# GENUS 2 FIELDS WITH DEGREE 3 ELLIPTIC SUBFIELDS

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Abstract. In this paper we study genus 2 function fields $K$ with degree 3 elliptic subfields. We compute an equation for the locus of these fields $K$ in the moduli space of genus 2 curves. From this we derive that the number of $Aut(K)$-classes of such subfields of fixed $K$ is 0,1,2 or 4.

## 1. Introduction

We study genus two curves $\mathcal{C}$ whose function fields have a degree 3 subfield of genus 1. Such subfields we call elliptic subfields. Such curves $\mathcal{C}$ have already occurred in the work of Hermite, Goursat, Burkhardt, Brioschi, and Bolza, see Krazer [6] (p. 479). More generally, degree $n$ elliptic subfields of genus 2 fields have been studied by Frey [2], Frey and Kani [3], Kuhn [8], Shaska [11], [13], and Shaska and Voelklein [14]. In the degree 3 case, explicit equations can be used to answer questions which remain open in the general case. This was the theme of the author's PhD thesis (see [12]) which also covered the case of degree 2 (which is subsumed in [14]).

Equation (5) gives a normal form for pairs $(K, E)$ where $E$ is a degree 3 elliptic subfield of $K$. This normal form depends on two parameters $a, b \in k$. Isomorphism classes of pairs $(K, E)$ are parameterized by parameters $u, v$ where $u := ab$ and $v := b^3$. This yields an expression for the invariants $(i_1, i_2, i_3)$ of $K$ (see Igusa [9]) in terms of $u$ and $v$. Our central object of study is the map

$$\theta : (u, v) \to (i_1, i_2, i_3)$$

From the general theory (see above) it is clear that this map has degree 2, which means that a general genus 2 field having elliptic subfields of degree 3 has exactly two such subfields. From the results of this paper we quickly derive that there are exactly four genus 2 fields which have four $Aut(K)$-classes of such subfields, and a 1-dimensional family with exactly one $Aut(K)$-class of such subfields.

The main goal of this paper is to compute an equation for the locus $\mathcal{L}_3$ of genus 2 fields with elliptic subfields of degree 3. This equation is in terms of the Igusa coordinates $i_1$, $i_2$, $i_3$ on the moduli space of genus 2 curves. It is displayed as equation (33) in Appendix A (because of its length). The direct approach to finding this equation would be to eliminate $u$ and $v$ from the explicit equations defining $\theta$ (see (13)). However, this direct approach exceeds available computer power, so we introduce auxiliary parameters $r_1$, $r_2$ that parameterize pairs of cubic polynomials. They arise from the fact that the subfield $E$ induces a particular sextic defining $K$ which splits naturally as a product of 2 cubic polynomials. We show that $\theta$ factorizes as

$$(u, v) \to (r_1, r_2) \to (i_1, i_2, i_3)$$

where the latter map is birational, and the former has degree 2. Thus $r_1$, $r_2$ yield a birational parameterization of the locus $\mathcal{L}_3$ of genus 2 fields having a degree 3 elliptic subfield. That $\mathcal{L}_3$ is a rational variety follows also from the general theory of "diagonal modular surfaces", see Kani [5].

The formulas expressing $i_1, i_2, i_3$ in terms of $r_1$, $r_2$ are much smaller than those in terms of $u$ and $v$. Now we can eliminate $r_1$ and $r_2$ to obtain the desired equation (33). We further find relations between the j-invariants of the degree 3 elliptic subfields of $K$ and classify all $K$ in $\mathcal{L}_3$ with extra automorphisms.

All the computations were done using Maple [10].

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## 2. Genus Two Fields With Degree 3 Elliptic Subfields

Let $k$ be an algebraically closed field of characteristic 0. All function fields will be over $k$. Let $K$ be a genus 2 function field and $E$ a degree 3 elliptic subfield of $K$.

*The associated extension $k(X)/k(U)$.* It is well-known that $K$ has exactly one genus zero subfield of degree 2, which we denote by $k(X)$. The generator of $Gal(K/k(X))$ is the hyperelliptic involution of $K$. It fixes each elliptic subfield of $K$, see Tamme [15]. Hence the field $E \cap k(X)$, denote it by $k(U)$, is a subfield of $E$ of degree 2.

$$\begin{array}{ccc} K & \overset{2}{-} & k(X) \\ 3\,| & & |\,3 \\ E & \overset{2}{-} & k(U) \end{array}$$

*Ramification of $K/E$.* Either $K/E$ is ramified at exactly 2 places of $K$, of ramification index 2, or at one place of $K$, of ramification index 3. It follows immediately from the Riemann-Hurwitz formula. The former (resp., latter) case we call the non-degenerate (resp., degenerate) case, as in [2].

**Definition 1.** A **non-degenerate pair** (resp., **degenerate pair**) is a pair $(K, E)$ such that $K$ is a genus 2 field with a degree 3 elliptic subfield $E$ where the extension $K/E$ is ramified in two (resp., one) places. Two such pairs $(K, E)$ and $(K', E')$ are called isomorphic if there is a $k$-isomorphism $K \to K'$ mapping $E \to E'$.

*Ramification of $k(X)/k(U)$.* In the non-degenerate (resp., degenerate ) case $k(X)/k(U)$ is ramified at exactly 4 (resp., 3) places of $k(X)$ each of ramification index 2 (resp., one of index 3 and the other two of index 2), see [2].

*Invariants of $K$.* We denote by $J_2, J_4, J_6, J_{10}$ the classical invariants of $K$, for their definitions see [9] or [14]. These are homogeneous polynomials (of degree indicated by subscript) in the coefficients of a sectix $f(X, Z)$ defining $K$

$$Y^2 = f(X, Z) = a_6X^6 + a_5X^5Z + \cdots + a_1XZ^5 + a_0$$

and they are a complete set of $SL_2(k)$-invariants (acting by coordinate change). The absolute invariants

$$i_1 := 144\frac{J_4}{J_2^2}, \quad i_2 := -1728\frac{J_2J_4 - 3J_6}{J_2^3}, \quad i_3 := 486\frac{J_{10}}{J_2^5} \tag{1}$$

are even $GL_2(k)$-invariants. Two genus 2 curves with $J_2 \neq 0$ are isomorphic if and only if they have the same absolute invariants.

**Main Theorem:** *Let $K$ be a genus 2 field and $e_3(K)$ the number of $Aut(K/k)$-classes of elliptic subfields of $K$ of degree 3. Then;*
*i)* $e_3(K) = 0, 1, 2,$ *or* $4$
*ii)* $e_3(K) \geq 1$ *if and only if the classical invariants of $K$ satisfy the irreducible equation $F(J_2, J_4, J_6, J_{10}) = 0$ displayed in Appendix A.*

There are exactly two genus 2 curves (up to isomorphism) with $e_3(K) = 4$, see 4.2. The case $e_3(K) = 1$ (resp., 2) occurs for a 1-dimensional (resp., 2-dimensional) family of genus 2 curves, see section 4.

