Transformation Groups
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ORIGINAL PAPER

Generalized Superelliptic Riemann Surfaces

Rubén A. Hidalgo[1] · Saúl Quispe[1] · Tony Shaska[2]

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Abstract

A conformal automorphism τ, of order $n \geqslant 2$, of a closed Riemann surface $\mathcal{X}$, of genus $g \geqslant 2$, which is central in $\mathrm{Aut}(\mathcal{X})$ and such that $\mathcal{X}/\langle\tau\rangle$ has genus zero, is called a superelliptic automorphism of level n. If $n = 2$, then τ is the hyperelliptic involution and it is known to be unique. In this paper, for the case $n \geqslant 3$, we investigate the uniqueness of the cyclic group $\langle\tau\rangle$. Let τ_1 and τ_2 be two superelliptic automorphisms of level n of $\mathcal{X}$. If $n \geqslant 3$ is odd, then $\langle\tau_1\rangle = \langle\tau_2\rangle$. If $n \geqslant 2$ is even, the same uniqueness result holds, up to some explicit exceptional cases. We also provide conditions for these surfaces to be definable over their field of moduli.

Keywords Generalized superelliptic curves · Cyclic gonal curves · Automorphisms · Riemann surfaces

Mathematics Subject Classification (2010) 14H37 · 14H45 · 30F10

1 Introduction

Let $\mathcal{X}$ be a closed Riemann surface of genus $g \geqslant 2$ and let $G = \mathrm{Aut}\,(\mathcal{X})$ be its group of conformal automorphisms. It is well known that $\mathrm{Aut}(\mathcal{X})$ is finite [20] of order at most $84(g-1)$ [13], and that the order of any conformal automorphism is bounded above by $4g+2$. This paper considers certain cyclic subgroups of $\mathrm{Aut}(\mathcal{X})$ which behave similarly to the cyclic subgroup generated by the hyperelliptic involution.

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✉ Rubén A. Hidalgo
ruben.hidalgo@ufrontera.cl

Saúl Quispe
saul.quispe@ufrontera.cl

Tony Shaska
tanush@umich.edu

1 Departamento de Matemática y Estadística, Universidad de La Frontera, Temuco, Chile

2 Department of Mathematics and Statistics, Oakland University, Rochester, MI 48386, USA

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Let $\tau \in G$ be an *n-gonal* automorphism, that is, it has order $n \geqslant 2$ and $\mathcal{X}/\langle\tau\rangle$ has genus zero. In this case, $H = \langle\tau\rangle \cong C_n$ is called an *n-gonal group* and $\mathcal{X}$ a *cyclic n-gonal* Riemann surface. Let N be the normalizer of $\langle\tau\rangle$ in G. It follows from the results in [7, 22], that generically $N = G$.

If $n = 2$, then τ is the hyperelliptic involution and it is known to be unique and central in G; in particular, it is central in $G = N$.

If $n \geqslant 3$ is a prime integer and $s \geqslant 3$ is the number of fixed points of τ, then H is known to be the unique n-Sylow subgroup of G if either (i) $2n < s$ [5] or (ii) $n \geqslant 5s - 7$ [9]. So, in this case, $N = G$; but it might be that τ is non-central.

If $n \geqslant 3$, not necessarily prime, such that: (i) every fixed point of a non-trivial power of τ is also a fixed point of τ, and (ii) the rotation number of τ at each of its fixed points is the same (some authors call τ a **superelliptic automorphism** and $\mathcal{X}$ a **superelliptic surface**), then τ is central in N (see Corollary 1), but in general $N \neq G$. In this case, under the extra condition that $g > (n-1)^2$, it is known that $N = G$ [15] (as a consequence of results in [1]). The computation of G has been done in [19]. Superelliptic Riemann surfaces have been studied in [4, 16, 17] and those with many conformal automorphisms and with CM structures have been considered in [18].

If τ is central in G (respectively, central in N), then we call it a **superelliptic automorphism of level n** (respectively, **generalized superelliptic automorphism of level n**); we also say that $H = \langle\tau\rangle$ is a **superelliptic group of level n** (respectively, **generalized superelliptic group of level n**), and that $\mathcal{X}$ is a **superelliptic curve of level n** (respectively, **generalized superelliptic curve of level n**). A superelliptic automorphism of level n is automatically a generalized one; the converse is in general false (but generically true). Also, as previously noted, a superelliptic automorphism of order n is a generalized superelliptic automorphism of level n (but it might not be a superelliptic automorphism of level n).

In this paper, (i) we provide necessary and sufficient conditions for an n-gonal automorphism to be a generalized superelliptic automorphism of level n (Theorem 1) and (ii) we provide conditions for a superelliptic curve of level n to have a unique superelliptic group of level n (Theorem 2 and Corollary 2).

Before stating the above two results, let us recall some facts on n-gonal automorphisms. Let us consider a pair $(\mathcal{X}, \tau)$, where τ is an n-gonal automorphism of $\mathcal{X}$, and set $H = \langle\tau\rangle \cong C_n$. Let $\pi : \mathcal{X} \to \widehat{\mathbb{C}}$ be a Galois branched covering, whose deck covering group is H, and let $p_1, \ldots, p_s \in \widehat{\mathbb{C}}$ be its branch values. Then there are integers $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$ satisfying that $l_1 + \cdots + l_s$ is a multiple of n and $\gcd(n, l_1, \ldots, l_s) = 1$, such that $\mathcal{X}$ can be described by an affine irreducible algebraic curve (which might have singularities) of the following form (called a **cyclic n-gonal curve**)

$$y^n = \textstyle\prod_{j=1}^{s}(x - p_j)^{l_j}. \tag{1}$$

If one of the branch values is ∞, say $p_s = \infty$, then we need to delete the factor $(x - p_s)^{l_s}$ from the above equation. In this algebraic model, τ and π are given respectively by $\tau(x, y) = (x, \omega_n y)$, where $\omega_n = e^{2\pi i/n}$, and $\pi(x, y) = x$.

