2-Weierstrass points of genus 3 hyperelliptic curves with extra involutions

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Abstract

We consider families of curves with extra automorphisms in $\mathcal{H}_3$, the moduli space of smooth hyperelliptic curves of genus $g=3$. Such families of curves are explicitly determined in terms of the absolute invariants of binary octavics. For each family of positive dimension where $|\mathrm{Aut}\ (C)| > 4$, we determine the possible distributions of weights of 2-Weierstrass points.

1 Introduction

In this paper, we focus primarily on 2-Weierstrass points on hyperelliptic curves of genus 3 with extra automorphisms. Our goal is to classify the 2-Weierstrass points in terms of the coordinates of the hyperelliptic moduli $\mathcal{H}_3$. This follows [9], where the authors classify 3-Weierstrass points on curves of genus 2 with extra automorphisms.

We fix a group G which acts on a genus 3 hyperelliptic curve as a full automorphism group. For a given signature, the locus of curves with automorphism group G is an irreducible locus in the hyperelliptic moduli $\mathcal{H}_3$. We focus on the groups G which determine a family of dimension $d > 0$ in $\mathcal{H}_3$. A complete list of such groups, signatures, and inclusions among the loci is given in [12]. The locus of curves C with the Klein four-group $V_4 \hookrightarrow \mathrm{Aut}\ (C)$ is a 3-dimensional locus in $\mathcal{H}_3$. The appropriate invariants in this case are the dihedral invariants $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ as defined in [15]. All the other cases can be described directly in terms of absolute invariants $t_1, \dots, t_6$ or their equivalents as defined in [13]. Our main result is the following:

Theorem. *Let G be a group with $|G| > 4$ such that the corresponding locus $\mathcal{H}_3(G)$ has dimension $d > 0$ in $\mathcal{H}_3$. Let $\mathfrak{p} \in \mathcal{H}_3(G)$, $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ its corresponding dihedral invariants, and C a representative for $\mathfrak{p}$. Then each branch point of the hyperelliptic projection $\pi : C \to \mathbb{P}^1$ has 2-weight 6 and one of the following holds:*

i) If Aut $(C) \cong \mathbb{Z}_2^3$, then C has non-branch 2-Weierstrass points of weight greater than one if and only if $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ satisfy

$$\left(16\,{\mathfrak{s}_2}^3 - 784\,{\mathfrak{s}_2}^2 + 56\,\mathfrak{s}_2\mathfrak{s}_3 - {\mathfrak{s}_3}^2\right)\, G(\mathfrak{s}_2, \mathfrak{s}_3) = 0,$$

where $G(\mathfrak{s}_2, \mathfrak{s}_3)$ is given as in Eq. (10).

ii) If Aut $(C) \cong \mathbb{Z}_2 \times D_8$, then C has exactly four non-branch 2-Weierstrass points of weight 3, unless C is isomorphic one of the curves

$$y^2 = tx^8 + tx^4 + 1,$$

for $t = 196$ (resp. $t = -\frac{196}{15}$) in which case it has in addition 8 other points of weight 3 (resp. 16 points of weight 2).

iii) If Aut $(C) \cong D_{12}$ then C has non-branch 2-Weierstrass points with weight greater than one if and only if C is isomorphic to one of the curves

$$y^2 = x(tx^6 + tx^3 + 1),$$

for $t = -\frac{49}{8}$ or $t = \frac{1728}{8} \pm \frac{621}{4}\sqrt{2}$. In the first case, the curve has twelve 2-Weierstrass points of weight 3 and in the other two cases twelve 2-Weierstrass points of weight 2.

iv) If Aut $(C) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ then C has 2-Weierstrass points if and only if C is isomorphic to one of the curves $y^2 = \left(tx^4 - 1\right)\left(tx^4 + tx^2 + 1\right)$, for $t = -8$ or it is a root of

$$t^8 + 600822\,t^7 + 71378609\,t^6 + 4219381768\,t^5 + 85080645104\,t^4 - 2272444082944\,t^3 + 16480136388352\,t^2 - 50330309965824\,t + 56693912375296 = 0.$$

In the first case, the curve has two 2-Weierstrass points of weight 3.

This paper is organized as follows. In Section 2 we give some basic preliminaries for Weierstrass points and their weights. We also give an introduction to genus 3 hyperelliptic curves with extra involutions and their dihedral invariants.

In Section 3 we focus on the q-Weierstrass points of genus 3 curves for $q = 1$ and 2. We show how to construct of basis for the space of holomorphic q-differentials. We use this basis for our computations in Section 4.

In Section 4 we compute the Wronskian Ω_q for $q = 2$ in terms of coordinates in the hyperelliptic moduli for each case when the group G has size $|G| > 2$ and the G-locus has dimension $d > 0$. Computations are challenging, especially in the case of $G \cong V_4$. In this case we make use of the dihedral invariants $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ which make such computations possible.

2 Preliminaries

Below we give definitions of ordinary and higher-order Weierstrass points and establish some of the basic facts about these points on curves over $\mathbb{C}$. For more information, the reader is encouraged to see [2, 10, 11, 14].

Let C be a non-singular projective curve over $k = \mathbb{C}$ of genus g, and let $k(C)$ be the associated function field. For any $f \in k(C)$, $\operatorname{div}(f)$ denotes the divisor associated to f, $\operatorname{div}(f)_0$ and $\operatorname{div}(f)_\infty$ respectively the zero and pole divisors of f. For any divisor D on C, we have $D = \sum_{P \in C} n_P P$ for $n_P \in \mathbb{Z}$ with almost all $n_P = 0$. Let $\nu_P(D) = n_P$, and let $\nu_P(f) = \nu_P(\operatorname{div}(f))$.

For any divisor D on C, let $\mathcal{L}(D) = \{f \in k(C) : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$ and $\ell(D) = \dim_k(\mathcal{L}(D))$. By the Riemann-Roch theorem, for any canonical divisor K, we have

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

Since the degree of a canonical divisor is $2g - 2$, and since $\mathcal{L}(D) = \{0\}$ for any divisor D with negative degree, if $\deg(D) \geq 2g - 1$, then $\deg(K - D) < 0$, so $\ell(K - D) = 0$. Thus, if $\deg(D) \geq 2g - 1$, then $\ell(D) = \deg(D) + 1 - g$.

Let P be a degree 1 point on C. Consider the chain of vector spaces

$$\mathcal{L}(0) \subseteq \mathcal{L}(P) \subseteq \mathcal{L}(2P) \subseteq \mathcal{L}(3P) \subseteq \cdots \subseteq \mathcal{L}\left((2g-1)P\right).$$

Since $\mathcal{L}(0) = k$, we have $\ell(0) = 1$. And $\ell\left((2g-1)P\right) = g$. We obtain the corresponding non-decreasing sequence of integers

$$\ell(0) = 1, \ell(P), \ell(2P), \ell(3P), \ldots, \ell\left((2g-1)P\right) = g.$$

If $\ell(nP) = \ell((n-1)P)$, then we call n a *Weierstrass gap number*. Weierstrass' *Lückensatz* ("gap theorem"), from the 1860s, states that for any point P there are exactly g Weierstrass gap numbers. If the gap numbers are $1, 2, \ldots, g$, then P is an *ordinary point*. Otherwise, we call P a *Weierstrass point*. (Equivalently, we call P a Weierstrass point if $\ell(gP) > 1$, which is the case when there is some $f \in k(C)^\times$ with $\mathrm{div}(f)_\infty = mP$ for $1 < m \leq g$.)

