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# CODES OVER RINGS OF SIZE FOUR, HERMITIAN LATTICES, AND CORRESPONDING THETA FUNCTIONS

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ABSTRACT. Let $K = Q(\sqrt{-\ell})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, where $\ell$ is a square free integer such that $\ell \equiv 3 \mod 4$ and $C = [n, k]$ be a linear code defined over $\mathcal{O}_K/2\mathcal{O}_K$. The level $\ell$ theta function $\Theta_{\Lambda_\ell(C)}$ of $C$ is defined on the lattice $\Lambda_\ell(C) := \{x \in \mathcal{O}_K^n : \rho_\ell(x) \in C\}$, where $\rho_\ell : \mathcal{O}_K \to \mathcal{O}_K/2\mathcal{O}_K$ is the natural projection. In this paper, we prove that: i) for any $\ell, \ell'$ such that $\ell \leq \ell'$, $\Theta_{\Lambda_\ell}(q)$ and $\Theta_{\Lambda_{\ell'}}(q)$ have the same coefficients up to $q^{\frac{\ell+1}{4}}$, ii) for $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$, $\Theta_{\Lambda_\ell}(C)$ determines the code $C$ uniquely, iii) for $\ell < \frac{2(n+1)(n+2)}{n} - 1$ there is a positive dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$.

## 1. Introduction

Let $K = Q(\sqrt{-\ell})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, where $\ell$ is a square free integer such that $\ell \equiv 3 \mod 4$. Then the image $\mathcal{O}_K/2\mathcal{O}_K$ of the projection $\rho_\ell : \mathcal{O}_K \to \mathcal{O}_K/2\mathcal{O}_K$ is $\mathbb{F}_4$ (resp., $\mathbb{F}_2 \times \mathbb{F}_2$) if $\ell \equiv 3 \mod 8$ (resp., $\ell \equiv 7 \mod 8$).

Let $\mathcal{R}$ be a ring isomorphic to $\mathbb{F}_4$ or $\mathbb{F}_2 \times \mathbb{F}_2$ and $C = [n, k]$ be a linear code over $\mathcal{R}$ of length $n$ and dimension $k$. An admissible level $\ell$ is an $\ell$ such that $\ell \equiv 3 \mod 8$ if $\mathcal{R}$ is isomorphic to $\mathbb{F}_4$ or $\ell \equiv 7 \mod 8$ if $\mathcal{R}$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$. Fix an admissible $\ell$ and define $\Lambda_\ell(C) := \{x \in \mathcal{O}_K^n : \rho_\ell(x) \in C\}$. Then, the **level $\ell$ theta function** $\Theta_{\Lambda_\ell(C)}(\tau)$ of the lattice $\Lambda_\ell(C)$ is given as the symmetric weight enumerator $swe_C$ of $C$, evaluated on the theta functions defined on cosets of $\mathcal{O}_K/2\mathcal{O}_K$. In this paper we study the following two questions:

i) How do the theta functions $\Theta_{\Lambda_\ell(C)}(\tau)$ of the same code $C$ differ for different levels $\ell$?

ii) Can non-equivalent codes give the same theta functions for all levels $\ell$?

In an attempt to study the second question Chua in [1] gives an example of two non-equivalent codes that give the same theta function for level $\ell = 7$ but not for higher level thetas. We will show in this paper how such an example is not a coincidence. Our main results are as follows:

**Theorem 1:** *Let $C$ be a code defined over $R$. For all admissible $\ell, \ell'$ such that $\ell > \ell'$, the following holds*

$$\Theta_{\Lambda_\ell}(C) = \Theta_{\Lambda_{\ell'}}(C) + \mathcal{O}(q^{\frac{\ell'+1}{4}}).$$

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**Theorem 2:** *Let $C$ be a code of size $n$ defined over $\mathcal{R}$ and $\Theta_{\Lambda_\ell}(C)$ be its corresponding theta function for level $\ell$. Then the following hold:*

- **i):** *For $\ell < \frac{2(n+1)(n+2)}{n} - 1$ there is a $\delta$-dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$, where $\delta \geq \frac{(n+1)(n+2)}{2} - \frac{n(\ell+1)}{4} - 1$.*
- **ii):** *For $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$ and $n < \frac{\ell+1}{4}$ there is a unique symmetrized weight enumerator polynomial which corresponds to $\Theta_{\Lambda_\ell}(C)$.*