Theorem 1 *Let $\mathcal{X}$ be a cyclic n-gonal Riemann surface, described by the cyclic n-gonal curve* (1)*, and N be the normalizer of $H = \langle\tau(x, y) = (x, \omega_n y)\rangle$ in* Aut $(\mathcal{X})$*. Let $\theta : N \to \overline{N} = N/H$ be the canonical projection homomorphism. Then τ is a* *generalized superelliptic automorphism of level n if and only if for all p_j and p_i in the same $\theta(N)$-orbit it holds that $l_j = l_i$.*

Corollary 1 *Let $\mathcal{X}$ be a cyclic n-gonal Riemann surface, described by the cyclic n-gonal curve (1). If $l_j = l$, for every j, where $\gcd(n, l) = 1$, then $\tau(x, y) = (x, \omega_n y) \in \mathrm{Aut}\,(\mathcal{X})$ is a generalized superelliptic automorphism of level n.*

Remark 1 The above corollary states that a superelliptic automorphism is always a generalized superelliptic automorphism of level n (but not necessarily a superelliptic automorphism of level n as the normalizer N might be smaller than the full group of automorphisms).

Theorem 2 *Let $\mathcal{X}$ be a cyclic n-gonal Riemann surface, admitting two superelliptic automorphisms τ and η, both of level n, such that $\langle\tau\rangle \neq \langle\eta\rangle$. Then*

(I) $\mathrm{Aut}(\mathcal{X})/\langle\tau\rangle$ is either a non-trivial cyclic group of even order or a dihedral group of order a multiple of four;
(II) there is an integer $d \geqslant 2$ such that $n = 2d$ and $\mathcal{X}$ can be represented by a cyclic n-gonal curve of the form

$$\mathcal{X}: \quad y^{2d} = x^2\left(x^2-1\right)^{l_1}\left(x^2-a_2^2\right)^{l_2}\prod_{j=3}^{L}\left(x^2-a_j^2\right)^{2\widehat{l_j}}, \tag{2}$$

where $l_1, l_2, 2\widehat{l_3}, \ldots, 2\widehat{l_L} \in \{1, \ldots, 2d-1\}$, l_1 is odd, and either one of the two conditions (a) or (b) below holds for l_2.

(a) If $l_2 = 2\widehat{l_2}$, then $\gcd\left(d, l_1, \widehat{l_2}, \ldots, \widehat{l_L}\right) = 1$.
(b) If l_2 is odd, then $\gcd\left(d, l_1, l_2, \widehat{l_3}, \ldots, \widehat{l_L}\right) = 1$.

In these cases, $\tau(x, y) = (x, \omega_{2d}y)$, $\eta(x, y) = (-x, \omega_{2d}y)$ (so $\tau^2 = \eta^2$) and $\langle\tau, \eta\rangle = \langle\tau, \eta : \tau^{2d} = 1, \tau^2 = \eta^2, \tau\eta = \eta\tau\rangle \cong C_{2d} \times C_2$.

Those superelliptic Riemann surfaces of level $n = 2d$, described by the cyclic $2d$-gonal curves in Theorem 2, will be called **exceptionals**.

Remark 2 The cyclic $2d$-gonal curves $\mathcal{X}$, defined by Eq. 2, are cyclic $2d$-gonal curves $\mathcal{X}$ admitting two commuting cyclic $2d$-gonal automorphisms, τ and η, such that $\langle\tau\rangle \neq \langle\eta\rangle$. We should note that not all of them need to be superelliptic of level n; the theorem only asserts that the exceptional ones are some of them. For instance, in the case (b) with $d = 2$, $l_1 = 1$ and $l_2 = 3$, the genus five curve $\mathcal{X}: \quad y^4 = x^2(x^2-1)(x^2+1)^3$ admits the automorphism $\rho(x, y) = \left(ix, \frac{\sqrt{i}y^3}{x(x^2+1)^2}\right)$, for which $\rho\tau\rho^{-1} = \tau^3$, where $\tau(x, y) = (x, iy)$.

Corollary 2 *Let $\mathcal{X}$ be a Riemann surface admitting a superelliptic group H of level n. Then H is the unique superelliptic group of level n of $\mathcal{X}$ if either: (1) $n = 2$, or (2) $n \geqslant 3$ is odd , or (3) $n \geqslant 4$ is even and $\mathcal{X}/H$ has no cone point of order $n/2$.*

Finally, in the last section, we provide some discussion on the field of moduli of these superelliptic Riemann surfaces (see Theorem 5).

Notation We denote by C_n the cyclic group of order n, by D_n the dihedral group of order $2n$, by A_n the alternating group, and by S_n the symmetric group.