2.1. **The non-degenerate case.** Let $(K, E)$ be a non-degenerate pair and $k(X)$ and $k(U)$ their associated genus 0 subfields. Both $k(X)/k(U)$ and $E/k(U)$ are ramified at 4 places of $k(U)$, three of which are in common. Take the common places to be $U = q_1, q_2, q_3$. Also, take $U = 0$ (resp., $U = \infty$) the place ramified in $k(X)$ but not in $E$ (resp., in $E$ but not $k(X)$). Take $X = 0$ (resp., $X = \infty$) the place over $U = 0$ of ramification index 2 (resp., 1). In the following figure bullets (resp., circles) represent places of ramification index 2 (resp., 1).

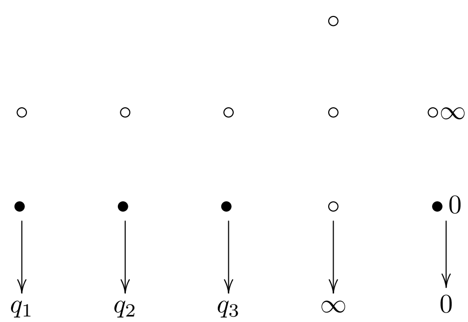


FIGURE 1. Ramification of $k(X)/k(U)$

Thus, $U = \phi(X) = l\frac{X^2}{X^3+aX^2+bX+c}$, where $l, a, b, c \in k$ such that $X^3 + aX^2 + bX + c$ has no multiple roots and $l, c \neq 0$. We can make $l = 1$ and $c = 1$ by replacing $U$ and $X$ by scalar multiples. Then,

$$U = \phi(X) = \frac{X^2}{X^3 + aX^2 + bX + 1} \tag{2}$$

**Remark 2.** *These conditions determine $X$ up to multiplication by a third root of unity $\xi_3 \in k$. Replacing $X$ by $\xi_3 X$ replaces $(a, b)$ by $(\xi_3^2 a, \xi_3 b)$. Invariants of this transformation are:*

$$\begin{aligned} u &= ab \\ v &= b^3 \end{aligned} \tag{3}$$

The derivative of $\phi(X)$ is

$$\phi'(X) = -\frac{X(X^3 - bX - 2)}{(X^3 + aX^2 + bX + 1)^2}$$

Taking the resultant with respect to $z$ of $z^2 - bz - 2$ and $\frac{\phi(x)-\phi(z)}{x-z}$ we get

$$(x^3 - bx - 2)(4x^3 + b^2x^2 + 2bx + 1) \tag{4}$$

The roots of this polynomial correspond to the 6 places of $k(X)$ over the places $U = q_1, q_2, q_3$. Thus, the 6 places of $k(X)$ ramified in $K$ correspond to the roots of the polynomial on the right side of

$$Y^2 = (X^3 + aX^2 + bX + 1)\,(4X^3 + b^2X^2 + 2bX + 1) \tag{5}$$

which gives an equation of $K$. The discriminant of the sextic is nonzero, hence

$$\Delta := (4a^3 + 27 - 18ab - a^2b^2 + 4b^3)^2(b^3 - 27) \neq 0 \tag{6}$$

One checks that the element

$$V = Y\,\frac{X^3 - bX - 2}{F(X)^2} \tag{7}$$

satisfies

$$V^2 = (U - q_1)(U - q_2)(U - q_3) = U^3 + 2\frac{ab^2 - 6a^2 + 9b}{R}\,U^2 + \frac{12a - b^2}{R}\,U - \frac{4}{R}$$

where $R := \frac{\Delta}{(b^3-27)}$. Thus, $E = k(U, V)$.

**Lemma 3.** *For $(a, b) \in k^2$ with $\Delta \neq 0$, equation (5) defines a genus 2 field $K_{a,b} = k(X, Y)$. It has a non-degenerate degree 3 elliptic subfield $E_{a,b} = k(U, V)$, where $U$ and $V$ are given in (2) and (7). Two such pairs $(K_{a,b}, E_{a,b})$ and $(K_{a',b'}, E_{a',b'})$ are isomorphic if and only if $u = u'$ and $v = v'$ (where $u, v$ and $u', v'$ are associated with $a, b$ and $a', b'$, respectively, by (3)).*

*Proof.* The first two statements follow by reversing the above arguments. If $u = u'$, $v = v'$ then $a' = \xi_3^i a$, $b' = \xi_3^i b$ for some $i$. Then clearly the two associated non-degenerate pairs are isomorphic. The converse follows from Remark 2.

□

**Proposition 4.** *The $(u, v) \in k^2$ with $\Delta \neq 0$ bijectively parameterize the isomorphism classes of non-degenerate pairs $(K, E)$ (via the parameterization defined in Lemma 3).*

*Proof.* The proof follows from Lemma 3.

□

From the normal form of $K$ in (5) we can compute the classical invariants $J_2, J_4, J_6, J_{10}$ in terms of $u, v$. These expressions satisfy equation (33) which proves one implication of claim ii) of the theorem for the non-degenerate pairs. In section 4 we explain how equation (33) was found.

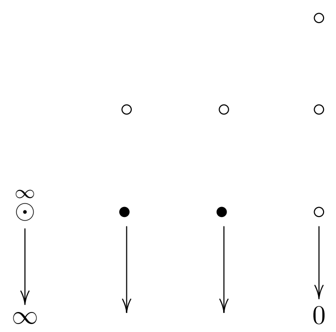

Figure 2. Ramification of $k(X)/k(U)$, degenerate case.

2.2. **The degenerate case.** Let $K/E$ be a degenerate pair. Then $k(X)/k(U)$ is ramified at exactly three places of $k(U)$ which are also ramified in $E$. Let $U=0$ be the place ramified in $E$ but not in $k(X)$. Take $U=\infty$ be the place of $k(U)$ that is totally ramified in $k(X)$ and $X=\infty$ the place over it.

Then, $U=l(X^3+aX^2+bX+c)$ for $a,b,c,l\in k$ such that $X^3+aX^2+bX+c$ has no multiple roots and $l\neq 0$. Replacing $X$ by $X+\frac{a}{3}$ and $U$ by a scalar multiple we get $U=X^3+bX+c$. Note that $b\neq 0$ (otherwise two of the places of $k(U)$ ramified in $k(X)$ coalesce). Thus, by replacing $X$ by a scalar multiple we can also get $b=1$. Then,

$$U=X^3+X+c \tag{8}$$

As in Lemma 3 we get $K=k(X,Y)$ such that

$$Y^2=(3X^2+4)(X^3+X+c) \tag{9}$$

where $c^2\neq -\frac{4}{27}$. Then, $V=Y(3X^2+1)$ satisfies the equation

$$V^2=U(27U^2-54cU+4+27c^2)$$

Thus, $E=k(U,V)$.