An integer n is a gap number when $\ell(nP) = \ell((n-1)P)$, which occurs exactly when $\ell(K - (n-1)P) - \ell(K - nP) = 1$, for K a canonical divisor of C. This means there is some $f \in k(C)^\times$ such that $\mathrm{div}(f) + K \geq (n-1)P$ but $\ngeq nP$. Thus, there's a differential dx with $\mathrm{div}(dx) = K$ so that $\nu_P(f \cdot dx) = n - 1$. Further, $f \cdot dx$ is a holomorphic differential.

Using Riemann-Roch, since $\ell(K) = g$, the space $H^0(C, (\Omega^1))$ of holomorphic differentials has dimension g. If two basis elements have the same order of vanishing, there is a linear combination of the two elements that has a higher order of vanishing. Thus, a basis can be chosen such that the orders of vanishing at P are all different, and P is an ordinary point when these orders of vanishing are $\{0, 1, 2, \ldots, g-1\}$; otherwise P is a Weierstrass point.

Above, we considered the spaces $\mathcal{L}(K - nP)$. Now, fix $q \in \mathbb{N}$ and consider $\mathcal{L}(qK - nP)$. Analogously, if $\ell(qK - (n-1)P) - \ell(qK - nP) = 1$, then there is a holomorphic q-differential with a zero of order $n - 1$ at P. Let $H^0(C, (\Omega^1)^q)$ denote the space of holomorphic q-differentials on C, with its dimension denoted by d_q. By Riemann-Roch,

$$d_q = \begin{cases} g & \text{if } q = 1, \\ (g-1)(2q-1) & \text{if } q > 1. \end{cases}$$

As before, we take a basis $\{\psi_1, \ldots, \psi_{d_q}\}$ of $H^0(C, (\Omega^1)^q)$ such that

$$\mathrm{ord}_P(\psi_1) < \mathrm{ord}_P(\psi_2) < \cdots < \mathrm{ord}_P(\psi_{d_q}).$$

For $i = 1, \ldots, d_q$, let $n_i = \mathrm{ord}_P(\psi_i) + 1$. The sequence of natural numbers $G^{(q)}(P) = \{n_1, n_2, \ldots, n_{d_q}\}$ is called the *q-gap sequence of P*. With such a gap sequence, we can calculate the *q-weight* of P, denoted $wt^{(q)}(P)$, given by $wt^{(q)}(P) = \sum_{i=1}^{d_q}(n_i - i)$. We call the point P a *q-Weierstrass point* if $wt^{(q)}(P) > 0$.

Given a basis $\{\psi_1, \ldots, \psi_{d_q}\}$ of $H^0(C, (\Omega^1)^q)$, where $\psi_i = f_i(x)dx$ for a holomorphic function f_i of a local coordinate x for each i, the *Wronskian* is the determinant of the following $d_q \times d_q$ matrix:

$$W = W(f_1(x), \ldots, f_{d_q}(x)) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_{d_q}(x) \\ f_1'(x) & f_2'(x) & \cdots & f_{d_q}'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(d_q-1)}(x) & f_2^{(d_q-1)}(x) & \cdots & f_{d_q}^{(d_q-1)}(x) \end{vmatrix}.$$

The Wronskian form is $\Omega_q = W(dx)^m$, for

$$m = q + (q+1) + (q+2) + \cdots + (q + d_q - 1) = \frac{d_q}{2}(2q - 1 + d_q).$$

The following result is due to Hurwitz. For a proof, see [11].

Theorem 1 (Hurwitz)**.** *P is a q-Weierstrass point with weight $wt^{(q)}(P) = r$ if and only if P is a zero of multiplicity r for the Wronskian form Ω_q (or, equivalently, in the support of div(Ω_q)).*

Since the Wronskian form is a holomorphic m-differential, $\operatorname{div}(\Omega_q)$ is effective. Thus, the q-Weierstrass points are the support of $\operatorname{div}(\Omega_q)$, and the sum of the q-weights of the q-Weierstrass points is the degree of $\operatorname{div}(\Omega_q)$, which is $m(2g-2) = d_q(2q - 1 + d_q)(g-1)$. In particular, this means there are a finite number of q-Weierstrass points.

Let $\mathcal{W}(C)$ denote the set of all Weierstrass points and $W_q(C)$ the set of all q-Weierstrass points on C. $W_1(C)$, the set of 1-Weierstrass points on C, is exactly the set of Weierstrass points described earlier. We summarize some properties in the following lemma; see [10, Section III.5] for details.

Lemma 1. *Let C be a genus $g \geq 2$ curve. The following hold:*

i) There are q-Weierstrass points for any $q \geq 1$.

ii) For $q > 1$

$$\sum_{P \in C} wt^{(q)}(P) = g(g-1)^2(2q-1)^2.$$

iii) $2g + 2 \leq |W_1(C)| \leq g^3 - g$.

Now we give some results specific to the $g = 3$ case.

Example 1. For $g = 3$ we have $d_q = 2(2q-1)$. The total weight is 24 for $q = 1$ and for $q > 1$ is

$$\sum_{P \in C} wt^{(q)}(P) = 12(2q-1)^2.$$

Notice that for $q = 2$ we have $d_2 = 6$ and the total weight is 108. For $q = 3$, $d_3 = 10$ and the total weight is 300. In these cases we have, respectively, a 6×6 and a 10×10 Wronskian.

In Section 3, we give the following result for $q = 2$, cf. Remark 2.

Remark 1. Let C be a genus 3 hyperelliptic curve. For any point $P \in C$, the 2-weight of P is $wt^{(2)}(P) \leq 6$. Further, if $wt^{(2)}(P) = 6$, then $P \in W_1(C)$. If $P \notin W_1(C)$, then $wt^{(2)}(P) \leq 3$.

Let C be a genus 3 hyperelliptic curve defined over $\mathbb{C}$, K its function field, and G be the full automorphism group $G := \mathrm{Aut}\ (K)$. All such groups G have distinct ramification structures and therefore there is no confusion to denote such locus $\mathcal{H}_3(G)$ for any fixed G. In this paper we will make use of the following facts, which are proven in [15, Sections 3 – 5].

Lemma 2. *Let C be a genus 3 hyperelliptic curve defined over a field k with a non-hyperelliptic involution. Then C is given by the equation $y^2 = x^8 + ax^6 + bx^4 + cx^2 + 1$ for some $a, b, c \in k$.*

The *dihedral invariants* of C are $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ where $\mathfrak{s}_2 = ac$, $\mathfrak{s}_3 = (a^2 + c^2)b$, and $\mathfrak{s}_4 = a^4 + c^4$.