2 Preliminaries

2.1 The Finite Groups of Möbius Transformations

Up to $\mathrm{PSL}_2(\mathbb{C})$-conjugation, the finite subgroups of the group $\mathrm{PSL}_2(\mathbb{C})$ of Möbius transformations are given by (see, for instance, [3])

$$
\begin{array}{l}
C_m := \langle a(x) = \omega_m x \rangle,\ D_m := \left\langle a(x) = \omega_m x, b(x) = \frac{1}{x} \right\rangle, \\
A_4 := \left\langle a(x) = -x, b(x) = \frac{i-x}{i+x} \right\rangle,\ S_4 := \left\langle a(x) = ix, b(x) = \frac{i-x}{i+x} \right\rangle, \\
A_5 := \left\langle a(x) = \omega_5 x, b(x) = \frac{(1-\omega_5^4)x + (\omega_5^4 - \omega_5)}{(\omega_5 - \omega_5^3)x + (\omega_5^2 - \omega_5^3)} \right\rangle,
\end{array} \tag{3}
$$

where ω_m is a primitive m-th root of unity. For each of the above finite groups A, a Galois branched covering $f_A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, with deck group A, is given as follows

$$
\begin{array}{lll}
f_{C_m}(x) = x^m; & & \text{branching: } (m, m). \\
f_{D_m}(x) = x^m + x^{-m}; & & \text{branching: } (2, 2, m). \\
f_{A_4}(x) = \dfrac{(x^4 - 2i\sqrt{3}x^2 + 1)^3}{-12i\sqrt{3}x^2(x^4-1)^2}; & & \text{branching: } (2, 3, 3). \\
f_{S_4}(x) = \dfrac{(x^8 + 14x^4 + 1)^3}{108x^4(x^4-1)^4}; & & \text{branching: } (2, 3, 4). \\
f_{A_5}(x) = \dfrac{(-x^{20} + 228x^{15} - 494x^{10} - 228x^5 - 1)^3}{1728x^5(x^{10} + 11x^5 - 1)^5}; & & \text{branching: } (2, 3, 5),
\end{array}
$$

see [11]. In the above, the branching corresponds to the tuple of branch orders of the cone points of the orbifold $\widehat{\mathbb{C}}/A$.

2.2 Fuchsian Groups

A *Fuchsian group* is a discrete subgroup Δ of $\mathrm{PSL}_2(\mathbb{R})$, the group orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$. It is called co-compact if the quotient orbifold $\mathbb{H}/\Delta$ is compact; its signature is the tuple $(g; n_1, \ldots, n_s)$, where g is the genus of the quotient orbifold $\mathbb{H}/\Delta$, s is the number of its cone points they having branch orders $n_1, \ldots, n_s$. The group Δ has a presentation as follows:

$$
\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_s : c_1^{n_1} = \cdots = c_s^{n_s} = 1, c_1 \cdots c_s [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle, \tag{4}
$$

where $[a, b] = aba^{-1}b^{-1}$.

If a co-compact Fuchsian group Γ has no torsion, then $\mathcal{X} = \mathbb{H}/\Gamma$ is a closed Riemann surface of genus $g \geqslant 2$ and its signature is $(g; -)$. Conversely, by the uniformization theorem, every closed Riemann surface of genus $g \geqslant 2$ can be represented as above. In this case, by Riemann's existence theorem, a finite group G acts faithfully as a group of conformal automorphisms of $\mathcal{X}$ if and only if there is a co-compact Fuchsian group Δ and a surjective homomorphism $\theta : \Delta \to G$ whose kernel is Γ.

2.3 Cyclic *n*-gonal Riemann Surfaces

Let $\mathcal{X}$ be a cyclic n-gonal Riemann surface of genus $g \geqslant 2$, $\tau \in \text{Aut}(\mathcal{X})$ be an n-gonal automorphism and $\pi : \mathcal{X} \to \widehat{\mathbb{C}}$ be a Galois branched cover whose deck group is the n-gonal group $H = \langle\tau\rangle \cong C_n$. Let $p_1, \ldots, p_s \in \widehat{\mathbb{C}}$ be the branch values of π and let us denote by $n_j \geqslant 2$ (which is a divisor of n) the branch order of π at p_j.

Let Δ be a Fuchsian group such that (up to biholomorphisms) $\mathbb{H}/\Delta = \mathcal{X}/\langle\tau\rangle$. Then Δ has signature $(0; n_1, \ldots, n_s)$ and a presentation

$$\Delta = \langle c_1, \ldots, c_s : c_1^{n_1} = \cdots = c_s^{n_s} = 1, c_1 \cdots c_s = 1\rangle. \tag{5}$$

The branched Galois covering π is determined by a surjective homomorphism $\rho : \Delta \to C_n = \langle\tau\rangle$ with a torsion-free kernel Γ such that $\mathcal{X} = \mathbb{H}/\Gamma$. (The homomorphism ρ is uniquely determined up to post-composition by automorphisms of C_n and pre-composition by an automorphism of Δ.) Let $\rho(c_j) = \tau^{l_j}$, where c_j is as in Eq. 4, for $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$. As a consequence of Harvey's criterion [8],

(a) $n = \text{lcm}(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_s)$ for all j;
(b) if n is even, then $\#\{j \in \{1, \ldots, s\} : n/n_j \text{ is odd}\}$ is even.

The equality $c_1 \cdots c_s = 1$ is equivalent to have $l_1 + \cdots + l_s \equiv 0 \bmod(n)$, and the condition for $\Gamma = \ker(\rho)$ to be torsion-free is equivalent to have $\gcd(n, l_j) = n/n_j$, for $j = 1, \ldots, s$. The surjectivity of ρ is equivalent to have $\gcd(n, l_1, \ldots, l_s) = 1$, which in our case is equivalent to condition (a). Condition (b) is equivalent to saying that for n even the number of l_j's being odd is even, which trivially holds. Summarizing all the above,

(1) $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$, (3) $\gcd(n, l_j) = n/n_j$, for all j,
(2) $l_1 + \cdots + l_s \equiv 0 \bmod(n)$, (4) $\gcd(n, l_1, \ldots, l_s) = 1$.

