SUBVARIETIES OF THE HYPERELLIPTIC MODULI DETERMINED BY GROUP ACTIONS

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Abstract. Let $\mathcal{H}_g$ be the moduli space of genus g hyperelliptic curves. In this note, we study the locus $\mathcal{H}_g(G, \sigma)$ in $\mathcal{H}_g$ of curves admitting a G-action of given ramification type σ and inclusions between such loci. For each genus we determine the list of all possible groups, the inclusions among the loci, and the corresponding equations of the generic curve in $\mathcal{H}_g(G, \sigma)$. The proof of the results is based solely on representations of finite subgroups of $PGL_2(\mathbb{C})$ and the Riemann-Hurwitz formula.

1. Introduction

Let $\mathcal{H}_g$ be the moduli space of genus g hyperelliptic curves. We study the locus $\mathcal{H}_g(G, \sigma)$ in $\mathcal{H}_g$ of curves admitting a G-action of given ramification type σ. All components of $\mathcal{H}_g(G, \sigma)$ have the same dimension which depends only on the signature of the G-action. Restricting the action to a subgroup H of G yields an inclusion of $\mathcal{H}_g(G, \sigma)$ into the corresponding locus $\mathcal{H}_g(H, \sigma)$ for the action of H. In this paper we study all possible loci $\mathcal{H}_g(G, \sigma)$, the inclusions between such loci, and determine an equation for a generic curve C in $\mathcal{H}_g(G, \sigma)$. This is the first part of the two paper sequence, the second of which will consider such problem for hyperelliptic curves defined over a field of characteristic $p > 0$. The main goal of this paper is twofold: first to give a unified treatment of automorphisms groups of hyperelliptic curves and the loci they determine in characteristic zero, and second to provide the motivation needed for studying such problem in characteristic $p > 0$. While some of the results of this paper are scattered in the literature we provide a unified approach which is algebraic and is based solely on representations of finite subgroups of $PGL_2(\mathbb{C})$ and the Riemann-Hurwitz formula.

In section 2, we give a brief introduction on hyperelliptic curves and their automorphism groups. This material can be found in [2, 3, 15, 14] among many other places in the literature. For a given hyperelliptic curve $\mathcal{X}$, defined over k, with automorphism group G, the reduced automorphism group is $\overline{G} := G/\langle w \rangle$, where w is the hyperelliptic involution. This group $\overline{G}$ is embedded in $PGL_2(k)$ and therefore is one of $\mathbb{Z}_n D_n, A_4, S_4, A_5$. $\overline{G}$ acts on e genus 0

1991 *Mathematics Subject Classification.* 14Q05, 14Q15, 14R20, 14D22.

Key words and phrases. Hyperelliptic curves, automorphism groups.

field $k(x)$. We determine a rational function $\phi(x)$ that generates the fixed field $k(x)^{\overline{G}}$ in all cases. Using only this rational function we are able to determine the parametric equation of each family $\mathcal{H}_g(G, \sigma)$ (cf. Section 4). Different decompositions of $\phi(x)$ give different decompositions of $f(x)$ in the equations of the hyperelliptic curve $y^2 = f(x)$. An equation of an hyperelliptic curve with an extra automorphism of order n can be written as $y^2 = f(x^n)$ or $y^2 = xf(x^n)$. This corresponds to a decomposition of the rational function $\phi(x)$ in x^n. For algorithms on decomposing rational functions one can check [6] and the references from there. Furthermore, for each fixed g we give a formula for the number of automorphism groups that occur.

In section 3, we discuss the locus $\mathcal{H}_g(G, \sigma)$. First, we determine the signatures of the covers $\psi : \mathcal{X}_g \to \mathbb{P}_1$. The moduli space of such covers with fixed group G and ramification σ is a Hurwitz space $\mathcal{H}$. There is an obvious map from $\mathcal{H}$ to the hyperelliptic moduli of curves $\mathcal{H}_g$. We denote the image of this map by $\mathcal{H}_g(G, \sigma)$ which is a subvariety of $\mathcal{H}_g$. The dimension of $\mathcal{H}_g(G, \sigma)$ is determined solely by the signature σ. For each $g \geq 2$ we list all possible groups, their signatures, and the dimension of the locus $\mathcal{H}_g(G, \sigma)$. In section 4, we determine the equations of the families of curves for a given group. This is determined using the rational function $\phi(x)$ from section 2. Weierstrass points of the curve are points in the fibers $\phi^{-1}(\lambda)$, where λ is a branch point of $\phi(x)$. Such branch points can be determined easily when $\phi(x)$ is known. For each group an equation for the family of curves is determined.

In section 5, we discuss the inclusions among the loci $\mathcal{H}_g(G, \sigma)$. We implement a program that for each genus $g \geq 2$ determines the list of groups which occur as full automorphism group of hyperelliptic curves of genus g and draws a the lattice of these groups.

There is plenty of literature on the automorphism groups of hyperelliptic curves. Among many papers we mention [2], [7]. Most of these papers have studied the automorphism groups of the hyperelliptic curve using the Fuchsian groups. Our goal is to provide a unified simple algebraic approach of the results with the list of all the groups G which occur as full automorphism groups of hyperelliptic curves, all possible signatures σ for each given group G, the dimension of each locus $\mathcal{H}_g(G, \sigma)$, and the lattice of the loci $\mathcal{H}_g(G, \sigma)$.

Notation: Throughout this paper k denotes an algebraically closed field of characteristic zero, g an integer ≥ 2, and $\mathcal{X}_g$ a hyperelliptic curve of genus g defined over k.

2. Hyperelliptic curves and their automorphisms

Let k be an algebraically closed field of characteristic zero and $\mathcal{X}_g$ be a genus g hyperelliptic curve given by the equation $y^2 = F(x)$, where $\deg(F) = 2g + 2$. Denote the function field of $\mathcal{X}_g$ by $K := k(x, y)$. Then, $k(x)$ is the unique degree 2 genus zero subfield of K. K is a quadratic extension field of $k(x)$ ramified exactly at $d = 2g + 2$ places $\alpha_1, \dots, \alpha_d$ of $k(x)$. The corresponding places of K are called the *Weierstrass points* of K. Let $\mathcal{W} := \{\alpha_1, \dots, \alpha_d\}$ and $G = Aut(K/k)$. Since $k(x)$ is the only genus 0 subfield of degree 2 of K, then G fixes $k(x)$. Thus, $G_0 := Gal(K/k(x)) = \langle z_0 \rangle$, with $z_0^2 = 1$, is central in G. We call *the reduced automorphism group* of K the group $\overline{G} := G/G_0$. Then, $\overline{G}$ is isomorphic to one of the following: $\mathbb{Z}_n$, D_n, A_4, S_4, A_5 with branching indices of the corresponding cover $\mathbb{P}^1_x \to \mathbb{P}^1/\overline{G}$ given respectively by (n,n), $(2,2,n)$, $(2,3,3)$, $(2,4,4)$, $(2,3,5)$.