This paper is organized as follows. In the second section, we give a basic introduction of lattices and theta functions. We define a lattice $\Lambda$ over a number field $K$ in general, the theta series of a lattice, and one dimensional theta series and its shadow. Then we discuss the lattices over imaginary quadratic fields $K = Q(\sqrt{-\ell})$ with ring of integers $\mathcal{O}_K$, where $\ell$ is a square free integer such that $\ell \equiv 3 \mod 4$. The ring $\mathcal{O}_K/(2\mathcal{O}_K)$ is equivalent to either the field of order 4 or a ring of order 4 depending on whether $\ell \equiv 3 \mod 8$ or $\ell \equiv 7 \mod 8$. We define bi-dimensional theta functions for the four cosets of $\mathcal{O}_K/(2\mathcal{O}_K)$.

In the third section, we define codes over $\mathbb{F}_4$ and $\mathbb{F}_2 \times \mathbb{F}_2$, the weight enumerators of a code, and recall the main result of [1]. We simplify the expressions for bi-dimensional theta series and prove Theorem 1.

In the fourth section, we study families of codes corresponding to the same theta function. We call an **acceptable theta series** $\Theta(q)$ a series for which there exists a code $C$ such that $\Theta(q) = \Theta_{\Lambda_\ell}(C)(q)$. For any given $\ell$ and an acceptable theta series $\Theta(q)$ we can determine a family of symmetrized weight enumerators which correspond to $\Theta(q)$. For small $\ell$ this is a positive dimensional family, where the dimension is given by Theorem 2, i). Hence, the example given in [1] is no surprise. For large $\ell$ (see Theorem 2, ii)) this is a 0-dimensional family of symmetrized weight enumerators which correspond to $\Theta(q)$. Therefore, the example that Chua provides can not occur for larger $\ell$.

## 2. Introduction to latices and theta functions

Let $K$ be a number field and $\mathcal{O}_K$ be its ring of integers. A lattice $\Lambda$ over $K$ is an $\mathcal{O}_K$-submodule of $K^n$ of full rank. The Hermitian dual is defined by

$$\Lambda^* = \{x \in K^n \mid x \cdot \bar{y} \in \mathcal{O}_K, \textit{for all } y \in \Lambda\}, \tag{2.1}$$

where $x \cdot y := \sum_{i=1}^n x_i y_i$. In the case that $\Lambda$ is a free $\mathcal{O}_K$ - module, for every $\mathcal{O}_K$ basis $\{v_1, v_2, ...., v_n\}$ we can associate a Gram matrix G($\Lambda$) given by $G(\Lambda) = (v_i . v_j)_{i,j=1}^n$ and the determinant $\det \Lambda := \det(G)$ defined up to squares of units in $\mathcal{O}_K$. If $\Lambda = \Lambda^*$ then $\Lambda$ is Hermitian self-dual (or unimodular) and integral if and only if $\Lambda \subset \Lambda^*$. An integral lattice has the property $\Lambda \subset \Lambda^* \subset \frac{1}{det\Lambda}\Lambda$. An integral lattice is called even if $x \cdot x \equiv 0 \mod 2$ for all $x \in \Lambda$, and otherwise it is odd. An odd unimodular lattice is called a Type 1 lattice and even unimodular lattice is called a Type 2 lattice.

The theta series of a lattice $\Lambda$ in $K^n$ is given by $\Theta_\Lambda(\tau) = \sum_{z \in \Lambda} e^{\pi i \tau z \bar{z}}$, where $\tau \in H = \{z \in \mathbb{C} : Im(z) > 0\}$. Usually we let $q = e^{\pi i \tau}$. Then, $\Theta_\Lambda(q) = \sum_{z \in \Lambda} q^{z\bar{z}}$. The 1-dimensional theta series and its **shadow** are given by

$$\theta_3(q) := \sum_{m \in \mathbb{Z}} q^{m^2}, \quad \theta_2(q) := \sum_{m \in \mathbb{Z}+1/2} q^{m^2}. \tag{2.2}$$

Let $\ell > 0$ be a square free integer and $K = Q(\sqrt{-\ell})$ be the imaginary quadratic field with discriminant $d_K$. Recall that $d_K = -\ell$ if $\ell \equiv 3 \mod 4$ and $d_K = -4\ell$ otherwise.

