REDUCTION OF BINARY FORMS VIA THE HYPERBOLIC CENTROID

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Abstract. In this paper we introduce a reduction theory based on the hyperbolic center of mass, which is different from the reduction introduced by Julia (1917). We show that the zero map via the Julia quadratic is different than the hyperbolic center of mass. Moreover, we discover some interesting formulas for computing the hyperbolic centroid.

1. Introduction

During the XIX-century, the mathematical community invested much efforts in developing a reduction theory of binary forms similar to that of quadratic forms, especially since invariant theory was at the forefront of mathematics. The idea of reduction on a set A with a right $\mathrm{SL}_2(\mathbb{Z})$-action is to associate to any element $a \in A$ a *covariant* point $\xi(a)$ in the upper half-plane $\mathcal{H}_2$, i.e to construct an $\mathrm{SL}_2(\mathbb{Z})$-equivariant map $\xi : A \to \mathcal{H}_2$. The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on binary forms $F(X,Z)$ via a linear change of variables and on the upper half-plane via Möbius transformations. A practical motivation for the reduction in this setting is: given a real binary form, can we find an $\mathrm{SL}_2(\mathbb{Z})$-equivalent with minimal coefficients? This question has a positive answer for quadratics but it is still not very well understood for higher degree forms.

In his thesis [6] of 1917, G. Julia introduced a reduction theory for binary forms with real coefficients (although explicit and complete answers were provided only in degrees three and four). To every binary form $F(X,Z)$ with real coefficients, Julia associated a positive definite quadratic $\mathcal{J}_F$ which is called the *Julia quadratic*. Cremona [4] showed that the coefficients of $\mathcal{J}_F$ are polynomial values of of the coefficients of F and this does not happen for higher degree forms. Since positive definite quadratics parametrize $\mathcal{H}_2$, one obtains a well defined map ξ from real binary forms to the upper half-plane. It is called *the zero map* and it is $\mathrm{SL}_2(\mathbb{Z})$-equivariant. If F is a real binary form, then $\xi(F)$ is a point in the hyperbolic convex hull of the roots of F with non-negative imaginary part. A binary form is called *reduced* if its image via the zero map is in the fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{Z})$.

In [8] Cremona and Stoll developed a reduction theory in a unified setting for binary forms with real or complex coefficients. A *unique* positive definite Hermitian quadratic $\mathcal{J}_F$ is associated to every binary complex form $F(X,Z)$. Since positive definite Hermitian forms parametrize the upper half-space $\mathcal{H}_3$, an extension of the zero map ξ from binary complex forms to $\mathcal{H}_3$ is obtained. The upper half-plane $\mathcal{H}_2$ is contained in $\mathcal{H}_3$ as a vertical cross section (see the following section). When the

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form $F(X,Z)$ has real coefficients, compatibility with complex conjugation forces $\xi(\mathcal{J}_F) \in \mathcal{H}_2$ (see Remark (1)). It is in this sense that working in $\mathcal{H}_3$ unifies the theory of real and complex binary forms. A degree n complex binary form $F(X,Z)$ is called *reduced* when its zero map value $\xi(\mathcal{J}_F)$ is in the fundamental domain of the action of the modular group $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{H}_3$.

In the works cited above, the term *reduced binary form* means reduced in the $\mathrm{SL}_2(\mathbb{Z})$ orbit. It is expected that the reduced forms have smallest size coefficients in such orbit. In [7] the concept of height was defined for forms defined over any ring of integers $\mathcal{O}_K$, for any number field K, and the notion of *minimal absolute height* was introduced. In [3], the author suggests an algorithm for determining the minimal absolute height for binary forms. Continuing with this idea, a database of binary sextics of minimal absolute height $\mathfrak{h} \leq 10$ together with many computational aspects of binary sextics are included in [2].

The genesis of this paper comes from our efforts to understand/explore the geometry behind the reductions of binary forms. In this process, we discovered and are presenting here an alternative reduction method. For real cubics and quartics, Julia ([6]) uses geometric constructions to establish the barycentric coordinates $t_1, \dots, t_n$ of $\xi(F)$ of $\xi(F)$ in the hyperbolic convex hull of the roots of F. Geometric arguments are also used in [8] for the reduction of binary complex forms. Our reduction is based solely on a very special geometric point $\xi_{\mathcal{C}}(F)$ inside the hyperbolic convex hull of the roots of F, namely the *hyperbolic centroid* of these roots. We will discuss whether such reduction has any benefits compared to the previous ones.

