On the generic curve of genus 3

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Abstract. We study genus g coverings of full moduli dimension of degree $d = [\frac{g+3}{2}]$. There is a homomorphism between the corresponding Hurwitz space $\mathcal{H}$ of such covers to the moduli space $\mathcal{M}_g$ of genus g curves. In the case $g = 3$, using the signature of such covering we provide an equation for the generic ternary quartic. Further, we discuss the degenerate subloci of the corresponding Hurwitz space of such covers from the computational group theory viewpoint. In the last section, we show that one of these degenerate loci corresponds to the locus of curves with automorphism group C_3. We give necessary conditions in terms of covariants of ternary quartics for a genus 3 curve to belong to this locus.

1. Introduction

In this brief note we study non-hyperelliptic curves of genus 3. The main idea is to determine an equation for the generic genus 3 curve starting from a covering of full moduli dimension.

Determining the monodromy group of a generic genus g curve covering $\mathbb{P}^1$ is a problem with a long history which goes back to Zariski and relates to Brill-Nother theory. Let $\mathcal{X}_g$ be generic curve of genus g and $f : \mathcal{X}_g \to \mathbb{P}^1$ a degree n cover. Denote by $G := Mon(f)$, the monodromy group of $f : \mathcal{X}_g \to \mathbb{P}^1$. Zariski showed that for $g > 6$, G is not solvable. For $g \leq 6$ the situation is more technical. This has been studied by many authors e.g., Fried, Guralnick, Neubauer, Magaard, Völklein et al. The problem is open for $g = 2$. The main question is to determine all possible signatures of the cover $f : \mathcal{X}_g \to \mathbb{P}^1$, where $\mathcal{X}_g$ is a generic curve. Covers with such signature are called covers of full moduli dimension. In section 2, we provide a brief description of the main situation and give at least one signature for each $g > 2$ which gives a cover of full moduli dimension. The degree of the corresponding cover is the well known gonality of the curve. We show that the locus of such curves in $\mathcal{M}_g$ does not intersect the hyperelliptic locus (cf. Proposition 4).

In section 3, we use this cover of full moduli dimension for $g = 3$ to find an equation of the generic curve given by

$$Y^3(X + a) + Y^2(bX + c) + Y(dX^2 + eX) + X^3 + fX^2 + X = 0.$$

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It is not clear whether the tuple (a, b, c, d, e, f) is determined uniquely once we fix the moduli point. In other words, we don't know the degree of the extension $k(a, b, c, d, e, f)/k(\mathfrak{p})$ where $k(\mathfrak{p})$ is the field of moduli. Investigating this question would give some insights on the problem of field of moduli versus field of definition which is classical in algebraic geometry. We prove computationally that $[k(a, b, c, d, e, f) : k(\mathfrak{p})] \leq 3$. Hence the obstruction has at most degree 3. We define six $GL_3(\mathbb{C})$-invariants $i_1, \dots, i_6$. However, it is not clear whether these invariants generate the field of invariants of $GL_3(\mathbb{C})$ on the space of ternary quartics. In other words, for a given curve it is not known if $[k(\mathfrak{p}) : k(i_1, i_2, i_3, i_4, i_5, i_6)] = 1$. If one were to show computationally that $[k(a, b, c, d, e, f) : k(i_1, i_2, i_3, i_4, i_5, i_6)] = 1$ then this would provide a proof that $(i_1, i_2, i_3, i_4, i_5, i_6)$ determines the moduli point in $\mathcal{M}_3$.

In section 4, we discuss briefly the Hurwitz space of these coverings of full moduli dimension. We show that this is an irreducible locus in $\mathcal{M}_3$ and compute the number of Nielsen classes. We show that there are 4 degenerate ways the branch points of the covering can collapse. In each case we compute the number of Nielsen classes. In the first three cases the monodromy group is S_3 and in the last case is the cyclic group C_3. The covering obtained in this case is a Galois covering and the curve has automorphism group C_3.

The generic curve of genus 3 has no automorphisms. In the last part we give a brief description of genus 3 non-hyperelliptic curves with automorphisms. We make some remarks on the invariants of these curves and the field of moduli and field of definition.

Notation: By a curve we mean an irreducible, smooth, projective curve defined over an algebraically closed field k. $\mathcal{M}_g$ denotes the moduli space of smooth curves of genus g. We denote by $[\mathcal{X}_g]$ or by $\mathfrak{p}$ the isomorphism class of $\mathcal{X}_g$, i.e. the corresponding point in $\mathcal{M}_g$. The cyclic group of order n is denoted by C_n and the Klein 4-group by V_4.