Theorem 2. *Let C be a genus 3 hyperelliptic curve such that $|G| > 2$ and $\dim \mathcal{H}\,(G) \geq 1$. Then, one of the following holds:*

i) $G \cong V_4$ and the locus $\mathcal{H}(V_4)$ is 3-dimensional. A generic curve in this locus has equation

$$y^2 = A\,x^8 + \frac{A}{\mathfrak{s}_4 + 2\mathfrak{s}_2^2}\ x^6 + \frac{\mathfrak{s}_3(A + \mathfrak{s}_2^2)}{(\mathfrak{s}_4 + 2\mathfrak{s}_2^2)^3}\ x^4 + \frac{\mathfrak{s}_2}{(\mathfrak{s}_4 + 2\mathfrak{s}_2^2)^3}\ x^2 + \frac{1}{(\mathfrak{s}_4 + 2\mathfrak{s}_2^2)^4} \quad (1)$$

where A satisfies $A^2 - \mathfrak{s}_4 A + \mathfrak{s}_2^4 = 0$.

ii) $G \cong \mathbb{Z}_2^3$ and the locus $\mathcal{H}(\mathbb{Z}_2^3)$ is 2-dimensional. A generic curve in this locus has equation

$$y^2 = \mathfrak{s}_2^2 x^8 + \mathfrak{s}_2^2 x^6 + \frac{1}{2}\,\mathfrak{s}_3 x^4 + \mathfrak{s}_2 x^2 + 1. \quad (2)$$

iii) $G \cong \mathbb{Z}_2 \times D_8$ and the locus $\mathcal{H}(\mathbb{Z}_2 \times D_8)$ is 1-dimensional. A generic curve in this locus has equation

$$y^2 = tx^8 + tx^4 + 1. \quad (3)$$

iv) $G \cong D_{12}$ and the locus $\mathcal{H}(D_{12})$ is 1-dimensional. A generic curve in this locus has equation

$$y^2 = x\,(tx^6 + tx^3 + 1). \quad (4)$$

v) $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and the locus $\mathcal{H}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is 1-dimensional. A generic curve in this locus has equation

$$y^2 = \left(tx^4 - 1\right)\,\left(tx^4 + tx^2 + 1\right). \quad (5)$$

Notice that in each case of the above, it is assumed that the discriminant of the polynomial in x is not zero.

3 2-Weierstrass points for genus 3 hyperelliptic curves

Let C be a hyperelliptic curve of genus $g = 3$ given by $y^2 = f(x)$ with $\deg(f) = 8$. Let $\{\alpha_1, \ldots, \alpha_8\}$ denote the eight distinct roots of $f(x)$, and for each i let $R_i = (\alpha_i, 0)$ denote the corresponding ramification points on C. Throughout this section, let $\omega \in \mathbb{C}$ denote any non-root of $f(x)$, and let P_1^ω and P_2^ω denote the two (distinct) points above ω. And let P_1^∞ and P_2^∞ denote the two points over ∞ in the non-singular model of C.

Here are the divisors associated to some functions and the differential dx:

- $\operatorname{div}(y) = \sum_{i=1}^{8} R_i - 4(P_1^\infty + P_2^\infty)$,
- $\operatorname{div}(x - \alpha_i) = 2R_i - (P_1^\infty + P_2^\infty)$,
- $\operatorname{div}(x - \omega) = P_1^\omega + P_2^\omega - (P_1^\infty + P_2^\infty)$.
- $\operatorname{div}(dx) = \sum_{i=1}^{8} R_i - 2(P_1^\infty + P_2^\infty)$,

In particular, note that $\operatorname{div}(dx/y) = 2(P_1^\infty + P_2^\infty)$, which is effective. With these divisors, we can explicitly construct bases of $H^0(C, (\Omega^1)^q)$ for all $q \geq 1$. For $d_q = \dim H^0(C, (\Omega^1)^q)$ and $g = 3$, we have $d_1 = 3$ and $d_q = 4q - 2$ for $q \geq 2$.

Theorem 3. *Let C be a hyperelliptic curve of genus $g = 3$ given by the equation $y^2 = f(x)$ with $\deg(f(x)) = 8$. For any $\beta \in \mathbb{C}$, one has the following bases of holomorphic q-differentials.*

For $q = 1$, a basis for $H^0(C, (\Omega^1)^1)$ is

$$B_{1,\beta} = \left\{(dx/y), (x - \beta)(dx/y), (x - \beta)^2(dx/y)\right\}.$$

For $q \geq 2$, a basis for $H^0(C, (\Omega^1)^q)$, is

$$B_{q,\beta} = \left\{(x - \beta)^j (dx/y)^q : 0 \leq j \leq 2q\right\} \cup \left\{(x - \beta)^k y(dx/y)^q : 0 \leq k \leq 2q - 4\right\}.$$

Note that the only poles occur at infinity, so to prove this, one needs to ensure that the pole orders are different and that there are d_q elements. For a proof, see [4, 2.1, Example (ii)].

Using these bases, we can calculate q-weights of ramification points.

Corollary 1. Let R be any ramification point on C. For $q = 1$, the 1-gap sequence of R is $\{1, 3, 5\}$, so $wt^{(1)}(R) = 3$. For any $q \geq 2$, the q-gap sequence of R is $\{1, 3, 5, \ldots, 4q + 1\} \cup \{2, 4, 6, \ldots, 4q - 6\}$, so $wt^{(q)}(R) = 6$.

Hence, for $q \geq 2$ the eight branch points contribute $8 \cdot 6 = 48$ to the total weight of q-Weierstrass points on the curve.

In particular, the 2-gap sequence for a branch point is $\{1, 2, 3, 5, 7, 9\}$. The corollary below gives the 2-gap sequence for a non-branch point.

Remark 2. Following from [8], the possible 2-gap sequences of 2-Weierstrass points on a curve of genus 3 are given in [1, Lemma 5]. From this, we see that if P_i^ω is a non-branch point on a hyperelliptic curve of genus 3, the 2-gap sequence contains 4 and 5, so $wt^{(2)}(P_i^\omega) \leq 3$.

We can use divisors to characterize the non-branch 2-Weierstrass points.

Proposition 1. *For the curve C given by $y^2 = f(x)$ and non-branch point P_i^ω above $x = \omega$, let $h(x) = f(x)^{1/2}$, chosen so that P_i^ω lies on the curve $y = h(x)$. Let*

$$N = \min\left\{n \in \mathbb{N} : n \geq 5, h^{(n)}(\omega) \neq 0\right\},$$

where $h^{(n)}(x)$ denotes the nth derivative of $h(x)$. Then $wt^{(2)}(P_i^\omega) = N - 5$ and $5 \leq N \leq 8$. Thus, P_i^ω is a 2-Weierstrass point if and only if $h^{(5)}(\omega) = 0$.

