2-Weierstrass points of genus 3 hyperelliptic curves with extra involutions

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Abstract

We consider families of curves with extra automorphisms in $H_3$, the moduli space of smooth hyperelliptic curves of genus $g = 3$. Such families of curves are explicitly determined in terms of the absolute invariants of binary octavics. For each family of positive dimension where $|\text{Aut } (C)| > 4$, we determine the possible distributions of weights of 2-Weierstrass points.

1 Introduction

In this paper, we focus primarily on 2-Weierstrass points on hyperelliptic curves of genus 3 with extra automorphisms. Our goal is to classify the 2-Weierstrass points in terms of the moduli of the hyperelliptic moduli $H_3$. This follows [9], where the authors classify 3-Weierstrass points on curves of genus 2 with extra automorphisms.

We fix a group $G$ which acts on a genus 3 hyperelliptic curve as a full automorphism group. For a given signature, the locus of curves with automorphism group $G$ is an irreducible locus in the hyperelliptic moduli $H_3$. We focus on the groups $G$ which determine a family of dimension $d > 0$ in $H_3$. A complete list of such groups, signatures, and inclusions among the loci is given in [12]. The locus of curves $C$ with the Klein four-group $V_4 \hookrightarrow \text{Aut } (C)$ is a 3-dimensional locus in $H_3$. The appropriate invariants in this case are the dihedral invariants $s_2, s_3, s_4$ as defined in [15]. All the other cases can be described directly in terms of absolute invariants $t_1, \ldots, t_6$ or their equivalents as defined in [13]. Our main result is the following:

**Theorem.** Let $G$ be a group with $|G| > 4$ such that the corresponding locus $H_3(G)$ has dimension $d > 0$ in $H_3$. Let $p \in H_3(G)$, $s_2, s_3, s_4$ its corresponding dihedral invariants, and $C$ a representative for $p$. Then each branch point of the hyperelliptic projection $\pi : C \to \mathbb{P}^1$ has 2-weight 6 and one of the following holds:

i) If $\text{Aut } (C) \cong \mathbb{Z}_2^3$, then $C$ has non-branch 2-Weierstrass points of weight greater than one if and only if $s_2, s_3, s_4$ satisfy

$$(16s_2^3 - 784s_2^2 + 56s_2s_3 - s_3^2)G(s_2, s_3) = 0,$$
where \( G(s_2, s_3) \) is given as in Eq. (10).

ii) If \( \text{Aut}(C) \cong \mathbb{Z}_2 \times D_8 \), then \( C \) has exactly four non-branch 2-Weierstrass points of weight 3, unless \( C \) is isomorphic one of the curves

\[
y^2 = tx^8 + tx^4 + 1,
\]

for \( t = 196 \) (resp. \( t = -\frac{196}{15} \)) in which case it has in addition 8 other points of weight 3 (resp. 16 points of weight 2).

iii) If \( \text{Aut}(C) \cong D_{12} \) then \( C \) has non-branch 2-Weierstrass points with weight greater than one if and only if \( C \) is isomorphic to one of the curves

\[
y^2 = x(tx^6 + tx^3 + 1),
\]

for \( t = -\frac{49}{8} \) or \( t = \frac{1728}{8} \pm \frac{621}{4} \sqrt{2} \). In the first case, the curve has twelve 2-Weierstrass points of weight 3 and in the other two cases twelve 2-Weierstrass points of weight 2.

iv) If \( \text{Aut}(C) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \) then \( C \) has 2-Weierstrass points if and only if \( C \) is isomorphic to one of the curves \( y^2 = (tx^4 - 1) (tx^4 + tx^2 + 1) \), for \( t = -8 \) or it is a root of

\[
t^8 + 600822 t^7 + 71378609 t^6 + 4219381768 t^5 + 85080645104 t^4 - 2272444082944 t^3 + 1648013638352 t^2 - 50330399658824 t + 56693912375296 = 0.
\]

In the first case, the curve has two 2-Weierstrass points of weight 3.

This paper is organized as follows. In Section 2 we give some basic preliminaries for Weierstrass points and their weights. We also give an introduction to genus 3 hyperelliptic curves with extra involutions and their dihedral invariants.

In Section 3 we focus on the \( q \)-Weierstrass points of genus 3 curves for \( q = 1 \) and 2. We show how to construct a basis for the space of holomorphic \( q \)-differentials. We use this basis for our computations in Section 4.

In Section 4 we compute the Wronskian \( \Omega_q \) for \( q = 2 \) in terms of coordinates in the hyperelliptic moduli for each case when the group \( G \) has size \( |G| > 2 \) and the \( G \)-locus has dimension \( d > 0 \). Computations are challenging, especially in the case of \( G \cong V_4 \). In this case we make use of the dihedral invariants \( s_2, s_3, s_4 \) which make such computations possible.

## 2 Preliminaries

Below we give definitions of ordinary and higher-order Weierstrass points and establish some of the basic facts about these points on curves over \( \mathbb{C} \). For more information, the reader is encouraged to see [2,10,11,14].

Let \( C \) be a non-singular projective curve over \( k = \mathbb{C} \) of genus \( g \), and let \( k(C) \) be the associated function field. For any \( f \in k(C) \), \( \text{div}(f) \) denotes the divisor associated to \( f \), \( \text{div}(f)_0 \) and \( \text{div}(f)_\infty \) respectively the zero and pole divisors of \( f \). For any divisor \( D \) on \( C \), we have \( D = \sum_{P \in C} n_P P \) for \( n_P \in \mathbb{Z} \) with almost all \( n_P = 0 \). Let \( \nu_P(D) = n_P \), and let \( \nu_P(f) = \nu_P(\text{div}(f)) \).
For any divisor $D$ on $C$, let $\mathcal{L}(D) = \{ f \in k(C) : \text{div}(f) + D \geq 0 \} \cup \{ 0 \}$ and $\ell(D) = \dim_k(\mathcal{L}(D))$. By the Riemann-Roch theorem, for any canonical divisor $K$, we have

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$  

Since the degree of a canonical divisor is $2g - 2$, and since $\mathcal{L}(D) = \{ 0 \}$ for any divisor $D$ with negative degree, if $\deg(D) \geq 2g - 1$, then $\deg(K - D) < 0$, so $\ell(K - D) = 0$. Thus, if $\deg(D) \geq 2g - 1$, then $\ell(D) = \deg(D) + 1 - g$.