We fix a coordinate z in $\mathbb{P}^1/\overline{G}$. Thus, $\overline{G}$ is the monodromy group of a cover $\phi : \mathbb{P}^1_x \to \mathbb{P}^1_z$. We denote by $q_1, \dots, q_r$ the corresponding branch points of ϕ. Let S be the set of branch points of $\Phi : \mathcal{X}_g \to \mathbb{P}^1_z$. Clearly $q_1, \dots, q_r \in S$. As above W denotes the images in $\mathbb{P}^1$ of Weierstrass points of $\mathcal{X}_g$ and $V := \cup_{i=1}^r \phi^{-1}(q_i)$. For each $q_1, \dots, q_r$ we have a corresponding permutation $\sigma_1, \dots, \sigma_r \in S_n$. The tuple $\bar{\sigma} := (\sigma_1, \dots, \sigma_r)$ is the signature of $\overline{G}$. Thus, $\overline{G} = \langle \sigma_1, \dots, \sigma_r \rangle$ and $\sigma_1 \cdots \sigma_r = 1$.

Since each of the above groups is embedded in $PGL_2(\mathbb{C})$ then we can have these generating systems $\sigma_1, \dots, \sigma_r$ as matrices in $PGL_2(\mathbb{C})$. Below we display all the cases:

$$
\begin{aligned}
&i) \quad \mathbb{Z}_n \cong \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_n^{n-1} & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \\
&ii) \quad D_n \cong \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \\
&iii) \quad A_4 \cong \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right\rangle \\
&iv) \quad S_4 \cong \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle \\
&v) \quad A_5 \cong \left\langle \begin{pmatrix} \omega & 1 \\ 1 & -\omega \end{pmatrix}, \begin{pmatrix} \omega & \xi_5^4 \\ 1 & -\xi_5^4\omega \end{pmatrix} \right\rangle
\end{aligned}
\tag{1}
$$

where $\omega = \frac{-1+\sqrt{5}}{2}$, ζ_n is a primitive n^{th} root of unity, ξ_5 is a primitive 5^{th} root of unity, and i is a primitive 4^{th} root of unity.

2.1. **Fixed fields of the reduced automorphism groups.**

The group $\overline{G}$ given above acts on $k(x)$ via the natural way. The fixed field is a genus 0 field, say $k(z)$. Thus, z is a degree $|\overline{G}|$ rational function in x, say $z = \phi(x)$. In this section we determine $\phi(x)$ and its decompositions.

Lemma 1. *Let H be a finite subgroup of $PGL_2(k)$. Let us identify each element of H with the corresponding Moebius transformation and let s_i be the i-th elementary symmetric polynomial in the elements of H, $i = 1, \dots, |H|$. Then any non-constant s_i generates $k(z)$.*

Proof. It is easy to check that the s_i are the coefficients of the minimum polynomial of x over $k(z)$. It is well-known that any non-constant coefficient of this polynomial generates the field. $\square$

Lemma 2. *The fixed field for each of the groups $\overline{G}$ in cases i) - v) is generated respectively by the function*

i) $z = x^n$

ii) $z = x^n + \frac{1}{x^n}$

iii) $z = \frac{x^{12}-33x^8-33x^4+1}{x^2(x^4-1)^2}$

iv) $z = \frac{(x^8+14x^4+1)^3}{108(x(x^4-1))^4}$

v) $z = \frac{\left(-x^{20}+228x^{15}-494x^{10}-228x^5-1\right)^3}{1728(x(x^{10}+11x^5-1))^5}$

Proof. Apply Lemma 1 to the embedding of $\overline{G}$ given above. □

Notice that the branch points of a rational function $\phi(x) = \frac{f(x)}{g(x)}$ are exactly the zeroes of the discriminant of the polynomial $r(x) := f(x) - t \cdot g(x)$ with respect to x. Then the branch points of each of the above functions are i) $\{0, \infty\}$, ii) $\{-2, 2, \infty\}$, iii) $\{\infty, -6i\sqrt{3}, 6i\sqrt{3}\}$, iv) $\{0, 1, \infty\}$, v) $\{0, 1728, \infty\}$. The above facts are well known in the literature, see for example Klein [9].

2.2. **Decomposition of $\phi(x)$.** The automorphism group of $k(x)/k(\phi)$ is the embedding of $\overline{G}$ detailed before. As $|\overline{G}| = [k(x) : k(\phi)]$, there is a degree-preserving correspondence between subgroups of $\overline{G}$ and intermediate fields in the extension. By Lüroth's Theorem, each of those fields is $k(h)$ for some rational function h. Now, it is clear that, in general, $k(f) \subset k(h) \Leftrightarrow f = g \circ h$ for some g. Thus, we can use computer algebra techniques to find all the decompositions of ϕ and describe the lattice of intermediate fields.

It is clear from the expression of ϕ that there is a decomposition $\phi = g(x^s)$ for s taking the values $n, n, 2, 4, 5$ respectively. This comes also from the fact that the subgroup $\langle \xi_5 \cdot x \rangle$ of $\overline{G}$ corresponds to the field generated by $x \cdot \xi_5 x \cdots \xi_5^{s-1} x = x^s$.

Finding different decompositions of $\phi(x)$ is not a trivial computational problem. There are algorithms available to do this; see [6] for details.

For example, for $\overline{G} = A_5$ it is also possible to find decompositions involving x^2 or x^3 for functions that are equivalent to ϕ. Namely, for any $\sigma \in PGL_2(k)$, a generator of the field fixed for the conjugate group $\sigma A_5 \sigma^{-1}$ is $\phi(\sigma^{-1})$. If σ is chosen in such a way as having $\{x, -x\} < \sigma A_5 \sigma^{-1}$, then $k(x \cdot (-x)) = k(x^2)$ will be an intermediate field by Lemma 1. This can be accomplished by conjugating any involution of A_5 into $-x$. In the same manner, if an element of order 3 in A_5 is conjugated into $\zeta_3 x$, where ζ_3 is a primitive cubic root of 1, the resulting function can be written in terms of $x \cdot \zeta_3 x \cdot \zeta_3^2 x = x^3$; see [13] for details.

2.3. **Automorphism groups and their signatures.** The automorphism groups of hyperelliptic curves have been classified by [2], [3]. Most of these results study automorphism groups in terms of the Fuchsian groups. Since, we take the algebraic approach we go over some of the results briefly.

The automorphism group G of the hyperelliptic curve is a degree 2 central extension of $\overline{G}$. The following lemma is proved in [7].

Lemma 3. *Let $p \geq 2$, $\alpha \in G$ and $\bar{\alpha}$ its image in $\overline{G}$ with order $|\,\bar{\alpha}\,| = p$. Then,*
i) $|\,\alpha\,| = p$ if and only if it fixes no Weierstrass points.
ii) $|\,\alpha\,| = 2p$ if and only if it fixes some Weierstrass point.