Proof. Let

$$T_{\omega,4,i}(x) = \sum_{n=0}^{4} \frac{h^{(n)}(\omega)}{n!}(x-\omega)^n,$$

the fourth degree Taylor polynomial for $h(x)$ at $x = \omega$. As in Theorem 3, the set

$$\{(x-\omega)^j (dx/y)^2 : 0 \leq j \leq 4\} \cup \{(y - T_{\omega,4,i}(x))\,(dx/y)^2\}$$

is a basis for $H^0(C, (\Omega^1)^2)$. The orders of vanishing at P_i^ω are

$$\nu_{P_i^\omega}((x-\omega)^j (dx/y)^2) = j \text{ for } 0 \leq j \leq 4,$$

and

$$\nu_{P_i^\omega}\left((y - T_{\omega,4,i}(x))(dx/y)^2\right) = \nu_{P_i^\omega}\left(\sum_{n=5}^{\infty} \frac{h^{(n)}(\omega)}{n!}(x-\omega)^n\right) = N.$$

Thus, the 2-gap sequence of P_i^ω is $\{1, 2, 3, 4, 5, N+1\}$, and so $wt^{(2)}(P_i^\omega) = N - 5$. Thus, P_i^ω is a 2-Weierstrass point precisely when $N > 5$. Finally, since $wt^{(q)}(P_i^\omega) \leq 3$ by Remark 2, we see $N \leq 8$. $\square$

Of course, we can perform these calculations with the Wronskian as well. With the basis $\{x^j (dx/y)^2 : 0 \leq j \leq 4\} \cup \{y(dx/y)^2\}$ of $H^0(C, (\Omega^1)^2)$, the Wronskian is

$$W = W\left(\frac{1}{y^2}, \frac{x}{y^2}, \frac{x^2}{y^2}, \frac{x^3}{y^2}, \frac{x^4}{y^2}, \frac{y}{y^2}\right) = \frac{1}{y^{12}} W(1, x, x^2, x^3, x^4, y).$$

Thus, $W = \frac{1}{y^{12}}\left(\prod_{i=0}^4 i!\right) y^{(5)}$, so the Wronskian form is $\Omega_2 = W(dx)^{27}$. Since $y^2 = f(x)$, five derivatives will yield $y^{(5)} = \phi(x)/y^9$ for some polynomial $\phi(x)$ of degree at most 29 (depending on $f(x)$). That is,

$$\Omega_2 = \left(\prod_{i=0}^{4} i!\right) \frac{\phi(x)}{y^{21}}(dx)^{27}.$$

Thus,

$$
\begin{aligned}
\operatorname{div}(\Omega_2) &= \operatorname{div}(\phi(x)) - \operatorname{div}(y^{21}) + \operatorname{div}((dx)^{27}) \\
&= \operatorname{div}(\phi(x))_0 + 6\left(\sum_{i=1}^{8} R_i\right) + (30 - \deg(\phi))(P_1^\infty + P_2^\infty).
\end{aligned}
$$

We see that the branch points have 2-weight 6 and the other 2-Weierstrass points are the zeros of $y^{(5)}$. Note that this result agrees with Corollary 1 and Proposition 1. Also, the points at infinity are 2-Weierstrass points with 2-weight $30 - \deg(\phi)$.

4 Computation of 2-Weierstrass points

In this section we will study the distributions of 2-Weierstrass points for curves in each family $\mathcal{H}_3(G)$ such that $\dim \mathcal{H}_3(G) > 0$; that is, for curves with full automorphism group isomorphic to $V_4, \mathbb{Z}_2^3, \mathbb{Z}_2 \times D_8, D_{12}$, or $\mathbb{Z}_2 \times \mathbb{Z}_4$. These families are described in Theorem 2. For our computations, we make use of the dihedral invariants and the results in [15]. We also need the following elementary result.

Lemma 3. *Let $f(x) = \sum_{i=0}^n a_i\, x^i$ and $g(x) = \sum_{i=0}^m b_i\, x^i$ be polynomials with no common roots. Then, the discriminant of $f(g(x))$ is given by*

$$
\Delta_{f\circ g} = (-1)^{\frac{mn(3mn-2m-1)}{2}} \cdot a_n^{m-1} \cdot b_m^{n(nm-m-1)} \Delta_f^m \cdot Res(f(g(x)), g'(x))
$$

Moreover, if $f(x) = \sum_{i=0}^n a_i\, x^i$ and $g(x) = x^m$. Then, the discriminant of $f \circ g$ is

$$
\Delta_{f\circ g} = (-1)^{\frac{mn(3mn-2m-1)}{2}} \cdot a_n^{m-1} \Delta_f^m \cdot Res(f(x^m), mx^{m-1}) \tag{6}
$$

Proof. The first part of the Lemma is proved by J. Cullinan in [5]. To prove the second part we have to compute $\operatorname{Res}(f(x^m), mx^{m-1})$. Indeed,

$$
\operatorname{Res}(f(x^m), mx^{m-1}) = ...
$$

This completes the proof. $\square$

Remark 3. Notice that if $f(x) = \sum_{i=0}^n a_i\, x^i$ and $g(x) = x^2$. Then, the discriminant of $f(x^2)$ is

$$
\Delta(f(x^2) = (-1)^n \cdot 2^{2n} \cdot a_0 a_n \cdot \Delta_f^2. \tag{7}
$$

4.1 The case Aut $(C) \cong V_4$.

Let C be a genus 3 hyperelliptic curve with a non-hyperelliptic involution. From Lemma 2, we know that the equation of C can be given by $y^2 = f(x)$, for

$$
f(x) = x^8 + ax^6 + bx^4 + cx^2 + 1. \tag{8}
$$

The Wronskian form is

$$
\Omega_2 = \frac{1}{y^{12}} W(1, x, x^2, x^3, x^4, y) = 4320 \frac{x\,\Phi(x^2)}{y^{21}} (dx)^{27},
$$

where $\Phi(x)$ is a polynomial of degree 14 which depends on a, b, c. We don't display its coefficients since they are large.

Let $\Phi(x^2) = \sum_{i=0}^{14} c_i x^{2i}$. The leading coefficient c_{14} and the constant term c_0 are

$$c_{14} = -3a^3 + 12ab - 24c \qquad \texttt{and} \qquad c_0 = -(-3c^3 + 12bc - 24a).$$

In general, the coefficients c_i and c_{14-i} differ by a permutation of a and c and a factor of -1. In other words the permutation of the curve $\tau_1 : (x, y) \to \left(\frac{1}{x}, \frac{y}{x^4}\right)$ which permutes coefficients a and c of te curve given in Eq. (8) acts on the coefficients of $\Phi(x)$ by

$$\tau_1(c_i) = -c_{14-i}.$$

Computing the discriminant $\Delta(\Phi, x)$ we get the following factors:

$$\Delta = 2^{16}\, c_0 c_{14} \cdot g(a, b, c)^2 \cdot \Delta(f, x)^{28},$$

where $g(a, b, c)$ is a degree 24, 28, 24 polynomial in terms of a, b, c respectively. We know that $\Delta(f, x) \neq 0$. Let us assume that $c_0 c_{14} \neq 0$. Then, the 2-Weierstrass points are those when $g(a, b, c) = 0$. The polynomial $g(a, b, c)$ can be easily computed. However, the triples (a, b, c) do not correspond uniquely to the isomorphism classes of curves. Naturally we would prefer to express such result in terms of the dihedral invariants $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$. One can take the equations $g(a, b, c) = 0$ and three equations from the definitions of $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ and eliminate a, b, c. It turns out that this is a challenging task computationally.

