_2,$$

with ϕ provided explicitly in [3, Theorem 1].

Using the notion of a normalized point as above we have the following:

Corollary

Normalized points in $\mathbb{WP}^3_{(2,4,6,10)}(\mathbb{Q})$ occur in pairs. In other words, for every normalized point $\mathfrak{p} = [J_2, J_4, J_6, J_{10}]$, there is another normalized point $\mathfrak{p}' = [-J_2, J_4, -J_6, -J_{10}]$ equivalent to $\mathfrak{p}$. Moreover, $\mathfrak{p}$ and $\mathfrak{p}'$ are isomorphic over the Gaussian integers.

Proof: Let $\mathcal{X}$ be a genus 2 curve with equation $y^2 = f(x)$ and $[J_2, J_4, J_6, J_{10}]$ its corresponding invariants. The transformation $x \mapsto \sqrt{-1} \cdot x$ with give a curve $\mathcal{X}'$ with invariants $[-J_2 : J_4 : -J_6 : -J_{10}]$ and the same weighted moduli height.

If two weighted moduli points have the same minimal absolute height, then they differ up to a multiplication by a unit. Hence,

$$[J_2' : J_4' : J_6' : J_{10}'] = [d^2 \cdot J_2 : d^4 \cdot J_4 : d^6 \cdot J_6 : d^{10} \cdot J_{10}]$$

such that d^2 is a unit. Then, $d^2 = \pm 1$. Hence, $d = \sqrt{-1}$. $\qquad\square$

So unfortunately for any genus 2 curve we have two corresponding normalized points $[\pm J_2, J_4, \pm J_6, \pm J_{10}]$. In [4] this problem is solved by taking always the point $[|J_2|, J_4, \pm J_6, \pm J_{10}]$ or by considering the space $\mathbb{WP}^3_{(1,2,3,5)}(\mathbb{Q})$ instead.

Example (Weighted projective space of binary octavics)

Every irreducible, smooth, hyperelliptic genus 3 curve has equation $y^2 z^6 = f(x, z)$, where $f(x, z)$ is a binary octavic with non-zero discriminant. The ring of invariants of binary octavics is generated by invariants $J_2, \ldots, J_8$, which satisfy an algebraic equation as in [5, Thm. 6]. Two genus 3 hyperelliptic curves $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic over a field K if and only if there exists some $\lambda \in k \setminus \{0\}$ such that

$$J_i(\mathcal{X}) = \lambda^i J_i(\mathcal{X}'), \text{ for } i = 2, \ldots, 7.$$

There is another invariant J_{14} given in terms of $J_2, \ldots J_7$ which is the discriminant of the binary octavic.

Hence, there is a bijection between the hyperelliptic locus in the moduli space of genus 3 curves and the weighted projective space $\mathbb{WP}^5_{(2,3,4,5,6,7)}(K) \setminus \{J_{14} \neq 0\}$. Since $d = \gcd(2, 3, 4, 5, 6, 7) = 1$ then we have:

Corollary

For every genus 3 hyperelliptic curve $\mathcal{X}$, defined over a field K, the corresponding normalized point

$$\mathfrak{p} = [J_2 : J_3 : J_4 : J_5 : J_6 : J_7] \in \mathbb{WP}^5_{(2,3,4,5,6,7)}(K)$$

is unique.

Example

Consider the curve $y^2 = x^8 - 1$. The moduli point in $\mathbb{WP}^5_{\mathfrak{w}}(\mathbb{Q})$ is

$$\mathfrak{p} = \left[-2^3 \cdot 5 \cdot 7, 0, 2^{10} \cdot 7^4, 0, 2^{15} \cdot 7^6, 0, -2^{19} \cdot 5 \cdot 7^8 \right]$$

For any point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}_\mathfrak{w}^n(\mathbb{Q})$ we may assume that $x_i \in \mathbb{Z}$ for $i = 0, \ldots, n$ and define

$$\overline{wgcd}(\mathfrak{p}) = \prod_{\lambda \in \bar{\mathbb{Q}}, \, \lambda^{q_i} | x_i} |\lambda|$$

as the product of all $\lambda \in \bar{\mathbb{Q}}$, such that for all $i = 0, \ldots, n$, $\lambda^i \in \mathbb{Z}$ and $\lambda^i | x_i$. A point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}_\mathfrak{w}^n(\mathbb{Q})$ is called **absolutely normalized** or **normalized over $\bar{\mathbb{Q}}$** if $\overline{wgcd}(\mathfrak{p}) = 1$.