Let $\mathcal{O}_K$ be ring of integers of $K$. The Hermitian lattice $\Lambda$ over $K$ is an $\mathcal{O}_K$ -submodule of $K^n$ of full rank. Let $\ell \equiv 3 \mod 4$ and $d$ be a positive number such that $\ell = 4d - 1$. Then, $-\ell \equiv 1 \mod 4$. This implies that the ring of integers is $\mathcal{O}_K = \mathbb{Z}[\omega_\ell]$, where $\omega_\ell = \frac{-1+\sqrt{-\ell}}{2}$ and $\omega_\ell^2 + \omega_\ell + d = 0$. The principal norm form of $K$ is given by $Q_d(x, y) = |x - y\omega_\ell|^2 = x^2 + xy + dy^2$. Since $\ell \equiv 3 \mod 4$, we can consider two cases:

(1) If $\ell \equiv 3 \mod 8$ then $-\ell \equiv 5 \mod 8$. Thus, the prime ideal $\langle 2 \rangle \subset \mathbb{Z}$ lifts to a prime $2\mathcal{O}_K \subset \mathcal{O}_K$. Since the ring of integers $\mathcal{O}_K$ is a Dedekind domain, $2\mathcal{O}_K$ is a maximal ideal. Therefore $\mathcal{O}_K/(2\mathcal{O}_K) \simeq \mathbb{F}_4$.

(2) If $\ell \equiv 7 \mod 8$ then $-\ell \equiv 1 \mod 8$. Then the prime ideal $\langle 2 \rangle \in \mathbb{Z}$ splits in $K$. Therefore $2\mathcal{O}_K$ splits in $\mathcal{O}_K$. Hence, $\mathcal{O}_K/(2\mathcal{O}_K) \simeq \mathbb{F}_2 \times \mathbb{F}_2$. In either case, a complete set of coset representatives is $\{0, 1, \omega_\ell, 1 + \omega_\ell\}$.

Let the following be the bi-dimensional theta series for the four cosets:

$$\begin{aligned} A_d(q) &:= \Theta_{2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m,n)} \\ C_d(q) &:= \Theta_{1+2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m+\frac{1}{2},n)} \\ G_d(q) &:= \Theta_{\omega_\ell+2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m,n+\frac{1}{2})} \\ H_d(q) &:= \Theta_{1+\omega_\ell+2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m+\frac{1}{2},n+\frac{1}{2})} \end{aligned} \tag{2.3}$$

Then we have the following lemma.

**Lemma 1.** *Bi-dimensional theta series can be further expressed in terms of the standard one dimensional theta series and its shadow.*

$$\begin{aligned} A_d(q) &= \theta_3(q^4)\theta_3(q^{4\ell}) + \theta_2(q^4)\theta_2(q^{4\ell}) \\ C_d(q) &= \theta_2(q^4)\theta_3(q^{4\ell}) + \theta_3(q^4)\theta_2(q^{4\ell}) \\ G_d(q) &= H_d(q) = \frac{\theta_2(q)\theta_2(q^\ell)}{2}. \end{aligned} \tag{2.4}$$

*Moreover,*

$$2G_d(q) = A_d(q^{1/4}) - A_d(q) - C_d(q). \tag{2.5}$$

*Proof.* See [3] for details. □

## 3. Codes over $\mathbb{F}_4$ and $\mathbb{F}_2 \times \mathbb{F}_2$

Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, where $\omega^2 + \omega + 1 = 0$, be the finite field of four elements. The conjugation is given by $\bar{x} = x^2$, $x \in \mathbb{F}_4$. In particular $\bar{\omega} = \omega^2 = \omega + 1$. Let $R_4 = \mathbb{F}_2 + \omega\mathbb{F}_2$ with the new equation for $\omega$ is $\omega^2 + \omega = 0$. Notice that $R_4$ has two maximal ideals namely $\langle \omega \rangle$ and $\langle \omega + 1 \rangle$. Furthermore, one can show that $R_4/\langle \omega \rangle$ and $R_4/\langle \omega + 1 \rangle$ are both isomorphic to $\mathbb{F}_2$. The Chinese remainder theorem tells us that $R_4 = \langle \omega \rangle \oplus \langle \omega + 1 \rangle$. Therefore, $R_4 \simeq \mathbb{F}_2 \times \mathbb{F}_2$. The conjugate of $\omega$ is $\omega + 1$. Let $\mathcal{R}$ be the field $\mathbb{F}_4$ if $\ell \equiv 3 \mod 8$ or the ring $R_4 \simeq \mathbb{F}_2 \times \mathbb{F}_2$ when $\ell \equiv 7 \mod 8$.