Hence, we continue with the following approach. From Theorem 2, we know that a curve C with Aut $(C) \cong V_4$ is isomorphic to a curve with equation

$$y^2 = A\, x^8 + \frac{A}{\mathfrak{s}_4 + 2\mathfrak{s}_2^2}\, x^6 + \frac{\mathfrak{s}_3(A + \mathfrak{s}_2^2)}{(\mathfrak{s}_4 + 2\mathfrak{s}_2^2)^3}\, x^4 + \frac{\mathfrak{s}_2}{(\mathfrak{s}_4 + 2\mathfrak{s}_2^2)^3}\, x^2 + \frac{1}{(\mathfrak{s}_4 + 2\mathfrak{s}_2^2)^4}$$

where A satisfies

$$A^2 - \mathfrak{s}_4 A + \mathfrak{s}_2^4 = 0, \tag{9}$$

for some $(\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4) \in k^3 \setminus \{\Delta_{\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4} = 0\})$.

The numerator in the Wronskian form is a degree 29 polynomial in x written as $x\, \phi(x^2)$. From Lemma 3, it is enough to compute the discriminant of the polynomial $\phi(t)$, where $t = x^2$. This is a degree 14 polynomial. Its discriminant is a polynomial $G(A, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4)$ in terms of $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ and A. Then, the relation between $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ is obtained by taking the resultant Res $(G, A^2 - \mathfrak{s}_4 A + \mathfrak{s}_2^4, A)$. The result is quite a large polynomial in terms of $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$. Fortunately, it turns out that the remaining cases are much easier.

Remark 4. The equivalent statement of Theorem 2, i) is proved in [3] for any genus $g \geq 3$. Also the $\mathfrak{s}$-invariants are defined for every $g > 3$. Hence, this method will work for any $g > 3$. With some modifications the method works for all superelliptic curves as in [12].

4.2 The case Aut $(C) \cong \mathbb{Z}_2^3$.

Proposition 2. *Let C be a genus 3 hyperelliptic curve with full automorphism group $\mathbb{Z}_2^3$. Then, C has non-branch 2-Weierstrass points of weight greater than one if and only if its corresponding dihedral invariants $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$ satisfy Eq. (10)*

$$\Delta = \left(-784\,\mathfrak{s}_2^2 + 16\,\mathfrak{s}_2^3 + 56\,\mathfrak{s}_2\mathfrak{s}_3 - \mathfrak{s}_3^2\right)\, G(\mathfrak{s}_2, \mathfrak{s}_3) = 0 \tag{10}$$

where

$$\begin{aligned}
G = {} & 617400\mathfrak{s}_3^9 + 180 s_2\,(315560 + 871 s_2)\,\mathfrak{s}_3^8 + 2\mathfrak{s}_2^2\left(4023040 s_2 + 970717440 + 31077\mathfrak{s}_2^2\right)\mathfrak{s}_3^7 \\
& + \mathfrak{s}_2^3\left(9\mathfrak{s}_2^3 - 3251241728 s_2 + 31937525760 + 5011112\mathfrak{s}_2^2\right)\mathfrak{s}_3^6 + 8\mathfrak{s}_2^4\,(-9204034560 s_2 \\
& -105048636\mathfrak{s}_2^2 + 15801\mathfrak{s}_2^3 + 41193015808)\,\mathfrak{s}_3^5 - 16\mathfrak{s}_2^5\left(22041513\mathfrak{s}_2^3 + 59872104320 s_2\right. \\
& \left.+11\mathfrak{s}_2^4 - 4535327496\mathfrak{s}_2^2 - 193117539328\right)\mathfrak{s}_3^4 - 256\mathfrak{s}_2^6\left(-2870647262\mathfrak{s}_2^2 - 73789452800\right. \\
& \left.+34876810752 s_2 - 47803959\mathfrak{s}_2^3 + 54199\mathfrak{s}_2^4\right)\mathfrak{s}_3^3 - 256\mathfrak{s}_2^7\left(5\mathfrak{s}_2^5 - 41807037944\mathfrak{s}_2^2\right. \\
& \left.+2624158985\mathfrak{s}_2^3 - 19769334\mathfrak{s}_2^4 + 283441853184 s_2 - 365995685888\right)\mathfrak{s}_3^2 - 2048\mathfrak{s}_2^8\left(308705831\mathfrak{s}_2^3\right. \\
& \left.+28144998\mathfrak{s}_2^4 - 39227605228\mathfrak{s}_2^2 + 123966280704 s_2 + 31711\mathfrak{s}_2^5 - 175618897664\right)\mathfrak{s}_3 \\
& + 4096\mathfrak{s}_2^9\left(455870765\mathfrak{s}_2^4 - 4058869\mathfrak{s}_2^5 + 7\mathfrak{s}_2^6 - 16649626455\mathfrak{s}_2^3 - 214358360384 s_2\right. \\
& \left.+85982595160\mathfrak{s}_2^2 + 144627327488\right).
\end{aligned}$$

Proof. The equation of the curve is given by Eq. (2), for $s_2, \mathfrak{s}_3 \neq 0, 4$. For $\Omega_2 = \dfrac{1}{y^{12}} W(1, x, x^2, x^3, x^4, y)(dx)^{27}$, we find

$$\Omega_2 = \frac{x\,(\mathfrak{s}_2 x^4 - 1) g(t)}{(4\,s_2^2 x^8 + 4\,s_2^2 x^6 + 2\,s_3 x^4 + 4\,s_2 x^2 + 4)^{21/2}}\,(dx)^{27}$$

where $g(t) = \sum_{i=0}^{12} c_i \cdot t^i$ is a degree 12 polynomial for $t = x^2$ with the following coefficients:

$$\begin{aligned}
c_0 &= 12\,\mathfrak{s}_2, \\
c_1 &= -4(-7\,\mathfrak{s}_3 + 4\,\mathfrak{s}_2^2 - 28\,\mathfrak{s}_2), \\
c_2 &= 12\,\mathfrak{s}_2\,(22\,\mathfrak{s}_2 - 3\,\mathfrak{s}_3)\,, \\
c_3 &= -4(-28\,\mathfrak{s}_2^2 + 9\,\mathfrak{s}_3^2 + 4\,\mathfrak{s}_2^3 + 29\,\mathfrak{s}_2\mathfrak{s}_3), \\
c_4 &= \mathfrak{s}_2\left(-\mathfrak{s}_3^2 - 1180\,\mathfrak{s}_2^2 + 16\,\mathfrak{s}_2^3 - 376\,\mathfrak{s}_2\mathfrak{s}_3\right), \\
c_5 &= 3\,\mathfrak{s}_3^3 - 24\,\mathfrak{s}_2\mathfrak{s}_3^2 - 48\,\mathfrak{s}_2^3\mathfrak{s}_3 - 1696\,\mathfrak{s}_2^4 - 1568\,\mathfrak{s}_2^3 - 536\,\mathfrak{s}_2^2\mathfrak{s}_3, \\
c_6 &= -26\,\mathfrak{s}_2^2\left(20\,\mathfrak{s}_2\mathfrak{s}_3 + 152\,\mathfrak{s}_2^2 - \mathfrak{s}_3^2 + 16\,\mathfrak{s}_2^3\right), \\
c_7 &= \mathfrak{s}_2\left(3\,\mathfrak{s}_3^3 - 24\,\mathfrak{s}_2\mathfrak{s}_3^2 - 48\,\mathfrak{s}_2^3\mathfrak{s}_3 - 1696\,\mathfrak{s}_2^4 - 1568\,\mathfrak{s}_2^3 - 536\,\mathfrak{s}_2^2\mathfrak{s}_3\right), \\
c_8 &= \mathfrak{s}_2^3\left(-\mathfrak{s}_3^2 - 1180\,\mathfrak{s}_2^2 + 16\,\mathfrak{s}_2^3 - 376\,\mathfrak{s}_2\mathfrak{s}_3\right), \\
c_9 &= -4\,\mathfrak{s}_2^3\left(-28\,\mathfrak{s}_2^2 + 9\,\mathfrak{s}_3^2 + 4\,\mathfrak{s}_2^3 + 29\,\mathfrak{s}_2\mathfrak{s}_3\right), \\
c_{10} &= 12\,\mathfrak{s}_2^5\,(22\,\mathfrak{s}_2 - 3\,\mathfrak{s}_3)\,, \\
c_{11} &= -4\,\mathfrak{s}_2^5\left(-7\,\mathfrak{s}_3 + 4\,\mathfrak{s}_2^2 - 28\,\mathfrak{s}_2\right), \\
c_{12} &= 12\,\mathfrak{s}_2^7.
\end{aligned}$$

