Curves of genus 2 with \((N,N)\) decomposable Jacobians

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Abstract
Let \(C\) be a curve of genus 2 and \(\psi_1 : C \rightarrow E_1\) a map of degree \(n\), from \(C\) to an elliptic curve \(E_1\), both curves defined over \(\mathbb{C}\). This map induces a degree \(n\) map \(\phi_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^1\) which we call a Frey-Kani covering. We determine all possible ramifications for \(\phi_1\). If \(\psi_1 : C \rightarrow E_1\) is maximal then there exists a maximal map \(\psi_2 : C \rightarrow E_2\), of degree \(n\), to some elliptic curve \(E_2\) such that there is an isogeny of degree \(n^2\) from the Jacobian \(J_C\) to \(E_1 \times E_2\). We say that \(J_C\) is \((n,n)\)-decomposable. If the degree \(n\) is odd the pair \((\psi_2, E_2)\) is canonically determined. For \(n = 3, 5,\) and 7, we give arithmetic examples of curves whose Jacobians are \((n,n)\)-decomposable.

1. Introduction
Curves of genus 2 with non-simple Jacobians are of much interest. Their Jacobians have large torsion subgroups, e.g. Howe, Leprévost, and Poonen have found a family of genus 2 curve with 128 rational points in its Jacobian, see (5). For other applications of genus 2 curves with \((n,n)\)-decomposable Jacobians see Frey (2). In this paper, we discuss genus 2 curves \(C\) whose function fields have maximal elliptic subfields. These elliptic subfields occur in pairs \((E_1, E_2)\) and we call each the complement of the other in \(J_C\). The Jacobian of \(C\) is isogenous to \(E_1 \times E_2\). Let \(\psi : C \rightarrow E\) be a maximal cover (cf. section 4) of odd degree \(n\). The moduli space parameterizing these covers is a surface, more precisely the product of modular curves \(X(n) \times X(n)/\Delta\), see Kani (6). When \(\psi : C \rightarrow E\) is degenerate (cf. section 2), this moduli space is a curve. Getting algebraic descriptions for these spaces is extremely difficult for large \(n\) (e.g. \(n \geq 7\)). Also, one would like to know how the elements of the pair \((E_1, E_2)\) relate to each other.

In sections 2 and 3 we define a Frey-Kani covering and determine all their possible ramifications. In section 4 we consider maximal covers. These covers allow us to determine the complement of \(E_1\) uniquely. The last section deals with some applications when \(n = 3, 5,\) or 7.
2. Frey - Kani covers

Let $C$ and $E$ be curves of genus 2 and 1, respectively. Both are smooth, projective curves defined over $\mathbb{C}$. Let $\psi : C \rightarrow E$ be a covering of degree $n$. We say that $E$ is an degree $n$ elliptic subcover of $C$. From the Riemann-Hurwitz formula, $\sum_{P \in C} (e_\psi(P) - 1) = 2$ where $e_\psi(P)$ is the ramification index of points $P \in C$, under $\psi$. Thus, we have two points of ramification index 2 or one point of ramification index 3. The two points of ramification index 2 can be in the same fiber or in different fibers. Therefore, we have the following cases of the covering $\psi$:

**Case I.** There are $P_1, P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) \neq \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

**Case II.** There are $P_1, P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) = \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

**Case III.** There is $P_1 \in C$ such that $e_\psi(P_1) = 3$, and $\forall P \in C \setminus \{P_1\}$, $e_\psi(P) = 1$.

In case I (resp. II, III) the cover $\psi$ has 2 (resp. 1) branch points in $E$.

Denote the hyperelliptic involution of $C$ by $w$. We choose $O$ in $E$ such that $w$ restricted to $E$ is the hyperelliptic involution on $E$, see (3) or (7). We denote the rest of the involution $\psi$ restricted to $E$.

**Lemma 2.1:**

1. $\psi(W) \subset E[2]$
2. If $n$ is an odd number then
   1) $\psi(W) = E[2]$
   2) $\psi(w(P)) = w(\psi(P))$.
3. If $n$ is an even number then for all $Q \in E[2]$, $\#(\psi^{-1}(Q) \cap W) = (2) \mod (2)$

Let $\pi_C : C \rightarrow \mathbb{P}^1$ and $\pi_E : E \rightarrow \mathbb{P}^1$ be the natural degree 2 projections. The hyperelliptic involution permutes the points in the fibers of $\pi_C$ and $\pi_E$. The ramified points of $\pi_C, \pi_E$ are respectively points in $W$ and $E[2]$ and their ramification index is 2. There is $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that the diagram commutes, see Frey (3) or Kuhn (7).

$\begin{array}{ccc}
C & \xrightarrow{\pi_C} & \mathbb{P}^1 \\
\psi & \downarrow & \phi \\
E & \xrightarrow{\pi_E} & \mathbb{P}^1
\end{array}$

The covering $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ will be called the corresponding Frey-Kani covering of $\psi : C \rightarrow E$. It has first appeared in (3) and (2). The term, Frey-Kani covering, has first been used by Fried in (4).
3. The ramification of Frey-Kani coverings

In this section we will determine the ramification of Frey-Kani coverings \( \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \). First we fix some notation. For a given branch point we will denote the ramification of points in its fiber as follows. Any point \( P \) of ramification index \( m \) is denoted by \((m)\). If there are \( k \) such points then we write \((m)^k\). We omit writing symbols for unramified points, in other words \((1)^k\) will not be written. Ramification data between two branch points will be separated by commas. We denote by \( \pi_E(E[2]) = \{q_1, \ldots, q_4\} \) and \( \pi_C(W) = \{w_1, \ldots, w_6\} \).

3.1. The case when \( n \) is odd

The following theorem classifies the ramification types for the Frey-Kani coverings \( \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) when the degree \( n \) is odd.

**Theorem 3.1:** If \( \psi : C \rightarrow E \) is a covering of odd degree \( n \) then the three cases of ramification for \( \psi \) induce the following cases for \( \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \).