The Riemann surface $\mathcal{X}$ can be described by the affine curve

$$\mathcal{X} : \quad y^n = \prod_{j=1}^{s} (x - p_j)^{l_j}, \tag{6}$$

where, if one of the branched values is infinity, say $p_s = \infty$, then we need to delete the factor $(x - p_s)^{l_s}$ in the above equation.

In such an algebraic model, $\tau(x, y) = (x, \omega_n y)$, where $\omega_n = e^{2\pi i/n}$, and $\pi(x, y) = x$. The branch order of π at p_j is $n_j = n/\gcd(n, l_j)$ and, by the Riemann-Hurwitz formula, the genus g of $\mathcal{X}$ is given by

$$g = 1 + \frac{1}{2}\left((s-2)n - \sum_{j=1}^{s} \gcd(n, l_j)\right). \tag{7}$$

3 Proof of Theorem 1

Let $\mathcal{X}$ be a curve given by Eq. 1, $\pi(x, y) = x$, and $\tau \in G = \text{Aut}(\mathcal{X})$. Let N be the normalizer of $H = \langle\tau\rangle$ in G. There is a short exact sequence

$$1 \to H = \langle\tau\rangle \to N \xrightarrow{\theta} \overline{N} = N/H \to 1, \tag{8}$$

where $\theta(\eta) \circ \pi = \pi \circ \eta$, for every $\eta \in N$.

The reduced group of automorphisms $\overline{N} = N/H < \mathrm{PSL}_2(\mathbb{C})$ is a finite group keeping invariant the set $\{p_1, \ldots, p_s\}$.

3.1 The following describes the form of those elements of N.

Lemma 1 *Let $\eta \in N$ and $l \in \{1, \ldots, n-1\}$ (necessarily relatively prime to n) such that $\eta\tau\eta^{-1} = \tau^l$. If $b = \theta(\eta)$, then $\eta(x, y) = (b(x), y^l Q(x))$, where $Q(x)$ is a suitable rational map.*

Proof Let us note that $\eta(x, y) = (b(x), L(x, y))$, where $L(x, y)$ is a suitable rational map. As $\eta(\tau(x, y)) = \eta(x, \omega_n y) = (b(x), L(x, \omega_n y))$ and $\tau^l(\eta(x, y)) = \tau^l(b(x), L(x, y)) = (b(x), \omega_n^l L(x, y))$, the condition $\eta\tau\eta^{-1} = \tau^l$ holds if and only if $L(x, \omega_n y) = \omega_n^l L(x, y)$, that is, $L(x, y) = Q(x)y^l$, for a suitable rational map $Q(x) \in \mathbb{C}(x)$. $\square$

Remark 3 (1) Lemma 1 asserts that those $\eta \in N$ commuting with τ have the form $\eta(x, y) = (b(x), Q(x)y)$. (2) If $t \in \mathrm{PSL}_2(\mathbb{C})$, then replacing π by $t \circ \pi$ only exchanges the set of branch points $\{p_1, \ldots, p_s\}$ for $\{t(p_1), \ldots, t(p_s)\}$ but keeps invariant the set of exponents $l_1, \ldots, l_s$.

3.2 Let $\eta \in N$ and assume $\theta(\eta)$ has order $m \geqslant 2$. As there is a suitable $t \in \mathrm{PSL}_2(\mathbb{C})$ such that $t\theta(\eta)t^{-1}(x) = \omega_m x$, we may assume (by post-composing π with t) that $\theta(\eta)(x) = \omega_m x$. So the cyclic n-gonal curve Eq. 6 can be written as

$$y^n = x^\alpha \prod_{j=1}^{L} (x - q_j)^{l_{j,1}} (x - \omega_m q_j)^{l_{j,2}} \cdots (x - \omega_m^{m-1} q_j)^{l_{j,m}}, \tag{9}$$

where

(a) the factor x^α only appears if one of the branch values $t(p_j)$ is equal to zero, and
(b) there is the following equality of sets of exponents in Eqs. 9 and 6 (where α needs to be deleted if none of the $t(p_j)$'s is equal to zero)

$$\{\alpha, l_{1,1}, \ldots, l_{1,m}, l_{2,1}, \ldots, l_{2,m}, \ldots, l_{L,1}, \ldots, l_{L,m}\} = \{l_1, \ldots, l_s\}.$$

In this model, $\tau(x, y) = (x, \omega_n y)$ and, by Lemma 1, $\eta(x, y) = (\omega_m x, Q(x)y^l)$, for a suitable rational map $Q(x) \in \mathbb{C}(x)$.

If $R(x)$ denotes the right side of Eq. 9, then $Q(x)^n y^{ln} = R(\omega_m x)$ on $\mathcal{X}$, where

$$\begin{aligned} R(\omega_m x) &= \omega_m^\alpha x^\alpha \prod_{j=1}^{L} \frac{\omega_m^{r_j} (x-q_j)^{l_{j,1}} (x-\omega_m q_j)^{l_{j,2}} \cdots (x-\omega_m^{m-1} q_j)^{l_{j,m}}}{(x-q_j)^{l_{j,1}-l_{j,2}} (x-\omega_m q_j)^{l_{j,2}-l_{j,3}} \cdots (x-\omega_m^{m-1} q_j)^{l_{j,m}-l_{j,1}}} \\ &= \frac{\omega_m^{(\alpha+\sum_{j=1}^{L} r_j)} y^n}{\prod_{j=1}^{L} (x-q_j)^{l_{j,1}-l_{j,2}} (x-\omega_m q_j)^{l_{j,2}-l_{j,3}} \cdots (x-\omega_m^{m-1} q_j)^{l_{j,m}-l_{j,1}}}, \end{aligned} \tag{10}$$