We note that $c_{12-i} = \mathfrak{s}_2^{6-i} c_i$ for $i = 0, \dots, 6$.

The discriminant of $g(t)$ factors as is written in Eq. (10). Each component can be expressed in terms of the absolute invariants $t_1, \dots t_6$ as defined in [13]. Since they are large expressions we do not display them. $\square$

The following determines a nice family of curves with automorphism group $\mathbb{Z}_2^3$.

Lemma 4. *Let C be a genus 3 curve with equation*

$$y^2 = \frac{t^4}{256}\, x^8 + \frac{t^4}{256}\, x^6 + \frac{t^2}{32}\, (t+28)\, x^4 + \frac{t^2}{16}\, x^2 + 1$$

such that $t \in \mathbb{C} \setminus \{-16, 0, 48\}$. Then, Aut $(C) \cong \mathbb{Z}_2^3$ and C has N_r 2-Weierstrass points of weight r as described in the table below.

	N_1	N_2	N_3
$t = -112/3$	24	0	12
$t = 14 \pm 14\sqrt{-15}$	16	16	4
$t \in \mathbb{C} \setminus \{-16, 0, 48, -112/3, 14 \pm 14\sqrt{-15}\}$	48	0	4

Proof. Let us assume that the dihedral invariants satisfy the first factor of the Eq. (10). Since this is a rational curve we can parametrize it as follows:

$$\mathfrak{s}_2 = \frac{1}{16}\, t^2, \quad \mathfrak{s}_3 = \frac{1}{16}\, (t+28)\, t^2.$$

In this case the curve C becomes

$$y^2 = \frac{t^4}{256}\, x^8 + \frac{t^4}{256}\, x^6 + \frac{t^2}{32}\, (t+28)\, x^4 + \frac{t^2}{16}\, x^2 + 1$$

with discriminant $\Delta = t^{28}\,(t-48)^4\,(t+16)^6 \neq 0$. The Wronskian form is

$$\begin{aligned}
\Omega_2 = {} & \frac{x\,(tx^2+4)(tx^2-4)^3}{(t^4x^8 + t^4x^6 + 8\,t^3x^4 + 224\,t^2x^4 + 16\,t^2x^2 + 256)^{21/2}} \left(t^2x^4 + 24\,tx^2 + 16\right) (3t^8x^{16} \\
& - 4\,t^6 \left(-16\,t + t^2 - 896\right) x^{14} - 16\,t^5 \left(5\,t^2 + 3584 + 220\,t\right) x^{12} - 192\,t^4 \left(9\,t^2 + 2688 + 368\,t\right) x^{10} \\
& - 512\,t^3 \left(487\,t + 3584 + 23\,t^2\right) x^8 - 3072\,t^2 \left(9\,t^2 + 2688 + 368\,t\right) x^6 - 4096\,t \left(5\,t^2 + 3584 + 220\,t\right) x^4 \\
& + (14680064 + 262144\,t - 16384\,t^2)x^2 + 196608)(dx)^{27}.
\end{aligned}$$

Hence, the curve has four 2-Weierstrass points of weight 3 which come from the two roots of the factor $(tx^2-4)^3 = 0$. Note that $x = 0$ is a root of order 1, so the points $(0, \pm 1)$ have weight 1. Removing these factors as well as the denominator, we obtain a polynomial in x^2 which we can write as

$$h(x^2) = \Omega_2 \cdot \frac{(t^4x^8 + t^4x^6 + 8\,t^3x^4 + 224\,t^2x^4 + 16\,t^2x^2 + 256)^{21/2}}{x(tx^2-4)^3},$$

for $\deg(h(x)) = 11$. We now check $h(x)$ for multiple roots. One finds that

$$\Delta(h, x) = 2^{289} 3^9 7^3 \cdot t^{93} (16+t)^{14} (3t+112)^6 (t-48)^6 (t^2 - 28t + 3136)^4.$$

Since we do not consider the cases where $t = 0, -16, 48$, to make $\Delta(h, x) = 0$, we look at $t = -112/3$ and $t = 14 \pm 14\sqrt{-15}$. When $t = -112/3$, then

$$h(x) = c(28x - 3)(81 + 168x + 784x^2)(784x^2 - 504x + 9)^3(3 + 56x + 2352x^2),$$

for some constant c. Thus, $h(x)$ has two roots of order 3 and five roots of order 1. Going back to Ω_2, these roots lead to eight 2-Weierstrass points of weight 3 and twenty 2-Weierstrass points of weight 1.

When $t = 14 \pm 14\sqrt{-15}$, $h(x)$ has four roots of order 2 and 3 roots of order 1. These lead to sixteen 2-Weierstrass points with weight 2 and twelve 2-Weierstrass points with weight 1.

Note that for any $t \neq 0$, the numerator of Ω_2 is a polynomial of degree 29, so the two points at infinity are 2-Weierstrass points with weight 1. □

The other component is also a genus 0 curve and the same method as above can also be used here.

Theorem 4. *The locus in $\mathcal{H}_3$ of curves with full automorphism group $\mathbb{Z}_2^3$ which have 2-Weierstrass points is a 1-dimensional variety with two irreducible components. Each component is a rational family. The equation of a generic curve in each family is given in terms of the parameter t.*

Next, we consider the 1-dimensional loci. There are three cases of groups which correspond to 1-dimensional loci in $\mathcal{H}_3$, namely the groups $\mathbb{Z}_2 \times D_8$, D_{12}, and $\mathbb{Z}_2 \times \mathbb{Z}_4$. Let us first consider the case Aut $(C) \cong \mathbb{Z}_2 \times D_8$.