Let $P$ be a degree 1 point on $C$. Consider the chain of vector spaces

$$\mathcal{L}(0) \subseteq \mathcal{L}(P) \subseteq \mathcal{L}(2P) \subseteq \mathcal{L}(3P) \subseteq \cdots \subseteq \mathcal{L}(2g - 1)P.$$  

Since $\mathcal{L}(0) = k$, we have $\ell(0) = 1$. And $\ell((2g - 1)P) = g$. We obtain the corresponding non-decreasing sequence of integers

$$\ell(0) = 1, \ell(P), \ell(2P), \ell(3P), \ldots, \ell((2g - 1)P) = g.$$  

If $\ell(nP) = \ell((n - 1)P)$, then we call $n$ a Weierstrass gap number. Weierstrass’ Löckensatz (“gap theorem”), from the 1860s, states that for any point $P$ there are exactly $g$ Weierstrass gap numbers. If the gap numbers are $1, 2, \ldots, g$, then $P$ is an ordinary point. Otherwise, we call $P$ a Weierstrass point. (Equivalently, we call $P$ a Weierstrass point if $\ell(gP) > 1$, which is the case when there is some $f \in k(C)^{\times}$ with $\text{div}(f)_{\infty} = mP$ for $1 < m \leq g$.)

An integer $n$ is a gap number when $\ell(nP) = \ell((n - 1)P)$, which occurs exactly when $\ell(K - (n - 1)P) - \ell(K - nP) = 1$, for $K$ a canonical divisor of $C$. This means there is some $f \in k(C)^{\times}$ such that $\text{div}(f) + K \geq (n-1)P$ but $\not\geq nP$. Thus, there’s a differential $dx$ with $\text{div}(dx) = K$ so that $nP(f \cdot dx) = n - 1$. Further, $f \cdot dx$ is a holomorphic differential.

Using Riemann-Roch, since $\ell(K) = g$, the space $H^0(C, (\Omega^1)^g)$ of holomorphic differentials has dimension $g$. If two basis elements have the same order of vanishing, there is a linear combination of the two elements that has a higher order of vanishing. Thus, a basis can be chosen such that the orders of vanishing at $P$ are all different, and $P$ is an ordinary point when these orders of vanishing are $\{0, 1, 2, \ldots, g - 1\}$; otherwise $P$ is a Weierstrass point.

Above, we considered the spaces $\mathcal{L}(K - nP)$. Now, fix $q \in \mathbb{N}$ and consider $\mathcal{L}(qK - nP)$. Analogously, if $\ell(qK - (n-1)P) - \ell(qK - nP) = 1$, then there is a holomorphic $q$-differential with a zero of order $n - 1$ at $P$. Let $H^0(C, (\Omega^1)^q)$ denote the space of holomorphic $q$-differentials on $C$, with its dimension denoted by $d_q$. By Riemann-Roch,

$$d_q = \begin{cases} 
  g & \text{if } q = 1, \\
  (g - 1)(2q - 1) & \text{if } q > 1.
\end{cases}$$  

As before, we take a basis $\{\psi_1, \ldots, \psi_{d_q}\}$ of $H^0(C, (\Omega^1)^q)$ such that

$$\text{ord}_P(\psi_1) < \text{ord}_P(\psi_2) < \cdots < \text{ord}_P(\psi_{d_q}).$$

For $i = 1, \ldots, d_q$, let $n_i = \text{ord}_P(\psi_i) + 1$. The sequence of natural numbers $G(q)(P) = \{ n_1, n_2, \ldots, n_{d_q} \}$ is called the $q$-gap sequence of $P$. With such a
gap sequence, we can calculate the \( q \)-weight of \( P \), denoted \( wt^{(q)}(P) \), given by 
\[
wt^{(q)}(P) = \sum_{i=1}^{d_q} (n_i - i).
\]
We call the point \( P \) a \( q \)-Weierstrass point if \( wt^{(q)}(P) > 0 \).

Given a basis \( \{\psi_1, \ldots, \psi_{d_q}\} \) of \( H^0(C, (\Omega^1)^q) \), where \( \psi_i = f_i(x)dx \) for a holomorphic function \( f_i \) of a local coordinate \( x \) for each \( i \), the Wronskian is the determinant of the following \( d_q \times d_q \) matrix:
\[
W = W(f_1(x), \ldots, f_{d_q}(x)) = \begin{vmatrix}
  f_1(x) & f_2(x) & \cdots & f_{d_q}(x) \\
  f_1'(x) & f_2'(x) & \cdots & f_{d_q}'(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(d_q-1)}(x) & f_2^{(d_q-1)}(x) & \cdots & f_{d_q}^{(d_q-1)}(x)
\end{vmatrix}.
\]
The Wronskian form is \( \Omega_q = W(dx)^m \), for
\[
m = q + (q + 1) + (q + 2) + \cdots + (q + d_q - 1) = \frac{d_q}{2} (2q - 1 + d_q).
\]
The following result is due to Hurwitz. For a proof, see [11].

**Theorem 1** (Hurwitz). \( P \) is a \( q \)-Weierstrass point with weight \( wt^{(q)}(P) = r \)
if and only if \( P \) is a zero of multiplicity \( r \) for the Wronskian form \( \Omega_q \) (or, equivalently, in the support of \( \text{div}(\Omega_q) \)).

Since the Wronskian form is a holomorphic \( m \)-differential, \( \text{div}(\Omega_q) \) is effective. Thus, the \( q \)-Weierstrass points are the support of \( \text{div}(\Omega_q) \), and the sum of the \( q \)-weights of the \( q \)-Weierstrass points is the degree of \( \text{div}(\Omega_q) \), which is \( m(2g - 2) = d_q(2q - 1 + d_q)(g - 1) \). In particular, this means there are a finite number of \( q \)-Weierstrass points.

Let \( \mathcal{W}(C) \) denote the set of all Weierstrass points and \( W_q(C) \) the set of all \( q \)-Weierstrass points on \( C \). \( W_1(C) \), the set of 1-Weierstrass points on \( C \), is exactly the set of Weierstrass points described earlier. We summarize some properties in the following lemma; see [10, Section III.5] for details.

**Lemma 1.** Let \( C \) be a genus \( g \geq 2 \) curve. The following hold:

i) There are \( q \)-Weierstrass points for any \( q \geq 1 \).

ii) For \( q > 1 \)
\[
\sum_{P \in C} wt^{(q)}(P) = g(g - 1)^2(2q - 1)^2.
\]

iii) \( 2g + 2 \leq |W_1(C)| \leq g^3 - g \).

Now we give some results specific to the \( g = 3 \) case.

**Example 1.** For \( g = 3 \) we have \( d_q = 2(2q - 1) \). The total weight is 24 for \( q = 1 \)
and for \( q > 1 \) is
\[
\sum_{P \in C} wt^{(q)}(P) = 12(2q - 1)^2.
\]
Notice that for \( q = 2 \) we have \( d_2 = 6 \) and the total weight is 108. For \( g = 3 \),
\( d_3 = 10 \) and the total weight is 300. In these cases we have, respectively, a \( 6 \times 6 \)
and a \( 10 \times 10 \) Wronskian.
In Section 3, we give the following result for $q = 2$, cf. Remark 2.

**Remark 1.** Let $C$ be a genus 3 hyperelliptic curve. For any point $P \in C$, the 2-weight of $P$ is $\text{wt} (2)(P) \leq 6$. Further, if $\text{wt} (2)(P) = 6$, then $P \in W_1 (C)$. If $P \notin W_1 (C)$, then $\text{wt} (2)(P) \leq 3$.