**Case I:** (the generic case)

\[
\left( (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right)
\]

Or the following degenerate cases:

**Case II:** (the 4-cycle case and the dihedral case)

\[
i) \left( (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (4)^{\frac{1}{2}} (2)^{\frac{n}{3}} \right)
\]

\[
ii) \left( (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}} \right)
\]

\[
iii) \left( (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (4)^{\frac{1}{2}} (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}} \right)
\]

**Case III:** (the 3-cycle case)

\[
i) \left( (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (3)^{\frac{1}{2}} (2)^{\frac{n}{3}} \right)
\]

\[
ii) \left( (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}}, (3)^{\frac{1}{2}} (2)^{\frac{n}{3}}, (2)^{\frac{n}{3}} \right)
\]

**Proof:** From lemma 2.2 we can assume that \( \phi(w_i) = q_i \) for \( i \in \{1, 2, 3\} \) and \( \phi(w_4) = \phi(w_5) = \phi(w_6) = q_4 \). Next we consider the three cases for the ramification of \( \psi : C \rightarrow E \) and see what ramifications they induce on \( \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \).

Suppose that \( P \in \psi^{-1}(E[2]) \setminus W \) and \( e_\psi(P) = 1 \). Then \( e_{\psi}(P) \cdot e_{\pi_E}(\psi(P)) = e_{\pi_C}(P) \cdot e_{\phi}(\pi_C(P)) = 2 \), so \( e_{\phi}(\pi_C(P)) = 2 \).

**Case I:** There are \( P_1 \) and \( P_2 \) in \( C \) such that \( e_{\psi}(P_1) = e_{\psi}(P_2) = 2 \) and \( \psi(P_1) \neq \psi(P_2) \). By lemma 2.1, \( e_{\psi}(w(P_1)) = 1 \). So \( w(P_1) = P_1 \) or \( w(P_1) = P_2 \).

Suppose that \( w(P_1) = P_1 \), so \( P_1 \in W \). If \( \pi_C(P_1) = w_i \) for \( i \in \{1, 2, 3\} \), say \( \pi_C(P_1) = w_1 \), then \( e_{\pi_C}(P_1) = e_{\psi_{\phi}}(P_1) = 4 \), which implies that \( e_{\phi}(w_1) = 2 \). All other points in the fiber of \( \pi_E \circ \psi(P_1) = q_1 \) have ramification index 2 under \( \phi \). So \( \phi \) has even degree, which is a contradiction. If \( \pi_C(P_1) = w_i \) for \( i \in \{4, 5, 6\} \), say \( \pi_C(P_1) = w_4 \), then in the fiber of \( q_4 \) are: \( w_4 \) of ramification index 2, \( w_5 \) and \( w_6 \) unramified, and all other points have ramification index 2.
So \( \#(\phi^{-1}(q_4)) = 2 + 1 + 1 + 2k \), is even. Thus \( P_1, P_2 \notin W \). Then \( P_1, P_2 \notin \psi^{-1}(E[2]) \), otherwise they would be in the same fiber.

Thus \( P_2 = w(P_1) \in C \setminus \psi^{-1}(E[2]) \) and \( \psi(P_1) = -\psi(P_2) \). Let \( \pi_E \circ \psi(P_1) = \pi_E \circ \psi(P_2) = \varphi \) and \( \pi_C(P_1) = \pi_C(P_2) = S \). So \( e_\varphi(P_1) \cdot e_\pi_\varphi(\psi(P_1)) = e_\pi_C(\pi_C(P_1)) \cdot e_\varphi(\pi_C(P_1)) \). Thus, \( e_\varphi(\pi_C(P_1)) = e_\varphi(S) = 2 \). All other points in \( \phi^{-1}(q_5) \) are unramified.

For \( P \in W, e_\pi_C(\pi_C(P)) = 2 \). Thus \( e_\varphi(\pi_C(P)) = 1 \). All \( w_1, \ldots, w_6 \) are unramified and other points in \( \phi^{-1}(E[2]) \) are of ramification index 2. By the Riemann–Hurwitz formula, \( \phi \) is unramified everywhere else.

Thus, there are \( \frac{n-1}{2} \) points of ramification index 2 in the fibers \( \phi^{-1}(q_1), \phi^{-1}(q_2), \phi^{-1}(q_3) \), \( \frac{n-1}{2} \) points of ramification index 2 in \( \phi^{-1}(q_4) \), and one point of index 2 in \( \phi^{-1}(q_5) \).

**Case II:** In this case, there are distinct \( P_1 \) and \( P_2 \) in \( C \) such that \( e_\psi(P_1) = e_\varphi(\pi_C(P_2)) = 2 \) and \( \psi(P_1) = \psi(P_2) \). Then \( P_2 = w(P_1) \) or \( w(P_1) = P_1 \), for \( i = 1, 2 \).

Let \( P_1 \) and \( P_2 \) be in the fiber which has three Weierstrass points.

i) Suppose that \( w \) permutes \( P_1 \) and \( P_2 \). So \( P_1 \) and \( P_2 \) are not Weierstrass points. Then \( e_\pi_\varphi(\psi(P_1)) = e_\varphi(P_1) \cdot e_\pi_\varphi(\psi(P_1)) = 4 \). Thus \( e_\pi_C(P_1) \cdot e_\pi_C(\pi_C(P_1)) = 4 \). Since \( e_\pi_C(P_1) = 1 \) then \( e_\pi_C(\pi_C(P_1)) = 4 \). So there is a point of index 4 in the fiber of \( q_4 \). The rest of the points are of ramification index 2, as in previous case, other then the \( w_1, \ldots, w_6 \) which are unramified.

ii) Suppose that \( w \) fixes \( P_1 \) and \( P_2 \). Thus \( P_1 \) and \( P_2 \) are Weierstrass points. Then \( e_\psi(P_1) \cdot e_\pi_\varphi(\psi(P_1)) = e_\pi_C(P_1) \cdot e_\varphi(\pi_C(P_1)) = 4 \). So \( e_\varphi(\pi_C(P_1)) = 2 \). Thus, \( \pi_C(P_1) \) have ramification index 2. The other points behave as in the previous case. So we have in each fiber of \( \phi \) one unramified point and everything else has ramification index 2.

Suppose that \( P_1 \) and \( P_2 \) are in one of the fibers which have only one Weierstrass point.

iii) Then \( w \) has to permutate them, so they are not Weierstrass points. As in case i) \( e_\varphi(\pi_C(P_1)) = 4 \). So there is a point of index 4 in one of \( \psi^{-1}(q_1) \), \( \psi^{-1}(q_2) \), \( \psi^{-1}(q_3) \) and everything else is of ramification index 2. The Weierstrass points are as in case i), unramified.