**Remark 5.** *In this case, those $w:=c^2$ with $w\neq -\frac{4}{27}$ bijectively parameterize the isomorphism classes of degenerate pairs $K/E$. The proof is analogous to the non-degenerate case.*

Since

$$\begin{aligned} J_2&=774\\ J_4&=36(268+837w)\\ J_6&=36(76760+290574w-729w^2)\\ J_{10}&=432(27w+4)\end{aligned} \tag{10}$$

the map

$$\begin{aligned} k\setminus\{-\frac{4}{27}\}&\to k^3\\ w&\to(i_1,i_2,i_3)\end{aligned}$$

is injective. Thus, a genus 2 field $K$ has at most one $Aut(K)$-orbit of subfields $E$ such that $(K,E)$ is a degenerate pair. One can check that $J_2,J_4,J_6,J_{10}$ satisfy equation (33). This completes the proof that the condition in part (ii) of Main Theorem is necessary.

From (10) we compute that the locus of those $K$ being part of a degenerate pair $(K, E)$ is given by

$$(11)\quad \begin{aligned} &3418801 i_1^2 - 2550732480 i_1 + 4023336960 + 611249816 i_2 = 0\\ &161446939560824832 i_3 + 23630752512 i_1^2 - 6321363049 i_1^3 - 29445783552 i_1\\ &+ 12230590464 = 0 \end{aligned}$$

(Note $J_2 = 774 \neq 0$ on this locus, so $i_1, i_2, i_3$ are everywhere defined). For later use we note the following: If $(K, E_0)$ is a non-degenerate pair with parameters $u, v$ then there exists a degenerate pair $(K, E_1)$ (for same $K$) if and only if

$$(12)\quad 2v - 9u + 27 = 0$$

This is obtained by expressing $i_1, i_2, i_3$ in (11) in terms of $u, v$.

## 3. Function Field of $\mathcal{L}_3$

The absolute invariants $i_1, i_2, i_3$ in terms of $u, v$ are

(13)

$$\begin{aligned} i_1 =& \frac{144}{v(-405 + 252u + 4u^2 - 54v - 12uv + 3v^2)^2}(1188u^3 - 8424uv + u^4 v - 24u^4\\ &+ 14580v - 66u^3 v + 138uv^2 + 297u^2 v + 945v^2 - 36v^3 + 9u^2 v^2)\\ i_2 =& -\frac{864}{v^2(-405 + 252u + 4u^2 - 54v - 12uv + 3v^2)^3}(-81v^3 u^4 + 2u^6 v^2 + 234u^5 v^2\\ &+ 3162402uv^2 - 21384v^3 u + 26676v^4 - 473121v^3 - 72u^6 v - 5832v^4 u + 14850v^3 u^2\\ &- 72v^3 u^3 + 324v^4 u^2 - 650268u^3 v - 5940u^3 v^2 - 3346110v^2 + 432u^6 - 1350u^4 v^2\\ &+ 136080u^4 v - 7020u^5 v - 307638u^2 v^2\\ i_3 =& -243\frac{(v-27)(4u^3 - u^2 v - 18uv + 4v^2 + 27v)^3}{v^3(-405 + 252u + 4u^2 - 54v - 12uv + 3v^2)^5} \end{aligned}$$

We will sometimes view $u, v$ as parameters from $k$ and sometimes as the corresponding coordinate functions on $k^2$. From the context it will be clear which point of view we are taking. If we could eliminate $u, v$ from equations (13) then the desired equation (33) would be obtained directly. However, this exceeds available computer power. We use additional invariants $r_1, r_2$ to overcome this problem in the next section.

As mentioned in the Introduction, the general theory of [2] and [8] implies that $[k(i_1, i_2, i_3) : k(u, v)] = 2$. We also obtain this as a by-product of our computations and the following Lemma, which we record for completeness.

**Lemma 6.** $[k(i_1, i_2, i_3) : k(u, v)] \leq 2$

*Proof.* From the resultants of equations in (13) we determine that $[k(v) : k(i_1, i_2)] = 16$, $[k(v) : k(i_2, i_3)] = 40$, and $[k(v) : k(i_1, i_3)] = 26$. We can show that $u \in k(i_1, i_2, i_3, v)$, the expression is large and we don't display it (the interested reader can check [12]). Since $[k(u, v) : k(i_1, i_2, i_3)]$ must be a common divisor of 16, 26, and 40, then the claim follows.

□

3.1. **Invariants of Two Cubics.** We define the following invariants of two cubic polynomials. For $F(X) = a_3X^3 + a_2X^2 + a_1X + a_0$ and $G(X) = b_3X^3 + b_2X^2 + b_1X + b_0$ define

$$H(F,G) := a_3b_0 - \frac{1}{3}a_2b_1 + \frac{1}{3}a_1b_2 - a_0b_3$$

We denote by $R(F,G)$ the resultant of $F$ and $G$ and by $D(F)$ the discriminant of $F$. Also,

$$r_1(F,G) = \frac{H(F,G)^3}{R(F,G)}, \quad r_2(F,G) = \frac{H(F,G)^4}{D(F)\,D(G)}$$

**Remark 7.** *In* [7] *it is shown that* $r_1, r_2$*, and* $r_3 = \frac{H(F,G)^2}{J_2(F\,G)}$ *form a complete system of invariants for unordered pairs of cubics.*

For $F(X) = X^3 + aX^2 + bX + 1$ and $G(X) = 4X^3 + b^2X^2 + 2bX + 1$ and $u, v$ as in (3) we have

$$\begin{aligned} r_1(F,G) &= 27\frac{v(v-9-2u)^3}{4v^2 - 18uv + 27v - u^2v + 4u^3} \\ r_2(F,G) &= -1296\frac{v(v-9-2u)^4}{(v-27)(4v^2 - 18uv + 27v - u^2v + 4u^3)} \end{aligned} \tag{14}$$