A linear code $C$ of length $n$ over $\mathcal{R}$ is an $\mathcal{R}$-submodule of $\mathcal{R}^n$. The dual is defined as $C^\perp = \{u \in \mathcal{R} : u \cdot \bar{v} = 0 \textit{ for all } v \in C\}$. If $C = C^\perp$ then $C$ is self-dual.

We define $\Lambda_\ell(C) := \{x \in \mathcal{O}_K^n : \rho_\ell(x) \in C\}$ where $\rho_\ell : \mathcal{O}_K \to \mathcal{O}_K/2\mathcal{O}_k \to \mathcal{R}$ . In other words, $\Lambda_\ell(C)$ consists of all vectors in $\mathcal{O}_K^n$ which when taken mod $2\mathcal{O}_K$ componentwise are in $\rho_\ell^{-1}(C)$. The following is immediate.

**Lemma 2.** *(1) $\Lambda_\ell(C)$ is an $\mathcal{O}_K$-lattice.*
*(2) $\Lambda_\ell(C^\perp) = 2\Lambda_\ell(C)^*$.*
*(3) $C$ is self dual if and only if $\frac{\Lambda_\ell(C)}{\sqrt{2}}$ is self dual.*

Let $u = (u_1, u_2, \cdots, u_n) \in \mathcal{R}^n$ be a codeword and $\alpha \in \mathcal{R}$. Then the counting function $n_\alpha(u)$ is defined as the number of elements in the set $\{j : u_j = \alpha\}$. For a code $C$ we define the complete weight enumerator (*cwe*), symmetrized weight enumerator (*swe*) and Hamming weight enumerator ($W$) to be

$$
\begin{aligned}
cwe_C(X,Y,Z,W) &:= \sum_{u\in C} X^{n_0(u)} Y^{n_1(u)} Z^{n_\omega(u)} W^{n_{1+\omega}(u)} \\
swe_C(X,Y,Z) &:= cwe_C(X,Y,Z,Z) \\
W_C(X,Y) &:= swe_C(X,Y,Y).
\end{aligned}
\tag{3.1}
$$

Then we have the following.

**Proposition 1.** *Let $\ell \equiv 3 \mod 4$, $C$ be a linear code over $\mathcal{R}$, and $\frac{\Lambda_\ell(C)}{\sqrt{2}}$ be a Hermitian lattice constructed via the construction A. Then*

$$\theta_{\Lambda_\ell(C)}(\tau) = swe_C(A_d(q), C_d(q), G_d(q)) \tag{3.2}$$

*where $A_d(q)$, $C_d(q)$, and $G_d(q)$ are given as in Eq. (2.4).*

For a proof of the above statement the reader can see [1]. From the definition of one dimensional theta series we have

$$\theta_2(q) = 2q^{1/4}\sum_{i\in S} q^i, \quad \theta_2(q^4) = 2q\sum_{i:odd} q^{i^2-1}, \quad \theta_3(q^4) = 1 + 2q^4\sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)},$$

where $S = \left\{\frac{j^2-1}{4} : j \equiv 1 \mod 2\right\}$. From Eq. (2.4) we can write

$$G_d(q) = \frac{\theta_2(q)\theta_2(q^\ell)}{2} = q^{\frac{(\ell+1)}{4}}\alpha_1,$$

where $\alpha_1 = \sum_{i\in S} q^i \sum_{j\in S} q^{\ell j}$. Then,

$$
\begin{aligned}
A_d(q) &= \theta_3(q^4)\theta_3(q^{4\ell}) + \theta_2(q^4)\theta_2(q^{4\ell}) \\
&= (1 + 2q^4\sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)})(1 + 2q^{4\ell}\sum_{j\in\mathbb{Z}^+} q^{4\ell(j^2-1)}) \\
&+ 4q^{\ell+1}\sum_{i:odd} q^{i^2-1}\sum_{j:odd} q^{(j^2-1)\ell} \\
&= \alpha_2 + q^{\ell+1}\alpha_3 + q^{4\ell}\alpha_4,
\end{aligned}
$$