**Case III:** Let \( P \) be the ramified point of index 3. By lemma 1, \( e_\psi(w(P)) = 3 \). But there is only one such point in \( C \), so \( P \in W \). Then \( e_\pi_\varphi(P) = e_\varphi(P) \cdot e_\pi(\psi(P)) = 6 \). So \( e_\pi_C(P) \cdot e_\varphi(\pi_C(P)) = 6 \). But \( e_\pi_C(P) = 2 \), because \( P \in W \). Thus, \( e_\varphi(\pi_C(P)) = 3 \).

i) \( Q \) is in the fiber that contains three Weierstrass points. Then we have a point of ramification index three in \( \psi^{-1}(q_4) \), two other Weierstrass points are unramified, and all the other points are of ramification index 2.

ii) \( Q \) is in one of the fibers that contains only one Weierstrass point. Then in one of \( \psi^{-1}(q_1) \), \( \psi^{-1}(q_2) \), \( \psi^{-1}(q_3) \) there is a point of index 3 and everything else is of index 2. \( \square \)

### 3.2. The case when \( n \) is even

Let us assume now that \( \deg(\psi) = n \) is an even number. The following theorem classifies the Frey–Kani coverings in this case.

**Theorem 3.2:** If \( n \) is an even number then the generic case for \( \psi : C \rightarrow E \) induce the following three cases for \( \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \):

I. \( (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2) \)

II. \( (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2) \)
Each of the above cases has the following degenerations (two of the branch points collapse to one)

I. 1. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

2. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (4)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

3. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (2)\frac{\hat{\varphi}}{2}, (4)(2)\frac{\hat{e}}{2}\)

4. \((3)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

II. 1. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

2. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

3. \((4)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

4. \((2)\frac{\hat{\varphi}}{2}, (4)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

5. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

6. \((3)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}, (4)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

7. \((2)\frac{\hat{\varphi}}{2}, (3)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{e}}{2}\)

III. 1. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (4)(2)\frac{\hat{\varphi}}{2}\)

2. \((2)\frac{\hat{\varphi}}{2}, (4)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}\)

3. \((2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (4)(2)\frac{\hat{\varphi}}{2}\)

4. \((3)(2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}, (2)\frac{\hat{\varphi}}{2}\)

Proof: We know that the number of Weierstrass points in the fibers of 2-torsion points is 0 mod \((2)\). Combining this with the Riemann - Hurwitz formula we get the three cases of the general case.

To determine the degenerate cases we consider cases when there is one branch point for \(\psi: C \rightarrow E\).

I) First, assume that the branch point has two points \(P_1\) and \(P_2\) of index 2 (Case II). Then \(w(P_1) = P_i\) for \(i = 1, 2\) or \(w(P_1) = P_2\). The first case implies that \(P_1, P_2 \in W\). Then \(e_\varphi(w(P_1)) = e_\varphi(w(P_2)) = 2\). So we have case I, 1. When \(w(P_1) = P_2\) then \(e_\varphi(w(P_1)) = 4\). Thus, we have a point of index 4 in \(\phi^{-1}(q)\) for \(q \in \{q_1, \ldots, q_l\}\). Therefore cases 2 and 3. If there is \(P \in C\) such that \(e_\varphi(P) = 3\), then \(P \in W\) and \(e_\varphi(w(P)) = 3\). So we have case 4.

II) As in case I, if \(P_1\) and \(P_2\) are Weierstrass points then they can be in the fiber of the point which has 4 or 2 Weierstrass points. So we get two cases, namely 1 and 2. Suppose now that \(P_1\) and \(P_2\) are not Weierstrass points, thus \(w(P_1) = P_2\) and \(e_\varphi(w(P_1)) = 4\). This point
of index 4 can be in the same fiber with 4, 2 or none Weierstrass points. So we get cases 3, 4, and 5 respectively. A point of index 3 is a Weierstrass point which can be in the fiber which has 4 or 2 Weierstrass points. So cases 6 and 7.

III) If $P_1$ and $P_2$ are Weierstrass points then they can be only in the fiber with 6 Weierstrass point so case 1. If they are not then we have a point of index 4 which can be in the fiber with all Weierstrass points or with none. Therefore, cases 2 and 3. The point of index 3 is a Weierstrass point so it can be in the fiber where all the Weierstrass points are, so case 4. This completes the proof.

4. Maximal coverings $\psi : C \rightarrow E$.

Let $\psi_1 : C \rightarrow E_1$ be a covering of degree $n$ from a curve of genus 2 to an elliptic curve. The covering $\psi_1 : C \rightarrow E_1$ is called a maximal covering if it does not factor over a nontrivial isogeny. A map of algebraic curves $f : X \rightarrow Y$ induces maps between their Jacobians $f^* : J_Y \rightarrow J_X$ and $f_* : J_X \rightarrow J_Y$. When $f$ is maximal then $f^*$ is injective and $\ker(f_*)$ is connected, see (9) (p. 158) for details.

Let $\psi_1 : C \rightarrow E_1$ be a covering as above which is maximal. Then $\psi_1^* : E_1 \rightarrow J_C$ is injective and the kernel of $\psi_1_* : J_C \rightarrow E_1$ is an elliptic curve which we denote by $E_2$, see (3) or (7).

For a fixed Weierstrass point $P \in C$, we can embed $C$ to its Jacobian via

$$i_P : C \rightarrow J_C$$

$$x \rightarrow [(x) - (P)]$$

Let $g : E_2 \rightarrow J_C$ be the natural embedding of $E_2$ in $J_C$, then there exists $g_* : J_C \rightarrow E_2$. Define $\psi_2 = g_* \circ i_P : C \rightarrow E_2$. So we have the following exact sequence

$$0 \rightarrow E_2 \xrightarrow{g} J_C \xrightarrow{\psi_1} E_1 \rightarrow 0$$

The dual sequence is also exact, see (3)

$$0 \rightarrow E_1 \xrightarrow{\psi_1^*} J_C \xrightarrow{g_*} E_2 \rightarrow 0$$

The following lemma shows that $\psi_2$ has the same degree as $\psi_1$ and is maximal.