2. Covers of full moduli dimension

Let $\mathcal{M}_g$ be the moduli space of curves of genus $g \geq 2$ and $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ the Riemann sphere. Let $\mathcal{X}$ be a curve of genus g and $\phi : \mathcal{X} \to \mathbb{P}^1$ be a degree n covering with r branch points. By covering space theory, there is a tuple $(\sigma_1, \dots, \sigma_r)$ in S_n such that $\sigma_1 \cdots \sigma_r = 1$ and $G := \langle \sigma_1, \dots, \sigma_r \rangle$ is a transitive group in S_n. We call such a tuple the signature of ϕ. Conversely, let $\sigma := (\sigma_1, \dots, \sigma_r)$ be a tuple in S_n such that $\sigma_1 \cdots \sigma_r = 1$ and $G := \langle \sigma_1, \dots, \sigma_r \rangle$ is a transitive group in S_n. We say a cover $\phi : \mathcal{X} \to \mathbb{P}^1$ of degree n is of type σ if it has σ as signature. The genus g of $\mathcal{X}$ depends only on σ (Riemann-Hurwitz formula). Let $\mathcal{H}_\sigma$ be the set of pairs $([f], (p_1, \dots, p_r)$, where $[f]$ is an equivalence class of covers of type σ, and $p_1, \dots, p_r$ is an ordering of the branch points of ϕ.

By [**Be**], the **Hurwitz space** $\mathcal{H}_\sigma$ carries a structure of quasi-projective variety (over $\mathbb{C}$). We have a morphism

$$\Phi_\sigma : \mathcal{H}_\sigma \to \mathcal{M}_g$$

mapping $([f], (p_1, \dots, p_r)$ to the class $[\mathcal{X}]$ in the moduli space $\mathcal{M}_g$.

Each component of $\mathcal{H}_\sigma$ has the same image in $\mathcal{M}_g$. As in [**MV**], we define **moduli dimension of** σ (denoted by $\dim(\sigma)$) as the dimension of $\Phi_\sigma(\mathcal{H}_\sigma)$; i.e., the dimension of the locus of genus g curves admitting a cover to $\mathbb{P}^1$ of type σ. We say σ has **full moduli dimension** if

$$\dim(\sigma) = \dim \mathcal{M}_g$$

We would like to explore the map Φ_σ for small g. The following two problems are significant.

Problem 1: Given a signature σ. Compute the moduli dimension of σ.

Problem 2: Given $g \geq 3$ and some σ which has full moduli dimension. What can you say about Φ_σ?

Fix $g \geq 3$. It is well known that the dimension of the moduli space $\mathcal{M}_g$ is $3g-3$. We want to find σ of full moduli dimension. It is known that $dim(\sigma) = r-3$, when the quotient space has genus 0; see [**MV**]. Thus, if a cover has full moduli dimension then it has $r = 3g$ branch points. By $[x]$ we denote the integer part of x. Then we have the following:

LEMMA 1. *For any $g \geq 3$ there is a degree $d = [\frac{g+3}{2}]$ cover*

$$\psi_g : \mathcal{X}_g \to \mathbb{P}^1$$

of full moduli dimension from a genus g curve $\mathcal{X}_g$ such that it has $r = 3g$ branch points and signature:

i) If g is odd, then $\sigma = (\sigma_1, \dots, \sigma_r)$ such that $\sigma_1, \dots, \sigma_{r-1} \in S_d$ are transpositions and $\sigma_r \in S_d$ is a 3-cycle.

ii) If g is even, then $\sigma = (\sigma_1, \dots, \sigma_r)$ such that $\sigma_1, \dots, \sigma_r \in S_d$ are transpositions.

REMARK 2. *Note that the degree $d = [\frac{g+3}{2}]$ is the minimum degree of a map from a generic curve of genus g to $\mathbb{P}^1$, see* [**Mu**]. *This is normally known as the gonality $gon(\mathcal{X}_g)$ of a generic curve $\mathcal{X}_g$ of genus g.*

DEFINITION 3. For a fixed $g \geq 3$ we call the cover $\psi_g : \mathcal{X}_g \to \mathbb{P}^1$ as above the **W-cover** associated to g.