Let $C$ be a genus 3 hyperelliptic curve defined over $\mathbb{C}$, $K$ its function field, and $G$ be the full automorphism group $G := \text{Aut} (K)$. All such groups $G$ have distinct ramification structures and therefore there is no confusion to denote such locus $H_3 (G)$ for any fixed $G$. In this paper we will make use of the following facts, which are proven in [15, Sections 3 – 5].

**Lemma 2.** Let $C$ be a genus 3 hyperelliptic curve defined over a field $k$ with a non-hyperelliptic involution. Then $C$ is given by the equation $y^2 = x^8 + ax^6 + bx^4 + cx^2 + 1$ for some $a, b, c \in k$.

The dihedral invariants of $C$ are $s_2, s_3, s_4$ where $s_2 = ac$, $s_3 = (a^2 + c^2)b$, and $s_4 = a^4 + c^4$.

**Theorem 2.** Let $C$ be a genus 3 hyperelliptic curve such that $|G| > 2$ and $\dim H (G) \geq 1$. Then, one of the following holds:

i) $G \cong V_4$ and the locus $H(V_4)$ is 3-dimensional. A generic curve in this locus has equation

$$y^2 = A x^8 + \frac{A}{s_4 + 2s_2^2} x^6 + \frac{s_3(A + s_2^2)}{(s_4 + 2s_2^2)^3} x^4 + \frac{s_2}{(s_4 + 2s_2^2)^3} x^2 + \frac{1}{(s_4 + 2s_2^2)^4}$$

where $A$ satisfies $A^2 - s_4 A + s_2^4 = 0$.

ii) $G \cong \mathbb{Z}_2^3$ and the locus $H(\mathbb{Z}_2^3)$ is 2-dimensional. A generic curve in this locus has equation

$$y^2 = s_2 x^8 + s_2^2 x^6 + \frac{1}{2} s_3 x^4 + s_2 x^2 + 1.$$  

iii) $G \cong \mathbb{Z}_2 \times D_8$ and the locus $H(\mathbb{Z}_2 \times D_8)$ is 1-dimensional. A generic curve in this locus has equation

$$y^2 = tx^8 + tx^4 + 1.$$  

iv) $G \cong D_{12}$ and the locus $H(D_{12})$ is 1-dimensional. A generic curve in this locus has equation

$$y^2 = x (tx^6 + tx^3 + 1).$$

v) $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and the locus $H(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is 1-dimensional. A generic curve in this locus has equation

$$y^2 = (tx^4 - 1) (tx^4 + tx^2 + 1).$$

Notice that in each case of the above, it is assumed that the discriminant of the polynomial in $x$ is not zero.
3 2-Weierstrass points for genus 3 hyperelliptic curves

Let $C$ be a hyperelliptic curve of genus $g = 3$ given by $y^2 = f(x)$ with $\text{deg}(f) = 8$. Let $\{\alpha_1, \ldots, \alpha_8\}$ denote the eight distinct roots of $f(x)$, and for each $i$ let $R_i = (\alpha_i, 0)$ denote the corresponding ramification points on $C$. Throughout this section, let $\omega \in \mathbb{C}$ denote any non-root of $f(x)$, and let $P_1^\omega$ and $P_2^\omega$ denote the two (distinct) points above $\omega$. And let $P_1^\infty$ and $P_2^\infty$ denote the two points over $\infty$ in the non-singular model of $C$.

Here are the divisors associated to some functions and the differential $dx$:

- $\text{div}(y) = \sum_{i=1}^{8} R_i - 4(P_1^\infty + P_2^\infty)$,
- $\text{div}(x - \alpha_i) = 2R_i - (P_1^\infty + P_2^\infty)$,
- $\text{div}(x - \omega) = P_1^\omega + P_2^\omega - (P_1^\infty + P_2^\infty)$.
- $\text{div}(dx) = \sum_{i=1}^{8} R_i - 2(P_1^\infty + P_2^\infty)$.

In particular, note that $\text{div}(dx/y) = 2(P_1^\infty + P_2^\infty)$, which is effective. With these divisors, we can explicitly construct bases of $H^0(C, (\Omega^1)^q)$ for all $q \geq 1$. For $d_q = \dim H^0(C, (\Omega^1)^q)$ and $g = 3$, we have $d_1 = 3$ and $d_q = 4q - 2$ for $q \geq 2$.

**Theorem 3.** Let $C$ be a hyperelliptic curve of genus $g = 3$ given by the equation $y^2 = f(x)$ with $\text{deg}(f(x)) = 8$. For any $\beta \in \mathbb{C}$, one has the following bases of holomorphic $q$-differentials.

For $q = 1$, a basis for $H^0(C, (\Omega^1)^1)$ is

$B_{1, \beta} = \{(dx/y), (x - \beta)(dx/y), (x - \beta)^2(dx/y)\}$.

For $q \geq 2$, a basis for $H^0(C, (\Omega^1)^q)$, is

$B_{q, \beta} = \{(x - \beta)^j(dx/y)^q : 0 \leq j \leq 2q\} \cup \{(x - \beta)^k y(dx/y)^q : 0 \leq k \leq 2q - 4\}$.

Note that the only poles occur at infinity, so to prove this, one needs to ensure that the pole orders are different and that there are $d_q$ elements. For a proof, see [4, 2.1, Example (ii)].

Using these bases, we can calculate $q$-weights of ramification points.

**Corollary 1.** Let $R$ be any ramification point on $C$. For $q = 1$, the 1-gap sequence of $R$ is $\{1, 3, 5\}$, so $wt^{(1)}(R) = 3$. For any $q \geq 2$, the $q$-gap sequence of $R$ is $\{1, 3, 5, \ldots, 4q + 1\} \cup \{2, 4, 6, \ldots, 4q - 6\}$, so $wt^{(q)}(R) = 6$.

Hence, for $q \geq 2$ the eight branch points contribute $8 \cdot 6 = 48$ to the total weight of $q$-Weierstrass points on the curve.

In particular, the 2-gap sequence for a branch point is $\{1, 2, 3, 5, 7, 9\}$. The corollary below gives the 2-gap sequence for a non-branch point.
Remark 2. Following from [8], the possible 2-gap sequences of 2-Weierstrass points on a curve of genus 3 are given in [1, Lemma 5]. From this, we see that if \( P_i \) is a non-branch point on a hyperelliptic curve of genus 3, the 2-gap sequence contains 4 and 5, so \( wt(2)(P_i) \leq 3 \).

We can use divisors to characterize the non-branch 2-Weierstrass points.

**Proposition 1.** For the curve \( C \) given by \( y^2 = f(x) \) and non-branch point \( P_i \) above \( x = \omega \), let \( h(x) = f(x)^{1/2} \), chosen so that \( P_i \) lies on the curve \( y = h(x) \). Let

\[
N = \min \left\{ n \in \mathbb{N} : n \geq 5, h^{(n)}(\omega) \neq 0 \right\},
\]

where \( h^{(n)}(x) \) denotes the nth derivative of \( h(x) \). Then \( wt(2)(P_i) = N - 5 \) and \( 5 \leq N \leq 8 \). Thus, \( P_i \) is a 2-Weierstrass point if and only if \( h^{(5)}(\omega) = 0 \).