Let W denote the images in $\mathbb{P}^1_x$ of Weierstrass places of $\mathcal{X}_g$ and $V := \cup_{i=1}^3 \phi^{-1}(q_i)$.

Let $z = \frac{\Psi(x)}{\Upsilon(x)}$, where $\Psi, \Upsilon \in k[x]$. For each branch point q_i, $i = 1, 2, 3$ we have the degree $|\overline{G}|$ equation $z \cdot \Upsilon(x) - q_i \cdot \Upsilon(x) = \Psi(x)$, where the multiplicity of the roots correspond to the ramification index for each q_i (i.e., the index of the normalizer in $\overline{G}$ of σ_i). We denote the ramification of $\phi : \mathbb{P}^1_x \to \mathbb{P}^1_z$, by $\varphi^r_m, \chi^s_n, \psi^t_p$, where the subscript denotes the degree of the polynomial.

Let $\lambda \in S \setminus \{q_1, q_2, q_3\}$. The points in the fiber of a non-branch point λ are the roots of the equation: $\Psi(x) - \lambda \cdot \Upsilon(x) = 0$. To determine the equation of the curve we simply need to determine the Weierstrass points of the curve. For each fixed ϕ there are the following eight cases:

$$
\begin{aligned}
&1) \quad V \cap W = \emptyset,\\
&2) \quad V \cap W = \phi^{-1}(q_1),\\
&3) \quad V \cap W = \phi^{-1}(q_2),\\
&4) \quad V \cap W = \phi^{-1}(q_3),\\
&5) \quad V \cap W = \phi^{-1}(q_1) \cup \phi^{-1}(q_2),\\
&6) \quad V \cap W = \phi^{-1}(q_2) \cup \phi^{-1}(q_3),\\
&7) \quad V \cap W = \phi^{-1}(q_1) \cup \phi^{-1}(q_3),\\
&8) \quad V \cap W = \phi^{-1}(q_1) \cup \phi^{-1}(q_2) \cup \phi^{-1}(q_3).
\end{aligned}
\tag{2}
$$

It turns out that the above cases also determine the full automorphism groups. We define the following groups as follows:

$$
\begin{aligned}
V_n :=& \langle\, x, y \,|\, x^4, y^n, (xy)^2, (x^{-1}y)^2 \,\rangle, \quad H_n := \langle\, x, y \;|\; x^4, y^2x^2, (xy)^n \,\rangle,\\
G_n :=& \langle\, x, y \;|\; x^2y^n, y^{2n}, x^{-1}yxy \,\rangle, \quad U_n := \langle\, x, y \,|\, x^2, y^n, xyxy^{n+1} \rangle,
\end{aligned}
\tag{3}
$$

Sometimes these groups are called **twisted dihedral, double dihedral, generalized quaternion**, and **semidihedral**. We warn the reader that these terms are not standard in the literature. They are all four degree 2 central extensions of the dihedral group D_n and therefore have order $4n$. Notice that V_2 is isomorphic with the dihedral group of order 8 and $H_2 \cong U_2 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_4$. Furthermore, we have the following result, the proof is elementary and we skip the details.

Remark 4. *i) If $n \equiv 1 \mod 2$ then $H_{4n} \cong G_{4n}$*
ii) If $n = 2^{s+1}$ then $G_n = Q_{2^{s+1}}$ for any $s \in \mathbb{Z}$.

Further, the following groups

$$W_2 := \langle x, y \,|\, x^4, y^3, yx^2y^{-1}x^2, (xy)^4 \rangle, \quad W_3 := \langle x, y \,|\, x^2, y^3, x^2(xy)^4, (xy)^8 \rangle$$

are degree 2 central extensions of S_4. Now we have the following result.

Theorem 5. *The full automorphism group of a hyperelliptic curve is isomorphic to one of the following $\mathbb{Z}_2 \otimes \mathbb{Z}_n$, $\mathbb{Z}_n$, $\mathbb{Z}_2 \otimes D_n$, V_n, D_n, H_n, G_n, U_n, $\mathbb{Z}_2 \otimes A_4$, $SL_2(3)$, $\mathbb{Z}_2 \otimes S_4$, $GL_2(3)$, W_2, W_3 $\mathbb{Z}_2 \otimes A_5$, $SL_2(5)$. Furthermore, the signature for each group is as in Table 1.*

Proof. The ramification of $\phi : \mathbb{P}^1_x \to \mathbb{P}^1/\overline{G}$ is one of the following (n, n), $(2, 2, n)$, $(2, 3, 3)$, $(2, 4, 4)$, $(2, 3, 5)$. Recall that $|G| = 2|\overline{G}|$. Each case of Eq. (2) determines a group of automorphisms.

Let $\overline{G} \cong \mathbb{Z}_n$ and $\overline{G} =< \alpha >$. If α fixes no Weierstrass points then, from the above Lemma, α lifts to an element of order n in G. Hence, $G = \mathbb{Z}_2 \otimes \mathbb{Z}_n$. If α fixes some Weierstrass points then α lifts to an element of order $2n$ in G and $|G| = 2n$ then G is the cyclic group of order $2n$.

The cases left have three branch points for the cover $\phi : \mathbb{P}^1_x \to \mathbb{P}^1/\overline{G}$. From the above lemma we have that if the places in the fiber $\phi^{-1}(q_1)$, $\phi^{-1}(q_2)$, $\phi^{-1}(q_3)$, are Weierstrass points then $\sigma_1, \sigma_2, \sigma_3$ lift in G to elements of orders $2|\sigma_1|$, $2|\sigma_2|$, and $2|\sigma_3|$ respectively.

Let $\overline{G} \cong D_n$ where D_n is given as in Eq. (1). Since the branching of q_1 and q_2 is the same then there are basically six distinct cases which could arise. In other words, cases 2 and 3 from Eq. (2) give the same group G. The same happens for cases 6 and 7 from Eq. (2).

If none of the places in the fibers $\phi^{-1}(q_1)$, $\phi^{-1}(q_2)$, $\phi^{-1}(q_3)$, are Weierstrass points then $\sigma_1, \sigma_2, \sigma_3$ lift in G to elements of orders $|\sigma_1|$, $|\sigma_2|$, and $|\sigma_3|$ respectively. Together, with the hyperelliptic involution we have $G = \mathbb{Z}_2 \otimes D_n$. When places in $\phi^{-1}(q_1)$ (or $\phi^{-1}(q_2)$) are Weierstrass points then the involution $\sigma_1 \in D_n$, as in Eq (1), lifts to an element of order 4. In this case, the group has generators

$$G = \langle\ \bar{\sigma}_1, \bar{\sigma}_3 \,|\, \bar{\sigma}_1^4, \bar{\sigma}_3^n, (\bar{\sigma}_1\bar{\sigma}_3)^2, (\bar{\sigma}_1^{(-1)}\bar{\sigma}_3)^2\ \rangle,$$

where $\bar{\sigma}_1, \bar{\sigma}_3$ are the lifts of σ_1, σ_3 in G. Thus, $G \cong V_n$.