and $r_j = l_{j,1} + \cdots + l_{j,m}$, that is,

$$Q(x)^n y^{ln} = \frac{\omega_m^{(\alpha+\sum_{j=1}^{L} r_j)} y^n}{\prod_{j=1}^{L} (x-q_j)^{l_{j,1}-l_{j,2}} (x-\omega_m q_j)^{l_{j,2}-l_{j,3}} \cdots (x-\omega_m^{m-1} q_j)^{l_{j,m}-l_{j,1}}}. \tag{11}$$

In particular,

$$Q(x)^n y^{(l-1)n} = \frac{\omega_m^{(\alpha+\sum_{j=1}^{L} r_j)}}{\prod_{j=1}^{L}(x-q_j)^{l_{j,1}-l_{j,2}}(x-\omega_m q_j)^{l_{j,2}-l_{j,3}}\cdots(x-\omega_m^{m-1}q_j)^{l_{j,m}-l_{j,1}}}. \tag{12}$$

3.3 Let us assume η commutes with τ, that is, $l=1$. We proceed to prove that the exponents $l_{j,i}$ are the same for every $i=1,\ldots,m$. As $\theta(\eta)^m=1$, it follows that $\eta^m \in \langle\tau\rangle$, from which we must have that

$$\left(\prod_{j=0}^{m-1} Q(\omega_m^j x)\right)^n = 1. \tag{13}$$

Claim 1 Equation 13 asserts that $Q(x)$ is either an nm-root of unity or it has the form

$$Q(x) = \lambda \prod_{u=1}^{A} \frac{x-\alpha_u}{x-\omega_m^{q_u}\alpha_u},$$

where $\lambda^{nm}=1$ and $q_u \in \{1,\ldots,m-1\}$.

Proof If we write

$$Q(x) = \lambda \frac{\prod_{u=1}^{A}(x-\alpha_u)}{\prod_{v=1}^{B}(x-\beta_v)},$$

then

$$\prod_{j=0}^{m-1} Q(\omega_m^j x) = \lambda^m \prod_{j=0}^{m-1} \omega_m^{(A-B)j} \frac{\prod_{u=1}^{A}(x-\omega_m^{m-j}\alpha_u)}{\prod_{v=1}^{B}(x-\omega_m^{m-j}\beta_v)} =$$

$$= \lambda^m \omega_m^{(A-B)m(m-1)/2} \frac{\prod_{u=1}^{A}(x^m-\alpha_u^m)}{\prod_{v=1}^{B}(x^m-\beta_v^m)}.$$

Equation 13 asserts that

$$A=B, \quad \prod_{v=1}^{B}(x^m-\beta_v^m) = \lambda^{nm}\prod_{u=1}^{A}(x^m-\alpha_u^m).$$

So, $\lambda^{nm}=1$ and, up to a permutation of indices, we may assume $\alpha_u^m=\beta_u^m$, for $u=1,\ldots,A$. □

By Claim 1, either $l_{j,i}-l_{j,i+1}=0$ or $\omega_m^{i-1}q_j$ must be either a zero or a pole of order n of the left side of Eq. 12, that is, each $l_{j,i}-l_{j,i+1} \in \{0,\pm n\}$. As $l_{j,i} \in \{1,\ldots,n-1\}$, it follows that $l_{j,1}=\cdots=l_{j,m}$.

3.4 In the other direction, let us assume that $l_{j,1}=\cdots=l_{j,m}=l_j$, for every $j=1,\ldots,L$. In this case, $\mathcal{X}$ has equation

$$y^n = x^\alpha \prod_{j=1}^{L}(x^m-q_j^m)^{l_j}. \tag{14}$$

A lifting of $\theta(\eta)$ under $\pi(x,y)=x$ is of the form $\widehat{\eta}(x,y)=(\omega_m x, \omega_m^{\alpha/n}y)$. This asserts that $\eta=\widehat{\eta}\tau^k$, for some $k\in\{0,\ldots,n-1\}$, i.e., $\eta(x,y)=(\omega_m x, \omega_n^k\omega_m^{\alpha/n}y)$, that is $l=1$.

3.1 A Consequence

The above permits us to observe that, if τ is a generalized superelliptic automorphism of level n, and the reduced group $\overline{N}$ admits an element of order m, then $\mathcal{X}$ can be represented by a cyclic n-gonal curve of the form

$$\mathcal{X}: \quad y^n = x^{l_0}(x^m - 1)^{l_1} \prod_{j=2}^{L}(x^m - a_j^m)^{l_j}, \tag{15}$$

where any one of the following Harvey's conditions is satisfied:

1. if $l_0 = 0$, then $m(l_1 + \cdots + l_L) \equiv 0 \operatorname{mod}(n)$ and $\gcd(n, l_1, \ldots, l_L) = 1$; or
2. if $l_0 \neq 0$, then $\gcd(n, l_0, l_1, \ldots, l_L) = 1$.

Note that, in (2) above, either: (2.1) $l_0 + m(l_1 + \cdots + l_L) \equiv 0 \operatorname{mod}(n)$ in case ∞ is not a branch value, or (2.2) $l_0 + m(l_1 + \cdots + l_L) \not\equiv 0 \operatorname{mod}(n)$ in case ∞ is a branch value.