Definition
A point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}_\mathfrak{w}^n(\mathbb{Q})$ is called **absolutely normalized** or **normalized over the algebraic closure** if $\overline{wgcd}(\mathfrak{p}) = 1$.

Lemma
For any point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}_\mathfrak{w}^n(\mathbb{Q})$ its normalization over the algebraic closure

$$\bar{\mathfrak{p}} = \frac{1}{\overline{wgcd}(\mathfrak{p})} \star \mathfrak{p}$$

is unique up to a multiplication by a d-th root of unity.

Proof: Let $\mathfrak{p} = [x_0 : \ldots, x_n] \in \mathbb{WP}_w^n(\mathbb{Q})$ and $\mathfrak{p}_1 = [\alpha_0 : \cdots : \alpha_n]$ and $\mathfrak{p}_2 = [\beta_0 : \cdots : \beta_n]$ two different normalizations of $\mathfrak{p}$ over $\bar{\mathbb{Q}}$. Then exists non-zero $\lambda_1, \lambda_2 \in \bar{\mathbb{Q}}$ such that

$$\mathfrak{p} = \lambda_1 \star \mathfrak{p}_1 = \lambda_2 \star \mathfrak{p}_2,$$

or in other words

$$(x_0, \ldots, x_n) = \left(\lambda_1^{q_0} \alpha_0, \ldots, \lambda_1^{q_i} \alpha_i, \ldots \right) = \left(\lambda_2^{q_0} \beta_0, \ldots, \lambda_2^{q_i} \beta_i, \ldots \right).$$

Thus,

$$(\alpha_0, \ldots, \alpha_i, \ldots, \alpha_n) = \left(r^{q_0} \beta_0, \ldots, r^{q_i} \beta_i, \ldots, r^{q_n} \beta_n \right).$$

for $r = \frac{\lambda_2}{\lambda_1}$. Then $r^{q_i} = 1$ for all $i = 0, \ldots, n$. Then $r = \frac{d}{\ldots} = 1$. This completes

Two points $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathbb{WP}^n_{\mathfrak{w}}(\mathbb{Q})$ are called **twists** of each other if they are equivalent in $\mathbb{WP}^n_{\mathfrak{w}}(\overline{\mathbb{Q}})$ but $Orb_{\mathbb{Q}}(\mathfrak{p})$ is not the same as $Orb_{\mathbb{Q}}(\mathfrak{q})$. Hence, we have the following.

Lemma

Let $\mathfrak{p}$ and $\mathfrak{p}'$ be normalized points in $\mathbb{WP}^n_{\mathfrak{w}}(\mathbb{Q})$. Then $\mathfrak{p}$ and $\mathfrak{p}'$ are twists of each other if and only if there exists $\lambda \in \overline{\mathbb{Q}}^\star$ such that $\lambda \star \mathfrak{p} = \mathfrak{p}'$.

Next we see another example from genus 2 curves.

Example

Let $\mathcal{X}$ be the genus two curve with equation $y^2 = x^6 - 1$ and J_2, J_4, J_6, and J_{10} its Igusa invariants. Then the isomorphism class of $\mathcal{X}$ is determined by the point $\mathfrak{p} = [240, 1620, 119880, 46656] \in \mathbb{WP}^3_{(2,4,6,10)}(\mathbb{Q})$. Thus,

$$\mathfrak{p} = [240, 1620, 119880, 46656] = [2^4 \cdot 3 \cdot 5;\ 2^2 \cdot 3^4 \cdot 5;\ 2^3 \cdot 3^4 \cdot 5 \cdot 37;\ 2^6 \cdot 3^6].$$

Therefore,

$$wgcd(240, 1620, 119880, 46656) = 1$$
$$\overline{wgcd}(240, 1620, 119880, 46656) = \sqrt{6}.$$