When places in $\phi^{-1}(q_3)$ are Weierstrass points then the element of order $\sigma_1 \in D_n$ of order n lifts to an element of order $2n$. In this case, the group G has generators of order $2n$ and the hyperelliptic involution. Thus, $G \cong D_{2n}$.

The other three cases from Eq. (2), namely case 5), 6) or 7), and case 8) give groups H_n, U_n, G_n with presentation as in Eq. (3).

If $\overline{G} \cong A_4$ then again we have two branch cycles which are the same. Hence, we have 6 cases. When the involution lifts to an element of order 4 then the degree 2 central extension of A_4 is $SL_2(3)$, otherwise the extension is $\mathbb{Z}_2 \otimes A_4$.

The proof when $\overline{G} \cong S_4$ or A_5 goes similarly and we skip the details. The signature for each group follows accordingly for each case.

□

#	G	$\overline{G}$	$\delta(G, \mathbf{C})$	δ, n, g	$\mathbf{C} = (C_1, \dots C_r)$	ϕ
1	$\mathbb{Z}_2 \otimes \mathbb{Z}_n$		$\frac{2g+2}{n} - 1$	$n < g+1$	$(n^2, n^2, 2^n, \dots, 2^n)$	
2	$\mathbb{Z}_{2n}$	$\mathbb{Z}_n$	$\frac{2g+1}{n} - 1$		$(n^2, 2n, 2^n, \dots, 2^n)$	(n, n)
3	$\mathbb{Z}_{2n}$		$\frac{2g}{n} - 1$	$n < g$	$(2n, 2n, 2^n, \dots, 2^n)$	
4	$\mathbb{Z}_2 \otimes D_n$		$\frac{g+1}{n}$		$(n^4, 2^{2n}, \dots, 2^{2n})$	
5	V_n		$\frac{g+1}{n} - \frac{1}{2}$		$(n^4, 4^n, 2^{2n}, \dots, 2^{2n})$	
6	D_{2n}	D_n	$\frac{g}{n}$		$((2n)^2, 2^{2n}, \dots, 2^{2n})$	$(2^n, 2^n, n^2)$
7	H_n		$\frac{g+1}{n} - 1$	$n < g+1$	$(4^n, 4^n, n^4, 2^{2n} \dots, 2^{2n})$	
8	U_n		$\frac{g}{n} - \frac{1}{2}$	$g \neq 2$	$(4^n, (2n)^2, 2^{2n}, \dots, 2^{2n})$	
9	G_n		$\frac{g}{n} - 1$	$n < g$	$(4^n, 4^n, (2n)^2, 2^{2n}, \dots, 2^{2n})$	
10	$\mathbb{Z}_2 \otimes A_4$		$\frac{g+1}{6}$		$(3^8, 3^8, 2^{12}, \dots, 2^{12})$	
11	$\mathbb{Z}_2 \otimes A_4$		$\frac{g-1}{6}$		$(3^8, 6^4, 2^{12}, \dots, 2^{12})$	
12	$\mathbb{Z}_2 \otimes A_4$	A_4	$\frac{g-3}{6}$	$\delta \neq 0$	$(6^4, 6^4, 2^{12}, \dots, 2^{12})$	$(2^6, 3^4, 3^4)$
13	$SL_2(3)$		$\frac{g-2}{6}$	$\delta \neq 0$	$(4^6, 3^8, 3^8, 2^{12}, \dots, 2^{12})$	
14	$SL_2(3)$		$\frac{g-4}{6}$		$(4^6, 3^8, 6^4, 2^{12}, \dots, 2^{12})$	
15	$SL_2(3)$		$\frac{g-6}{6}$	$\delta \neq 0$	$(4^6, 6^4, 6^4, 2^{12}, \dots, 2^{12})$	
16	$\mathbb{Z}_2 \otimes S_4$		$\frac{g+1}{12}$		$(3^{16}, 4^{12}, 2^{24}, \dots, 2^{24})$	
17	$\mathbb{Z}_2 \otimes S_4$		$\frac{g-3}{12}$		$(6^8, 4^{12}, 2^{24}, \dots, 2^{24})$	
18	$GL_2(3)$		$\frac{g-2}{12}$		$(3^{16}, 8^6, 2^{24}, \dots, 2^{24})$	
19	$GL_2(3)$	S_4	$\frac{g-6}{12}$		$(6^8, 8^6, 2^{24}, \dots, 2^{24})$	$(2^{12}, 3^8, 4^6)$
20	W_2		$\frac{g-5}{12}$		$(4^{12}, 4^{12}, 3^{16}, 2^{24}, \dots, 2^{24})$	
21	W_2		$\frac{g-9}{12}$		$(4^{12}, 4^{12}, 6^8, 2^{24}, \dots, 2^{24})$	
22	W_3		$\frac{g-8}{12}$		$(4^{12}, 3^{16}, 8^6, 2^{24}, \dots, 2^{24})$	
23	W_3		$\frac{g-12}{12}$		$(4^{12}, 6^8, 8^6, 2^{24}, \dots, 2^{24})$	
24	$\mathbb{Z}_2 \otimes A_5$		$\frac{g+1}{30}$		$(3^{40}, 5^{24}, 2^{60}, \dots, 2^{60})$	
25	$\mathbb{Z}_2 \otimes A_5$		$\frac{g-5}{30}$		$(3^{40}, 10^{12}, 2^{60}, \dots, 2^{60})$	
26	$\mathbb{Z}_2 \otimes A_5$		$\frac{g-15}{30}$		$(6^{20}, 10^{12}, 2^{60}, \dots, 2^{60})$	
27	$\mathbb{Z}_2 \otimes A_5$	A_5	$\frac{g-9}{30}$		$(6^{20}, 5^{24}, 2^{60}, \dots, 2^{60})$	$(2^{30}, 3^{20}, 5^{12})$
28	$SL_2(5)$		$\frac{g-14}{30}$		$(4^{30}, 3^{40}, 5^{24}, 2^{60}, \dots, 2^{60})$	
29	$SL_2(5)$		$\frac{g-20}{30}$		$(4^{30}, 3^{40}, 10^{12}, 2^{60}, \dots, 2^{60})$	
30	$SL_2(5)$		$\frac{g-24}{30}$		$(4^{30}, 6^{20}, 5^{24}, 2^{60}, \dots, 2^{60})$	
31	$SL_2(5)$		$\frac{g-30}{30}$		$(4^{30}, 6^{20}, 10^{12}, 2^{60}, \dots, 2^{60})$	

TABLE 1. $\mathrm{Aut}(\mathcal{X}_g)$ and the corresponding signatures

The above theorem was first proven in [3] using Fuchsian groups. Notice that representations of groups are given in that paper.

Remark 6. *i) In cases 4, 5, and 7-9 we have* $n \equiv 0 \mod 2$.

2.4. **The number of automorphism groups for a fixed genus.** For a fixed g we denote by N_g the number of groups that occur as automorphism groups of genus g curves. We would like to determine what happens to N_g as g increases.