4 Proof of Theorem 2 and Corollary 2

Let us assume $\mathcal{X}$ admits two superelliptic automorphisms τ and η, both of level n, that is, each one being central in $G = \mathrm{Aut}\,(\mathcal{X})$. Let $H = \langle \tau \rangle$ and the reduced group $\overline{G} = G/H$. We proceed to investigate when it is possible to have that $\eta \notin H$.

4.1 Proof of Theorem 2

As the case $n = 2$ corresponds to the hyperelliptic situation, and the hyperelliptic involution is unique, necessarily $n \geqslant 3$.

Proposition 1 *If $\overline{G}$ is either trivial, a dihedral group of order not divisible by* 4 *or* A_4 *or* S_4 *or* A_5*, then* $\eta \in H$.

Proof Assume, to the contrary, that $\eta \notin H$. Then η induces a non-trivial central element of the reduced group $\overline{G}$. As the Platonic groups and the dihedral groups of order not divisible by 4, have no nontrivial central element, this is a contradiction.

□

Let us assume that $\eta \notin H$. So, by the above, $n \geqslant 3$ and $\overline{G}$ is either a non-trivial cyclic group or a dihedral group of order a multiple of 4. Let us consider, as before, the canonical quotient homomorphism $\theta : G \to \overline{G}$, and let $\pi : \mathcal{X} \to \widehat{\mathbb{C}}$ be a Galois branched cover with deck group H. As τ is central, $K = \langle \tau, \eta \rangle < G$ is an abelian group and $\overline{K} = K/H = \langle \theta(\eta) \rangle \cong C_m$, where $n = md$ and $m \geqslant 2$. Since $\theta(\eta)$ has order m, $\eta^m \in H$ and it has order d. So, replacing τ by a suitable power (still being a generator of H) we may assume that $\eta^m = \tau^m$. Now, as noted in Section 3.1, we may assume $\mathcal{X}$ to be represented by a cyclic n-gonal curve of the form

$$\mathcal{X}: \quad y^n = x^{l_0}(x^m - 1)^{l_1} \prod_{j=2}^{L}(x^m - a_j^m)^{l_j}, \tag{16}$$

where one of the following Harvey's conditions is satisfied:

(C1) $l_0 = 0, m(l_1 + \cdots + l_L) \equiv 0 \mod (n)$ and $\gcd(n, l_1, \ldots, l_L) = 1$; or
(C2) $l_0 \neq 0$ and $\gcd(n, l_0, l_1, \ldots, l_L) = 1$.

In this algebraic model, $\tau(x, y) = (x, \omega_n y)$, $\pi(x, y) = x$ and $\theta(\eta)(x) = \omega_m x$ (where $\omega_t = e^{2\pi i/t}$). In this way, $\eta(x, y) = (\omega_m x, \omega_m^{l_0/n} y)$. Since $\eta^m = \tau^m$ and η has order n, we may assume the following

$$\begin{cases} \text{if } l_0 \neq 0 : \eta(x, y) = (\omega_m x, \omega_n y) \text{ and } l_0 = m, \\ \text{if } l_0 = 0 : \eta(x, y) = (\omega_m x, y) \quad \text{ and } n = m. \end{cases} \tag{17}$$

i): *Case* $l_0 = m$. In this case, $\eta(x, y) = (\omega_m x, \omega_n y)$ and we are in case (C2) above. The η-invariant algebra $\mathbb{C}[x, y]^{\langle \eta \rangle}$ is generated by the monomials $u = x^m$, $v = y^n$ together with those of the form $x^a y^b$, where $a \in \{0, 1, \ldots, m-1\}$ and $b \in \{0, 1, \ldots, n-1\}$ (where the case $a = b = 0$ is not considered) satisfy that $a + b/d \equiv 0 \mod (m)$. In particular, $b = dr$ for $r \in \{0, 1, \ldots, [(n-1)/d]\}$ so that $a + r \equiv 0 \mod (m)$. As $0 \leqslant a + r \leqslant (m-1) + [(n-1)/d] \leqslant (m-1) + [(md-1)/d] < 2m$, it follows that $a + r \in \{0, m\}$. As the case $a + r = 0$ asserts that $a = b = 0$, which is not considered, we must have $a + r = m$, from which we see that the other generators are given by $t_1, \ldots, t_{m-1}$, where $t_j = x^{m-j} y^{dj}$. As consequence of invariant theory, the quotient curve $\mathcal{X}/\langle \eta \rangle$ corresponds to the algebraic curve

$$\mathcal{Y} : \quad \begin{cases} t_1^m & = u^{m-1} v, \\ t_2^m & = u^{m-2} v^2, \\ & \vdots \\ t_{m-1}^m & = u v^{m-1}, \\ v & = u(u-1)^{l_1} \prod_{j=2}^{L} (u - a_j^m)^{l_j}. \end{cases} \tag{18}$$