Hence, $\mathfrak{p}$ is normalized but not absolutely normalized. The point $\mathfrak{p}$ has twists,

$$\mathfrak{p}_1 = \frac{1}{\sqrt{2}} \star \mathfrak{p}_1 = [120, 405, 14985, 1458] = [2^3 \cdot 3 \cdot 5\ :\ 3^4 \cdot 5\ :\ 3^4 \cdot 5 \cdot 37\ :\ 2 \cdot 3^6],$$

$$\mathfrak{p}_2 = \frac{1}{\sqrt{3}} \star \mathfrak{p}_1 = [80, 180, 4440, 192] = [2^4 \cdot 5\ :\ 2^2 \cdot 3^2 \cdot 5\ :\ 2^3 \cdot 3 \cdot 5 \cdot 37\ :\ 2^6 \cdot 3],$$

and the absolutely normalized point of $\mathfrak{p}$ which is

We can do better even with the genus 3 curve from Example 5.

Example

The normalized moduli point in $\mathbb{WP}^5_{\mathfrak{w}}(\mathbb{Q})$ the curve $y^2 = x^8 - 1$ is

$$\frac{1}{2} \star \mathfrak{p} = \left[-2 \cdot 5 \cdot 7, 0, 2^6 \cdot 7^4, 0, 2^9 \cdot 7^6, 0, -2^{11} \cdot 5 \cdot 7^8 \right].$$

Then, $\overline{wgcd}(\mathfrak{p}) = \frac{\mathfrak{i}}{\sqrt{14}}$, for $\mathfrak{i}^2 = -1$. Then its absolutely normalized form is

$$\bar{\mathfrak{p}} = \left[5, 0, 2^4 \cdot 7^2, 0, 2^6 \cdot 7^3, 0, -2^7 \cdot 5 \cdot 7^4 \right].$$

$\square$

In the next section we will introduce some measure of the magnitude of points in weighted moduli spaces $\mathbb{WP}^n_{\mathfrak{w}}(K)$ and show that the process of normalization and absolute normalization lead us to the representation of points in $\mathbb{WP}^n_{\mathfrak{w}}(K)$ with smallest possible coordinates.

Outline

Let K be an algebraic number field and $[K : \mathbb{Q}] = n$ and its ring of integers $\mathcal{O}_K$. With M_K we denote the set of all absolute values in K. For $v \in M_K$, the **local degree at** v, denoted n_v is $n_v = [K_v : \mathbb{Q}_v]$, where $K_v, \mathbb{Q}_v$ are the completions with respect to v. As above $\mathbb{WP}^n(K)$ is the projective space with weights $w = (q_0, \dots, q_n)$, and $\mathfrak{p} \in \mathbb{WP}^n(K)$ a point with coordinates $\mathfrak{p} = [x_0, \dots, x_n]$, for $x_i \in K$. The **multiplicative height** of $\mathfrak{p}$ is defined as follows

$$\mathfrak{h}_K(\mathfrak{p}) := \prod_{v \in M_K} \max \left\{ |x_0|_v^{n_v/q_0}, \dots, |x_n|_v^{n_v/q_n} \right\}$$

Let $\mathfrak{p} = [x_0, \dots, x_n] \in \mathbb{WP}^n(\mathbb{Q})$ with weights $w = (q_0, \dots, q_n)$. It is clear that $\mathfrak{p}$ will have a representative $[y_0, \dots, y_n]$ such that $y_i \in \mathbb{Z}$ for all i and $wgcd(y_0, \dots, y_n) = 1$. With such representative for the coordinates of $\mathfrak{p}$, the non-Archimedean absolute values give no contribution to the height, and we obtain

$$\mathfrak{h}_{\mathbb{Q}}(\mathfrak{p}) = \max_{0 \le j \le n} \left\{ |x_j|_\infty^{1/q_j} \right\}$$

So for a tuple $\mathbf{x} = (x_0 : \cdots : x_n)$ the height of the corresponding point $\mathfrak{p} = [\mathbf{x}]$ is

$$\mathfrak{h}(\mathfrak{p}) = \frac{1}{wgcd(\mathbf{x})} \max \left\{ |x_0|^{1/q_0}, \ldots, |x_n|^{1/q_n} \right\}.$$

We combine some of the properties of $\mathfrak{h}(\mathfrak{p})$ in the following:

Proposition

Then the following are true:
i) The function $\mathfrak{h} : \mathbb{WP}^n_{\mathfrak{w}}(\mathbb{Q}) \to \mathbb{R}$ is well-defined.
ii) A normalized point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}^n_{\mathfrak{w}}(\mathbb{Q})$ is the point with smallest coordinates in its orbit $Orb(\mathfrak{p})$.
iii) For any constant $c > 0$ there are only finitely many points $\mathfrak{p} \in \mathbb{WP}^n_w(\mathbb{Q})$ such that $\mathfrak{h}(\mathfrak{p}) \leq c$.