Taking the resultants from the above equations we get the following equations for $u$ and $v$ over $k(r_1, r_2)$:

$$\begin{aligned} &65536r_1r_2^3\,u^2 + (42467328r_2^4 + 21233664r_2^4r_1 + 480r_2r_1^4 + 2r_1^5 + 41472r_2^2r_1^3 \\ &+1548288r_2^3r_1^2 - 294912r_2^3r_1)u - 382205952r_2^4 + 238878720r_2^4r_1 - 2654208r_2^3r_1 \\ &\qquad +13934592r_2^3r_1^2 + 285696r_2^2r_1^3 + 2400r_2r_1^4 + 7r_1^5 = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} &16384v^2r_2^3 + (221184r_2^3r_1 + r_1^4 + 11520r_2^2r_1^2 - 442368r_2^3 + 192r_2r_1^3)v \\ &\qquad -5971968r_2^3r_1 - 864r_2r_1^3 - 124416r_2^2r_1^2 - 2r_1^4 = 0 \end{aligned} \tag{16}$$

Roots of equation (15) (resp., equation (16)) are $u$ and $\beta(u)$ (resp., $v$ and $\beta(v)$) where,

$$\begin{aligned} \beta(u) &= \frac{(v-3u)(324u^2 + 15u^2v - 378uv - 4uv^2 + 243v + 72v^2)}{(v-27)(4u^3 + 27v - 18uv - u^2v + 4v^2)} \\ \beta(v) &= -\frac{4(v-3u)^3}{4u^3 + 27v - 18uv - u^2v + 4v^2} \end{aligned} \tag{17}$$

It follows that either $[k(u,v) : k(r_1,r_2)] = 2$ or $k(u,v)/k(r_1,r_2)$ is Galois with Klein 4-group as Galois group. In the latter case, the map

$$\begin{aligned} \tau : k(u,v) &\to k(u,v) \\ u &\to u \\ v &\to \beta(v) \end{aligned} \tag{18}$$

is an involutory automorphism of $k(u,v)$. But by plugging in equation (17) we see that it is not involutory. Thus, $[k(u,v) : k(r_1,r_2)] = 2$ and $Gal_{k(u,v)/k(r_1,r_2)} = \langle \beta \rangle$.

**Lemma 8.** *The fields $k(i_1,i_2,i_3)$ and $k(r_1,r_2)$ coincide, hence $[k(u,v):k(i_1,i_2,i_3)]=2$. Moreover;*

(19)

$$i_1 = \frac{9}{4}\frac{(13824r_1^3r_2^2+442368r_1^2r_2^3+5308416r_1r_2^4+192r_1^4r_2+r_1^5+786432r_1r_2^3+9437184r_2^4)}{r_1(-1152r_2^2+96r_2r_1+r_1^2)^2}$$

$$i_2 = \frac{27}{8r_1^2(-1152r_2^2+96r_2r_1+r_1^2)^3}(+79626240r_1^4r_2^4-4076863488r_1^2r_2^5+34560r_1^6r_2^2$$
$$+12230590464r_1^2r_2^6+32614907904r_1r_2^6+14495514624r_2^6+288r_1^7r_2+2211840r_1^5r_2^3$$
$$+r_1^8-212336640r_1^3r_2^4+1528823808r_1^3r_2^5-2359296r_1^4r_2^3)$$

$$i_3 = -521838526464\frac{r_2^9}{r_1^2(-1152r_2^2+96r_2r_1+r_1^2)^5}$$

*Proof.* Equations (19) are verified by expressing all variables in terms of $u,v$ (These equations were found by taking suitable resultants). It follows that $k(i_1,i_2,i_3)\subset k(r_1,r_2)$. Equality follows since $[k(u,v):k(r_1,r_2)]=2$ and $[k(u,v):k(i_1,i_2,i_3)]\le 2$ (see Lemma 6).

□

**Finding equation (33):** To find equation (33) we eliminate $r_1$ and $r_2$ from equations (19). The resulting equation (33) has degree 8, 13, and 20 in $i_1,i_2,i_3$ respectively.

## 4. Completing the proof of the Main Theorem

The map

$$\theta:(u,v)\to(i_1,i_2,i_3)$$

given by (13) has degree 2 and it is defined when $J_2\neq 0$. For now we assume that $J_2\neq 0$ (The case $J_2=0$ is treated in section 4.2). Denote the minors of the Jacobian matrix of $\theta$ by $M_1(u,v),M_2(u,v),M_3(u,v)$. The solutions of

$$(20)\qquad \begin{cases} M_1(u,v)=0\\ M_2(u,v)=0\\ M_3(u,v)=0\end{cases}$$

consist of the (non-singular) curve

$$(21)\qquad 8v^3+27v^2-54uv^2-u^2v^2+108u^2v+4u^3v-108u^3=0$$

and 7 isolated solutions which we display in table (1) together with the corresponding values $(i_1,i_2,i_3)$ and properties of the corresponding genus 2 field $K$.

Let $\bar{u},\bar{v},\bar{i}_1,\bar{i}_2,\bar{i}_3$ denote the restrictions of the corresponding functions to the curve (21). Eliminating $u$ from (21) and the defining equation for $i_1$ in (13) we get a relation between $v$ and $i_1$ which shows $[k(\bar{v},\bar{i}_1):k(\bar{i}_1)]=8$; also we get $\bar{u}$ expressed as a rational function of $\bar{v}$ and $\bar{i}_1$. Hence $k(\bar{v},\bar{i}_1)=k(\bar{u},\bar{v})$ and so $[k(\bar{u},\bar{v}):k(\bar{i}_1)]=8$. Similarly we get a relation between $\bar{v}$ and $\bar{i}_2$. Eliminating $\bar{v}$ gives

$$(22)\qquad G(\bar{i}_1,\bar{i}_2)=0$$

an irreducible equation between $\bar{i}_1$ and $\bar{i}_2$ of degree 8 (resp., 12) in $\bar{i}_1$ (resp., $\bar{i}_2$). It is fully displayed as equation (34) in Appendix B. Thus, $[k(\bar{i}_1,\bar{i}_2):k(\bar{i}_1)]=8$. Since also $[k(\bar{u},\bar{v}):k(\bar{i}_1)]=8$, it follows that $k(\bar{u},\bar{v})=k(\bar{i}_1,\bar{i}_2)$. Hence, $e_3(K)=1$

| $(u,v)$ | $(i_1,i_2,i_3)$ | $Aut(K)$ | $e_3(K)$ |
|---|---|---|---|
| $(-\frac{7}{2},2)$ | $J_{10}=0$, no associated genus 2 field K | | |
| $(-\frac{775}{8},\frac{125}{96})$, $(\frac{25}{2},\frac{250}{9})$ | $-\frac{8019}{20},-\frac{1240029}{200},\frac{531441}{100000}$ | $D_4$ | 2 |
| $(27-\frac{77}{2}\sqrt{-1},23+\frac{77}{9}\sqrt{-1})$, $(27+\frac{77}{2}\sqrt{-1},23-\frac{77}{9}\sqrt{-1})$ | $(\frac{729}{2116},\frac{1240029}{97336},\frac{531441}{13181630464}$ | $D_4$ | 2 |
| $(-15+\frac{35}{8}\sqrt{5},\frac{25}{2}+\frac{35}{6}\sqrt{5})$, $(-15-\frac{35}{8}\sqrt{5},\frac{25}{2}-\frac{35}{6}\sqrt{5})$ | $81,-\frac{5103}{25},-\frac{729}{12500}$ | $D_6$ | 2 |

TABLE 1. Exceptional points where $det(Jac(\theta))=0$

for any $K$ such that the associated $u$ and $v$ satisfy equation (21) except possibly those $(u,v)$ that lie over singular points of the curve (22). We check that all these singular points have multiplicity 2, hence there are at most two $(u,v)$ points over each of them.