Let $n \in \mathbb{Z}$ such that $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$. Denote by $\mathfrak{d}(n)$ the number of divisors of n. It is well known that $\mathfrak{d}(n) = \prod_{i=1}^{s}(\alpha_i + 1)$. Further, we denote by $\bar{\mathfrak{d}}(n)$ the number of even divisors of n. We have the following lemma:

Lemma 7. *Let g be fixed. Then the number of automorphism groups that can occur as automorphism groups $Aut(\mathcal{X}_g)$ of genus g hyperelliptic curves is*

i) if $\overline{Aut}(\mathcal{X}_g) \cong \mathbb{Z}_n$ *then* $n_1 = \mathfrak{d}(g+1) + \mathfrak{d}(2g+1) + \mathfrak{d}(2g) - 1$

ii) if $\overline{Aut}(\mathcal{X}_g) \cong D_n$ *then* $n_2 = 3\bar{\mathfrak{d}}(g+1) + 2\bar{\mathfrak{d}}(g) + \mathfrak{d}(g) - 2$
iii) if $\overline{Aut}(\mathcal{X}_g) \cong A_4$ *and* $g > 6$ *then* $n_3 = 1$
iv) if $\overline{Aut}(\mathcal{X}_g) \cong S_4$ *then* $n_4 = 1$ *or 0.*
v) if $\overline{Aut}(\mathcal{X}_g) \cong A_5$ *then* $n_5 = 1$ *or 0.*

Proof. The proof is elementary and we skip the details. □

3. Moduli spaces of covers

Fix an integer $g \geq 2$ and a finite group G. Let $C_1, ..., C_r$ be nontrivial conjugacy classes of G. Let $\mathbf{C} = (C_1, ..., C_r)$, viewed as unordered tuple, repetitions are allowed. We allow r to be zero, in which case $\mathbf{C}$ is empty. Consider pairs $(\mathcal{X}, \mu)$, where $\mathcal{X}$ is a curve and $\mu : G \to \mathrm{Aut}(\mathcal{X})$ is an injective homomorphism. We will suppress μ and just say $\mathcal{X}$ is a curve with G-action, or a G-curve. Two G-curves $\mathcal{X}$ and $\mathcal{X}'$ are called equivalent if there is a G-equivariant isomorphism $\mathcal{X} \to \mathcal{X}'$.

We say a G-curve $\mathcal{X}$ is **of ramification type** $(g, G, \mathbf{C})$ if the following holds: Firstly, g is the genus of $\mathcal{X}$. Secondly, the points of the quotient $\mathcal{X}/G$ that are ramified in the cover $\mathcal{X} \to \mathcal{X}/G$ can be labeled as $p_1, ..., p_r$ such that C_i is the conjugacy class in G of distinguished inertia group generators over p_i (for $i = 1, ..., r$). (Distinguished inertia group generator means the generator that acts in the tangent space as multiplication by $\exp(2\pi\sqrt{-1}/e)$, where e is the ramification index). For short, we will just say $\mathcal{X}$ is of type $(g, G, \mathbf{C})$.

If $\mathcal{X}$ is a G-curve of type $(g, G, \mathbf{C})$ then the genus g_0 of $\mathcal{X}/G$ is given by the Riemann-Hurwitz formula. Define $\mathcal{H} = \mathcal{H}(g, G, \mathbf{C})$ to be the set of equivalence classes of G-curves of type $(g, G, \mathbf{C})$. By covering space theory, $\mathcal{H}$ is non-empty if and only if G can be generated by elements $\alpha_1, \beta_1, ..., \alpha_{g_0}, \beta_{g_0}, \gamma_1, ..., \gamma_r$ with $\gamma_i \in C_i$ and $\prod_j \, [\alpha_j, \beta_j] \, \prod_i \gamma_i = 1$, where $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$.

Let $\mathcal{M}_g$ be the moduli space of genus g curves, and $\mathcal{M}_{g_0,r}$ the moduli space of genus g_0 curves with r distinct marked points, where we view the marked points as unordered (contrary to usual procedure). Consider the map $\Phi : \mathcal{H} \to \mathcal{M}_g$, forgetting the G-action, and the map $\Psi : \mathcal{H} \to \mathcal{M}_{g_0,r}$ mapping (the class of) a G-curve $\mathcal{X}$ to the class of the quotient curve $\mathcal{X}/G$ together with the (unordered) set of branch points $p_1, ..., p_r$. If $\mathcal{H} \neq \emptyset$ then Ψ is surjective and has finite fibers, by covering space theory. Also Φ has finite fibers, since the automorphism group of a curve of genus ≥ 2 is finite. By [1], the set $\mathcal{H}$ carries a structure of quasi-projective variety (over $\mathbb{C}$) such that the maps Φ and Ψ are finite morphisms. If $\mathcal{H} \neq \emptyset$ then all components of $\mathcal{H}$ map surjectively to $\mathcal{M}_{g_0,r}$ (through a finite map), hence they all have the same dimension

$$\delta(g, G, \mathbf{C}) := \dim \mathcal{M}_{g_0,r} = 3g_0 - 3 + r$$

Since also Φ is a finite map, we get

Lemma 8. *Let* $\mathcal{M}(g, G, \mathbf{C})$ *denote the image of* Φ*, i.e., the locus of genus* g *curves admitting a* G*-action of type* $(g, G, \mathbf{C})$*. If this locus is non-empty then each of its components has dimension* $\delta(g, G, \mathbf{C})$*.*

For a description of some of the loci $\mathcal{M}(g, G, \mathbf{C})$, not in the hyperelliptic locus, in terms of the theta nulls see [10].

Next we focus on the hyperelliptic locus. Let $\phi_0 : \mathcal{X}_g \to \mathbb{P}^1$ be the cover which corresponds to the degree 2 extension $K/k(X)$. Then, $\psi := \phi \circ \phi_0$ has monodromy group $G := \mathrm{Aut}(\mathcal{X}_g)$. From basic covering theory, the group G is embedded in the group S_n, where $n = \deg \psi$. There is an r-tuple $\bar{\sigma} := (\sigma_1, \dots, \sigma_r)$, where $\sigma_i \in S_n$ such that $\sigma_1, \dots, \sigma_r$ generate G and $\sigma_1 \cdots \sigma_r = 1$. The signature of ψ is an r-tuple of conjugacy classes $\mathbf{C} := (C_1, \dots, C_r)$ in S_n such that C_i is the conjugacy class of σ_i. We use the notation n^p to denote the conjugacy class of permutations which are a product of p cycles of length n. Using the signature of $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ and the Riemann-Hurwitz formula, one finds out the signature of $\psi : \mathcal{X}_g \to \mathbb{P}^1$ for any given g and G.

The following theorem describes the list of all the subvarieties of the hyperelliptic moduli which are determined by group actions.