**Proof.** Let

\[
T_{\omega,4,i}(x) = \sum_{n=0}^{4} \frac{h^{(n)}(\omega)}{n!} (x - \omega)^n,
\]

the fourth degree Taylor polynomial for \( h(x) \) at \( x = \omega \). As in Theorem 3, the set

\[
\{(x - \omega)^j (dx/y)^2 : 0 \leq j \leq 4\} \cup \{(y - T_{\omega,4,i}(x))(dx/y)^2\}
\]

is a basis for \( H^0(C, (\Omega^1)^2) \). The orders of vanishing at \( P_i \) are

\[
\nu_{P_i}((x - \omega)^j (dx/y)^2) = j \text{ for } 0 \leq j \leq 4,
\]

and

\[
\nu_{P_i}((y - T_{\omega,4,i}(x))(dx/y)^2) = \nu_{P_i} \left( \sum_{n=5}^{\infty} \frac{h^{(n)}(\omega)}{n!} (x - \omega)^n \right) = N.
\]

Thus, the 2-gap sequence of \( P_i \) is \( \{1, 2, 3, 4, 5, N + 1\} \), and so \( wt(2)(P_i) = N - 5 \). Thus, \( P_i \) is a 2-Weierstrass point precisely when \( N > 5 \). Finally, since \( wt(5)(P_i) \leq 3 \) by Remark 2, we see \( N \leq 8 \). \( \Box \)

Of course, we can perform these calculations with the Wronskian as well. With the basis \( \{x^j(dx/y)^2 : 0 \leq j \leq 4\} \cup \{y(dx/y)^2\} \) of \( H^0(C, (\Omega^1)^2) \), the Wronskian is

\[
W = W \left( \frac{1}{y^2}, \frac{x}{y^2}, \frac{x^2}{y^2}, \frac{x^3}{y^2}, \frac{x^4}{y^2}, \frac{y}{y^2} \right) = \frac{1}{y^{12}} W(1, x, x^2, x^3, x^4, y).
\]

Thus, \( W = \frac{1}{y^{12}} \left( \prod_{i=0}^{4} i! \right) y^5 \), so the Wronskian form is \( \Omega_2 = W(dx)^{27} \). Since \( y^2 = f(x) \), five derivatives will yield \( y^5 = \phi(x)/y^9 \) for some polynomial \( \phi(x) \) of degree at most 29 (depending on \( f(x) \)). That is,

\[
\Omega_2 = \left( \prod_{i=0}^{4} i! \right) \frac{\phi(x)}{y^{21}} (dx)^{27}.
\]

Thus,
\[
\text{div}(\Omega_2) = \text{div}(\phi(x)) - \text{div}(y^{21}) + \text{div}((dx)^{27})
\]
\[
= \text{div}(\phi(x)) + 6 \sum_{i=1}^{8} R_i + (30 - \deg(\phi))(P_1^\infty + P_2^\infty).
\]

We see that the branch points have 2-weight 6 and the other 2-Weierstrass points are the zeros of \(y^{(5)}\). Note that this result agrees with Corollary 1 and Proposition 1. Also, the points at infinity are 2-Weierstrass points with 2-weight \(30 - \deg(\phi)\).

4 Computation of 2-Weierstrass points

In this section we will study the distributions of 2-Weierstrass points for curves in each family \(H_3(G)\) such that \(\dim H_3(G) > 0\); that is, for curves with full automorphism group isomorphic to \(V_4, \mathbb{Z}_2^3, \mathbb{Z}_2 \times D_8, D_{12}\), or \(\mathbb{Z}_2 \times \mathbb{Z}_4\). These families are described in Theorem 2. For our computations, we make use of the dihedral invariants and the results in [15]. We also need the following elementary result.

**Lemma 3.** Let \(f(x) = \sum_{i=0}^{n} a_i x^i\) and \(g(x) = \sum_{i=0}^{m} b_i x^i\) be polynomials with no common roots. Then, the discriminant of \(f(g(x))\) is given by

\[
\Delta_{f \circ g} = (-1)^{\frac{mn(3mn - 2m - 1)}{2}} a_n^{m-1} b_m^{n(m - m - 1)} \Delta_f \cdot \text{Res}(f(g(x)), g'(x))
\]

Moreover, if \(f(x) = \sum_{i=0}^{n} a_i x^i\) and \(g(x) = x^m\). Then, the discriminant of \(f \circ g\) is

\[
\Delta_{f \circ g} = (-1)^{\frac{mn(3mn - 2m - 1)}{2}} a_n^{m-1} \Delta_f \cdot \text{Res}(f(x^m), mx^{m-1})
\]

**Proof.** The first part of the Lemma is proved by J. Cullinan in [5]. To prove the second part we have to compute \(\text{Res}(f(x^m), mx^{m-1})\). Indeed,

\[
\text{Res}(f(x^m), mx^{m-1}) = ...
\]

This completes the proof. \(\square\)

**Remark 3.** Notice that if \(f(x) = \sum_{i=0}^{n} a_i x^i\) and \(g(x) = x^2\). Then, the discriminant of \(f(x^2)\) is

\[
\Delta(f(x^2)) = (-1)^n \cdot 2^{2n} \cdot a_0 a_n \cdot \Delta_f^2.
\]

4.1 The case Aut \((C) \cong V_4\).

Let \(C\) be a genus 3 hyperelliptic curve with a non-hyperelliptic involution. From Lemma 2, we know that the equation of \(C\) can be given by \(y^2 = f(x)\), for

\[
f(x) = x^8 + ax^6 + bx^4 + cx^2 + 1.
\]

The Wronskian form is

\[
\Omega_2 = \frac{1}{y^{12}} W(1, x, x^2, x^3, x^4, y) = 4320 \frac{x^\Phi(x^2)}{y^{21}} (dx)^{27},
\]
where $\Phi(x)$ is a polynomial of degree 14 which depends on $a, b, c$. We don’t display its coefficients since they are large.