PROPOSITION 4. *Let $\mathcal{X}_g$ be a hyperelliptic curve of odd genus $g \geq 3$. Then, there is no W-covering $\psi : \mathcal{X}_g \to \mathbb{P}^1$.*

PROOF. Let w be the hyperelliptic involution of $\mathcal{X}_g$. We assume that there exists a W-covering $\psi_g : \mathcal{X}_g \to \mathbb{P}^1$. Then there is $P \in \mathcal{X}_g$ such that the ramification index $e_\psi(P)$ of P under ψ is 3. Denote the corresponding function fields by K and $k(z)$. If w does not fix $k(z)$, then $s = w(z)$ and $K = k(z, w)$. Thus, s and w satisfy a symmetric polynomial $g(s,t) = 0$ which has degree 3 in both s and degree 2 in w. Then the curve $g(s,w)$ has genus $[\frac{d-1}{2}] \neq g$. Assume that w fixes $k(x)$. Let v denote its restriction in $k(x)$. Then, there exists a degree d covering $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{X}_g & \xrightarrow{w} & \mathbb{P}^1 \\ \psi \downarrow & & \downarrow \phi \\ k(z) & \xrightarrow{v} & \mathbb{P}^1 \end{array}$$

We denote by $a_0 = v \circ \psi(P) = \phi \circ w(P)$. Then, in $(v \circ \psi)^{-1}(a_0)$ there could be one point of index 6, one point of index 3 and three unramified points, or one point of index 3 and one point of index 2. In $(\phi \circ w)^{-1}(a_0)$ there could be one point of index 6 or two points of index 3. Thus, $e_{\phi\circ w}(P) = 6$. Hence, $e_w(P) = 2$ and P is a Weierstrass point. Since $\mathcal{X}_g$ is hyperelliptic then the Weierstrass gap sequence is

$$1 = n_1 < 3 < \cdots < 2g - 1.$$

Thus, there are no functions with a pole of order three at a Weierstrass point. Hence, there is no W-cover for $\mathcal{X}_g$. □

3. The case of genus 3

For the rest of this paper we will focus on the case $g = 3$. Let $\mathcal{C}$ be a genus 3 curve defined over $k = \mathbb{C}$ and K its function field. It is well known that if $\mathcal{C}$ is a hyperelliptic curve then it corresponds to a binary octavic, otherwise to a ternary quartic. Thus, the isomorphic classes of genus 3 curves correspond to (projective) equivalence classes of binary octavics or ternary quartics. Let $S(n, r)$ denote the graded ring of (projective) invariants of homogeneous polynomials of order r in n variables with coefficients in $\mathbb{C}$. In this section we will describe briefly $S(3, 4)$ (ternary quartics).

We denote by V be the set of ternary quartics. V is a complex vector space of dimension 15. Let $\mathbb{C}[V]$ be the algebra of complex polynomial functions on V and by $\mathbb{C}[V]_n$ the set of homogeneous elements of degree n in $\mathbb{C}[V]$. Then, $\mathbb{C}[V] = \oplus_{n\geq 0}\, \mathbb{C}[V]_n$ is a graded algebra.

The group $SL_3(\mathbb{C})$ acts in a natural way in V. The invariant space under this action is the algebra $\mathcal{A}$ of projective invariants of quartic plane curves. $\mathcal{A}$ admits a homogeneous system of parameters of degree 3, 6, 9, 12, 15, 18, 27. We denote these system of parameters by $I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}$, where I_j's are defined as in Dixmier [**Di**]. They are homogenous polynomials of degree 3, 6, 9, 12, 15, 18, 27 respectively and generate the ring of invariants $SL(3, 4)$ of ternary quartics. For a general ternary quartic given by

$$\begin{aligned} g(x, y, z) = {} & a_1x^4 + 4a_2x^3y + 6a_3x^2y^2 + 4a_4xy^3 + a_5y^4 + 4a_6x^3z + 12a_7x^2yz + 12a_8xy^2z \\ & + 4a_9y^3z + 6a_{10}x^2z^2 + 12a_{11}xyz^2 + 6a_{12}y^2z^2 + 4a_{13}xz^3 + 4a_{14}yz^3 + a_{15}z^4. \end{aligned}$$

I_3 and I_6 are defined as follows:

$$\begin{aligned} I_3(g) = {} & a_1a_5a_{15} + 3(a_1a_{12}^2 + a_5a_{10}^2a_{15}a_3^2) + 4(a_2a_9a_{13} + a_6a_4a_{14} \\ & - 4(a_1a_9a_{14} + a_5a_6 \end{aligned}$$

$$I_6(g) = \det(A), \ \textit{where } A = \begin{bmatrix} a_1 & a_3 & a_{10} & a_7 & a_6 & a_2 \\ a_3 & a_5 & a_{12} & a_9 & a_8 & a_4 \\ a_{10} & a_{12} & a_{15} & a_{14} & a_{13} & a_{11} \\ a_7 & a_9 & a_{14} & a_{12} & a_{11} & a_8 \\ a_6 & a_8 & a_{13} & a_{11} & a_{10} & a_7 \\ a_2 & a_4 & a_{11} & a_8 & a_7 & a_3 \end{bmatrix}$$

see Dixmier [**Di**] for definitions of $I_9, \ldots, I_{27}$.