In section 2, we describe in detail the reduction relevant features of the hyperbolic geometry of the upper half plane $\mathcal{H}_2$ and upper half-space $\mathcal{H}_3$. These spaces are shown to parameterize respectively the positive definite quadratics and the positive definite Hermitian forms. We prove that these parameterizations respect the corresponding structures: for any n points $w_1, \dots, w_n \in \mathcal{H}_3$, the hyperbolic convex hull of these points parametrizes the positive linear combinations $\sum_{i=1}^n \lambda_i H_{w_i}(x)$, where $H_{w_i}(x)$ is the positive definite Hermitian form corresponding to w_i.

In section 3, we summarize the reduction theory developed in [6] and [8]. We focus especially on the geometrical aspects of the zero map and the reduction, as these are of special interest to us.

In section 4 we define the *hyperbolic centroid* of a collection. For a finite subset $\{w_1, ... w_n\} \subset \mathcal{H}_2$, it is the unique point x inside their hyperbolic convex hull which minimizes $\sum_{i=1}^n \cosh(d_H(\mathbf{x}, w_i)$ (here d_H is the hyperbolic distance). To each real binary form $F(X,Z)$ with no real roots, our alternative zero map associates the hyperbolic centroid of its roots. We show that this map is $SL_2(\mathbb{R})$ equivariant and different from Julia's, hence it defines a new reduction algorithm. We also provide examples of reductions of binary forms with no real roots (see also [3] for totally complex forms). Although zero maps are different, it seems that the effects of both reductions in decreasing the height are similar. Naturally, one would like to determine how different the zero map are, or whether one can get examples where the reductions give different results. Since the zero maps of a binary form F are points in the convex hull of the roots of F, a natural example would be one where these roots are far so that their convex hull is relatively large.

2. The hyperbolic geometry of positive definite binary forms

In this section we present some features of hyperbolic geometry that are not only relevant for the reduction theory of binary forms, but are also interesting on their own. We also establish a correspondence between hyperbolic spaces and positive definite quadratic forms.

2.1. The hyperbolic plane $\mathcal{H}_2$.

The upperhalf-plane equipped with the Riemanian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

is one of the models of the two dimensional hyperbolic space. It is denoted by $\mathcal{H}_2$. The geodesics of the Riemaniann manifold $\mathcal{H}_2$, i.e the hyperbolic equivalents of Euclidean straight lines, are either semicircles $C_{a,b}$ with diameter from $A(a, 0)$ to $B(b, 0)$ on the real axis, or the vertical rays C_a with origin at $x = a$. In the standard literature, the points $A(a, 0), B(b, 0)$ are called *the ideal points* of the geodesic $C_{a,b}$, likewise $A(a, 0)$ and ∞ are the ideal points of C_a. They live in the boundary of $\mathcal{H}_2$.

The hyperbolic distance between two points $z = x + \mathbf{i}y$ and $w = u + \mathbf{i}v$ is computed as follows. Let z_∞, w_∞ be the ideal points of the geodesic through z, w, where z_∞ is the one closer to z.

The hyperbolic distance is defined in terms of the cross-ratio or Euclidean distances

$$d_H(z, w) = \log[z, w, w_\infty, z_\infty] = \log\left(\frac{z - w_\infty}{w - w_\infty}\,\frac{w - z_\infty}{z - z_\infty}\right) = \ln\left(\frac{|z - w_\infty|}{|w - w_\infty|}\,\frac{|w - z_\infty|}{|z - z_\infty|}\right).$$

Notice that for $x = u$ and $y < v$, the geodesic is the vertical ray C_x. In this case $z_\infty = (x, 0), w_\infty = \infty$ and

$$d_H(z, w) = \ln\left(\frac{v}{y}\right).$$

For $A(a, 0)$ and $z = x + \mathbf{i}y \in \mathcal{H}_2$, define

$$d_H(A, z) := \ln\left(\frac{(x - a)^2 + y^2}{y}\right).$$

An additive property of this distance is claimed and used in [8]. To make the paper self-contained and for the benefit of the reader, we state and prove it below.

Proposition 1. *Let A be one of the ideal points of a geodesic that passes through $z = x + y\mathbf{i}, w = u + v\mathbf{i} \in \mathcal{H}_2$. Then $d_H(z, w) = |d_H(A, z) - d_H(A, w)|$.*

Proof. Assume first that $x \neq u$, i.e. the geodesic through z and w is a semicircle. Without loss of generality, assume that $A(0, 0)$. Let $(x - r)^2 + y^2 = r^2$ be the equation of the geodesic and $B(2r, 0)$ the other ideal point. If $z(x, y), w(u, v)$, then $x^2 + y^2 = 2rx,\ u^2 + v^2 = 2ru,\ v^2 = u(2r - u),\ y^2 = x(2r - x)$. Now

$$\begin{aligned}|d_H(A, z) - d_H(A, w)| &= \left|\ln\left(\frac{x^2 + y^2}{y}\right) - \ln\left(\frac{u^2 + v^2}{v}\right)\right| \\ &= \left|\ln\left(\frac{2rx}{y}\right) - \ln\left(\frac{2ru}{v}\right)\right| = \left|\ln \frac{xv}{yu}\right|\end{aligned}$$