**Lemma 4.1:** a) $\deg(\psi_2) = n$

b) $\psi_2$ is maximal

**Proof:** For every $D \in \text{Div}(E_2)$, $\deg(\psi_2^* D) = \deg(\psi_2) \cdot \deg(D)$. Take $D = \mathcal{O}_2 \in E_2$, then $\deg(\psi_2^* \mathcal{O}_2) = \deg(\psi_2)$. Also $\psi_2^* (\mathcal{O}_2) = (\psi_2^* \mathcal{O}_2)$ as divisor and

$$\psi_2^* \mathcal{O}_2 = i_P^* g(\mathcal{O}_2) = i_P^* \mathcal{O}_J = \psi_1^* \mathcal{O}_1$$

So $\deg(\psi_2^* \mathcal{O}_2) = \deg(\psi_1^* \mathcal{O}_1) = \deg(\psi_1) = n$

To prove the second part suppose $\psi_2 : C \rightarrow E_2$ is not maximal. So there exists an elliptic curve $E_0$ and morphisms $\psi_0$ and $\beta$, such that the following diagram commutes
Proof: Let \( \psi : C \rightarrow E \) be the identity of \( E_0 \). Then exists \( \psi_0 \) such that \( \psi_0 = \psi_0 \circ i_p \). Thus, \( \psi_2 = \beta \circ \psi_0 \). So ker \( \psi_0 \) is a proper subgroup of ker \( \psi_2 \), since \( \deg \beta > 1 \). Thus,

\[
\psi_0 : E_1 \rightarrow \ker \beta
\]

is a surjective homomorphism. Therefore, \( E_1 \) has a proper subgroup of finite index. So, there exists an intermediate field between function fields \( \mathbb{C}(C) \) and \( \mathbb{C}(E_1) \). This contradicts the fact that \( \psi_1 \) is maximal.

If \( \deg(\psi_1) \) is an odd number then the maximal covering \( \psi_2 : C \rightarrow E_2 \) is unique (up to isomorphism of elliptic curves), see Kuhn (7).

To each of the covers \( \psi_i : C \rightarrow E_i, i = 1, 2 \), correspond Frey-Kani covers \( \phi_i : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \). If the cover \( \psi_1 : C \rightarrow E_1 \) is given, and therefore \( \phi_1 \), we want to determine \( \psi_2 : C \rightarrow E_2 \) and \( \phi_2 \). The study of the relation between the ramification structures of \( \phi_1 \) and \( \phi_2 \) provides information in this direction. The following lemma (see (3), p. 160) answers this question for the set of Weierstrass points \( W = \{P_1, \ldots, P_6\} \) of \( C \) when the degree of the cover is odd.

Let \( \psi_i : C \rightarrow E_i, i = 1, 2 \), be maximal of odd degree \( n \). Let \( \mathcal{O}_i \in E_i[2] \) be the points which have three Weierstrass points in its fiber. Then we have the following:

**Lemma 4.2 (Frey-Kani):** The sets \( \psi_1^{-1}(\mathcal{O}_1) \cap W \) and \( \psi_2^{-1}(\mathcal{O}_2) \cap W \) form a disjoint union of \( W \).

When \( n \) is even the ramification of \( \psi \), is more precise.

**Lemma 4.3:** Let \( \psi : C \rightarrow E \) is maximal of even degree \( n \), and \( Q \in E[2] \). Then \( \psi^{-1}(Q) \) has either none or two Weierstrass points.

**Proof:** If there are no Weierstrass points in \( \psi^{-1}(Q) \) there is nothing to prove. Suppose there is one, from lemma 3.2 we know there are at least 2, say \( P_1, P_2 \). We embed \( C \rightarrow J_C \) via \( x \rightarrow [(x) - (P_1)] \) and \( E \rightarrow J_E \) via \( x \rightarrow [(x) - (Q)] \).

\[
\begin{array}{ccc}
C & \overset{i_1}{\rightarrow} & J_C \\
\psi \downarrow & & \downarrow \psi_* \\
E & \overset{i_2}{\rightarrow} & J_E
\end{array}
\]

Then \( \psi_*([(x) - (P_1)]) = [(\psi(x)) - (Q)] \).

Also, \( \psi_* \psi^* = [n] \) is the multiplication by \( n \) in \( E \). Since \( 2|n \) then \( E[2] \) is a subgroup of \( E[n] \). So \( \psi^*(E[2]) = \ker(\psi_*,J_E[2]) \), we call this group \( H \). Suppose \( P_3 \in \psi^{-1}(Q) \). Then \( \psi_*(i_1(P_3)) = \mathcal{O}_E \), so \( (P_1, P_3) \in H \), where the unordered pair \( (P_1, P_3) \) denotes the point \( [(P_1) - (P_3)] \) of order 2 in \( J_C \). By addition of points of order 2 in \( J_C \), \( (P_2, P_3) \in H \). So
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\[ H = \{0_J, (P_1, P_2), (P_1, P_3), (P_2, P_3)\} \] can’t have any other points, therefore \( \psi^{-1}(Q) \) has three Weierstrass points, which contradicts theorem 3.2. Thus, there are only two Weierstrass points in \( \psi^{-1}(Q) \).

The above lemma says that if \( \psi \) is maximal of even degree then the corresponding Frey-Kani covering can have only type \( I \) ramification, see theorem 3.1.

5. Arithmetic Applications

In this section, we characterize genus 2 curves with degree 3 elliptic subcovers and determine the j-invariants of these elliptic subcovers in terms of coefficients of the genus 2 curve. If the elliptic subcover is of degenerate ramification type, then its j-invariant is determined in terms of the absolute invariants of the genus 2 curve. We find two isomorphism classes of genus 2 curves which have both elliptic subcovers of degenerate type.

When \( n = 5 \) or 7 we discuss only Case II, iii), and Case II, i) of theorem 3.1, respectively. In both cases we determine the j-invariants of elliptic subcovers in terms of the coefficients of the genus 2 curves. Other types of ramifications are computationally harder and results are very large for display.