Theorem 9. *For each $g \geq 2$, the groups G that occur as automorphism groups and their signatures $\mathbf{C}$ are given in Table 1. Moreover; the locus $\mathcal{H}(G, \mathbf{C})$ in $\mathcal{H}_g$ of curves with automorphism group G and signature $\mathbf{C}$ is an irreducible algebraic variety of dimension $\delta(G, \mathbf{C})$ as given in Table 1.*

Proof. The dimension of each locus is an immediate consequence of Lemma 8. Next, we will show the irreducibility of the Hurwitz space $\mathcal{H}(G, \mathbf{C})$ in each case.

The cases 1-3, 10-15, and 24-31 of Table 1 follow from the results of [14] and [13] respectively. It is left to prove cases 4-9 and 16-23 which correspond to the cases when $\overline{G}$ is isomorphic to D_n and A_5 respectively. To prove this we make use the GAP and the Braid program written by K. Magaard. For each case we construct the group G as a subgroup of $S_{|G|}$. In each case we find a generating tuple and compute its braid action. There is only one braid orbit which shows that the corresponding space is irreducible. This completes the proof.

□

4. Determining equations of families of curves

In this section we state the equations of curves in each case of Table 1. For a more detailed treatment of these spaces, including proofs, the reader can check results in [3], [13], [14], [15], among others. Recall that $\overline{G}$ is the monodromy group of a cover $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ with signature $\bar{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ as in Table 1. We fix the coordinates in $\mathbb{P}^1$ as x and z respectively and from now on denote the cover $\phi : \mathbb{P}^1_x \to \mathbb{P}^1_z$. Thus, z is a rational function in x of degree $|\overline{G}|$. We denote by q_1, q_2, q_3 the corresponding branch points of ϕ. Let S be the set of branch points of $\Phi : \mathcal{X}_g \to \mathbb{P}^1$. Clearly $q_1, q_2, q_3 \in S$. As above W denotes the images in $\mathbb{P}^1_x$ of Weierstrass points of $\mathcal{X}_g$ and $V := \cup_{i=1}^3 \phi^{-1}(q_i)$.

Let

$$z = \frac{\Psi(x)}{\Upsilon(x)}$$

where $\Psi, \Upsilon \in k[x]$. For each branch point q_i, $i = 1, 2, 3$ we have $z - q_i = \frac{p(x)}{\Upsilon(x)}$. Hence, we have the degree $|\overline{G}|$ equation

$$p(x) = \Psi(x) - q_i \cdot \Upsilon(x) = 0,$$

where the multiplicity of all the roots is the same and correspond to the ramification index of q_i (i.e., the index of the normalizer in $\overline{G}$ of σ_i). We denote the ramification of $\phi : \mathbb{P}^1_x \to \mathbb{P}^1_z$, by

$$\begin{aligned} \varphi^r_m(x) &:= \Psi(x) - q_1 \cdot \Upsilon(x), \\ \chi^s_n(x) &:= \Psi(x) - q_2 \cdot \Upsilon(x) \\ \psi^t_p(x) &:= \Psi(x) - q_3 \cdot \Upsilon(x), \end{aligned} \tag{4}$$

where the subscript denotes the degree of the polynomial and the superscript is the index of the normalizer in $\overline{G}$ of σ_i.

It is obvious that

$$\phi^{-1}\left(S \setminus \{q_1, q_2, q_3\}\right) \subset W.$$

Let $\lambda \in S \setminus \{q_1, q_2, q_3\}$. The points in the fiber $\phi^{-1}(\lambda)$ are the roots of the equation:

$$\Psi(x) - \lambda \cdot \Upsilon(x) = 0 \tag{5}$$

To determine the equation of the curve we simply need to determine the Weierstrass points of the curve. Let

$$G(x) := \prod_{\lambda \in S \setminus \{q_1, q_2, q_3\}} \left(\Psi(x) - \lambda \cdot \Upsilon(x)\right).$$

For each fixed ϕ there are the following cases and the corresponding equation of the curve is $y^2 = f(x)$ where $f(x)$ is as below:

$$\begin{array}{lll} 1) & V \cap W = \emptyset, & G(x), \\ 2) & V \cap W = \Phi^{-1}(q_1), & \varphi(x) \cdot G(x), \\ 3) & V \cap W = \Phi^{-1}(q_2), & \chi(x) \cdot G(x), \\ 4) & V \cap W = \Phi^{-1}(q_3), & \psi(x) \cdot G(x), \\ 5) & V \cap W = \Phi^{-1}(q_1) \cup \Phi^{-1}(q_2), & \varphi(x) \cdot \chi(x) \cdot G(x), \\ 6) & V \cap W = \Phi^{-1}(q_2) \cup \Phi^{-1}(q_3), & \chi(x) \cdot \psi(x) \cdot G(x), \\ 7) & V \cap W = \Phi^{-1}(q_1) \cup \Phi^{-1}(q_3), & \varphi(x) \cdot \psi(x) \cdot G(x), \\ 8) & V \cap W = \Phi^{-1}(q_1) \cup \Phi^{-1}(q_2) \cup \Phi^{-1}(q_3), & \varphi(x) \cdot \chi(x) \cdot \psi(x) \cdot G(x) \end{array} \tag{6}$$

Since we know $z = \frac{\Psi(x)}{\Upsilon(x)}$ in each case, then we can easily compute the equation of the curve for all cases of Table 1.

Remark 10. *When $\overline{G} = \mathbb{Z}_n, D_n, A_4$, then two of the branch points of $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ correspond to the same conjugacy class. Then, cases 2) and 3) are the same. So are also the cases 6) and 7). This explains the number of cases in Table 1.*

Remark 11. *The polynomial $G(x)$ can also be computed by computing an orbit of $\overline{G}$ using the generators in Eq. (1). This follows from the fact that $\overline{G}$ is the monodromy group of $\phi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ and therefore has a complete orbit on the fiber $\phi^{-1}(\lambda)$ for each $\lambda \neq q_1, q_2, q_3$.*

4.1. $\overline{G} \cong \mathbb{Z}_n$**.** The branch points of the cover $\phi(x) = x^n$ are 0 and ∞. For $\lambda \in S \setminus \{0, \infty\}$, the points $\phi^{-1}(\lambda)$ satisfy the equation $G_\lambda(x) = x^n - \lambda$.

We have

$$G(x) = \prod_{\lambda \in S \setminus \{0,\infty\}} G_\lambda(x) = x^{nt} + \cdots + a_i x^{n(t-i)} + \dots a_{t-1} x^n + 1, \tag{7}$$

Then $\mathcal{X}_g$ belongs to cases 1, 2, 3 in Table 1. The equation of each family is $y^2 = F(x)$, where $F(x)$ is $G(x), G(x), x \cdot G(x)$ and t respectively $\frac{2g+2}{n}, \frac{2g+1}{n}, \frac{2g}{n}$. See [14] for details.