where $\alpha_2, \alpha_3$ and $\alpha_4$ have the following forms

$$\begin{aligned}
\alpha_2 &= 1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)} \\
\alpha_3 &= 4 \sum_{i:odd} q^{i^2-1} \sum_{j:odd} q^{(j^2-1)\ell} \\
\alpha_4 &= 2 \sum_{j \in \mathbb{Z}^+} q^{4\ell(i^2-1)} (1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)}).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
C_d(q) &= \theta_2(q^4)\theta_3(q^{4\ell}) + \theta_3(q^4)\theta_2(q^{4\ell}) \\
&= 2q \sum_{i:odd} q^{i^2-1}(1 + 2q^{4\ell} \sum_{i \in \mathbb{Z}^+} q^{4\ell(i^2-1)}) \\
&+ (1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)})(2q^\ell \sum_{i:odd} q^{(i^2-1)\ell} \\
&= \alpha_5 + q^\ell \alpha_6 + q^{4\ell+1}\alpha_7,
\end{aligned}$$

where $\alpha_5, \alpha_6$ and $\alpha_7$ have the following forms

$$\begin{aligned}
\alpha_5 &= 2 \sum_{i:odd} q^{i^2-1} \\
\alpha_6 &= 2 \sum_{j:odd} q^{(j^2-1)\ell}(1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)}) \\
\alpha_7 &= 4 \sum_{i:odd} q^{i^2-1} \sum_{j \in \mathbb{Z}^+} q^{4\ell(j^2-1)}.
\end{aligned}$$

The next result shows that for large enough admissible $\ell$ and $\ell'$ the theta functions $\Theta_{\Lambda_\ell}(C)$ and $\Theta_{\Lambda_{\ell'}}(C)$ are virtually the same.

**Theorem 1.** *Let $C$ be a code defined over $R$. For all admissible $\ell, \ell'$ such that $\ell > \ell'$, the following holds*

$$\Theta_{\Lambda_\ell}(C) = \Theta_{\Lambda_{\ell'}}(C) + \mathcal{O}(q^{\frac{\ell'+1}{4}}). \tag{3.3}$$

*Proof.* Let

$$swe_C(X, Y, Z) = \sum_{i+j+k=n} a_{i,j,k} \cdot X^i Y^j Z^k$$

be a degree $n$ polynomial. Write this as a polynomial in $Z$. Then

$$swe_C(Z) = \sum_{k=0}^{n} L_k Z^k = L_0 + Z(\sum_{k=1}^{n} L_k Z^{k-1}).$$

Terms in $L_0$ are of the form of $a_{i,j}X^iY^j$, where $i + j = n$. From the above we have

$$\begin{aligned}
A_d(q)^i \cdot C_d(q)^j &= (\alpha_2 + q^{\ell+1}\alpha_3 + q^{4\ell}\alpha_4)^i \cdot (\alpha_5 + q^\ell\alpha_6 + q^{4\ell+1}\alpha_7)^j \\
&= (\textit{terms independent from } \ell) + q^\ell(\cdots)
\end{aligned}$$

Also we have seen that $G_d(q) = q^{(\ell+1)/4}\alpha_1$. This gives

$$\begin{aligned}
\Theta_{\Lambda_\ell}(C) &= swe_C(A_d(q), C_d(q), G_d(q)) \\
&= (\textit{terms independent from } \ell) + \mathcal{O}(q^{\frac{\ell+1}{4}}).
\end{aligned}$$

Then the result follows. □

**Example 1.** *Let $C$ be a code defined over $R_4$ which has symmetrized weight enumerator*

$$swe_C(X,Y,Z) = X^3 + X^2Z + XY^2 + 2XZ^2 + Y^2Z + 2Z^3.$$

*Then we have the following:*

$$\begin{aligned} &\Theta_{\Lambda_{63}}(C) = 1 + 6q^4 + 12q^8 + 8q^{12} + 12q^{16} + 6q^{18} + 48q^{20} + 30q^{22} + \cdots \\ (3.4) \qquad &\Theta_{\Lambda_{79}}(C) = 1 + 6q^4 + 12q^8 + 8q^{12} + 6q^{16} + 30q^{20} + 6q^{22} + 48q^{24} + \cdots \\ &\Theta_{\Lambda_{79}}(C) = \Theta_{\Lambda_{63}}(C) + \mathcal{O}(q^{16}). \end{aligned}$$

## 4. A family of codes corresponding to the same theta function

If we are given the code over $\mathcal{R}$ and its symmetrized weight enumerator polynomial, then by Eq. (3.2) we can find the theta function of the lattice constructed from the code by using the construction $A$. Now, we would like to give a way to construct families of codes corresponding to the same theta function.