Two algebraic curves $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic if there exists an $\alpha \in GL_3(\mathbb{C})$ such that $\mathcal{C} \cong \mathcal{C}^\alpha$. We define

$$i_1 := \frac{I_6}{I_3^2}, \quad i_2 := \frac{I_9}{I_3^3}, \quad i_3 := \frac{I_{12}}{I_3^4}, \quad i_4 := \frac{I_{15}}{I_3^5}, \quad i_5 := \frac{I_{18}}{I_3^6}, \quad i_6 := \frac{I_{27}}{I_3^9}.$$

Since I_j's are $SL_3(\mathbb{C})$-invariants then $i_1, \ldots, i_6$ are invariants under the $GL_3(\mathbb{C})$-action. If $C \cong \mathcal{C}'$, then $i_j(\mathcal{C}) = i_j(\mathcal{C}')$, for $j = 1, \ldots, 6$.

REMARK 5. *It is a difficult task of computing explicitly the invariants I_j of a ternary quartic. This is because their expressions are rather large. We will see that using the canonical form found in the following lemma, this is much easier.*

3.1. The generic curve of genus 3. In this section we use the structure of W-covers to obtain an equation for the generic curve of genus 3. This equation is given with 6 parameters (the moduli dimension is also 6), hence the name generic.

For $g = 3$ a W-cover has degree three and 9 branch points. The signature is $\sigma = (\sigma_1, \dots, \sigma_9)$ where $\sigma_i \in S_3$ is an transposition for $i = 1, \dots, 8$ and σ_9 is the 3-cycle.

LEMMA 6. *Let $\mathcal{C}$ be a generic curve of genus 3 defined over a field L such that $char\, L \neq 2, 3$. Then, there is a degree 3 covering $\psi : \mathcal{C} \to \mathbb{P}^1$ of full moduli dimension. Moreover, $\mathcal{C}$ is isomorphic to a curve with affine equation*

$$Y^3(X+a) + Y^2(bX+c) + Y(dX^2+eX) + X^3 + fX^2 + X = 0$$

for $a, b, c, d, e, f \in L$ such that $\Delta \neq 0$, where Δ is the discriminant of the quartic.

PROOF. Let P be a Weierstrass point on $\mathcal{C}$ and K the function field of $\mathcal{C}$. Then exists a meromorphic function x which has P as a triple pole and no other poles. Thus, $[K : L(x)] = 3$. Consider x as a mapping of $\mathcal{C}$ to the Riemann sphere. We call this mapping $\psi : \mathcal{C} \to \mathbb{P}^1$ and let ∞ be $\psi(P)$. From the Riemann-Hurwitz formula we have that ψ has at most 8 other branch points. There is also a meromorphic function y which has P as a triple pole and no other poles. Thus the equation of K is given by

$$F(x,y) := \gamma_1(x)\, y^3 + \gamma_2(x)\, y^2 + \gamma_3(x)\, y + \gamma_0(x) = 0 \tag{1}$$

where $\gamma_0(x), \dots, \gamma_3(x) \in L[x]$ and $deg(\gamma_i) = i$ for $i = 1, 2, 3$, $deg(\gamma_0) = 3$. The discriminant of $F(x,y)$ with respect to y

$$D(F,y) := -27\,(\gamma_1\,\gamma_0)^2 + 18\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3 + (\gamma_2\,\gamma_3)^2 - 4\,\gamma_0\,\gamma_2^3 - 4\,\gamma_1\,\gamma_3^3,$$

must have at most degree 8 since its roots are the branch points of $\psi : \mathcal{C} \to \mathbb{P}^1$. Thus, we have

$$deg\,(\gamma_3\gamma_2) \leq 4, \quad deg\,(\gamma_0\gamma_2^3) \leq 8, \quad deg\,(\gamma_3^3\,\gamma_1) \leq 8.$$