4.2. $\overline{G} \cong D_n$**.** In this case, the branch points of $z(x)$ are ∞, and ± 2. We have $G(x)$ as below:

$$G(X) = \prod_{i=1}^{\delta} (X^{2n} - \lambda_i X^n + 1).$$

Then,

$$G(X) = X^{2nt} + a_1 X^{2nt-n} + \cdots + a_t X^{nt} + a_{t-1} X^{(n-1)t} + \cdots + a_1 X^n + 1 \tag{8}$$

where $a_i, i = 1, \dots t$ are polynomials in terms of the symmetric polynomials $s_1, \dots, s_t$ of λ_i (i.e., etc.).

$$a_1 = s_1, a_2 = t + s_2, a_3 = (t-1)s_1 + s_3, a_4 := \binom{t}{n/2} + (t-2)s_2 + s_4,$$

Then, each family is parameterized as in Table 2.

4.3. $\overline{G} \cong A_4$**.** The branch points of the cover ϕ are $\{\infty, 6i\sqrt{3}, -6i\sqrt{3}\}$. The polynomials over these branch points are

$$\begin{aligned} \varphi_m(x) &:= x^4 + 2i\sqrt{3}x^2 + 1, \\ \chi_n(x) &:= x^8 + 14x^4 + 1, \\ \psi_p(x) &:= x(x^4 - 1) \end{aligned} \tag{9}$$

For $\lambda \in S \setminus \{\infty, 6i\sqrt{3}, -6i\sqrt{3}\}$ (equivalently $\lambda^2 + 108 \neq 0$) we have

$$G_\lambda(x) = x^{12} - \lambda x^{10} - 33x^8 + 2\lambda x^6 - 33x^4 - \lambda x^2 + 1, \tag{10}$$

There are $\delta = \frac{g+1}{6}$ points in $S \setminus \{\infty, \pm 6i\sqrt{3}\}$. Denote by

$$G(x) := \prod_{i=1}^{\delta} \left(x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1 \right)$$

Then, each family is parameterized as cases 10-15 in Table 2.

4.4. $\overline{G} \cong S_4$**.** The branch points of $\phi(x)$ are $\{0, 1, \infty\}$. Then,

$$\begin{aligned} \varphi(x) &:= x^{12} - 33x^8 - 33x^4 + 1, \\ \chi(x) &:= x^8 + 14x^4 + 1, \\ \psi(x) &:= x^4 - 1. \end{aligned} \tag{11}$$

For $\lambda \in S \setminus \{0, 1, \infty\}$, points in $\phi^{-1}(\lambda)$ are roots of the polynomial

$$G_\lambda(x) = x^{24} + \lambda x^{20} + (759 - 4\lambda)x^{16} + 2(3\lambda + 1288)x^{12} + (759 - 4\lambda)x^8 + \lambda x^4 + 1$$

There are δ points in $S \setminus \{0, 1, \infty\}$, where δ is given as in Table 1; see [14] for details. We denote

$$M(x) := \prod_{i=1}^{\delta} G_{\lambda_i}(x) \tag{12}$$

4.5. $\overline{G} \cong A_5$**.** The branch points of ϕ are 0, 1728 and ∞. At the place $z = 1728$ the function has the following ramification:

$$\phi(x) - 1728 = -\frac{\left(x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1\right)^2}{x^5\,\left(x^{10} + 11x^5 - 1\right)^5}.$$

Then,

$$\begin{aligned} \varphi(x) &= x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1 \\ \chi(x) &= x\,\left(x^{10} + 11x^5 - 1\right) \\ \psi(x) &= x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1. \end{aligned} \tag{13}$$

Then for each $\lambda_i \in S \setminus \{0, 1728, \infty\}$ the places in $\phi^{-1}(\lambda_i)$ are the roots of the following polynomial

$$\begin{aligned} G_i(x) &= -x^{60} + (684 - \lambda_i)x^{55} - (55\lambda_i + 157434)x^{50} - (1205\lambda_i - 12527460)x^{45} \\ &- (13090\lambda_i + 77460495)x^{40} + (130689144 - 69585\lambda_i)x^{35} + (33211924 - 134761\lambda_i)x^{30} \\ &+ (69585\lambda_i - 130689144)x^{25} - (13090\lambda_i + 77460495)x^{20} - (12527460 - 1205\lambda_i)x^{15} \\ &- (157434 + 55\lambda_i)x^{10} + (\lambda_i - 684)x^5 - 1 \end{aligned}$$

Then, we let

$$\Lambda(x) := \prod_{i=1}^{\delta} G_i(x) \tag{14}$$

where $M(x)$ is as in Eq. (12) and $\Lambda(x)$ as in Eq. (14).

Further, we notice that curves with automorphism do have something in common. Let I_4 be the invariant of binary forms defined in terms of transvections as in [15].

Lemma 12. *Let $\mathcal{X}_g$ be a hyperelliptic curve given with equation $y^2 = f(x)$ such that $|Aut(\mathcal{X}_g)| > 2$. Then, $I_4(f) = 0$.*

#	$y^2 = f(x)$
1	$x^{2g+2} + a_1 x^{n(t-1)} + \cdots + a_\delta x^n + 1, \quad t = \frac{2g+2}{n}$
2	$x^{2g+1} + a_1 x^{n(t-1)} + \cdots + a_\delta x^n + 1, \quad t = \frac{2g+1}{n}$
3	$x(x^{nt} + a_1 x^{n(t-1)} + \cdots + a_\delta x^n + 1), \quad t = \frac{2g}{n}$
4	$F(x) := \prod_{i=1}^{t} (x^{2n} + \lambda_i x^n + 1), \quad t = \frac{g+1}{n}$
5	$(x^n - 1) \cdot F(x)$
6	$x \cdot F(x)$
7	$(x^{2n} - 1) \cdot F(x)$
8	$x(x^n - 1) \cdot F(x)$
9	$x(x^{2n} - 1) \cdot F(x)$
10	$G(x) := \prod_{i=1}^{\delta} \left(x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1\right)$
11	$(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
12	$(x^8 + 14x^4 + 1) \cdot G(x)$
13	$x(x^4 - 1) \cdot G(x)$
14	$x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
15	$x(x^4 - 1)(x^8 + 14x^4 + 1) \cdot G(x)$
16	$M(x)$
17	$S(x) \cdot M(x)$
18	$T(x) \cdot M(x)$
19	$S(x) \cdot T(x) \cdot M(x)$
20	$R(x) \cdot M(x)$
21	$R(x) \cdot S(x) \cdot M(x)$
22	$R(x) \cdot T(x) \cdot M(x)$
23	$R(x) \cdot S(x) \cdot T(x) \cdot M(x)$
24	$\Lambda(x)$
25	$\left(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1\right) \cdot \Lambda(x)$
26	$\left(x\ (x^{10} + 11x^5 - 1)\right) \cdot \Lambda(x)$
27	$\psi \cdot \Lambda(x)$
28	$\left(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1\right) \cdot \left(x\ (x^{10} + 11x^5 - 1)\right) \cdot \Lambda(x)$
29	$\left(x\ (x^{10} + 11x^5 - 1)\right) \cdot \psi \cdot \Lambda(x)$
30	$\left(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1\right) \cdot \psi \cdot \Lambda(x)$
31	$\left(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1\right) \cdot \left(x\ (x^{10} + 11x^5 - 1)\right) \cdot \psi \cdot \Lambda(x)$

TABLE 2. The equation of the generic hyperelliptic curve with group G.