4.3 The case Aut $(C) \cong \mathbb{Z}_2 \times D_8$.

Proposition 3. *Let C be a genus 3 hyperelliptic curve with full automorphism group $\mathbb{Z}_2 \times D_8$. Then C is isomorphic to a curve of the form $y^2 = tx^8 + tx^4 + 1$ for some $t \neq 0, 4$. For any other $t \neq -140, -980/3$, C has N_r 2-Weierstrass points of weight r as described in the table below.*

	N_1	N_2	N_3
$t = 196$	24	0	12
$t = -196/5$	16	16	4
$t \in \mathbb{C} \setminus \{0, 4, -140, -980/3\}$	48	0	4

Proof. In this case the curve has equation $y^2 = tx^8 + tx^4 + 1$, with discriminant $\Delta = 2^{16} \cdot t^7 \,(t - 4)^4 \neq 0$, where $t = -28 \frac{5t_4 + 28}{t_4 - 4}$; see [15, Lemma 7]. Ω_2 is the product of the two following factors,

$$\frac{34560\, t(t-4)\, x^3 \left(tx^8 - 1\right)}{\left(tx^8 + tx^4 + 1\right)^{21/2}} \text{ and}$$
$$\left(7\, t^2 x^{16} - 18\, t^2 x^{12} + 3\, t^2 x^8 - 98\, tx^8 - 18\, tx^4 + 7\right) (dx)^{27}.$$

Since $x = 0$ is of multiplicity 3, then the points $(0, \pm 1)$ have each weight 3.

The other factors of the Wronskian, namely

$$\left(tx^8-1\right)\left(7\,t^2x^{16}-18\,t^2x^{12}+3\,t^2x^8-98\,tx^8-18\,tx^4+7\right)$$

have double roots if the discriminant is zero. This happens if $t=196$ or $t=-\frac{196}{15}$. If $t=196$ then

$$\Omega_2 = 9103933440\,\frac{x^3\left(14\,x^4+1\right)\left(196\,x^8-476\,x^4+1\right)\left(14\,x^4-1\right)^3}{\left(196\,x^8+196\,x^4+1\right)^{9/2}}(dx)^{27}.$$

Hence, there are 24 points of weight 1, and 8 other points of weight 3 which come from the roots of $14x^4=1$.

If $t=-\frac{196}{15}$, then the curve C becomes

$$y^2 = -\frac{196}{15}\,x^8-\frac{196}{15}\,x^4+1$$

and

$$\Omega_2 = -614515507200000\,\frac{x^3\left(15+196\,x^8\right)\left(-15-252\,x^4+196\,x^8\right)^2}{\left(-15\,\left(14\,x^4+15\right)\left(14\,x^4-1\right)\right)^{9/2}}(dx)^{27}.$$

Hence, there are 16 points of weight 1 and 16 points of weight 2.

Finally, observe that since the numerator of Ω_2 is a polynomial in x of degree 27, the two points at infinity have 2-weight equal to $30-27=3$. $\square$

4.4 The case Aut $(C)\cong D_{12}$

Let us now assume that C has full automorphism group D_{12}. In this case the curve has equation

$$y^2 = x\,(tx^6+tx^3+1)$$

for $t=\frac{7}{2}\,\frac{5t_4+7}{t_4-2}$ and discriminant $\Delta = 3^6\cdot t^5\,(t-4)^3\neq 0$; see [15, Lemma 8] for details.

In particular, for a curve C given by the equation $y^2=f(x)$, with $\deg(f)=7$, there is one point at infinity, which is singular. This point is a branch point, and in the desingularization remains as one point, which we will denote here by P^∞. Let $\{\alpha_i\}$ denote the roots of $f(x)$, and $R_i=(\alpha_i,0)$ the affine branch points. Let $\omega\in\mathbb{C}\setminus\{\alpha_i\}$ and let P_1^ω and P_2^ω denote the points over ω. One has the following divisors.

- $\operatorname{div}(y) = \left(\sum_{i=1}^{7} R_i\right) - 7P^\infty$,
- $\operatorname{div}(x-\omega) = P_1^\omega + P_2^\omega - 2P^\infty$,
- $\operatorname{div}(x-\alpha_i) = 2R_i - 2P^\infty$.

- $\operatorname{div}(dx) = \left(\sum_{i=1}^{7} R_i\right) - 3P^{\infty}$,

Working with these divisors, as in Theorem 3 one finds that a basis of holomorphic 2-differentials is given by

$$\left\{(x-\beta)^j (dx/y)^2 : 0 \leq j \leq 4\right\} \cup \left\{y(dx/y)^2\right\},$$

for any $\beta \in \mathbb{C}$. Letting $\beta = \alpha_i$, the 2-Weierstrass weight for the affine branch point R_i is 6. And using any value of β, one finds orders of vanishing $8, 6, 4, 2, 0, 1$ at P^{∞}, so $wt^{(2)}(P^{\infty}) = 6$ as well.

Proposition 4. *Let C be a genus 3 hyperelliptic curve with full automorphism group D_{12}. By [15, Lemma 8], C has equation $y^2 = x(tx^6 + tx^3 + 1)$. Then, C has non-branch points with 2-Weierstrass weight greater than 1 if and only if $t = -\frac{49}{8}$ or $t = \frac{1787}{8} \pm \frac{621}{4}\sqrt{2}$.*

In particular, for each value of t, C has N_r 2-Weierstrass points of weight r as described in the table below.

	N_1	N_2	N_3
$t = -49/8$	24	0	12
$t = 1787/8 \pm 621/4\sqrt{2}$	36	12	0
$t \in \mathbb{C} \setminus \{0, 4, -49/8, 1787/8 \pm 621/4\sqrt{2}\}$	60	0	0

Proof. In this case the curve has equation $y^2 = x\left(tx^6 + tx^3 + 1\right)$ for $t = \frac{7}{2}\,\frac{5t_4+7}{t_4-2}$ and discriminant $\Delta = 3^6 \cdot t^5\,(t-4)^3 \neq 0$; see [15, Lemma] for details. The Wronskian is

$$\begin{aligned}\Omega_2 = -135\,&\frac{\left(tx^6-1\right)}{\left(x\left(tx^6+tx^3+1\right)\right)^{9/2}}\left(7\,t^4x^{24} + 28\,t^4x^{21} - 336\,t^4x^{18} + 1216\,t^4x^{15}\right.\\ &-128\,t^4x^{12} + 1540\,t^3x^{18} - 4668\,t^3x^{15} + 6672\,t^3x^{12} + 1216\,t^3x^9 - 24150\,t^2x^{12}\\ &\left.-4668\,t^2x^9 - 336\,t^2x^6 + 1540\,tx^6 + 28\,tx^3 + 7\right)(dx)^{27}.\end{aligned}$$

Its discriminant factors as

$$\Delta(\Omega_2, x) = t^{145}\,(t-4)^{42}\,\left(64\,t^2 - 28592\,t + 108241\right)^9\,(8\,t+49)^{12}.$$

Since $t \neq 0, 4$, then the Ω_2 form has multiple roots if and only if

$$t = -\frac{49}{8}, \quad t = \frac{1787}{8} + \frac{621}{4}\sqrt{2}, \quad \text{or } t = \frac{1787}{8} - \frac{621}{4}\sqrt{2}.$$

For each one of these values of t, Ω_2 has multiple zeros and hence 2-Weierstrass points of weight at least 2.

For $t = -\frac{49}{8}$ the numerator of Ω_2 is the polynomial

$$\left(49\,x^6 + 8\right)\left(49\,x^6 + 616\,x^3 - 8\right)\left(49\,x^6 - 140\,x^3 - 8\right)^3,$$

which has six roots of multiplicity 3. Hence, the curve $y^2 = x\left(-\frac{49}{8}x^6 - \frac{49}{8}x^3 + 1\right)$ has twelve 2-Weierstrass points of weight 3. There are twelve simple roots of this polynomial and therefore twenty-four points of weight 1.