Let $\Phi(x^2) = \sum_{i=0}^{14} c_i x^{2i}$. The leading coefficient $c_{14}$ and the constant term $c_0$ are

$$c_{14} = -3a^3 + 12ab - 24c \quad \text{and} \quad c_0 = -(3c^3 + 12bc - 24a).$$

In general, the coefficients $c_i$ and $c_{14-i}$ differ by a permutation of $a$ and $c$ and a factor of $-1$. In other words the permutation of the curve $\tau_1: (x, y) \to (\frac{1}{x}, \frac{y}{x^2})$ which permutes coefficients $a$ and $c$ of the curve given in Eq. (8) acts on the coefficients of $\Phi(x)$ by

$$\tau_1(c_i) = -c_{14-i}.$$

Computing the discriminant $\Delta(\Phi, x)$ we get the following factors:

$$\Delta = 2^{16} c_0 c_{14} \cdot g(a, b, c)^2 \cdot \Delta(f, x)^{28},$$

where $g(a, b, c)$ is a degree 24, 28, 24 polynomial in terms of $a, b, c$ respectively. We know that $\Delta(f, x) \neq 0$. Let us assume that $c_0 c_{14} \neq 0$. Then, the 2-Weierstrass points are those when $g(a, b, c) = 0$. The polynomial $g(a, b, c)$ can be easily computed. However, the triples $(a, b, c)$ do not correspond uniquely to the isomorphism classes of curves. Naturally we would prefer to express such result in terms of the dihedral invariants $s_2, s_3, s_4$. One can take the equations $g(a, b, c) = 0$ and three equations from the definitions of $s_2, s_3, s_4$ and eliminate $a, b, c$. It turns out that this is a challenging task computationally.

Hence, we continue with the following approach. From Theorem 2, we know that a curve $C$ with $\text{Aut}(C) \cong V_4$ is isomorphic to a curve with equation

$$y^2 = Ax^8 + \frac{A}{s_4 + 2s_2^2} x^6 + \frac{s_3(A + s_2^2)}{(s_4 + 2s_2^2)^3} x^4 + \frac{s_2}{(s_4 + 2s_2^2)^3} x^2 + \frac{1}{(s_4 + 2s_2^2)^4}$$

where $A$ satisfies

$$A^2 - s_4 A + s_2^4 = 0,$$

for some $(s_2, s_3, s_4) \in k^3 \setminus \{\Delta_{s_2, s_3, s_4} = 0\})$.

The numerator in the Wronskian form is a degree 29 polynomial in $x$ written as $x \phi(x^2)$. From Lemma 3, it is enough to compute the discriminant of the polynomial $\phi(t)$, where $t = x^2$. This is a degree 14 polynomial. Its discriminant is a polynomial $G(A, s_2, s_3, s_4)$ in terms of $s_2, s_3, s_4$ and $A$. Then, the relation between $s_2, s_3, s_4$ is obtained by taking the resultant $\text{Res}(G, A^2 - s_4 A + s_2^4, A)$. The result is quite a large polynomial in terms of $s_2, s_3, s_4$. Fortunately, it turns out that the remaining cases are much easier.

Remark 4. The equivalent statement of Theorem 2, i) is proved in [3] for any genus $g \geq 3$. Also the $s$-invariants are defined for every $g > 3$. Hence, this method will work for any $g > 3$. With some modifications the method works for all superelliptic curves as in [12].
4.2 The case \( \text{Aut}(C) \cong \mathbb{Z}_2^3 \).

**Proposition 2.** Let \( C \) be a genus 3 hyperelliptic curve with full automorphism group \( \mathbb{Z}_2^3 \). Then, \( C \) has non-branch 2-Weierstrass points of weight greater than one if and only if its corresponding dihedral invariants \( s_2, s_3, s_4 \) satisfy Eq. (10)

\[
\Delta = (-784 s_2^2 + 16 s_2^4 + 56 s_2 s_3 - s_3^3) G(s_2, s_3) = 0
\]

where

\[
G = 617400 s_2^3 + 180 s_2 (315560 + 871 s_2) s_3^3 + 2s_2^2 (4023040 s_2 + 97071440 + 310774 s_3) s_3
\]

\[+ s_2^3 (9 s_2^2 - 325124728 s_2 + 31937525760 + 5011112 s_3^2) s_3^3 + 8s_2^5 (-9204034560 s_2^2
\]

\[- 10548636 s_2^4 + 15861 s_3^3 + 41193015800) s_3^5 - 16 s_2^6 (22041513 s_3^3 + 59872104320 s_2
\]

\[+ 11 s_2^7 - 453327496 s_2^5 - 193117539328) s_3^7 - 256 s_2^8 (-2870647262 s_2^3 - 73789542800
\]

\[+ 34875810752 s_2^4 - 478935992 s_2^2 + 541999 s_2^2 s_3^3 - 256 s_2^2 (4s_2^2 - 41807037944 s_2
\]

\[+ 2826159858 s_2^2 s_3^3 - 10769334 s_2^3 + 283441853184 s_2^2 - 365995658888) s_3^5 - 2048 s_2^5 (308705831 s_2
\]

\[+ 281449598 s_2^4 - 39227605228 s_2^2 - 129966220704 s_2^3 + 31711 s_2^3 - 175618597664) s_3
\]

\[+ 4096 s_2^2 (458707653 s_2^2 - 4058869 s_2^2 + 7 s_2^6 - 1664926455 s_3^2 + 21435836038 s_2
\]

\[+ 859825951600 s_3^2 + 14627327488)
\]

**Proof.** The equation of the curve is given by Eq. (2), for \( s_2, s_3 \neq 0, 4 \). For \( \Omega_2 = \frac{1}{y^{12}} W(1, x, x^2, x^3, x^4, y)(dx)^{27} \), we find

\[
\Omega_2 = \frac{x (s_2 x^4 - 1) g(t)}{(4 s_2^3 x^8 + 4 s_2^3 x^6 + 2 s_3 x^2 + 4 s_2 x^2 + 4)^{21/2}} (dx)^{27}
\]

where \( g(t) = \sum_{i=0}^{12} c_i \cdot t^i \) is a degree 12 polynomial for \( t = x^2 \) with the following coefficients:

\[
c_0 = 12 s_2, \\
c_1 = -4 (-7 s_2 + 4 s_2^2 - 28 s_2), \\
c_2 = 12 s_2 (22 s_2 - 3 s_3), \\
c_3 = -4 (-28 s_2^2 + 9 s_2^2 + 4 s_3^2 + 29 s_2 s_3), \\
c_4 = 5 s_2 (-s_3^2 - 1180 s_2^2 + 16 s_3^3 - 376 s_2 s_3), \\
c_5 = 3 s_3^3 - 24 s_2 s_3^2 - 48 s_3 s_3^2 - 1696 s_2^3 - 1568 s_3^3 - 536 s_2^2 s_3, \\
c_6 = -26 s_2^2 (20 s_2 s_3 + 152 s_2^2 - s_3^2 + 16 s_3^3), \\
c_7 = s_3 (3 s_3^2 - 24 s_2 s_3^2 - 48 s_3 s_3^2 - 1696 s_2^2 - 1568 s_3^3 - 536 s_2^2 s_3), \\
c_8 = s_3^2 (-s_3^2 - 1180 s_2^2 + 16 s_3^3 - 376 s_2 s_3), \\
c_9 = -4 s_3^3 (-28 s_2^2 + 9 s_2^2 + 4 s_3^2 + 29 s_2 s_3), \\
c_{10} = 12 s_3^2 (22 s_2 - 3 s_3), \\
c_{11} = -4 s_3^2 (-7 s_3 + 4 s_3^2 - 28 s_2), \\
c_{12} = 12 s_3^2.
\]
We note that $c_{12-i} = s_2^{6-i} c_i$ for $i = 0, \ldots, 6$.