If $deg\,(\gamma_2) = 2$ then $deg\,(\gamma_3) \leq 2$ and $deg\,(\gamma_0) = 0$. Thus, $deg\,(F, x) = 2$. Then, $F(x,y) = 0$ is not the equation of an genus 3 curve. Hence, $deg\,(\gamma_2) \leq 1$. Clearly, $deg\,(\gamma_3) \leq 2$. We denote:

$$\begin{aligned} \gamma_1(x) &:= ax + b, & \gamma_2(x) &:= cx + d \\ \gamma_3(x) &:= ex^2 + fx + g, & \gamma_0(x) &:= hx^3 + kx^2 + lx + m \end{aligned} \tag{2}$$

Then we have

$$F(x,y) = (ax+b)y^3 + (cx+d)y^2 + (ex^2+fx+g)y + (hx^3+kx^2+lx+m) = 0$$

We can make $g = 0$ and $m = 0$ by the transformation

$$x \to x + r, \quad y \to y + s$$

such that

$$\begin{aligned} 3bs^2 + 3as^2r + 2csr + er^2 + 2ds + fr + g &= 0 \\ as^3r + bs^3 + hr^3 + cs^2r + ds^2 + er^2s + kr^2fsr + lr + gs + m &= 0 \end{aligned} \tag{3}$$

Thus we have

$$F(x,y) = (ax+b)y^3 + (cx+d)y^2 + (ex^2+fx)y + x^3 + kx^2 + x = 0.$$

We can make $a = 1$ and $h = 1$ by the transformation

$$x \to r\,x, \quad y \to s\,y, \ \textit{such that} \quad r^3 = \frac{1}{h}, \quad s^3 = \frac{1}{ar}.$$

Then,

$$F(x,y) = (x+b)y^3 + (cx+d)y^2 + (ex^2+fx)y + x^3 + kx^2 + lx = 0.$$

This completes the proof of the lemma.

□

LEMMA 7. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-hyperelliptic genus 3 projective curves defined over $\mathbb{C}$. If $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic then exists $\lambda \in k^*$ such that*

$$I'_j = -\lambda^{\frac{j}{3}} \cdot I_j,$$

for $j = 3, 6, 9, , 12, 15, 18, 27$.

PROOF. Let $\mathcal{C}$ be a genus 3 curve given by $F(X,Y,Z) = 0$, where

$$F = Y^3(X+aZ) + Y^2(bXZ + cZ^2) + Y(dX^2Z + eXZ^2) + X^3Z + fX^2Z^2 + XZ^3$$

defined over a $\mathbb{C}$. Using some computer algebra system we can compute $I_3, \dots, I_{27}$. The expressions are very large so we display only I_3 and I_6.

$$
\begin{aligned}
I_3 &= \frac{1}{144} \cdot (2fb^2 + 2cd^2 - dbe - 6bc - 6ef - 6adf + 9d + 9ea)\\
I_6 &= \frac{1}{2^{12} \cdot 3^6} \cdot (24cfb^3 - 4c^2d^4 - 4b^4f^2 + 36cd^3 - 216c^2e - 36c^2b^2 - 36f^2e^2\\
&\quad + 144f^3c - 81a^2e^2 - 648cf + 108dfe - 81d^2 + 24af^2b^2d + 24cd^3fa - 8cd^2fb^2\\
&\quad + 4ced^3b + 4b^3dfe - 144af^3b + 24c^2d^2b + 24b^2f^2e - b^2e^2d^2 - 36a^2f^2d^2\\
&\quad + 144c^2fd - 108cbd - 162aed - 108ae^2f + 72ce^2d - 36b^2df + 18bed^2 + 648abf\\
&\quad - 48cd^2fe + 108ceab + 18ae^2bd + 144af^2ed - 12cedb^2 - 36cead^2 - 12be^2df\\
&\quad - 36afeb^2 + 108a^2fed - 72cfabd - 12aed^2fb - 108ad^2f)
\end{aligned}
$$

Let $\alpha \in GL_3(\mathbb{C})$, such that $D := det(\alpha) \neq 0$ and $\mathcal{C}' \cong \mathcal{C}^\alpha$. Then, $\mathcal{C}'$ has equation $F(X', Y', Z') = 0$ where

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \alpha \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

The covariants $I_3, \dots, I_{27}$ of $F(X', Y', Z')$ are exactly

$$I'_j = D^4 \cdot I_j,$$

for $j = 3, 6, 9, , 12, 15, 18, 27$.