The curve $\mathcal{Y}$ admits the automorphisms $T_1, \ldots, T_{m-1}$, where T_j is just an amplification of the t_j-coordinate by ω_m and acts as the identity on all the other coordinates. The group generated by all of these automorphisms is

$$\mathcal{U} = \langle T_1, \ldots, T_{m-1} \rangle \cong C_m^{m-1}. \tag{19}$$

The Galois branched cover map $\pi_{\mathcal{U}} : \mathcal{Y} \to \widehat{\mathbb{C}} : (u, v, t_1, \ldots, t_{m-1}) \mapsto u$ has $\mathcal{U}$ as its deck group. Let us observe that the values $0, a_1^m, \ldots, a_L^m$ belong to the branch set of $\pi_{\mathcal{U}}$. Since $\mathcal{Y} = \mathcal{X}/\langle \eta \rangle$ has genus zero and the finite abelian groups of automorphisms of the Riemann sphere are either the trivial group, a cyclic group or $V_4 = C_2^2$, the group $\mathcal{U}$ is one of these three types. As $m \geqslant 2$, the group $\mathcal{U}$ cannot be the trivial group nor can it be isomorphic to the Klein group $V_4 = C_2^2$. It follows that $\mathcal{U}$ is a cyclic group; so $m = 2$ and, in particular, $n = 2d$, where $d \geqslant 2$, and

$$\mathcal{X} : \quad y^{2d} = x^2 (x^2 - 1)^{l_1} \prod_{j=2}^{L} (x^2 - a_j^2)^{l_j}. \tag{20}$$

Harvey's condition (a) is equivalent to have $\gcd(2d, 2, l_1, \ldots, l_L) = 1$, which is satisfied if some of the exponents l_j is odd. Without loss of generality, we may assume that l_1 is odd. In this case, the curve $\mathcal{Y}$ is given by

$$\mathcal{Y}: \quad \begin{cases} t_1^2 = uv, \\ v \ = u(u-1)^{l_1} \prod_{j=2}^{L}(u-a_j^2)^{l_j}, \end{cases} \tag{21}$$

which is isomorphic to the curve

$$w^2 = (u-1)^{l_1} \prod_{j=2}^{L}(u-a_j^2)^{l_j}. \tag{22}$$

As this curve must have genus zero, and l_1 is odd, the number of indices $j \in \{2, \dots, L\}$ for which l_j is odd must be at most one.

(i) If l_1 is the only odd exponent and $l_j = 2\widehat{l_j}$, for $j = 2, \dots, L$, then the condition $\gcd(2d, 2, l_1, 2\widehat{l_2}, \dots, 2\widehat{l_L}) = 1$ is equivalent to $\gcd(d, l_1, \widehat{l_2}, \dots, \widehat{l_L}) = 1$.

(ii) If there are exactly two of the exponents being odd, then we may assume, without loss of generality, that l_1 and l_2 are the only odd exponents. This means that the curve in Eq. 22 is isomorphic to $\widehat{w}^2 = (u-1)^{l_1}(u-a_2^2)^{l_2}$, where $\widehat{w} = w / \prod_{j=3}^{L}(u-a_j)^{l_j/2}$. If we write $l_j = 2\widehat{l_j}$, for $j = 3, \dots, L$, then the condition $\gcd(2d, 2, l_1, l_2, 2\widehat{l_3}, \dots, 2\widehat{l_L}) = 1$ is equivalent to $\gcd(d, l_1, l_2, \widehat{l_3}, \dots, \widehat{l_L}) = 1$.

ii): *Case* $l_0 = 0$. In this case, $m = n$, $\eta(x, y) = (\omega_n x, y)$ and we are in case (C1) above. The η-invariants algebra $\mathbb{C}[x, y]^{\langle \eta \rangle}$ is generated by the monomials $u = x^n$, $v = y$. As a consequence of the invariant theory, the quotient curve $\mathcal{X}/\langle \eta \rangle$ corresponds to one of the following algebraic curves

$$\mathcal{Y}_1: \quad v^n = (u-1)^{l_1}, \tag{23}$$

or

$$\mathcal{Y}_2: \quad v^n = (u-1)^{l_1} \prod_{j=2}^{L}(u-a_j^n)^{l_j}. \tag{24}$$

As $\mathcal{Y}$ must have genus zero and $n \geqslant 3$, we should have either $\mathcal{Y}_1$ or $\mathcal{Y}_2$ with $L = 2$ and $l_1 + l_2 \equiv 0 \mod (n)$. In particular, we have one of the two cases below for $\mathcal{X}$:

$$\begin{array}{ll} (1)\ \mathcal{X}: & y^n = (x^n-1)^{l_1}. \\ (2)\ \mathcal{X}: & y^n = (x^n-1)^{l_1}(x^n-a_2^n)^{l_2},\ l_1 + l_2 \equiv 0 \mod (n). \end{array} \tag{25}$$

Note that, for situation (1) above, we may assume $l_1 = 1$ (this is the classical Fermat curve of degree n). As the group of automorphisms of the classical Fermat curve of degree n is $C_n^2 \rtimes S_3$, we may see that τ is not central; that is, it is not a generalized superelliptic Riemann surface of level n. In case (2), Harvey's conditions hold exactly when $\gcd(n, l_1, l_2) = 1$. As $l_1 + l_2 \equiv 0 \mod (n)$ and $l_1, l_2 \in \{1, \dots, n-1\}$, we have that $l_1 + l_2 = n$. If we write $l_2 = n - l_1$, then

$$\left(\frac{x^n-1}{x^n-a_2^n}\right)^{l_1} = \frac{y^n}{(x^n-a_2^n)^n}, \tag{26}$$

and by writing $l_1 = n - l_2$ we also have that

$$\left(\frac{x^n - a_2^n}{x^n - 1}\right)^{l_2} = \frac{y^n}{(x^n - 1)^n}. \tag{27}$$

Then the Möbius transformation $M(x) = a_2/x$ induces the automorphism

$$\alpha(x, y) = \left(\omega_n \frac{a_2}{x}, \frac{-a_2^{l_2}(x^n - 1)(x^n - a_2^n)}{x^n y}\right), \tag{28}$$

which does not commute with $\eta(x, y) = (\omega_n x, y)$ since $n \geqslant 3$, a contradiction.