Let $\Theta(q) = \sum_{i=0}^{\infty} \lambda_i q^i$ be an acceptable theta series for level $\ell$ and

$$f(x,y,z) = \sum_{i+j+k=n} c_{i,j,k} x^i y^j z^k$$

be a degree $n$ generic ternary homogeneous polynomial. We want to find out how many polynomials $f(x,y,z)$ correspond to $\Theta(q)$ for a fixed $\ell$.

We have the following lemma.

**Lemma 3.** *Let $C$ be a code of size n defined over $\mathcal{R}$ and $\Theta(q)$ be its theta function for level $\ell$. Then, $\Theta(q)$ is uniquely determined by its first $\frac{n(\ell+1)}{4}$ coefficients.*

*Proof.* Let $C$ be a code over $\mathcal{R}$, $\Theta(q) = \sum_{i=0}^{\infty} \lambda_i q^i$ be its theta series, $s = \frac{n(\ell+1)}{4}$ and

$$f(x,y,z) = \sum_{i+j+k=n} c_{i,j,k} x^i y^j z^k$$

be a degree $n$ generic ternary homogeneous polynomial. Find $A_d(q), C_d(q)$,$G_d(q)$ for the given $\ell$ and substitute in $f(x,y,z)$. Hence $f(x,y,z)$ is now written as a series in $q$. Recall that a generic degree $n$ ternary polynomial has $r = \frac{(n+1)(n+2)}{2}$ coefficients. So, the corresponding coefficients of the two sides of the equation are equal:

$$f(A_d(q), C_d(q), G_d(q)) = \sum_{i=0}^{\infty} \lambda_i q^i.$$

Consider the term

$$c_{i,j,k}(\alpha_2 + q^{\ell+1}\alpha_3 + q^{4\ell}\alpha_4)^i (\alpha_5 + q^{\ell}\alpha_6 + q^{4\ell+1}\alpha_7)^j (q^{\frac{(\ell+1)}{4}}\alpha_1)^k.$$

Then $c_{i,j,k}$ appears first as a coefficient of $q^{j+\frac{k(\ell+1)}{4}}$. For all such $j,k$ we have $j + \frac{k(\ell+1)}{4} \leq \frac{n(\ell+1)}{4}$. Consider the equations where $c_{i,j,k}$ appears first. This is a system of equations with $\leq \frac{(n+1)(n+2)}{2}$ equations. Let us denote this system of equations as $\Xi$. Solve this system for $c_{i,j,k}$. Hence, $c_{i,j,k}$ is a function of $l_0, \cdots, l_s$. For each $\mu > s, l_\mu$ is a function of $c_{i,j,k}$ for $i,j,k = 0, \cdots, n$, and therefore a rational function on $l_0, \cdots, l_s$. This completes the proof. □

Next we have the following theorem:

**Theorem 2.** *Let $C$ be a code of size $n$ defined over $\mathcal{R}$ and $\Theta_{\Lambda_\ell}(C)$ be its corresponding theta function for level $\ell$. Then the following hold:*

- **i):** *For $\ell < \frac{2(n+1)(n+2)}{n} - 1$ there is a $\delta$-dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$, where $\delta \geq \frac{(n+1)(n+2)}{2} - \frac{n(\ell+1)}{4} - 1$*
- **ii):** *For $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$ and $n < \frac{\ell+1}{4}$ there is a unique symmetrized weight enumerator polynomial which corresponds to $\Theta_{\Lambda_\ell}(C)$.*

*Proof.* We want to find out how many polynomials $f(x, y, z)$ correspond to $\Theta_{\Lambda_\ell}(C)$ for a fixed $\ell$. $\Theta_{\Lambda_\ell}(C)$ and $f(x, y, z)$ are defined as above. Consider the system of equations $\Xi$.

If $\frac{n(\ell+1)}{4} < r$ then our system has more variables than equations. Since the system is linear, the solution space is a family of positive dimension.