**Remark 9.** *If $K$ has one subfield of degree 3, then it has at least two distinct such subfields. This is actually true for degree $n$ elliptic subfields, for any $n$, see [2] and [8]). (For $n=3$ it also follows from the explicit construction in* [12]*, Lemma 5.4.) Since $e_3(K)=1$ for a generic curve $K$ of the locus* (22)*, such $K$ has $|Aut(K)|>2$. Actually $Aut(K)=V_4$. We verified this by using equation (17) from* [14] *which gives a necessary and sufficient condition for a genus 2 field $K$ to have $V_4\le Aut(K)$ (in terms of the classical invariants of $K$). Taking the resultant of this equation and the one in Appendix A yields an expression that has* (22) *(the projectivized version) as a factor. From the conditions given in* [14] *for $K$ to have $Aut(K)$ strictly bigger than $V_4$, one checks that actually $Aut(K)=V_4$ for all $K$ in locus* (22)*.*

4.1. **The general case.** By previous section $\theta$ is generically a covering of degree 2. So there exists a Zariski open subset $\mathcal{U}$ of $k^2$ with the following properties: Firstly, $\theta$ is defined everywhere on $\mathcal{U}$ and is a covering of degree 2 from $\mathcal{U}$ to $\theta(\mathcal{U})$. Further, if $\mathfrak{u}\in\mathcal{U}$ then all $\mathfrak{u}'\in k^2$ with $\theta$ defined at $\mathfrak{u}'$ and $\theta(\mathfrak{u}')=\theta(\mathfrak{u})$ also lie in $\mathcal{U}$. Suppose $\underline{i}\in k^3$ such that $|\theta^{-1}(\underline{i})|>2$ and $det(Jac(\theta))$ does not vanish at any point of $\theta^{-1}(\underline{i})$. Then by implicit function theorem, there is an open ball $B$ around each element of $\theta^{-1}(\underline{i})$ such that each point in $\theta(B)$ has $>2$ inverse images under $\theta$. But $B$ has to intersect the Zariski open set $\mathcal{U}$. This is a contradiction. Thus, if $\underline{i}\in k^3$ and $|\theta^{-1}(\underline{i})|>2$, then $det(Jac(\theta))=0$ at some point of $\theta^{-1}(\underline{i})$ and so $\underline{i}$ satisfies (22). It follows that if $K$ is a genus 2 field with $J_2\neq 0$ and $G\neq 0$ then $e_3(K)=2$.

Note that if there is a degenerate pair $(K,E_1)$ and a non-degenerate pair $(K,E_2)$ (for the same $K$) then for the $(u,v)$-invariants of the latter we have (12), hence $\beta(v)=27$ and so the $\beta$-image of $(u,v)$ doesn't correspond to a non-degenerate pair.

We can now complete the proof of part (ii) of Main Theorem. We noted before that the condition in (ii) is necessary. This condition (see Appendix A) defines an irreducible sublocus of $\mathcal{M}_2$ because the corresponding equation is irreducible (by Maple). The locus $e_3(K)\ge 1$ lies in this sublocus and contain a dense subset of it (image of $\theta$). By Lange [4], the locus $e_3(K) \geq 1$ is Zariski closed, hence equals the locus defined by the equation in Appendix A.

4.2. **Exceptional Cases for $J_2 = 0$.** We consider the case when $J_2 = 0$, where

$$
\begin{aligned}
J_2 &= -2(3v^2 + 4u^2 - 12uv + 252u - 54v - 405) \\
&= \frac{2}{3}(3v - 6u + 2u\sqrt{6} - 27 - 18\sqrt{6})(3v - 27 + 18\sqrt{6} - 6u - 2u\sqrt{6})
\end{aligned} \tag{23}
$$

Genus 2 fields $K$ with $J_2 = J_4 = 0$ and $J_6 \neq 0$ (resp., $J_2 = J_6 = 0$ and $J_4 \neq 0$) are classified up to isomorphism by the invariant $\frac{J_6^5}{J_{10}^3}$ (resp., $\frac{J_4^5}{J_{10}^2}$), see Igusa [9]. In the first (resp., second) case there are exactly two (resp., four) such $K$ with $e_3(K) \geq 1$ and they all have $e_3(K) = 2$.

The invariants

$$
a_1 := \frac{J_4 \cdot J_6}{J_{10}}, \quad a_2 := \frac{J_6 \cdot J_{10}}{J_4^4} \tag{24}
$$

determine genus two fields with $J_2 = 0$, $J_4 \neq 0$, and $J_6 \neq 0$ up to isomorphism, see Igusa [9]. A field $K$ with such invariants has $e_3(K) \geq 1$ if and only if

$$
\begin{aligned}
&46656a_1^5a_2^3 + 7558272a_1^4a_2^3 + 1259712\sqrt{6}a_1^4a_2^3 - 15552a_1^4a_2^2 - 12427478784a_1^3a_2^2 + 4917635712\sqrt{6}a_1^3a_2^2 \\
&+1728a_1^3a_2 - 656217531654480a_1^2a_2^2 + 267571209034080\sqrt{6}a_1^2a_2^2 - 1844125056a_1^2a_2 + 743525568\sqrt{6}a_1^2a_2 \\
&-64a_1^2 - 6334497449472117312a_1a_2^2 + 2585860435265558832\sqrt{6}a_1a_2^2 + 230833239838992a_2a_1 - \\
&94237227087840a_2a_1\sqrt{6} - 6012444299758050307777a_2^2 + 2454295392577643805 72a_2^2\sqrt{6} = 0
\end{aligned} \tag{25}
$$

or

$$
\begin{aligned}
&230833239838992a_2a_1 + 94237227087840a_2a_1\sqrt{6} - 12427478784a_1^3a_2^2 - 4917635712\sqrt{6}a_1^3a_2^2 \\
&+1728a_1^3a_2 - 15552a_1^4a_2^2 - 656217531654480a_1^2a_2^2 - 267571209034080\sqrt{6}a_1^2a_2^2 + 46656a_1^5a_2^3 - 64a_1^2 \\
&-6012444299758050307777a_2^2 - 2454295392577643805 72a_2^2\sqrt{6} - 1844125056a_1^2a_2 - 743525568\sqrt{6}a_1^2a_2 \\
&-6334497449472117312a_1a_2^2 - 2585860435265558832\sqrt{6}a_1a_2^2 + 7558272a_1^4a_2^3 - 1259712\sqrt{6}a_1^4a_2^3 = 0
\end{aligned} \tag{26}
$$