□

COROLLARY 8. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-hyperelliptic genus 3 curves defined over $\mathbb{C}$. If $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic then*

$$i_j(\mathcal{C}) = i_j(\mathcal{C}'), \qquad \textit{for } j = 1, \dots, 6$$

Since $i_1, \dots, i_6$ are $GL_2(\mathbb{C})$ invariants then $\mathbb{C}(i_1, \dots, i_6) < \mathbb{C}(a,b,c,d,e,f)$. The degrees of $i_1, \dots, i_6$ are 6, 9, 12, 15, 18, and 27 respectively. Since the degree $[\mathbb{C}(i_1,i_2,i_3,i_4,i_5,i_6) : \mathbb{C}(a,b,c,d,e,f)]$ must be a common factor of the above degrees, we have:

Corollary 9. $[\mathbb{C}(i_1,i_2,i_3,i_4,i_5,i_6) : \mathbb{C}(a,b,c,d,e,f)] \leq 3$

One can attempt to find $[\mathbb{C}(a,b,c,d,e,f) : \mathbb{C}(i_1,i_2,i_3,i_4,i_5,i_6)]$ computationally. However, we could not accomplish this even using sophisticated computer algebra packages and a lot of computer power. The upshot would be to show that $\mathbb{C}(a,b,c,d,e,f) = \mathbb{C}(i_1,i_2,i_3,i_4,i_5,i_6)$, which we believe is true. This would have some very important consequences. First, it would show that i_1,i_2,i_3,i_4,i_5,i_6 generate the field of invariants and therefore describe a moduli point, which is still an open problem. Second, for every moduli point $\mathfrak{p} = (i_1,i_2,i_3,i_4,i_5,i_6)$ it would provide a rational model of the curve over its field of moduli.

Remark 10. *In this section we focused on curves defined over $\mathbb{C}$ instead of a general field k. The main reason was that the invariants of ternary quartics in* [**Di**] *were defined over $\mathbb{C}$. One could make the necessary adjustments and define Dixmier invariants over any field of characteristic $p \neq 2, 3$.*

4. Hurwitz space of W-covers and moduli space $\mathcal{M}_3$

In this section we discuss the corresponding Hurwitz space and degenerations of the covering $\phi : \mathcal{X}_g \to \mathbb{P}^1$.

Let $\mathcal{M}_g$ be the moduli space of curves of genus $g \geq 2$ and $\mathbb{P}^1 = \mathbb{P}^1(k)$ the Riemann sphere. Let $\phi : \mathcal{X}_g \to \mathbb{P}^1$ be a degree n covering with r branch points. By covering space theory, there is a tuple $(\sigma_1, \dots, \sigma_r)$ in S_n such that $\sigma_1 \cdots \sigma_r = 1$ and $G :=< \sigma_1, \dots, \sigma_r >$ is a transitive group in S_n. We call such a tuple the *signature* of ϕ. We say that a permutation is of type n^p if it is a product of p disjoint n-cycles.

Conversely, let $\sigma := (\sigma_1, \dots, \sigma_r)$ be a tuple in S_n such that $\sigma_1 \cdots \sigma_r = 1$ and $G :=< \sigma_1, \dots, \sigma_r >$ is a transitive group in S_n. We say that a cover $\phi : \mathcal{X} \to \mathbb{P}^1$ of degree n is of type σ if it has σ as signature. The genus g of $\mathcal{X}$ depends only on σ (Riemann-Hurwitz formula). Let $\mathcal{H}_\sigma$ be the set of pairs $([f], (p_1, \dots, p_r))$, where $[f]$ is an equivalence class of covers of type σ, and $p_1, \dots, p_r$ is an ordering of the branch points of ϕ. The **Hurwitz space** $\mathcal{H}_\sigma$ is a quasiprojective variety. We have a morphism

$$\Phi_\sigma : \mathcal{H}_\sigma \to \mathcal{M}_g$$

mapping $([f], (p_1, \dots, p_r))$ to the class $[\mathcal{X}]$ in the moduli space $\mathcal{M}_g$. Each component of $\mathcal{H}_\sigma$ has the same image in $\mathcal{M}_g$.