5.1. Curves of genus 2 with a degree 3 elliptic subfield.

Let \( \psi : C \rightarrow E_1 \) be a covering of degree 3, where \( C \) is a genus 2 curve given by

\[ C : Y^2 = x(x - 1)(x - d)(x^3 - ax^2 + bx - c) \]

and \( E_1 \) an elliptic curve. Denote the 2-torsion points of \( E_1 \) by 0, 1, t, s. Let \( \phi_1 \) be the Frey-Kani covering with \( \deg(\phi_1) = 3 \) such that \( \phi_1(0) = 0, \phi_1(1) = 1, \phi_1(d) = t, \) and the roots of \( f(x) = x^3 - ax^2 + bx - c \) are in the fiber of s. The fifth branch point is infinity and in its fiber is \( u \) of index 1 and infinity of index 2. So \( \phi_1 \) is of generic type (Theorem 3.1). Points of index 2 in the fibers of 0, 1, t are m, n, p respectively. Then the cover is given by

\[ z = k \frac{x(x - m)^2}{x - u} \]

Then from equations:

\[ z - 1 = k(x - 1)(x - n)^2, \quad z - t = k(x - d)(x - p)^2, \quad z - s = f(x) \]

we compare the coefficients of \( x \) and get a system of 9 equations in the variables \( a, b, c, d, k, m, n, p, t, s, u \).

Using the Buchberger’s Algorithm (see (1), p. 86-91) and a computational symbolic package (as Maple) we get;

**Lemma 5.1:** Let \( E_1 \) be the elliptic curve given by \( y^2 = z(z - 1)(z - t)(z - s) \). Then the genus 2 curve

\[ C : Y^2 = x(x - 1) \left( x - \frac{a(a - 2)}{2a - 3} \right) \left( x^3 - ax^2 + \left( \frac{(2a - 3)c}{(a - 1)^2} + \frac{a^2}{4} \right) x - c \right) \]
covers $E_1$ with a maximal cover of degree 3 of generic case (Theorem 3.1). Moreover $s$ and $t$ are given by,
\[
t = \frac{a^3(a - 2)}{(2a - 3)^4}, \quad s = \frac{4c}{(a - 1)^5}
\]

Next, we find the $j$-invariants of $E_1$ and $E_2$. The $j$-invariant of $E_1$ is as follows,
\[
 j(E_1) = \frac{16}{C^2} \cdot \frac{A^3}{a^6c^2(a - 1)^2(a - 2)^2(a - 3)^6((a - 1)^2 - 4c)^2}
\]
where $A$ and $C$ are:
\[
 A = a^{12} - 8a^{11} + 16c^2a^8 + 11664c^2 + 36720c^2a^4 - 69984c^2a^3 - 192c^2a^7 + 7776c^2a^2 - 4656c^2a + 1920c^2a^6 - 11232c^2a^5 - 4a^{10}c + 26a^{10} - 44a^9 + 41a^8 - 20a^7 + 220a^6c - 904a^7c + 1740a^6c - 1800a^5c - 8a^6c - 216ca^3 + 4a^6 + 972ca^4
\]
\[
 C = a^6 - 4a^5 + 5a^4 - 2a^3 - 32a^3c + 144ca^2 - 216ca + 108c
\]

To find $j_2$ we take $\phi_2 : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi_2(0) = \phi_2(1) = \phi_2(d) = \infty$. Three roots of $f_3(x) = x^3 - ax^2 + bx - c$ go to 2-torsion points $s_1, s_2, s_3$ of $E_2$ and 0 is the fifth branch point of $\phi_2$. Solving the corresponding system we get $s_1, s_2, s_3$ in terms of $a$ and $c$. Then $j_2$ is
\[
 j(E_2) = \frac{16}{C} \cdot \frac{B^3}{c((a - 1)^2 - 4c)}
\]
where $A$ is as above and $B = a^4 - 2a^3 + a^2 - 24ca + 36c$.

5.2. Degenerate Cases

Notice that only one degenerate case can occur when $n = 3$. In this case, one of the Weierstrass points has ramification index 3, so the cover is totally ramified at this point, see theorem 3.1.

**Lemma 5.2:** Let $E$ be an elliptic curve given by $y^2 = z(z - 1)(z - s)$. Suppose that the genus two curve $C$ with equation
\[
 Y^2 = x(x - 1)(x - w_1)(x - w_2)(x - w_3)
\]
covers $E$, of degree 3, such that the covering is degenerate. Then $w_3$ is given by
\[
 w_3 = \frac{(4w_1^3 - 7w_1^2 + 4w_1 - w_2)^3(4w_1^3 - 3w_1^2 - w_2)}{16w_1^3(w_1 - 1)^3(4w_1^3 - 6w_1^2 + 3w_1 - w_2)}
\]
and $w_1$ and $w_2$ satisfy the equation,
\[
 w_1^4 - 4w_1^3w_2 + 6w_1^2w_2 - 4w_1w_2 + w_2^2 = 0
\]
Moreover,
\[
 s = -27 \left( w_1(w_1 - 1) \frac{(4w_1^3 - 7w_1^2 + 4w_1 - w_2)(4w_1^3 - 5w_1^2 + 2w_1 - w_2)}{(4w_1^3 - 9w_1^2 - w_2 + 6w_1)(4w_1^3 - 3w_1^2 - w_2)(4w_1^3 - 6w_1^2 + 3w_1 - w_2)} \right)^2
\]
**Proof:** We take $\psi : C \to E$ and $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ its corresponding Frey-Kani covering. To compute $\phi$, let $w_1$ be the point of ramification index 3. Take a coordinate in the lower $\mathbb{P}^1$ such that $\phi(w_1) = 0$, $\phi(w_2) = s$, $\phi_2(w_3) = 1$, and $\phi(0) = \phi(1) = \phi(\infty) = \infty$. We denote points of ramification index 2 in the fibers of $s$ and $1$ by $p$ and $q$, respectively. Then, $\phi$ is given as $z = k_2 \frac{x-w_1}{z(x-w_1)}$. From the corresponding system we get the above result.