As one can check by discussing the map

$$\vartheta : (u, v) \to (a_1, a_2)$$

these fields $K$ generically have $e_3(K) = 2$. It remains to check the singular points of the curves (25) and (26). The intersection of the two components is empty. Each component has four singular points two of which have $e_3(K) = 2$ and the other two have $e_3(K) = 4$. Thus there are exactly four genus 2 fields with $e_3(K) = 4$. Two of them satisfy (25), we call them $P_1$ and $P_2$. The other two are obtained from $P_1, P_2$ by applying the automorphism $\sqrt{6} \to -\sqrt{6}$.

| | |
|---|---|
| $P_1$ | $a_1 = -\frac{77169}{8} + \frac{30759}{8}\sqrt{6}, \quad a_2 = \frac{13783592}{23149125} + \frac{5629912}{23149125}\sqrt{6}$ |
| $P_2$ | $a_1 = -\frac{650835}{8} + \frac{268785}{8}\sqrt{6}, \quad a_2 = -\frac{2984}{20002028625} - \frac{144872}{60006085875}\sqrt{6}$ |

TABLE 2. Singular points of (25) with $e_3(K) = 4$

The four values of $(u, v)$ corresponding to $P_1$ (resp., $P_2$) are obtained as follows: $u$ is determined by $v$ via

$$u = \frac{1}{2}(3+\sqrt{6})(v-9-6\sqrt{6})$$

and $v$ is one of the 4 roots of

$$9v^4 + (-693-27\sqrt{6})v^3 + (15141+1749\sqrt{6})v^2 + (-66414-33396\sqrt{6})v - 1368\sqrt{6} + 3388 = 0 \tag{27}$$

(resp.)

$$3v^4 + (-153+33\sqrt{6})v^3 + (3129-1219\sqrt{6})v^2 + (-10854-11556\sqrt{6})v + 176904\sqrt{6} + 450036 = 0 \tag{28}$$

This completes the proof of the Main Theorem.

□

## 5. j-invariants

A genus 2 field $K$ corresponding to a generic point of locus $e_3(K) \geq 1$ has exactly 2 elliptic subfields $E_1$ and $E_2$ of degree 3. We can take $E_1 = E$ from lemma 3. Its j-invariant is

$$j_1 = 16v \frac{(vu^2 + 216u^2 - 126vu - 972u + 12v^2 + 405v)^3}{(v-27)^3(4v^2 + 27v + 4u^3 - 18vu - vu^2)^2} \tag{29}$$

The automorphism $\beta \in Gal_{k(u,v)/k(r_1,r_2)}$ permutes $E_1$ and $E_2$, therefore switches $j_1$ and $j_2$. Then $j_2 = \beta(j_1)$ is

$$j_2 = -256 \frac{(u^2-3v)^3}{v(4v^2+27v+4u^3-18vu-vu^2)} \tag{30}$$

For the equation of $E_2$ using a different approach see [12], chapter 5.

### 5.1. Isomorphic Elliptic Subfields.

Suppose that $E_1 \cong E_2$. Then, $j_1 = j_2$ implies that

$$8v^3 + 27v^2 - 54uv^2 - u^2v^2 + 108u^2v + 4u^3v - 108u^3 = 0 \tag{31}$$

or

$$\begin{aligned} &324v^4u^2 - 5832v^4u + 37908v^4 - 314928v^3u - 81v^3u^4 + 255879v^3 + 30618v^3u^2 \\ &- 864v^3u^3 - 6377292uv^2 + 8503056v^2 - 324u^5v^2 + 2125764u^2v^2 - 215784u^3v^2 \\ &+ 14580u^4v^2 + 16u^6v^2 + 78732u^3v + 8748u^5v - 864u^6v - 157464u^4v + 11664u^6 = 0 \end{aligned} \tag{32}$$

**Remark 10.** *The former equation is the condition that $det(Jac(\theta)) = 0$ see equation (21).*

If $e_3(K) \geq 1$ then the automorphism group $Aut(K/k)$ is one of the following: $\mathbb{Z}_2, V_4$, $D_8$, or $D_{12}$. $\mathbb{Z}_2$ is the generic case, $V_4$ is a 1-dimensional family. There are exactly 6 genus 2 fields with automorphism group $D_8$ (resp., $D_{12}$). Thus, there are exactly 6 genus 2 curves (defined over $\mathbb{C}$) in the locus (33) and with automorphism group $D_8$ (resp., $D_{12}$). Exactly four (resp., three) ov these curves with automorphism group $D_8$ (resp., $D_{12}$) are defined over $\mathbb{Q}$. This is studied in [13] where the rational points of these genus 2 curves are also disscussed.

## Appendix A:

Here is the equation which defines the locus $\mathcal{L}_3$ of fields $K$ with $e_3(K) \geq 1$.

$$C_8 J_{10}^8 + C_7 J_{10}^7 + \cdots + C_1 J_{10} + C_0 = 0 \tag{33}$$

where $C_0, \ldots, C_8$ are

$C_8 = 2^4 \cdot 3^{31} \cdot 5^5 \cdot 19^{10} \cdot 29^5$

$C_7 = 2^4 \cdot 3^{27} \cdot 19^5 (-194894640029511 J_2^5 - 55588661819356000 J_4^2 J_2 - 12239149540657725 J_2^3 J_4$
$+ 223103526505680000 J_4 J_6 + 40811702108053500 J_2^2 J_6)$

$C_6 = 2^2 \cdot 3^{21} (-358022844687577658584 32 J_4^5 - 1756270399106587730391 J_4^2 J_2^6 - 2863899185952300 6654 J_4 J_2^8$
$- 840912252037601594 41286 J_4^3 J_2^4 + 40089595939100695356 1032 J_4^4 J_2^2 - 61773685738999443 J_2^10$
$- 36732013960722596037 56160 J_4^3 J_2 J_6 + 78794917552182649843 87200 J_4^2 J_6^2 + 15251447355608658629952 J_2^5 J_4 J_6$
$+ 1179903008384844066250272 J_2^3 J_4^2 J_6 - 5566672398589809889658760 J_2^2 J_4 J_6^2 + 1120242893725541836 80 J_2^7 J_6$
$- 3211676940972271618 2888 J_2^4 J_6^2 + 851217187775496224915 5200 J_2 J_6^3)$