We denote by $C := (C_1, \dots, C_r)$, where C_i is the conjugacy class of σ_i in G. The set of Nielsen classes $\mathcal{N}(G, C)$ is

$$\mathcal{N}(G,C) := \{(\sigma_1, \dots, \sigma_r) \mid \sigma_i \in C_i,\, G =< \sigma_1, \dots, \sigma_r >,\ \sigma_1 \cdots \sigma_r = 1\}.$$

Fix a base point $\lambda_0 \in \mathbb{P}^1 \setminus S$ where S is the set of branch points. Then $\pi_1(\mathbb{P}^1 \setminus S)$ is generated by homotopy classes of loops $\gamma_1, \dots, \gamma_r$. The braid group acts on $\mathcal{N}(G, C)$ as

$$[\gamma_i] : \quad (\sigma_1, \dots, \sigma_r) \to (\sigma_1, \dots, \sigma_{i-1}, \sigma_i \sigma_{i+1} \sigma_i^{-1}, \sigma_i, \sigma_{i+2}, \dots, \sigma_r).$$

The orbits of this action are called the *braid orbits* and correspond to the irreducible components of $\mathcal{H}(G, C) := \mathcal{H}_\sigma$.

LEMMA 11. *Let $C = (2^8, 3)$ and $\mathcal{H}_\sigma := \mathcal{H}(S_3, C)$ be the corresponding Hurwitz space. Then, $\mathcal{H}_\sigma$ is an irreducible locus in $\mathcal{M}_3$*

PROOF. The proof is elementary and is based on the fact that there is one braid orbit of the braid action. The computationally minded reader can use the braid program to compute the braid orbits, see [**MSV**]. □

Degenerate cases: If some of the branch points of ψ coalesce we have the following signatures:

$$(3,3,2,2,2,2,2,2), \quad (3,3,3,2,2,2,2), \quad (3,3,3,3,2,2), \quad (3,3,3,3,3).$$

The information of each Hurwitz space is compiled in Table 4.

Signature	Mon. group	# Nielsen cl.	mod. dim.
$(3,\, 2^8)$	S_3	729	6
$(3^2, 2^5)$	S_3	36	5
$(3^3, 2^4)$	S_3	6	4
$(3^4, 2^2)$	S_3	6	3
(3^5)	C_3	5	2

TABLE 1. The number of Nielsen classes in each Hurwitz space

Proofs of the data in the Table 4 are simple exercises in combinatorial group theory. Moreover, one can use the braid package in GAP written by K. Magaard.

REMARK 12. *These cases are studied in* [**KK**]. *The main goal of that paper is to study the Weierstrass points in each case and provide equations for the curve. The equation of the curve in the last case is mistakenly identified.*

We find conditions on a, b, c, d, e, f for each case. The discriminant of curve given in Lemma 6 with respect to Y is a polynomial $\Delta(X)$ in X given by

$$\Delta(X) = -X(27X^7 + A_6X^6 + A_5X^5 + A_4X^4 + A_3X^3 + A_2X^2 + A_1X + 4c^3) \tag{4}$$

where $A_1, \dots A_6$ are as follows:

$$A_1 = 12bc^2 + 4c^3f - c^2e^2 - 18eca + 27a^2$$

$$A_2 = 12b^2c + 54a - 18dca - 2be^2c - 18ec + 54fa^2 - 18eba + 4c^3 - 18ecfa + 12bc^2f + 4e^3a - 2c^2de$$

$$A_3 = 27 - 18dcfa - 18ebfa - 18dc - 18eb + 108af + 27f^2a^2 - b^2e^2 + 12bc^2 - c^2d^2 - 18eca - 18dba - 18ecf + 12b^2fc + 12de^2a + 54a^2 + 4b^3 + 4e^3 - 4bdce$$

$$A_4 = -2bd^2c - 18dcf + 54f^2a + 54fa^2 - 18ebf - 18dbfa + 12de^2 + 12d^2ae - 18db - 2b^2de + 4b^3f + 54f - 18dca - 18ec - 18eba + 108a + 12b^2c$$

$$A_5 = 4d^3a + 27a^2 + 54 - 18eb + 108af - 18dc - 18dbf - b^2d^2 + 12d^2e - 18dba + 27f^2 + 4b^3$$

$$A_6 = 4d^3 + 54a - 18db + 54f$$

The branch points of the cover $\phi : \mathcal{C} \to \mathbb{P}^1$ coalesce when $\Delta(X)$ has multiple roots. Thus, its discriminant Δ in X is $\Delta = 0$. There are four factors of the discriminant

$$\Delta = \Delta_1 \cdot \Delta_2 \cdot \Delta_3 \cdot \Delta_4 = 0,$$

each corresponding to one of the degenerate cases in the previous table. We don't display them since they are easily computed using Maple or any other computer algebra packages.