Proof: i) It is enough to show that two normalizations of the same point $\mathfrak{p} \in \mathbb{WP}^n_{\mathfrak{w}}(\mathbb{Q})$ have the same height. Let $\mathfrak{p}$ and $\mathfrak{q}$ be such normalizations. Then from Lemma 2 we have $\mathfrak{p} = r \star \mathfrak{q}$, where $r^d = 1$. Thus,

$$\mathfrak{h}(\mathfrak{p}) = \mathfrak{h}(r \star \mathfrak{q}) = |r| \cdot \mathfrak{h}(\mathfrak{q}) = \mathfrak{h}(\mathfrak{q}).$$

ii) This is obvious from the definition.
iii) Let $\mathfrak{p} \in \mathbb{WP}^n_{\mathfrak{w}}(\mathbb{Q})$. It is enough to count only normalized points $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}^n_w(\mathbb{Z})$ such that $\mathfrak{h}(\mathfrak{p}) \leq c$. For every coordinate x_i there are only finitely values in $\mathbb{Z}$ such that $|x_i|_v^{1/q_i} \leq c$. Hence, the result holds.

$\square$

Part iii) of the above is the analogue of the Northcott's theorem in projective spaces.

Remark

If the set of weights $\mathfrak{w} = (1, \ldots 1)$ *then* $\mathbb{WP}^n_{\mathfrak{w}}(\mathbb{Q})$ *is simply the projective space* $\mathbb{P}^n(\mathbb{Q})$ *and the height* $\mathfrak{h}(\mathfrak{p})$ *correspond to the height of a projective point as defined in* [7].

Let's see an example how to compute the height of a point.

Example

Let $\mathfrak{p} = (2^2, 2 \cdot 3^4, 2^6 \cdot 3, 2^{10} \cdot 5^{10}) \in \mathbb{WP}^3_{(2,4,6,10)}(\mathbb{Q})$. *Notice that* $\mathfrak{p}$ *is normalized, which implies that*

$$\mathfrak{h}(\mathfrak{p}) = \max \left\{ 2, 2^{1/4} \cdot 3, 3^{1/6}, 2 \cdot 5 \right\} = 10$$

However, the point $\mathfrak{q} = (2^2, 2^4 \cdot 3^4, 2^6 \cdot 3, 2^{10} \cdot 5^{10}) \in \mathbb{WP}^3_{(2,4,6,10)}(\mathbb{Q})$ *can be normalized to* $(1, 3^4, 3, 5^{10})$ *which has height*

$$\mathfrak{h}(\mathfrak{q}) = \max \left\{ 1, 3, 3^{1/6}, 5 \right\} = 5.$$

A proof for the following will be provided in [8].

Lemma

Let $\mathfrak{p} \in \mathbb{WP}^n(K)$ *with weights* $w = (q_0, \ldots, q_n)$ *and* L/K *be a finite extension. Then,*

$$\mathfrak{h}_L(P) = \mathfrak{h}_K(P)^{[L:K]}.$$

We can define the height on $\mathbb{WP}^n(\bar{\mathbb{Q}})$. The height of a point on $\mathbb{WP}^n(\bar{\mathbb{Q}})$ is called the **weighted absolute (multiplicative) height** and is the function

$$\tilde{\mathfrak{h}} : \mathbb{WP}^n(\bar{\mathbb{Q}}) \to [1, \infty)$$
$$\tilde{\mathfrak{h}}(\mathfrak{p}) = \mathfrak{h}_K(P)^{1/[K:\mathbb{Q}]},$$

where $\mathfrak{p} \in \mathbb{WP}^n(K)$, for any K. Then, the absolute weight height is given by

$$\tilde{\mathfrak{h}}_{\mathbb{Q}}(\mathfrak{p}) = \frac{1}{wgcd(\mathfrak{p})} \max \left\{ |x_0|^{1/q_0}, \ldots, |x_n|^{1/q_n} \right\} \tag{1}$$

Let's see an example which compares the height of a point with the absolute height.