4.2 Proof of Corollary 2

Let $\mathcal{X}$ be a cyclic n-gonal Riemann surface admitting two superelliptic automorphisms τ and η, both of level n, such that $\langle \tau \rangle \neq \langle \eta \rangle$. By Theorem 2, $\mathcal{X}$ has an equation of the form as in Eq. 20, where $n = 2d \geqslant 4$. The factor x^2 in such an equation asserts that 0 is a branch value of order $d = n/2$.

5 A Remark on the Field of Moduli of Superelliptic Curves

5.1 Field of Definitions and the Field of Moduli

As a consequence of the Riemann-Roch theorem, every closed Riemann surface $\mathcal{X}$ can be described as a complex projective irreducible algebraic curve, say defined as the common zeros of the homogeneous polynomials $P_1, \ldots, P_r$. If $\sigma \in \mathrm{Gal}(\mathbb{C})$, the group of field automorphisms of $\mathbb{C}$, then X^σ will denote the curve defined as the common zeros of the polynomials $P_1^\sigma, \ldots, P_r^\sigma$, where P_j^σ is obtained from P_j by applying σ to its coefficients. The new algebraic curve $\mathcal{X}^\sigma$ is again a closed Riemann surface of the same genus. Let us observe that, if $\sigma, \tau \in \mathrm{Gal}(\mathbb{C})$, then $X^{\sigma\tau} = (X^\sigma)^\tau$ (we multiply the permutations from left to right). A subfield $\mathbb{L}$ of $\mathbb{C}$ is called a *field of definition* of $\mathcal{X}$ if there is a curve $\mathcal{Y}$, defined over $\mathbb{L}$, which is isomorphic to $\mathcal{X}$ over $\mathbb{C}$. Weil's descent theorem [23] provides sufficient conditions for a given subfield of $\mathbb{C}$ to be a field of definition of $\mathcal{X}$. These conditions hold if $\mathcal{X}$ has no non-trivial automorphisms (a generic situation for $g \geqslant 3$).

If $G_{\mathcal{X}}$ is the subgroup of $\mathrm{Gal}(\mathbb{C})$ consisting of those σ so that $\mathcal{X}^\sigma$ is isomorphic to $\mathcal{X}$, then the fixed field $M_{\mathcal{X}}$ of $G_{\mathcal{X}}$ is called *the field of moduli* of $\mathcal{X}$. The notion of the field of moduli was originally introduced by Shimura [21] for the case of abelian varieties and later extended to more general algebraic varieties by Koizumi [14]. In that same paper, Koizumi observed that: (i) $M_{\mathcal{X}}$ is the intersection of all the fields of definition of $\mathcal{X}$, and (ii) $\mathcal{X}$ has a field of definition being a finite extension of $M_{\mathcal{X}}$.

There are examples for which the field of moduli is not a field of definition [6, 21]. In [10], the following sufficient condition for a surface to be definable over its field of moduli was obtained.

Theorem 3 *Let $\mathcal{X}$ be a Riemann surface of genus $g \geqslant 2$ admitting a subgroup $L <$ Aut $(\mathcal{X})$ so that $\mathcal{X}/L$ has genus zero. If L is unique in* Aut $(\mathcal{X})$ *and the reduced group* Aut $(\mathcal{X})/L$ *is different from trivial or cyclic, then $\mathcal{X}$ is definable over its field of moduli.*

If $\mathcal{X}$ is hyperelliptic and L is the cyclic group generated by the hyperelliptic involution, then the above result is due to Huggins [12].

Another sufficient condition on a curve $\mathcal{X}$ to be definable over its field of moduli, which in particular contains the case of quasiplatonic curves, was provided in [2]. We say that $\mathcal{X}$ has *odd signature* if $\mathcal{X}/$ Aut $(\mathcal{X})$ has genus zero and in its signature one of the cone orders appears an odd number of times.

Theorem 4 *Let $\mathcal{X}$ be a Riemann surface of genus $g \geqslant 2$. If $\mathcal{X}$ has an odd signature, then it can be defined over its field of moduli.*

5.2 Minimal Fields of Definition of Superelliptic Curves

Let $\mathcal{X}$ be a superelliptic curve of level n and $H = \langle \tau \rangle \leqslant$ Aut $(\mathcal{X})$ be a superelliptic group of level n. If $\mathcal{X}$ is non-exceptional, then H is unique (Corollary 2). So, if Aut $(\mathcal{X})/H$ is different from trivial or cyclic, then $\mathcal{X}$ is definable over its field of moduli by Theorem 3. By Theorem 4, the same result holds if $\mathcal{X}$ has odd signature.

At the level of the exceptional ones, we have seen that H is not unique. But, if η is another superelliptic automorphism of level n, then there is a power of η inside H. In this case, we have seen that the quotient of $\mathcal{X}$ by the abelian group $K = \langle \tau, \eta \rangle$ has an odd signature. If Aut $(\mathcal{X}) = K$, then again $\mathcal{X}$ is definable over its field of moduli.

Theorem 5 *Let $H \cong C_n$ be a superelliptic group of a superelliptic curve $\mathcal{X}$. Then $\mathcal{X}$ is definable over its field of moduli if $\mathcal{X}$ is non-exceptional with either (i)* $\mathrm{Aut}(\mathcal{X})/H$ *different from trivial or cyclic or (ii)* $\mathrm{Aut}(\mathcal{X})/H$ *either trivial or cyclic and $\mathcal{X}$ has an odd signature.*

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