4.1. Curves with automorphisms. The generic curve has no automorphisms. For sake of completeness we will briefly describe non-hyperelliptic genus 3 curves with automorphisms. Genus 3 hyperelliptic curves and their automorphisms are treated in [**GS**].

There are several papers written on automorphism groups of genus 3 curves. The following table is taken from [**MS**] and classifies all such families.

G	sig.	equation	Group ID
V_4	(2^6)	$x^4 + y^4 + ax^2y^2 + bx^2 + cy^2 + 1 = 0$	(4,2)
D_8	(2^5)	take $b = c$	(8,3)
S_4	$(2^3, 3)$	take $a = b = c$	(24,12)
$C_4^2 \rtimes S_3$	$(2, 3, 8)$	take $a = b = c = 0$ or $y^4 = x(x^2 - 1)$	(96,64)
16	$(2^3, 4)$	$y^4 = x(x - 1)(x - t)$	(16,13)
48	$(2, 3, 12)$	$y^4 = x^3 - 1$	(48,33)
C_3	(3^5)	$y^3 = x(x - 1)(x - s)(x - t)$	(3,1)
C_6	$(2, 3, 3, 6)$	take $s = 1 - t$	(6,2)
C_9	$(3, 9, 9)$	$y^3 = x(x^3 - 1)$	(9,1)
$L_3(2)$	$(2, 3, 7)$	$x^3y + y^3z + z^3x = 0$	(168,42)
S_3	$(2^4, 3)$	$a(x^4 + y^4 + z^4) + b(x^2y^2 + x^2z^2 + y^2z^2)+$ $c(x^2yz + y^2xz + z^2xy) = 0$	(6,1)
C_2	(2^4)	$x^4 + x^2(y^2 + az^2) + by^4 + cy^3z + dy^2z^2$ $+eyz^3 + gz^4 = 0$, either $e = 1$ or $g = 1$	(2,1)

TABLE 2. Automorphism groups of genus 3 non-hyperelliptic curves

In the table the cyclic group if order n is denoted by C_n and V_4 denotes the Klein 4-group. Each group is identified also with the Gap identity number, see [**MS**] for details. Each of the above cases is an irreducible locus in $\mathcal{M}_3$ whose equation in terms of $i_1, \dots, i_6$ can be determined. Such equations are for most cases large and we don't display them. The followings remarks are rather easy to check computationally.

Remark 13. *If the automorphism group of a non-hyperelliptic genus 3 curve is isomorphic to V_4 and equation as in the above table, then*

$$\begin{aligned} I_3 &= \frac{1}{36}\left(36 + 3c^2 + 3b^2 + 3a^2 + abc\right) \\ I_6 &= \frac{abc}{2^5\, 3^6}\left(108 - 3(a^2 + b^2 + c^2) + abc\right) \end{aligned} \tag{5}$$

The following three cases $D_8, S_4, C_4^2 \rtimes S_3$ are easily obtained.

Remark 14. *If the automorphism group of a non-hyperelliptic genus 3 curve is isomorphic to one of the following (16, 3), (48, 33), (3, 1), (6, 2), and (9, 1) then $I_3 = I_6 = 0$. If the automorphism group is isomorphic to the group with Gap identity (96, 64) then $I_3 = 1$ and $I_6 = 0$.*

Remark 15. *The case when the automorphism group is isomorphic to $L_3(2)$ corresponds to the Hurwitz curve (i.e., obtains the maximum number of automorphisms). In this case, $i_1 = -\frac{1}{16}$.*

Remark 16. *In the last degenerate case, the covering $\phi : \mathcal{X} \to \mathbb{P}^1$ with signature*

$$\sigma = (3, 3, 3, 3, 3)$$

is a Galois covering. The curve $\mathcal{X}$ has automorphism group isomorphic to C_3. The Hurwitz space corresponding to these coverings is the locus in $\mathcal{M}_3$ of genus 3 non-hyperelliptic curves with automorphism group C_3. These curves have equation

$$y^4 = x(x-1)(x-s)(x-t).$$

The equation of this family is misidentified in [**KK**]. *The covariants of such curves are*

$$I_3 = I_6 = 0.$$

It is an interesting question to investigate the field of moduli of these curves and see if that is a field of definition. This is an open problem with a long history. In general the generic curve has no automorphisms and by Weil's criterion its field of moduli is a field od definition. Hence the curves on the above table are of special interest. For the field of moduli of genus 3 hyperelliptic curves see [**GS**], [**Sh9**].

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