The discriminant of $g(t)$ factors as is written in Eq. (10). Each component can be expressed in terms of the absolute invariants $t_1, \ldots, t_6$ as defined in [13]. Since they are large expressions we do not display them.

The following determines a nice family of curves with automorphism group $\mathbb{Z}_2^3$.

**Lemma 4.** Let $C$ be a genus 3 curve with equation

$$y^2 = \frac{t^4}{256} x^8 + \frac{t^4}{256} x^6 + \frac{t^2}{32} (t + 28) x^4 + \frac{t^2}{16} x^2 + 1$$

such that $t \in \mathbb{C} \setminus \{-16, 0, 48\}$. Then, $\text{Aut}(C) \cong \mathbb{Z}_2^3$ and $C$ has $N_r$ 2-Weierstrass points of weight $r$ as described in the table below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = -112/3$</td>
<td>24</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>$t = 14 \pm 14\sqrt{-15}$</td>
<td>16</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>$t \in \mathbb{C} \setminus {-16, 0, 48, -112/3, 14 \pm 14\sqrt{-15}}$</td>
<td>48</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

**Proof.** Let us assume that the dihedral invariants satisfy the first factor of the Eq. (10). Since this is a rational curve we can parametrize it as follows:

$$s_2 = \frac{1}{16} t^2, \quad s_3 = \frac{1}{16} (t + 28) t^2.$$ 

In this case the curve $C$ becomes

$$y^2 = \frac{t^4}{256} x^8 + \frac{t^4}{256} x^6 + \frac{t^2}{32} (t + 28) x^4 + \frac{t^2}{16} x^2 + 1$$

with discriminant $\Delta = t^{28} (t - 48)^4 (t + 16)^6 \neq 0$. The Wronskian form is

$$\Omega_2 = \frac{x (t x^2 + 4) (t x^2 - 4)^3}{(t^3 x^8 + t^4 x^6 + 8 t^3 x^4 + 224 t^2 x^2 + 16 t^2 x^2 + 256)^{21/2}} \cdot \frac{(t^2 x^4 + 24 t x^2 + 16) (3 t^2 x^2 + 16) (3 t^2 x^2 + 16)}{t^8 x^6 - 4 t^6 (-16 t + t^2 - 896) x^{14} + 16 t^5 (-10 t^3 + 3584 + 220 t) x^{12} - 192 t^4 (9 t^2 + 2688 + 368 t) x^{10} - 512 t^3 (8 t^2 + 3584 + 23 t^2) x^8 - 3072 t^2 (9 t^2 + 2688 + 368 t) x^6 - 4096 t (5 t^2 + 3584 + 220 t) x^4 + (1468096 + 262144 t - 16384 t^2) x^2 + 196608) x^2}.$$ 

Hence, the curve has four 2-Weierstrass points of weight 3 which come from the two roots of the factor $(tx^2 - 4)^3 = 0$. Note that $x = 0$ is a root of order 1, so the points $(0, \pm 1)$ have weight 1. Removing these factors as well as the denominator, we obtain a polynomial in $x^2$ which we can write as

$$h(x^2) = \Omega_2 \cdot \frac{(t^4 x^8 + t^4 x^6 + 8 t^3 x^4 + 224 t^2 x^2 + 16 t^2 x^2 + 256)^{21/2}}{x (t x^2 - 4)^3},$$

for $\deg(h(x^2)) = 11$. We now check $h(x)$ for multiple roots. One finds that

$$\Delta(h, x) = 2^{28} 3^9 t^3 \cdot t^{93} (16 + t)^{14} (3 t + 112)^6 (t - 48)^6 (t^2 - 28 t + 3136)^4.$$
Since we do not consider the cases where \( t = 0, -16, 48 \), to make \( \Delta(h, x) = 0 \), we look at \( t = \frac{-112}{3} \) and \( t = 14 \pm \sqrt{-15} \). When \( t = \frac{-112}{3} \), then

\[
h(x) = c(28x - 3)(81 + 168x + 784x^2)(784x^2 - 504x + 9)^3(3 + 56x + 2352x^2),
\]

for some constant \( c \). Thus, \( h(x) \) has two roots of order 3 and five roots of order 1. Going back to \( \Omega_2 \), these roots lead to eight 2-Weierstrass points of weight 3 and twenty 2-Weierstrass points of weight 1.

When \( t = 14 \pm \sqrt{-15} \), \( h(x) \) has four roots of order 2 and 3 roots of order 1. These lead to sixteen 2-Weierstrass points with weight 2 and twelve 2-Weierstrass points with weight 1.

Note that for any \( t \neq 0 \), the numerator of \( \Omega_2 \) is a polynomial of degree 29, so the two points at infinity are 2-Weierstrass points with weight 1.

The other component is also a genus 0 curve and the same method as above can also be used here.

**Theorem 4.** The locus in \( H_3 \) of curves with full automorphism group \( \mathbb{Z}_3 \) which have 2-Weierstrass points is a 1-dimensional variety with two irreducible components. Each component is a rational family. The equation of a generic curve in each family is given in terms of the parameter \( t \).

Next, we consider the 1-dimensional loci. There are three cases of groups which correspond to 1-dimensional loci in \( H_3 \), namely the groups \( \mathbb{Z}_2 \times \mathbb{D}_8 \), \( \mathbb{D}_{12} \), and \( \mathbb{Z}_2 \times \mathbb{Z}_4 \). Let us first consider the case \( \text{Aut} (C) \cong \mathbb{Z}_2 \times \mathbb{D}_8 \).

### 4.3 The case \( \text{Aut} (C) \cong \mathbb{Z}_2 \times \mathbb{D}_8 \).

**Proposition 3.** Let \( C \) be a genus 3 hyperelliptic curve with full automorphism group \( \mathbb{Z}_2 \times \mathbb{D}_8 \). Then \( C \) is isomorphic to a curve of the form \( y^2 = tx^8 + tx^4 + 1 \) for some \( t \neq 0, 4 \). For any other \( t \neq -140, -980/3 \), \( C \) has \( N_r \) 2-Weierstrass points of weight \( r \) as described in the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N_1 )</th>
<th>( N_2 )</th>
<th>( N_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 196 )</td>
<td>24</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>( t = -196/5 )</td>
<td>16</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>( t \in \mathbb{C} \setminus {0, 4, -140, -980/3} )</td>
<td>48</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

*Proof.* In this case the curve has equation \( y^2 = tx^8 + tx^4 + 1 \), with discriminant \( \Delta = 2^{16} \cdot t^7 (t - 4)^4 \neq 0 \), where \( t = -28 \frac{3t^4 + 28}{t^4 - 4} \); see [15, Lemma 7]. \( \Omega_2 \) is the product of the two following factors,

\[
\frac{34560 t(t - 4) x^3 (tx^8 - 1)}{(tx^8 + tx^4 + 1)^{21/2}}(7 t^2 x^{16} - 18 t^2 x^{12} + 3 t^2 x^8 - 98 tx^8 - 18 tx^4 + 7) (dx)^{27}.
\]