For $t = \frac{1787}{8} \pm \frac{621}{4}\sqrt{2}$, the numerator of Ω_2 is the polynomial

$$\left(108241x^6 - (60536 \pm 35532\sqrt{2})x^3 + (14296 \pm 9936\sqrt{2})\right)^2 g(x),$$

for $g(x)$ a degree-18 polynomial with coefficients in $\mathbb{Z}[\sqrt{2}]$ and distinct roots. The numerator of Ω_2 has six double roots which lead to twelve 2-Weierstrass points of weight 2. The remaining eighteen roots are single roots, leading to thirty-six 2-Weierstrass points of weight 1.

Finally, note that in both cases, the 2-Weierstrass points we have calculated make a contribution of 60 to the total weight. The eight branch points (including the point at infinity) each have 2-Weierstrass weight 6, thus making a contribution of 48 to the total weight, which is 108. $\square$

Remark 5. Notice that in the case of the curve $y^2 = x\left(-\frac{49}{8}x^6 - \frac{49}{8}x^3 + 1\right)$, even though the curve is defined over $\mathbb{Q}$ the 2-Weierstrass points are defined over a degree 6 extension of $\mathbb{Q}$.

4.5 The case Aut $(C) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$

Proposition 5. *Let C be a genus 3 hyperelliptic curve with full automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_4$. Then, C has 2-Weierstrass points if and only if C is isomorphic to one of the curves $y^2 = \left(tx^4 - 1\right)\left(tx^4 + tx^2 + 1\right)$, for $t = -8$ or it is a root of*

$$\begin{aligned} t^8 &+ 600822\,t^7 + 71378609\,t^6 + 4219381768\,t^5 + 85080645104\,t^4 - 2272444082944\,t^3 \\ &+16480136388352\,t^2 - 50330309965824\,t + 56693912375296 = 0. \end{aligned} \tag{11}$$

In the first case, the curve has two 2-Weierstrass points of weight 3.

Proof. The equation of this curve is given by

$$y^2 = \left(tx^4 - 1\right)\left(tx^4 + tx^2 + 1\right)$$

with discriminant $\Delta = -2^{12} \cdot t^{14}(t-4)^6$. The numerator of the Wronskian is a degree 29 polynomial in x, given by $x\phi(x)$, where

$$\begin{aligned} \phi(x) = &\left(24\,t^7 - 3\,t^8\right)x^{28} + \left(-4\,t^7 + 4\,t^8 + 224\,t^6\right)x^{26} + \left(63\,t^7 + 504\,t^6\right)x^{24} + 1368\,t^6x^{22} \\ &+ \left(4\,t^7 + 2888\,t^5 + 1045\,t^6\right)x^{20} + \left(3360\,t^4 + 588\,t^6 + 3780\,t^5\right)x^{18} + (3375\,t^5 + 108\,t^6 \\ &+ 5544\,t^4)x^{16} + \left(7632\,t^4 + 1056\,t^5\right)x^{14} + \left(5544\,t^3 + 3375\,t^4 + 108\,t^5\right)x^{12} + (3780\,t^3 \\ &+ 3360\,t^2 + 588\,t^4)x^{10} + \left(1045\,t^3 + 4\,t^4 + 2888\,t^2\right)x^8 + 1368\,t^2x^6 + \left(504\,t + 63\,t^2\right)x^4 \\ &+ \left(-4\,t + 4\,t^2 + 224\right)x^2 + 24 - 3t. \end{aligned}$$

Its discriminant is

$$\Delta = t^{275}\,(t-4)^{90}\,\,(t-8)^4\,\big(t^8+600822\,t^7+71378609\,t^6+4219381768\,t^5+85080645104\,t^4 \\ -2272444082944\,t^3+16480136388352\,t^2-50330309965824\,t+56693912375296\big)^4.$$

Hence, for $t=8$ or t satisfying the degree 8 polynomial the corresponding curve has 2-Weierstrass points. In the first case, $t=8$, the curve becomes

$$y^2=\left(8\,x^4-1\right)\left(8\,x^4+8\,x^2+1\right).$$

The Wronskian Ω_2 has $x=0$ as a triple root. Hence, the points $(0,i)$ and $(0,-i)$, for $i^2=-1$ are 2-Weierstrass points of weight 3. If t is a root of the second factor, then the Galois group of this degree 8 polynomial is S_8 and therefore not solvable by radicals. $\square$

Summarizing we have the following theorem.

Theorem 5. *Let G be a group such that $|G|>4$ and $\mathcal{H}(G)$ is a locus of dimension $d>0$ in $\mathcal{H}_3$. Let C be a curve in the locus $\mathcal{H}(G)$, $\mathfrak{s}_2,\mathfrak{s}_3,\mathfrak{s}_4$ its corresponding dihedral invariants and $\pi: C\to\mathbb{P}^1$ the hyperelliptic projection. Then each branch point of π has 2-weight 6 and one of the following holds:*

i) If Aut $(C)\cong\mathbb{Z}_2^3$, then C has non-branch 2-Weierstrass points of weight greater than one if and only if $\mathfrak{s}_2,\mathfrak{s}_3,\mathfrak{s}_4$ satisfy Eq. (10).

ii) If Aut $(C)\cong\mathbb{Z}_2\times D_8$ then C has at least four non-branch 2-Weierstrass points of weight 3. Moreover, if C is isomorphic to the curve

$$y^2=tx^8+tx^4+1,$$

for $t=196$ (resp. $t=-\frac{196}{15}$) then C has in addition 8 other points of weight 3 (resp. 16 points of weight 2).

iii) If Aut $(C)\cong D_{12}$ then C has non-branch 2-Weierstrass points with weight greater than one if and only if C is isomorphic to one of the curves

$$y^2=x(tx^6+tx^3+1),$$

for $t=-\frac{49}{8}$ or $t=\frac{1728}{8}\pm\frac{621}{4}\sqrt{2}$. In the first case, the curve has twelve 2-Weierstrass points of weight 3 and in the other two cases twelve 2-Weierstrass points of weight 2.

iv) If Aut $(C)\cong\mathbb{Z}_2\times\mathbb{Z}_4$ then C has 2-Weierstrass points if and only if C is isomorphic to one of the curves $y^2=\left(tx^4-1\right)\left(tx^4+tx^2+1\right)$, for $t=-8$ or it is a root of

$$t^8+600822\,t^7+71378609\,t^6+4219381768\,t^5+85080645104\,t^4-2272444082944\,t^3 \\ +16480136388352\,t^2-50330309965824\,t+56693912375296=0.$$

In the first case, the curve has two 2-Weierstrass points of weight 3.

5 Concluding remarks

In this paper we explicitly determined the 2-Weierstrass points of genus 3 hyperelliptic curves with extra automorphisms. Similar methods can be used for 3-Weierstrass points even though the computations are longer and more difficult.

The method, especially the result of Lemma 3 can be used for q-Weierstrass points of all superelliptic curves. The automorphism groups of such curves are fully classified and their equations are $y^n = f(x^m)$ for different values of n and m, see [6, 7, 12] among other papers.

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