Example

Let $\mathfrak{p} = [0 : 2 : 0 : 0] \in \mathbb{WP}^3_{(2,4,6,10)}(\mathbb{Q})$. Then $\mathfrak{p}$ is normalized and therefore $\mathfrak{h}(\mathfrak{p}) = 2$. However, it absolute normalization is $\mathfrak{q} = \frac{1}{2^{1/4}} \star \mathfrak{p} = [0 : 0 : 1 : 0]$. Hence, $\tilde{\mathfrak{h}}(\mathfrak{p}) = 1$.

Remark

As a consequence of the above results it is possible to "sort" the points in $\mathbb{WP}^n_{\mathfrak{w}}(\bar{K})$ according to the absolute height and even determine all the twists for each point when the weighted projective space is not well-formed. This is used in [4] to create a database of genus 2 curves and similarly in [6] for genus 3 hyperelliptic curves.

The **weighted absolute height** of $\mathfrak{p} = [\mathbf{x}] \in \mathbb{WP}^n_\mathfrak{w}(K)$, where $\mathbf{x} = (x_0 : \cdots : x_n)$, for any number field K, is

$$\tilde{\mathfrak{h}}_K(\mathfrak{p}) = \frac{1}{wgcd(\mathbf{x})} \prod_{v \in M_K} \max \left\{ |x_0|^{1/q_0}, \ldots, |x_n|^{1/q_n} \right\} \tag{2}$$

The concept of weighted absolute height correspond to that of absolute height in [7]. In [7] a curve with minimum absolute height has an equation with the smallest possible coefficients. In this paper, the absolute height says that there is a representative tuple of $\mathfrak{p} \in \mathbb{WP}^n_\mathfrak{w}(K)$ with smallest magnitude of coordinates.
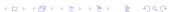Then we have the following:

Proposition

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Then the following are true:
i) The absolute height function $\tilde{\mathfrak{h}}_K : \mathbb{WP}^n_\mathfrak{w}(K) \to \mathbb{R}$ is well-defined.
ii) $\tilde{\mathfrak{h}}(\mathfrak{p})$ is the minimum of heights of all twists of $\mathfrak{p}$.
iii) For any constant $c > 0$ there are only finitely many points $\mathfrak{p} \in \mathbb{WP}^n_w(K)$ such that $\tilde{\mathfrak{h}}(\mathfrak{p}) \leq c$.

Proof: Part ii) and iii) are obvious. We prove part i). We have to show that two different normalizations over the algebraic closure have the same absolute height. Let $\mathfrak{p}$ and $\mathfrak{q}$ be such normalizations. Then from Lemma 3 we have $\mathfrak{p} = r \star \mathfrak{q}$, where $r^d = 1$. Thus,

$$\tilde{\mathfrak{h}}(\mathfrak{p}) = \tilde{\mathfrak{h}}(r \star \mathfrak{q}) = |r| \cdot \tilde{\mathfrak{h}}(\mathfrak{q}) = \mathfrak{h}(\mathfrak{q}).$$

This completes the proof. $\qquad\square$

For more details we direct the reader to [8]. Let's revisit again our example from genus 2 curves.

Example

Let $\mathcal{X}$ be the genus two curve with equation $y^2 = x^6 - 1$ and moduli point
$\mathfrak{p} = [240, 1620, 119880, 46656] \in \mathbb{WP}^3_{(2,4,6,10)}(\mathbb{Q})$. *We showed that $\mathfrak{p}$ is normalized and therefore has height $\mathfrak{h}(\mathfrak{p}) = 4\sqrt{15}$. Its absolute normalization is*

$$\bar{\mathfrak{p}} = [40, 45, 555, 6] = [2^3 \cdot 5, \, 3^2 \cdot 5, \, 3 \cdot 5 \cdot 37, \, 2 \cdot 3]$$

Hence, the absolute height is $\tilde{\mathfrak{h}}(\mathfrak{p}) = 2\sqrt{10}$.

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