If $\frac{n(\ell+1)}{4} \geq r$ then for each equation in $\Xi$ (see the proof of the previous Lemma) we have only one $c_{i,j,k}$ appearing for the first time. Otherwise suppose $c_{i,j,k}$ and $c_{i',j',k'}$ appear for the first time in an equation of $\Xi$. Then $j + \frac{k(\ell+1)}{4} = j' + \frac{k'(\ell+1)}{4}$. This implies

$$4(j - j') = (k' - k)(\ell + 1). \tag{4.1}$$

Without loss of generality, assume $k' \geq k$. We can consider three cases.

case 1: If $k' - k \geq 2$, then from Eq. (4.1) we have $4n(j - j') = n(k' - k)(\ell + 1) \geq 4r(k' - k)$. Then we have $n(j - j') \geq (n + 1)(n + 2)$. Since $n \geq (j - j')$, we have a contradiction.

case 2: If $k' - k = 1$, then by Eq. (4.1) $j - j' = \frac{\ell+1}{4}$. Since $j - j' \leq n$ and $\frac{\ell+1}{4} > n$, we get a contradiction.

case 3: $k' - k = 0$, then by Eq. (4.1) we have $j = j'$. Hence $i = i'$.

Notice that $c_{n,0,0}$ is uniquely determined by the equation corresponding to the equation of coefficient of $q^0$. Solve the system $\Xi$ in the order of the equation that corresponds to the power of $q$. We have a unique solution for $c_{i,j,k}$. □

4.1. **Families of codes of length 3.** In this section we discuss the codes of length 3 for different levels $\ell$. Our main goal is to investigate the example provided in [1] and provide some computational evidence for the above two cases. We assume that the symmetrized weight enumerator polynomial is a generic homogenous polynomial of degree three.

Let $P(x, y, z)$ be a generic ternary cubic homogeneous polynomial given as below

$$\begin{aligned} P(x, y, z) = c_1x^3 + c_2y^3 + c_3z^3 + c_4x^2y + c_5x^2z + c_6y^2x + c_7y^2z \\ + c_8z^2x + c_9z^2y + c_{10}xyz. \end{aligned} \tag{4.2}$$

Assume that there is a code $C$, of length 3, defined over $\mathcal{R}$ such that $swe_C(x, y, z) = P(x, y, z)$. First we have to fix the level $\ell$. When we fix the level, we can find $A_d(q), C_d(q), G_d(q)$. By equating both sides of

$$p(A_d(q), C_d(q), G_d(q)) = \sum_{i=0}^{\infty} \lambda_i q^i,$$

we can get a system of equations. When $\ell = 7$, we are in the first case of the previous theorem. The system of equations is given by the following.

$$\begin{cases} c_1 - \lambda_0 = 0 \\ 2c_4 - \lambda_1 = 0 \\ 4c_6 + 2c_5 - \lambda_2 = 0 \\ 8c_2 + 4c_{10} - \lambda_3 = 0 \end{cases} \qquad \begin{cases} 6c_1 + 4c_8 + 2c_5 + 8c_7 - \lambda_4 = 0 \\ 8c_4 + 8c_9 + 4c_{10} - \lambda_5 = 0 \\ 8c_5 + 8c_3 + 8c_7 + 8c_8 + 8c_6 - \lambda_6 = 0. \end{cases} \tag{4.3}$$

The solution for the above system is given by $c_1 = \lambda_0, c_4 = \frac{1}{2}\lambda_1$, and

$$\begin{aligned} & c_2 = \frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_3 - \frac{1}{8}\lambda_5 + c_9, \quad c_3 = \frac{3}{2}\lambda_0 - \frac{1}{4}\lambda_2 - \frac{1}{4}\lambda_4 + \frac{1}{8}\lambda_6 + c_7, \\ & c_5 = -3\lambda_0 + \frac{1}{2}\lambda_4 - 4c_7 - 2c_8, \quad c_6 = \frac{3}{2}\lambda_0 + \frac{1}{4}\lambda_2 - \frac{1}{4}\lambda_4 + 2c_7 + c_8, \\ & c_{10} = -\lambda_1 + \frac{1}{4}\lambda_5 - 2c_9 \end{aligned} \tag{4.4}$$

where $c_7, c_8, c_9$ are free variables. By giving different triples $(c_7, c_8, c_9)$, we can construct different polynomials $P(x, y, z)$ for the same $\sum_{i=0}^{\infty} \lambda_i q^i$.

Consider the following theta function. From [1] there are two non isomorphic codes that give this theta function for level $\ell = 7$:

$$\Theta_{\sqrt{2}\mathcal{O}_{K_7^3}} = 1 + 6q^2 + 24q^4 + 56q^6 + 114q^8 + 168q^{10} + 280q^{12} + 294q^{14} + \cdots \tag{4.5}$$

For this particular theta function, We can rewrite the solution (Eq. (4.4)) as follows: $c_1 = 1, c_2 = c_9, c_3 = 1 + c_7, c_4 = 0, c_5 = 9 - 4c_7 - 2c_8, c_6 = -3 - 2c_7 + c_8, c_{10} = -2c_9$.