$C_5 = 2^2 \cdot 3^{19} (-1263000438269546265 3 J_4^4 J_2^7 + 3208392527642873625 60 J_4^7 J_2 - 1876069272397136886448 J_4^6 J_2^3$
$+ 6067428662204563565 80 J_4^5 J_2^5 - 12417348571905271 5 J_4^3 J_2^9 + 222410345121014389440 00 J_4^5 J_2^2 J_6$
$- 885467367039388263045 12 J_4^4 J_2 J_6^2 - 1071207892642075344998 4 J_2^4 J_4^4 J_6 + 689046353233036645112 64 J_2^3 J_4^3 J_6^2$
$- 192353895694677164016384 J_2^2 J_4^2 J_6^3 + 197449733923926905783808 J_2 J_4 J_6^4 - 19161730473716452239 36 J_4^6 J_6$
$+ 116211018774997425051648 J_4^3 J_6^3 - 13214359769778617241 6 J_6^5 + 1361403542457288 J_2^10 J_4 J_6$
$- 50057651187404926 56 J_2^7 J_4 J_6^2 + 23281906163948343072 0 J_2^6 J_4^3 J_6 - 15763198946945851784 52 J_2^5 J_4^2 J_6^2$
$+ 46552394592087645530 88 J_2^4 J_4 J_6^3 - 226900590409548 J_2^{11} J_4^2 - 2042105313685932 J_2^9 J_6^2 + 62616327559678728 00 J_2^6 J_6^3$
$- 50657347964782755761 76 J_2^3 J_6^4 + 1352318109350828796 J_2^8 J_4^2 J_6)$

$C_4 = 3^{15} (1417825317153277312 J_4^9 J_2^2 + 2391308818408811717 J_4^6 J_2^8 + 718590303030600 J_2^{10} J_4^5$
$- 638760745337170544640 J_4 J_6^6 + 440759275303802880 J_4^{10} - 811871728077168654 0192 J_4^5 J_2^4 J_6^2$
$+ 42668434906863398019072 J_4^2 J_2 J_6^5 - 5705466481402064057 4336 J_4^3 J_2^2 J_6^4 + 305467747401585816760 32 J_4^4 J_2^3 J_6^3$
$+ 860181421512383127590 4 J_4^6 J_2^2 J_6^2 - 144956270068219591680 0 J_4^7 J_2^3 J_6 + 10679283541242493031 04 J_4^6 J_2^5 J_6$
$- 1044389626302431630131 2 J_2^3 J_4 J_6^5 + 7247970315150439028112 J_2^4 J_4^2 J_6^4 - 21769241176751736619008 J_4^5 J_2 J_6^3$
$- 2640201919999154595648 J_2^5 J_4^3 J_6^3 - 5589356242444526131 2 J_2^7 J_4^5 J_6 + 5314096352411911193 04 J_2^6 J_4^4 J_6^2$
$- 1012614205133520 J_2^8 J_6^4 - 2454855015326199552 J_2^5 J_6^5 - 12501409939920 J_2^{12} J_4^4 - 675076136755680 J_2^{10} J_4^2 J_6^2$
$- 33390518666828400 J_2^9 J_4^4 J_6 + 318836356802749843 2 J_2^6 J_4 J_6^4 - 1569001498547402304 J_2^7 J_4^2 J_6^3$
$- 275375222428239820800 J_4^7 J_6^2 + 1980984909551805033062 4 J_4^4 J_6^4 + 617951606198374018368 0 J_2^2 J_6^6$
$+ 150016919279040 J_2^{11} J_4^3 J_6 + 8779948140633562113 6 J_4^8 J_2^4 + 1350152273511360 J_2^9 J_4 J_6^3$
$- 554966111861328006 48 J_4^7 J_2^6 + 3991180985584255795 2 J_4^8 J_2 J_6 + 353362680242481096 J_2^8 J_4^3 J_6^2)$

$C_3 = 2^4 \cdot 3^{12} (-19225816442103600 J_4^{10} J_2^5 + 6433952690394144 J_2^4 J_6^7 - 2917203075615 J_2^{11} J_4^7$
$+ 62951605613640 J_2^{10} J_4^6 J_6 + 7900854051362368 J_4^{11} J_2^3 - 873165547551982 J_2^9 J_4^8 + 13234982161044480 J_4^{12} J_2$
$+ 407790286455018700 8 J_4^5 J_6^5 + 7506792545698293 J_4^9 J_2^7 - 55019014994202624 J_4^{11} J_6 - 3415519987075510272 J_4^2 J_6^7$
$- 932605137272623104 J_4^8 J_6^3 - 6607177263254292480 J_2 J_6^8 - 1394785406520 J_2^7 J_6^6 - 1913285880 J_2^{13} J_4^6$
$- 258293593800 J_2^{11} J_4^4 J_6^2 - 1976299597616301504 J_4^8 J_2^3 J_6^2 + 1337598192058041744 J_4^7 J_2^5 J_6^2$
$- 2324642344200 J_2^9 J_4^2 J_6^4 + 2789570813040 J_2^8 J_4 J_6^5 - 243015467955111198 J_2^7 J_4^6 J_6^2 + 22136761801348668 J_2^8 J_4^7 J_6$
$-1554638964372636 12 J_4^8 J_2^6 J_6 + 1610103379618300435 2 J_4^5 J_2^3 J_6^4 + 16367298631796450304 J_4^3 J_2 J_6^6 + 8433152 J_4^7 J_2^2 J_6^3$
$- 825496517802146918 4 J_4^6 J_2 J_6^4 + 34439145840 J_2^{12} J_4^5 J_6 - 576988130682378 J_2^9 J_4^5 J_6^2 + 2912934238489260 J_2^8 J_4^4 J_6^3$
$-8749875412454175 J_2^7 J_4^3 J_6^4 + 15637511592200340 J_2^6 J_4^2 J_6^5 - 127105068829245696 J_4^{10} J_2^2 J_6 + 61490858151742156 8 J_4^9 J_2 J_6^2$
$- 2337441943136020761 6 J_4^4 J_2^2 J_6^5 + 1508868948605946984 J_2^6 J_4^5 J_6^3 - 5795040294470623824 J_2^5 J_4^4 J_6^4$
$+ 14094983896511630112 J_2^4 J_4^3 J_6^5 + 1033174375200 J_2^{10} J_4^3 J_6^3 + 314069798204069472 J_4^9 J_2^4 J_6 - 61501104 J_4^6 J_2^4 J_6^3$
$- 21194163080222025024 J_2^3 J_4^2 J_6^6 + 18002402119176332544 J_2^2 J_4 J_6^7 - 15392091937240080 J_2^5 J_4 J_6^6)$