Denote the $J$-invariant of $E$ by $j_1$. Using the above expression of $s$ in terms of $w_1$ and $w_2$, we get an equation in terms of $j_1$, $w_1$, and $w_2$. Taking the resultant of this expression and equation (2) we get,

$$2617344w_1^7 + 386371w_1^7 - 17496j_1w_1^6 - 29207808w_1^5 - 7569408w_1^4 - 7569408w_1^3 5$$

$$-729w_1^4j_1 + 5103j_1w_1^3 + 69984j_1w_1^3 - 60507j_1w_1^3 + 65536 - 589824w_1 + 16411392w_1^2$$

$$-29207808w_1^13 + 44960208w_1^12 - 60666336w_1^11 + 72010800w_1^10 + 44960208w_1^9$$

$$-60666336w_1^7 + 72010800w_1^6 - 75998272w_1^5 + 16411392w_1^4 + 2617344w_1^3 - 589824w_1^2$$

$$-60507j_1w_1^10 + 38637j_1w_1^11 - 17496j_1w_1^12 + 5103j_1w_1^13 - 729j_1w_1^14 + 65536w_1^18 = 0$$

We denote with $j$ the $j$-invariant of the elliptic curve $y^2 = (x - w_1)(x - w_2)(x - w_3)$. Then, proceeding as above, $j$ can be expressed in terms of $w_1$ as below,

$$65536w_1^6 - 19608w_1^5 + 35632w_1^4 - 35024w_1^3 + (289536 - 9j)w_1^2$$

$$+(-129972 + 9j)w_1 + 35152 - 9j = 0$$

(4)

Taking the resultants of the two previous equations we have

$$256 A(j) j_1^3 + 3 B(j) j_1^2 + 6 C(j) j_1 - D(j) = 0$$

(5)

where

$$A(j) = (9j - 35152)^4$$

$$B(j) = -2187j^7 + 3896640j^6 - 277882258176j^5 + 998642127618048j^4$$

$$- 1868045010870009856j^3 + 1665950950848367910912j^2$$

$$- 543484043691057422896448j + 166124820657244821172518912$$

(6)

$$C(j) = 27j^7 + 1125216j^7 + 9650655872j^6 - 31593875152896j^5 + 2774880499728384j^4$$

$$+ 1114515284358510673920j^3 - 606198995609093246100480j^2$$

$$+ 8346397859247767524611194880j + 353019691006036487376293855232$$

$$D(j) = (j^3 + 33120j^2 + 290490624j - 310747594752)^3$$

For the genus 2 curve $C$ we compute the Igusa invariants $J_2, J_4, J_6, J_{10}$ in terms of the coefficients of the curve, see Igusa (8) for their definitions. The absolute invariants of $C$ are defined it terms of Igusa invariants as follows,

$$i_1 := 144 J_1 J_2^2, \quad i_2 := -1728 J_2 J_4 - 3 J_6, \quad i_3 := 48 J_{10} J_2^3$$

(7)

Two genus 2 curves with $J_2 \neq 0$ are isomorphic if and only if they have the same absolute
invariants. The absolute invariants can be expressed in terms of \( w_1 \) and \( w_2 \). Taking the resultant of the first two equations in (7) we get an equation \( F(i_1, i_2, w_1) = 0 \). The resultant of \( F(i_1, i_2, w_1) \) and equation (4) we get \( j = 13824 \frac{S}{T} \) where \( S \) and \( T \) are:

\[
S = 247945848003i_1^3 - 409722141024i_1^2 - 7591354214400i_1 + 17736744960000 \\
+ 61379512488i_2i_1^2 + 64268527400i_2^2i_1 - 2031496516224i_2
\]

\[
T = 1034723291140i_1^2i_2 - 317545076512i_2i_1^2 - 7250280129792i_2 + 1670535171334i_2^2
\]

\[
+ 366156782208i_2^3 - 67382113075200i_1 + 141893959680000
\]

The conjugate solutions of (5) are j-invariants of \( E_1 \) and \( E_2 \). For \( j = 0 \) the equation (3) has one triple root \( j_1 = \frac{1213857792}{28561} \). Then, \( C \) and \( E \) are given by,

\[
Y^2 = x^5 - x^4 + 216x^2 - 216x \\
y^2 = x^3 - 668644200x + 6788828143125
\]

For \( j = 1728 \) the values for \( j_1 \) are

\[
j_1 = 1728, \quad \frac{942344950464}{1500625}, \quad \frac{942344950464}{1500625}
\]

This value of \( j \) does not give a genus 2 curve since the discriminant \( J_{10} \) of \( C \) is 0.

Next we will see what happens when both \( \phi_1 \) and \( \phi_2 \) are degenerate. We find only two triples \((C, E_1, E_2)\) such that the corresponding \( \phi_1 : C \to E_i, i = 1, 2, \) degenerate. It is interesting that in both cases \( E_1 \) and \( E_2 \) are isomorphic.

**Lemma 5.3:** Let \( E : y^2 = z(z - 1)(z - t) \) be an elliptic curve. Then the genus 2 curve

\[
Y^2 = x(x - 1) \left( x^3 - \frac{3}{2}x^2 + \frac{9}{16}x - \frac{t}{16} \right)
\]

covers \( E \), such that the covering is of degree 3 and the corresponding Frey-Kani covering of type II, iii) (Theorem 3.1), for \( t \neq 0, 1 \).

**Proof:** Let \( \phi_1 \) be the Frey-Kani covering with \( \text{deg}(\phi_1) = 3 \) such that \( \phi_1(w_1) = \phi_1(w_2) = \phi_1(w_3) = t \), \( \phi_1(0) = 0 \), \( \phi_1(1) = 1 \), \( \phi_1(\infty) = \infty \). Let \( t \infty \) be the point of ramification index 3, and denote the points of ramification index 2 in the fibers of 0 and 1 with \( m \) and \( n \) respectively. If \( z \) is a coordinate in the lower \( \mathbb{P}^1 \) then \( \phi_1 \) is given by \( z = k_1(x - m)^2 \). The relations \( z - 1 = k_1(x - 1)(x - n)^2 \), \( z - t = k_1(x^3 - ax^2 + bx - c) \) hold, where \( x^3 - ax^2 + bx - c = (x - w_1)(x - w_2)(x - w_3) \). Comparing the coefficients and solving the system, we get

\[
(a, b, c, k_1, m, n) = \left( \frac{3}{2}, \frac{9}{16}, \frac{t}{16}, \frac{3}{4}, \frac{1}{4} \right)
\]