Since \( x = 0 \) is of multiplicity 3, then the points \((0, \pm 1)\) have each weight 3.
The other factors of the Wronskian, namely
$$(tx^8 - 1) (7t^2 x^{16} - 18 t^2 x^{12} + 3t^2 x^8 - 98tx^8 - 18tx^4 + 7)$$
have double roots if the discriminant is zero. This happens if $t = 196$ or $t = -\frac{196}{15}$. If $t = 196$ then
$$\Omega_2 = 9103933440 \frac{x^3 (14x^4 + 1) (196x^8 - 476x^4 + 1) (14x^4 - 1)^3}{(196x^8 + 196x^4 + 1)^{9/2}} (dx)^{27}.$$  
Hence, there are 24 points of weight 1, and 8 other points of weight 3 which come from the roots of $14x^4 = 1$.
If $t = -\frac{196}{15}$, then the curve $C$ becomes
$$y^2 = -\frac{196}{15} x^8 - \frac{196}{15} x^4 + 1$$
and
$$\Omega_2 = -614515507200000 \frac{x^3 (15 + 196x^8) (-15 - 252x^4 + 196x^8)^2}{(-15 (14x^4 + 15)(14x^4 - 1))^{9/2}} (dx)^{27}.$$  
Hence, there are 16 points of weight 1 and 16 points of weight 2.

Finally, observe that since the numerator of $\Omega_2$ is a polynomial in $x$ of degree 27, the two points at infinity have 2-weight equal to 30 $- 27 = 3$.

4.4 The case $\text{Aut}(C) \cong D_{12}$

Let us now assume that $C$ has full automorphism group $D_{12}$. In this case the curve has equation
$$y^2 = x (tx^6 + tx^3 + 1)$$
for $t = \frac{7}{2} \frac{5x^4 + 7}{x^4 - 2}$ and discriminant $\Delta = 3^6 \cdot 7^5 (t - 4)^3 \neq 0$; see [15, Lemma 8] for details.

In particular, for a curve $C$ given by the equation $y^2 = f(x)$, with $\deg(f) = 7$, there is one point at infinity, which is singular. This point is a branch point, and in the desingularization remains as one point, which we will denote here by $P^\infty$. Let $\{\alpha_i\}$ denote the roots of $f(x)$, and $R_i = (\alpha_i, 0)$ the affine branch points. Let $\omega \in \mathbb{C} \setminus \{\alpha_i\}$ and let $P_1^\omega$ and $P_2^\omega$ denote the points over $\omega$. One has the following divisors.

- $\text{div}(y) = \left( \sum_{i=1}^7 R_i \right) - 7P^\infty$,
- $\text{div}(x - \omega) = P_1^\omega + P_2^\omega - 2P^\infty$,
- $\text{div}(x - \alpha_i) = 2R_i - 2P^\infty$. 

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\[ \text{div}(dx) = \left( \sum_{i=1}^{7} R_i \right) - 3P^\infty, \]

Working with these divisors, as in Theorem 3 one finds that a basis of holomorphic 2-differentials is given by
\[
\{(x - \beta)^j(dx/y)^2 : 0 \leq j \leq 4\} \cup \{y(dx/y)^2\},
\]
for any \( \beta \in \mathbb{C} \). Letting \( \beta = \alpha_i \), the 2-Weierstrass weight for the affine branch point \( R_i \) is 6. And using any value of \( \beta \), one finds orders of vanishing 8, 6, 4, 2, 0, 1 at \( P^\infty \), so \( wt^2(P^\infty) = 6 \) as well.

**Proposition 4.** Let \( C \) be a genus 3 hyperelliptic curve with full automorphism group \( D_{12} \). By [15, Lemma 8], \( C \) has equation \( y^2 = x(t x^6 + t x^3 + 1) \). Then, \( C \) has non-branch points with 2-Weierstrass weight greater than 1 if and only if \( t = -\frac{49}{8} \) or \( t = \frac{1787}{8} \pm \frac{621}{4} \sqrt{2} \).

In particular, for each value of \( t \), \( C \) has \( N_r \) 2-Weierstrass points of weight \( r \) as described in the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N_1 )</th>
<th>( N_2 )</th>
<th>( N_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = -\frac{49}{8} )</td>
<td>24</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>( t = \frac{1787}{8} \pm \frac{621}{4} \sqrt{2} )</td>
<td>36</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>( t \in \mathbb{C} \setminus {0, 4, -\frac{49}{8}, \frac{1787}{8} \pm \frac{621}{4} \sqrt{2} } )</td>
<td>60</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Proof.** In this case the curve has equation \( y^2 = x(t x^6 + t x^3 + 1) \) for \( t = \frac{7}{2} \frac{54t+7}{t^4-\frac{9}{2}} \) and discriminant \( \Delta = 3^6 \cdot t^5 (t - 4)^3 \neq 0 \); see [15, Lemma ] for details. The Wronskian is
\[
\Omega_2 = -135 \frac{(t x^6 - 1)}{(x(t x^6 + t x^3 + 1))^{9/2}} \left( 7 t^4 x^{24} + 28 t^4 x^{21} - 336 t^4 x^{18} + 1216 t^4 x^{15} - 128 t^4 x^{12} + 1540 t^3 x^{18} - 4668 t^3 x^{15} + 6672 t^3 x^{12} + 1216 t^3 x^9 - 24150 t^2 x^{12} - 4668 t^2 x^9 - 336 t^2 x^6 + 1540 t x^6 + 28 t x^3 + 7 \right) (dx)^{27}.
\]

Its discriminant factors as
\[
\Delta(\Omega_2, x) = t^{145} (t - 4)^{42} \left( 64 t^2 - 28592 t + 108241 \right)^9 \left( 8 t + 49 \right)^{12}.
\]

Since \( t \neq 0, 4 \), then the \( \Omega_2 \) form has multiple roots if and only if
\[
t = -\frac{49}{8}, \quad t = \frac{1787}{8} + \frac{621}{4} \sqrt{2}, \quad \text{or} \quad t = \frac{1787}{8} - \frac{621}{4} \sqrt{2}.
\]

For each one of these values of \( t \), \( \Omega_2 \) has multiple zeros and hence 2-Weierstrass points of weight at least 2.

For \( t = -\frac{49}{8} \) the numerator of \( \Omega_2 \) is the polynomial
\[
(49 x^6 + 8) (49 x^6 + 616 x^3 - 8) (49 x^6 - 140 x^3 - 8)^3,
\]
which has six roots of multiplicity 3. Hence, the curve $y^2 = x \left(-\frac{49}{8} x^6 - \frac{49}{8} x^3 + 1\right)$ has twelve 2-Weierstrass points of weight 3. There are twelve simple roots of this polynomial and therefore twenty-four points of weight 1.