$C_2 = 2^5 \cdot 3^8 (-159732958548480 J_4^{13} J_2 J_6 - 27945192968593920 J_2 J_4 J_6^9 + 238596124150086 J_4^{10} J_2^7 J_6 + 3224288 J_2^3 J_6^9$
$-36311136215244 J_2^9 J_4^9 J_6 - 996173640 J_2^{12} J_4^6 J_6^2 + 5977041840 J_2^{11} J_4^5 J_6^3 - 22413906900 J_2^{10} J_4^4 J_6^4 - 37501414009508 J_4^{13} J_2^4$

$$
\begin{aligned}
&+14210312049697149J_2^6J_4^6J_6^4+86354918885580768J_2^4J_4^4J_6^6-111444977082978432J_2^3J_4^3J_6^7+27908893977856J_4^{14}J_2^2\\
&+83768141083825152J_2^2J_4^2J_6^8-42942980968765488J_2^5J_4^5J_6^5-10736445647473J_4^{11}J_2^8+61746352553318400J_4^4J_2J_6^7\\
&+410958880454688J_4^{12}J_2^3J_6-14059252057660416J_4^3J_6^8-1643659809866496J_4^{11}J_2^2J_6^2+441832778741790J_2^8J_4^8J_6^2\\
&-3128599551108636J_2^7J_4^7J_6^3+2815950495430656J_4^{10}J_2J_6^3+12291244885171152J_4^8J_2^5J_6^3+1579225145J_2^{12}J_4^9\\
&-2286353789913249J_4^9J_2^6J_6^2-40300476525629352J_4^7J_2^4J_6^4+81707043798929088J_4^6J_2^3J_6^5-25936092270J_2^6J_6^8\\
&-98257765274489088J_4^5J_2^2J_6^6+3318887207480832J_4^{10}J_2^4J_6^2-478511899451856J_4^{11}J_2^5J_6+28476287051677J_4^1 2J_2^6\\
&-12153253649302656J_4^9J_2^3J_6^3+24757975700165376J_4^8J_2^2J_6^4-26570902457981952J_4^7J_2J_6^5-6755065089024J_4^{15}\\
&+10883911680J_6^{10}+53793376560J_2^9J_4^3J_6^5-80690064840J_2^8J_4^2J_6^6+69162912720J_2^7J_4J_6^7+94873680J_2^{13}J_4^7J_6\\
&+227109129291J_2^{10}J_4^7J_6^2-628213747356J_2^9J_4^6J_6^3-1389130574661J_2^8J_4^5J_6^4+16465793988870J_2^7J_4^4J_6^5\\
&-56794191944715J_2^6J_4^3J_6^6+102713329135152J_2^5J_4^2J_6^7-98529746457492J_2^4J_4J_6^8-30650938650J_2^{11}J_4^8J_6\\
&-1716480768J_4^9J_6^4+11718053954519040J_4^6J_6^6+220752428322816J_4^{12}J_6^2+1322792799725J_2^{10}J_4^{10}-3953070J_2^{14}J_4^8)
\end{aligned}
$$

$$
\begin{aligned}
C_1 = {}&-2^8\cdot 3^5(61736960J_4^8J_2-182135808J_4^7J_6+16021872J_4^7J_2^3-211022400J_4^6J_2^2J_6-26594919J_4^6J_2^5\\
&+899159040J_4^5J_2J_6^2+330458928J_4^5J_2^4J_6-215198J_2^7J_4^5-1227405312J_4^4J_6^3-1535734368J_4^4J_2^3J_6^2\\
&+2930532J_4^4J_2^6J_6-363J_2^9J_4^4+3162070656J_4^3J_2^2J_6^3-16471998J_4^3J_2^5J_6^2+4356J_4^3J_2^8J_6-19602J_4^2J_2^7J_6^2\\
&-2433162240J_4^2J_2J_6^4+47961936J_4^2J_2^4J_6^3+39204J_4J_2^6J_6^3-72369936J_4J_2^3J_6^4+746496J_4J_6^5-29403J_2^5J_6^4\\
&+45116352J_2^2J_6^5)(J_4^3-J_2^2J_4^2+6J_2J_6J_4-9J_6^2)^3
\end{aligned}
$$

$$C_0 = 2^8(768J_4^2-416J_4J_2^2-J_2^4+1536J_2J_6)(J_4^3-J_2^2J_4^2+6J_2J_6J_4-9J_6^2)^6$$

## Appendix B:

The equation of the branch locus of the map

$$\theta : k^2 \setminus \{\Delta = 0\} \to \mathcal{L}_3$$

$$(u,v) \to (i_1, i_2, i_3)$$

(34)

$$
\begin{aligned}
&3507505273398025i_1^{12}-4880484817793862073548480i_1^{11}+192(11302504938388489628125i_2\\
&+3464520392376896505811 92)i_1^{10}-4(894399050462789643193589 76+6030715704603053299722 5i_2^2\\
&-418898106606911323464864 0i_2)i_1^9+192(492467235580956800488550 4+3358888775389041314351 5i_2^2\\
&-892903738745402453566608i_2)i_1^8-192(636013959123538338132787 2+593587805135845078438632i_2^2\\
&+849397678885114696829952i_2-155377016701633403297775i_2^3)i_1^7+2(406606074742962841916289024i_2^2\\
&-277115195110992004206521 60i_2^3+305526006347596356290347008+18304980860695577479407 5i_2^4\\
&+255320765313220782576893952i_2)i_1^6-192i_2(136540237119466922802892 80i_2+6037722845335050737631 5i_2^3\\
&-146435002277144299726579 2i_2^2+22457545304665379297034 24)i_1^5+192i_2^2(19906166147692916436664320\\
&+5244658469917784551174 32i_2^2-23333601385906920314818 56i_2-10694743521252049023175i_2^3)i_1^4\\
&+20i_2^2(25359283200486542884819 84i_2^3-5763664926963183845376000i_2-2826269156665784550688358 4i_2^2\\
&-99413657833844087193600000-92803044099032574023 25i_2^4)i_1^3-192i_2^3(18076023264213617319826 56i_2^2\\
&-7640175202256773859162 5i_2^3-28676270196566138880000 00-99189527154240669959116 80i_2)i_1^2\\
&+1728i_2^4(1262371230708245434125i_2^3-4497031891936327675228 0i_2^2-17528823669935877120000 00\\
&+50687044955460223588454 4i_2)i_1+i_2^4(1273853956404329093756 25i_2^4-6059765135286837968377600i_2^3\\
&-71714832485837325926400000 0i_2+1028383875477725986413527 04i_2^2+1617386925920256\cdot 10^{12})=0
\end{aligned}
$$

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