For the triple $(1, 2, 0)$ (resp., $(0, 3, 0)$) we get the symmetrized weight enumerator polynomial for the code $C_{3,2}$ (resp. $C_{3,3}$). That is $swe_{C_{3,2}}(X, Y, Z) = X^3 + X^2Z + XY^2 + 2XZ^2 + Y^2Z + 2Z^3$ (resp., $swe_{C_{3,3}}(X, Y, Z) = X^3 + 3X^2Z + 3XZ^2 + Z^3$), where $C_{3,2}$ and $C_{3,3}$ are given by:

$$\begin{aligned} C_{3,2} &= \omega < [0,1,1] > + (\omega + 1) < [0,1,1] >^{\perp} \\ C_{3,3} &= \omega < [0,0,1] > + (\omega + 1) < [0,0,1] >^{\perp} . \end{aligned} \tag{4.6}$$

When $\ell = 15$, we are in the second case of the above theorem. The system of equations is as follows:

$$\begin{cases} c_1 - \lambda_0 = 0 \\ 2c_4 - \lambda_1 = 0 \\ 4c_6 - \lambda_2 = 0 \\ 8c_2 - \lambda_3 = 0 \\ 6c_1 + 2c_5 - \lambda_4 = 0 \end{cases} \qquad \begin{cases} 8c_4 + 4c_{10} - \lambda_5 = 0 \\ 2c_5 + 8c_7 + 8c_6 - \lambda_6 = 0 \\ 4c_8 + 8c_7 + 12c_1 + 8c_5 - \lambda_8 = 0 \\ 10c_4 + 8c_9 + 8c_{10} - \lambda_9 = 0 \\ 8c_7 + 8c_5 + 12c_8 + 8c_3 + 8c_1 - \lambda_{12} = 0. \end{cases} \tag{4.7}$$

Each $c_i$ appears first in exactly one equation. For example consider the seventh equation. $c_7$ is the only variable that appears first in the seventh equation. Solve the system in given order. The solution for the above system is given by; $c_1 = \lambda_0$, $c_2 = \frac{1}{8}\lambda_3$, $c_4 = \frac{1}{2}\lambda_1$, $c_6 = \frac{1}{4}\lambda_2$, and

$$\begin{aligned} & c_3 = -\lambda_0 - \frac{1}{2}\lambda_2 + \frac{3}{4}\lambda_4 + \frac{1}{4}\lambda_6 - \frac{3}{8}\lambda_8 + \frac{1}{8}\lambda_{12}, \quad && c_5 = -3\lambda_0 + \frac{1}{2}\lambda_4 \\ & c_7 = \frac{3}{4}\lambda_0 - \frac{1}{4}\lambda_2 - \frac{1}{8}\lambda_4 + \frac{1}{8}\lambda_6, && c_9 = \frac{3}{8}\lambda_1 - \frac{1}{4}\lambda_5 + \frac{1}{8}\lambda_9 \\ & c_8 = \frac{3}{2}\lambda_0 + \frac{1}{2}\lambda_2 - \frac{3}{4}\lambda_4 - \frac{1}{4}\lambda_6 + \frac{1}{4}\lambda_8, && c_{10} = -\lambda_1 + \frac{1}{4}\lambda_5 \end{aligned} \tag{4.8}$$

We have a unique solution. This implies that two non equivalent codes cannot give the same theta function for $\ell = 15$ and $n = 3$.

## 5. Concluding remarks

The main goal of this paper was to find out how theta functions determine the codes over a ring of size 4. First we have shown how the theta functions of the same code $C$ differ for different levels $\ell$. The first $\frac{\ell+1}{4}$ terms of the theta functions for levels $\ell$ and $\ell'$ are the same, where $\ell' \geq \ell$.

In [1], two non-isomorphic codes that give the same theta function for level $\ell = 7$ but not under higher level constructions are given. We justified the reason why we don't have a similar situation for higher level constructions. In this note we have addressed a method that we can use for finding a family of polynomials that correspond to a given acceptable theta series for some fixed level $\ell$. We have studied two cases depending upon $\ell$ that give either a positive dimensional family of polynomials or a unique polynomial.

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