5. LATTICE OF GROUPS AND INCLUSION AMONG THE LOCI

In this section we will study the inclusions between the automorphism groups for a fixed genus g. This was also studied in [16], however, the author focuses only on automorphism groups with reduced automorphism groups isomorphic to a dihedral group.

For any genus g we determine completely the lattice of loci $\mathcal{H}_g^G$ in $\mathcal{H}_g$. Using GAP we can determine the list of all groups that occur as automorphism groups of genus g. This can be done for any fixed genus $g \leq 299$.

We identify groups in GAP by their identity in the library of SmallGroups. This library contains only groups of order up to 2400. We know that the order of the automorphism group is $\leq 8(g+1)$. Hence, we can determine the list of groups for all $g \leq 299$.

Let L be the list of all groups occurring as automorphism groups of genus g hyperelliptic curves. Each entry in L is an ordered pair $[m, n]$ where m denotes the order of the group and n the position that this group is stored in the Gap library. We order L as follows

$$[m, n] < [r, s] \quad \textit{if} \quad m \leq r \textit{ and } n \leq s$$

Consider $L = \{G_1, \dots G_N\}$ such that

$$G_1 < G_2 < \cdots < G_N$$

with respect to the above ordering. The incidence matrix of L is

$$M = [m_{i,j}]$$

where

$$m_{i,j} = \begin{cases} 1, & \textit{if } G_i \textit{ is a subgroup of } G_j \\ 0, & \textit{otherwise} \end{cases}$$

Then M is an upper triangular $N \times N$ matrix. Such matrix can be easily determined for any g. We have implemented a program in GAP which gives the lattice and the incidence matrix for any $g \geq 2$.

Example 1. *The lattice of the groups for genus 4 is given in Fig. 1. Each group is presented by its* GAP *identity. Each level contains cases with the same dimension.*

6. Concluding remarks

This main goal of this paper was to give a more unified algebraic approach of the case of automorphism groups of algebraic curves in zero characteristic. As part II of this project we intend to ask the same questions on characteristic $p > 0$. Such questions are very much unexplored for algebraic curves defined over fields of positive characteristic.

One would like to describe such loci in terms of invariants of the curves as already done for genus $g = 2, 3$; see [7, 8]. There have been attempts to do this in [14, 15, 13] using invariant of binary forms. Since such invariants are unknown for degree ≥ 7 which makes this method unlikely to succeed. Further, invariants of binary forms are huge polynomials in terms of the coefficients of

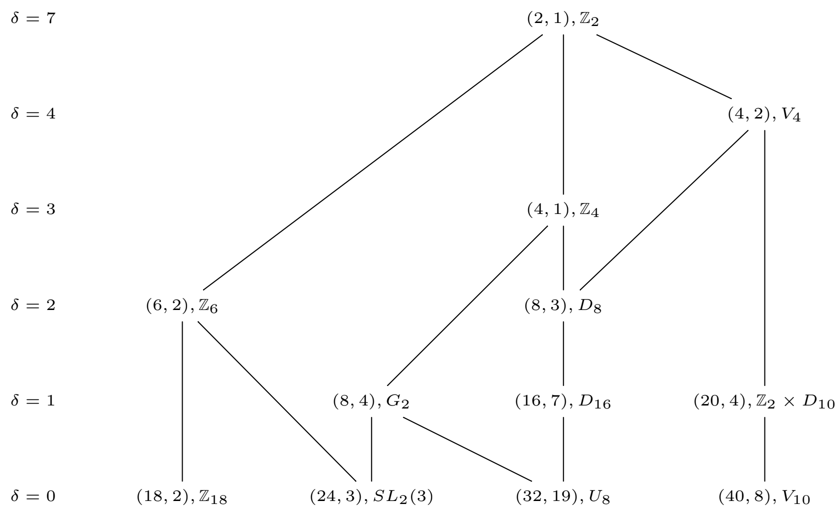

FIGURE 1. The lattice of automorphism groups for hyperelliptic curves of $g = 4$.

the curve. Even, if they are completely known they would be computationally not efficient. Hence, one is tempted to try to describe such loci in terms of theta nulls; see [10, 11, 12].

REFERENCES

[1] JOSÉ BERTIN, Compactification des schémas de Hurwitz, *C.R. Acad. Sci. Paris I*, vol. 322 (1996), 1063–1066.
[2] R. BRANDT AND H. STICHTENOCH, Die Automorphismengrupenn Hyperelliptischer Kurven. *Manuscripta Math* **55** (1986), no. 1, 83–92.
[3] E. BUJALANCE, J. M. GAMBOA, G. GROMADZKI, The full automorphism groups of hyperelliptic Riemann surfaces, *Manuscripta Math.* **79** (1993), no. 3-4, 267–282.
[4] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2006, (http://www.gap-system.org).
[5] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.4; 2006, package *SmallGroups*, by E. A. O'Brien, B. Eick, and H. U. Besche (http://www-gap.dcs.st-and.ac.uk/ gap/Packages/sgl.html)
[6] J. GUTIERREZ, A polynomial decomposition algorithm over factorial domains, Comptes Rendues Mathematiques, de Ac. de Sciences, 13 (1991), 81-86.
[7] J. GUTIERREZ AND T. SHASKA, Hyperelliptic curves with extra involutions, *LMS J. of Comput. Math.*, 8 (2005), 102-115.
[8] J. GUTIERREZ, D. SEVILLA, AND T. SHASKA,, Hyperelliptic curves of genus 3 and their automorphisms, Lect. Notes Comp, vol 13.(2005),109–123.
[9] F. KLEIN, Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree. Dover Publications, Inc., New York, N. Y., 1956.
[10] E. PREVIATO AND T. SHASKA, Vanishing theta nulls and algebraic curves with automorphisms, (work in progress).

[11] E. Previato and T. Shaska, Theta functions and automorphisms of curves, (work in progress).
[12] E. Previato, T. Shaska, and S. Wijesiri, Vanishing theta-nulls on genus two curves with split Jacobians, (work in progress).
[13] D. Sevilla and T. Shaska, Hyperelliptic curves with reduced automorphism group A_5, *Appl. Algebra Engrg. Comm. Comput.*, (to appear).
[14] T. Shaska, Some special families of hyperelliptic curves, *J. Algebra Appl.*, vol **3**, No. 1 (2004), 75-89.
[15] T. Shaska, Determining the automorphism group of hyperelliptic curves, *Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation*, ACM Press, pg. 248 - 254, 2003.
[16] A. Weaver, Hyperelliptic surfaces and their moduli. Geom. Dedicata 103 (2004), 69–87.

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