To compute \( \phi_2 \), let \( w_1 \) be the point of ramification index 3. Take a coordinate in the lower \( \mathbb{P}^1 \) such that \( \phi_2(w_1) = 0 \), \( \phi_2(w_2) = s \), \( \phi_2(w_3) = 1 \), and \( \phi_2(0) = \phi_2(1) = \phi_2(\infty) = \infty \). The
points of ramification index 2 in the fibers of $s$ and 1 we denote by $p$ and $q$, respectively. Then $\phi_2$ is given as $z_2 = k_2 \frac{z-w_1}{(z-1)^2}$. Then from the corresponding system we get
\[
\begin{align*}
w_1 &= -q(q-2)/(2q-1), \quad w_2 = -q^3(q-2)/(2q-1), \quad w_3 = -q(12q-8-6q^2+q^3)/(2q-1)^2, \\
k_2 &= \frac{1}{27} \left( -1 + 2q \right)^3, \quad s = \frac{-1 - (1 + 2q)^3}{27} q^2(q-1)^2,
\end{align*}
\] (9)

Using the fact that the $a, b, c$ are the symmetric polynomials of $w_1, w_2, w_3$ we have;
\[
(t, q) = \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{3} \right), \quad \left( \frac{-241 + 221 \sqrt{2}}{2 + 221 \sqrt{2}}, \pm \frac{1}{2} \sqrt{2} \right), \quad \left( \frac{243}{2 + 221 \sqrt{2}}, \pm \frac{1}{2} \sqrt{2} \right)
\] (10)

where $I = \sqrt{-1}$. So we have three pairs of elliptic curves
\[
E_1 : y^2 = z(z-1)(z-\frac{1}{2}) \quad \text{and} \quad E_2 : y^2 = z(z-1)(z+1)
\]

with $j(E_1) = j(E_2) = 1728$.

$E_1 : y^2 = z(z-1) \left( z - \frac{241 + 221 \sqrt{2}}{-2 + 221 \sqrt{2}} \right)$, \quad $E_2 : y^2 = z(z-1) \left( z - \frac{241 + 221 \sqrt{2}}{243} \right)$

with $j(E_1) = j(E_2) = \frac{-873722816}{99049}$.

$E_1 : y^2 = z(z-1) \left( z - \frac{243}{1 + 2(11 \sqrt{2})} \right)$, \quad $E_1 : y^2 = z(z-1) \left( z - \frac{241 - 221 \sqrt{2}}{243} \right)$

and $j(E_1) = j(E_2) = \frac{-873722816}{99049}$. The last two cases correspond to the same isomorphism class of genus 2 curves. Thus, when $\phi_1$ and $\phi_2$ are both degenerate then we get two isomorphism classes of elliptic curves. Summarizing everything above we have the following table:

<table>
<thead>
<tr>
<th>$f_3(x)$</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$j_1 = j_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3 - \frac{3}{2} x^2 + \frac{9}{8} x - \frac{1}{4}$</td>
<td>$z(z-1)(z-\frac{1}{2})$</td>
<td>$z(z-1)(z+1)$</td>
<td>1728</td>
</tr>
<tr>
<td>$x^3 - \frac{3}{2} x^2 + \frac{9}{8} x - \frac{241 + 221 \sqrt{2}}{321(1+11 \sqrt{2})}$</td>
<td>$t_1 = \frac{241 + 221 \sqrt{2}}{243}$</td>
<td>$t_2 = \frac{241 + 221 \sqrt{2}}{243}$</td>
<td>$\frac{-873722816}{99049}$</td>
</tr>
</tbody>
</table>

where $C : y^2 = x(x-1)f_3(x)$, $E_1 : y^2 = z(z-1)(z-t_1)$. One can check, using the absolute invariants of the genus two curves, that they are not isomorphic to each other. Moreover, an equation for $E_1 \cong E_2$ in the second case is as follows:

\[
y^2 = z^3 + z^2 - 277520614451197z + 1880509439898307064603
\]

and its conductor $N = 2^8 \cdot 3 \cdot 11^2 \cdot 239^2 \cdot 251^2$. 


5.3. Curves of genus 2 with degree 5 elliptic subfields, the 4-cycle case.

Notice that the case II, i) does not occur when \( n = 5 \). So we will consider only case II, iii). We will prove the following lemma:

**Lemma 5.4:** Let \( \psi : C \rightarrow E_1 \) be a covering of degree 5 such that the corresponding Frey-Kani cover is of ramification type II, iii) (theorem 3.1). Then the genus two curve can be given by

\[
Y^2 = x(x - 1)(x - d)(x^3 - ux^2 + vx - w)
\]

where

\[
d = \frac{(3u^2 - 4u - 4v + 1)^2}{(2u - 3)(6u^2 - 10u + 5 - 8v)}, \quad w = -\frac{(u^2 - 6u + 4v)(u^2 - 4v)}{8(2u - 3)}
\]

and \( u \) and \( v \) satisfy

\[
15u^4 - 82u^3 - 8vu^2 + 159u^2 - 140u + 56vu - 16v^2 - 52v + 50 = 0
\]

Moreover, an equation of \( E_1 \) is \( y^2 = z(z - 1)(z - t) \), where

\[
t = \frac{(u^2 - 4v)(-8u^4 + 24u^3 + 63u^2 + 64v^2 - 192uv + 196v + 16u^2v - 180u + 100)}{(2u - 3)(6u^2 - 10u + 5 - 8v)}
\]

**Proof:** Take the genus 2 curve to be

\[
Y^2 = x(x - 1)(x - d)(x^3 - ux^2 + v - w)
\]

Let \( \phi_1 \) be the Frey-Kani covering with \( \text{deg} (\phi_1) = 5 \) such that \( \phi_1 (w_1) = \phi_1 (w_2) = \phi_1 (w_3) = t \), \( \phi_1 (0) = 0 \), \( \phi_1 (1) = 1 \), and \( \phi_1 (d) = \infty \). Take \( \infty \) to be the point of ramification index 4 such that \( \phi_1 (\infty) = \infty \). Then \( \phi_1 \) is given by

\[
z = k_1 \frac{(x^2 - ax + b)^2}{(x - d)}
\]

Solving the corresponding system we get the above result.

\[
F(u, v)j + G(u, v) = 0
\]

From the previous lemma, the j-invariant of the elliptic curve satisfies

\[
F(u, v)j + G(u, v) = 0
\]

Taking the resultant of the previous two equations, the j-invariant satisfies an equation of degree 2:

\[
A(u)j^2 + B(u)j + C(u) = 0 \quad (11)
\]

where

\[
A(u) = (u - 1)^2(u - 2)^2(3u - 4)^6(3u - 5)^6(2u^2 - 6u + 5)^8\quad (12)
\]
The solutions of (11) give the j-invariants of $E_1$ and its complement $E_2$.