For $t = \frac{1787}{8} \pm \frac{621}{4} \sqrt{2}$, the numerator of $\Omega_2$ is the polynomial

$$
\left(108241 x^6 - (60536 \pm 35532 \sqrt{2}) x^3 + (14296 \pm 9936 \sqrt{2})\right)^2 g(x),
$$

for $g(x)$ a degree-18 polynomial with coefficients in $\mathbb{Z}[\sqrt{2}]$ and distinct roots. The numerator of $\Omega_2$ has six double roots which lead to twelve 2-Weierstrass points of weight 2. The remaining eighteen roots are single roots, leading to thirty-six 2-Weierstrass points of weight 1.

Finally, note that in both cases, the 2-Weierstrass points we have calculated make a contribution of 60 to the total weight. The eight branch points (including the point at infinity) each have 2-Weierstrass weight 6, thus making a contribution of 48 to the total weight, which is 108.

**Remark 5.** Notice that in the case of the curve $y^2 = x \left(-\frac{49}{8} x^6 - \frac{49}{8} x^3 + 1\right)$, even though the curve is defined over $\mathbb{Q}$ the 2-Weierstrass points are defined over a degree 6 extension of $\mathbb{Q}$.

### 4.5 The case $\text{Aut}(C) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$

**Proposition 5.** Let $C$ be a genus 3 hyperelliptic curve with full automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_4$. Then, $C$ has 2-Weierstrass points if and only if $C$ is isomorphic to one of the curves $y^2 = (tx^4 - 1) \left(tx^4 + tx^2 + 1\right)$, for $t = -8$ or it is a root of

$$
t^8 + 600822 t^7 + 71378609 t^6 + 4219381768 t^5 + 85080645104 t^4 - 2272444082944 t^3 + 164801368352 t^2 - 50330309965824 t + 56693912375296 = 0.
$$

(11)

In the first case, the curve has two 2-Weierstrass points of weight 3.

**Proof.** The equation of this curve is given by

$$
y^2 = (tx^4 - 1) \left(tx^4 + tx^2 + 1\right)
$$

with discriminant $\Delta = -2^{12} \cdot t^{14}(t - 4)^6$. The numerator of the Wronskian is a degree 29 polynomial in $x$, given by $x\phi(x)$, where

$$
\phi(x) = (24 t^7 - 3 t^6) x^{28} + (-4 t^7 + 4 t^8 + 224 t^6) x^{26} + (63 t^7 + 504 t^6) x^{24} + 1368 t^6 x^{22} + (4 t^7 + 2888 t^6 + 1045 t^5) x^{20} + (3360 t^4 + 588 t^6 + 3780 t^5) x^{18} + (3375 t^5 + 108 t^6 + 5544 t^4) x^{16} + (7632 t^4 + 1056 t^5) x^{14} + (5544 t^3 + 3375 t^4 + 108 t^5) x^{12} + (3780 t^3 + 3360 t^2 + 588 t^4) x^{10} + (1045 t^3 + 4 t^4 + 2888 t^2) x^8 + 1368 t^2 x^6 + (504 t + 63 t^2) x^4 + (-4 t + 4 t^2 + 224) x^2 + 24 - 3 t.
$$
Its discriminant is
\[
\Delta = t^{275} (t - 4)^90 (t - 8)^4 \left( t^8 + 600822 t^7 + 71378609 t^6 + 4219381768 t^5 + 85080645104 t^4 \\
-2272444082944 t^3 + 1648013638352 t^2 - 5033030965824 t + 56693912375296 \right)^4.
\]

Hence, for \( t = 8 \) or \( t \) satisfying the degree 8 polynomial the corresponding curve has 2-Weierstrass points. In the first case, \( t = 8 \), the curve becomes
\[
y^2 = (8 x^4 - 1) \left( 8 x^4 + 8 x^2 + 1 \right).
\]

The Wronskian \( \Omega_2 \) has \( x = 0 \) as a triple root. Hence, the points \((0, i)\) and \((0, -i)\), for \( i^2 = -1 \) are 2-Weierstrass points of weight 3. If \( t \) is a root of the second factor, then the Galois group of this degree 8 polynomial is \( S_8 \) and therefore not solvable by radicals. \( \square \)

Summarizing we have the following theorem.

**Theorem 5.** Let \( G \) be a group such that \( |G| > 4 \) and \( \mathcal{H}(G) \) is a locus of dimension \( d > 0 \) in \( \mathcal{H}_3 \). Let \( C \) be a curve in the locus \( \mathcal{H}(G) \), \( s_2, s_3, s_4 \) its corresponding dihedral invariants and \( \pi : C \to \mathbb{P}^1 \) the hyperelliptic projection. Then each branch point of \( \pi \) has 2-weight 6 and one of the following holds:

i) If \( \text{Aut} \ (C) \cong \mathbb{Z}_2^3 \), then \( C \) has non-branch 2-Weierstrass points of weight greater than one if and only if \( s_2, s_3, s_4 \) satisfy Eq. (10).

ii) If \( \text{Aut} \ (C) \cong \mathbb{Z}_2 \times D_8 \) then \( C \) has at least four non-branch 2-Weierstrass points of weight 3. Moreover, if \( C \) is isomorphic to the curve
\[
y^2 = tx^8 + tx^4 + 1,
\]
for \( t = 196 \) (resp. \( t = -\frac{196}{19} \)) then \( C \) has in addition 8 other points of weight 3 (resp. 16 points of weight 2).

iii) If \( \text{Aut} \ (C) \cong D_{12} \) then \( C \) has non-branch 2-Weierstrass points with weight greater than one if and only if \( C \) is isomorphic to one of the curves
\[
y^2 = x(tx^6 + tx^3 + 1),
\]
for \( t = -\frac{49}{8} \) or \( t = \frac{1728}{8} + \frac{621}{4} \sqrt{2} \). In the first case, the curve has twelve 2-Weierstrass points of weight 3 and in the other two cases twelve 2-Weierstrass points of weight 2.

iv) If \( \text{Aut} \ (C) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \) then \( C \) has 2-Weierstrass points if and only if \( C \) is isomorphic to one of the curves \( y^2 = (tx^4 - 1) \left( tx^4 + tx^2 + 1 \right) \), for \( t = -8 \) or it is a root of
\[
t^8 + 600822 t^7 + 71378609 t^6 + 4219381768 t^5 + 85080645104 t^4 \\
-2272444082944 t^3 + 1648013638352 t^2 - 5033030965824 t + 56693912375296 = 0.
\]
In the first case, the curve has two 2-Weierstrass points of weight 3.
5 Concluding remarks

In this paper we explicitly determined the 2-Weierstrass points of genus 3 hyperelliptic curves with extra automorphisms. Similar methods can be used for 3-Weierstrass points even though the computations are longer and more difficult.

The method, especially the result of Lemma 3 can be used for $q$-Weierstrass points of all superelliptic curves. The automorphism groups of such curves are fully classified and their equations are $y^n = f(x^m)$ for different values of $n$ and $m$, see [6,7,12] among other papers.

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References