**Example 5.1:** The two elliptic curves are isomorphic when the equation

$$A(u)j^2 + B(u)j + C(u) = 0$$

of the above lemma has a double root. This happens for $u = \frac{\alpha}{2} \pm \frac{1}{4}\sqrt{-5}$. Then

$$j_1 = j_2 = \frac{28849701763}{16941456}$$

The elliptic curve with $j$-invariant as above has equation,

$$y^2 + yz = x^3 + 6388018241406303862z - 7543791818526000444980292108$$

**5.4. Curves of genus 2 with degree 7 elliptic subfields, 4-cycle case**

The case $n = 7$ is the first case that all degenerations occur. However, it is very difficult to compute the space of genus 2 curves with degree 7 elliptic subcovers. We discuss only one degenerate case, namely case II. iii) of theorem 3.1. We will assume that the genus two curve is given by

$$C : Y^2 = x(x-1)(x-d)(x^3 - ax^2 + bx - c)$$
and the elliptic curve in Legendre form $E_1: y^2 = z(z-1)(z-t)$. Moreover, let’s assume that the corresponding Frey-Kani covering $\phi: \mathbb{P}^1 \to \mathbb{P}^1$ is of type II, i) of theorem 3.1. Take the coordinates such that, $\phi(0) = 0$, $\phi(1) = 1$, $\phi(d) = t$, and three distinct roots of $x^3 - ax^2 + bx - c$ are in the fiber of infinity. Let the point of ramification index 4 be infinity, which is in the same fiber as roots of $x^3 - ax^2 + bx - c$. Then the cover is given by,

$$z = k \frac{x P_1(x)}{x^3 - ax^2 + bx - c}$$

where $P_1(x)$ is a cubic polynomial which represents the three points of order 2 in the fiber of 0. Solving the corresponding system we get,

$$a = -\frac{1}{4A} (7d^{20} + 424t^3d^8 - 11072d^{12}t^3 + 2368t^2d^{13} - 872d^{16}t^2 - 1532d^{17}t - 21568d^{14}t^2 - 56d^{19}t + 478d^{12}t + 365t^8d^4 - 42t^6d^2 + 18160t^3d^{11} - 43564t^4d^3 - 624t^5d^6 + 8t^6d^7 - 736d^8d^4 - 52594t^4d^2 + 624td^{14} - 2576td^{15} + 2725td^{16} + 736td^{13} - 36d^{19} - 2368t^2d^8 + 42d^{18} + 6112d^{15}t^2 - 29576t^3d^7 - 7t^4 + 52594t^3d^8 - 44496t^3d^7 + 2576td^8 - 2725td^4 + 1532t^4d^3 + 56td^5d + 872td^3 - 6112td^4 - 478td^5d - 18160td^t - 242d^2t + 1072d^t - 8d^{17} + 44496t^2d^{13} + 21568t^2d^9 + 4356d^{10}t^2 + 29576t^2d^{13})$$

$$b = \frac{1}{16A} (-14d^{21} + 77d^{20} + 400d^3t^4 - 3496td^9 + 94280d^{12}t^5 + 1680t^7d^{14} - 21232t^9d^{13})$$

$$+ 1008d^{17}t^2 + 35d^{17}t + 31612d^{14}t^2 + 84d^{20}t - 616d^{19}t + 1313td^{18}t - 77t^6d + 121t^5d^2$$

$$- 103561t^4d^3 - 721t^6d^3 + 901td^4d^3 + 20t^5d^4 + 139344t^3d^3 + 269886d^2t^2 - 901td^4d^3$$

$$- 5222t^{16} + 3496t^{12} - 121td^9 - 1680t^2d^2 - 20t^6d^2 + 72d^{18} + 5352t^5d^6$$

$$+ 139344t^3d^9 - 31612t^4d^7 + 5222td^8 - 35t^4d^6 - 3532td^3d^6 + 1313td^4d^3 - 84td^6$$

$$+ 616d^4d^9 - 94280d^2t^2 - 400d^{12}t + 21232td^t + 219712d^{10}t^2 - 308478t^3d^{11} + 308478t^3d^{10}$$

$$- 219712t^3d^{11} + 5080td^3 - 5080td^3 + 10356td^{15} + 14t^5)$$

$$c = -\frac{1}{448A} (28d^{21} - 77d^{20} - 561d^{12}t^2 - 1800td^t + 84d^{16}t + 122d^2d + 364td^3 - 118t^2d + t^3)$$

$$+ 20d^6 + 120td^4 - 608td^5 + 1400d^6 + 1311td^8 - 42d^{10} - 140d^t - 504td^9 + 440d^t)$$

where,

$$A = d(90d^4d^2 - 36d^3t - 9t^2d - 84t^4d^2 + 36t^2d^2 + t^8 - d^9 - 36td^8 + 9td^8$$

$$- 36d^2t^2) (168td^9 - t^2 - 168td^5 - 20td^3 + 64d^2d - 10d^7d^2 + 5t^2d^3 + 90td^4 - 90t^2d + 20td^8)$$

$$- 6d^{10} + d^{11} + 10d^9 - 5d^8)$$

Also, $t$ and $d$ satisfy the equation,

$$d^{16} - 16(td^{15} + t^3d) + 120td^{14} - 560td^{13} + (400t^2 + 1420d)td^{12} - (240d^2 + 1968)d^{11}$$

$$+ (6608t^2 + 1400d^10 - (110040t^2 + 400)d^9 + 12870d^6d^8 - (400t^3 + 11040d^2)d^7 + 120t^3d^2$$

$$+ (1400t^4 + 6608d^3)d^6 - (1908t^4 + 2400d^3)d^5 + (1420t^3 + 400t^2)d^2 - 560d^2d^2 + t^4 = 0$$
Thus, we can express the coefficients of $C$ in terms of $t$ and $d$. Absolute invariants $i_1, i_2, i_3$ of $C$ can be expressed in terms of $t$ and $d$. Using resultants and a symbolic computational package as Maple we are able to get an equation in terms of $i_1, i_2, i_3$. The equation is quite large for display. This is the moduli space of genus two curves whose Jacobian is the product of two elliptic curves and the Frey-Kani coverings are of degree 7 and ramification as above.

References