On the other hand,

$$d_H(z,w) = \left| \ln \left(\frac{|z||w-B|}{|w||z-B|} \right) \right| = \left| \ln \left(\frac{\sqrt{x^2+y^2}\sqrt{(2r-u)^2+v^2}}{\sqrt{(2r-x)^2+y^2}\sqrt{u^2+v^2}} \right) \right|$$

$$= \left| \ln \left(\frac{\sqrt{2rx}\sqrt{4r^2-2ru}}{\sqrt{4r^2-2rx}\sqrt{2ru}} \right) \right| = \left| \ln \sqrt{\frac{x(2r-u)}{u(2r-x)}} \right| = \left| \ln \sqrt{\frac{x^2v^2}{y^2u^2}} \right| = \left| \ln \frac{xv}{yu} \right|.$$

When $x = u$ the geodesic through z, w is the ray C_x with an ideal point at $A(x, 0)$. Then, $d_H(A, z) = \ln y$ and $d_H(A, w) = \ln v$. Hence,

$$d_H(z,w) = \left| \ln \frac{v}{y} \right| = |\ln v - \ln y| = |d_H(A,w) - d_H(A,z)|.$$

This completes the proof. ◻

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the right on $\mathcal{H}_2$: if $M \in \mathrm{SL}_2(\mathbb{R})$ and $M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$z \cdot M := M^{-1} z = \frac{az+b}{cz+d}$$

2.2. **The upper half-plane $\mathcal{H}_2$ as a parameter space for positive definite quadratics.** Let

$$Q(X,Z) = aX^2 - 2bXZ + cZ^2$$

be a binary quadratic form with real coefficients and homogeneous variables $[X, Z] \in \mathbb{RP}^1$. Let $\Delta = ac - b^2$ be its discriminant. Then

$$Q(X,Z) = a[X - (b/a)Z]^2 + (\Delta/a)Z^2.$$

For both $\Delta > 0$ and $a > 0$, $Q(X, Z)$ is always positive (note that $(X, Z) \neq (0, 0)$ since $[X, Z] \in \mathbb{RP}^1$). Such a quadratic form Q is called *positive definite*. It has two complex roots $[\omega, 1], [\bar{\omega}, 1]$ where $\omega = b/a + (\sqrt{\Delta}/a)\mathbf{i} \in \mathcal{H}_2$. Let $V^+_{2,\mathbb{R}}$ be the space of positive definite real quadratic forms. To each $Q(X, Z) \in V^+_{2,\mathbb{R}}$, we associate the complex number ω in $\mathcal{H}_2$.

Definition 1. *The map*

$$\xi : V^+_{2,\mathbb{R}} \to \mathcal{H}_2$$

which sends a positive definite quadratic to its root in $\mathcal{H}_2$ is called the zero map.

The hyperbolic plane $\mathcal{H}_2$ is a parameter space for positive definite quadratic forms (up to a constant factor) via the inverse

$$\xi^{-1}(\omega) = Q_\omega := (X - \omega Z)(X - \bar{\omega} Z).$$

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $V^+_{2,\mathbb{R}}$ via the linear change of variables: for a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$(M \cdot Q)(X,Z) = Q^M(X,Z) := Q(aX + bZ, cX + dZ).$$

Note that the $\mathrm{SL}_2(\mathbb{R})$ action does not change the discriminant. One can easily verify the following

Proposition 2. *The zero map $\xi : V^+_{2,\mathbb{R}} \to \mathcal{H}_2$ is $\mathrm{SL}_2(\mathbb{R})$-equivariant, i.e.*

$$\xi(M \cdot Q) = M^{-1} \xi(Q).$$

When $\Delta = 0$, the quadratic form $Q(X, Z) = a[X - (b/a)Z]^2$ has a real, double root $[b/a, 1]$. If a is a real number, we let $Q_a = (X - aZ)^2$ be the quadratic with a double root at $[a, 1]$. We also let $Q_\infty = Z^2$ be the quadratic form with a double root at ∞. It has thus been established that the boundary $\mathbb{RP}^1 = \mathbb{R} \cup \infty$ of $\mathcal{H}_2$ parametrizes quadratic forms (up to a constant factor) with discriminant $\Delta = 0$.

To recap: the hyperbolic plane $\mathcal{H}_2$ parametrizes binary quadratic forms with discriminant $\Delta > 0$ and $a > 0$, while its boundary parametrizes those with discriminant $\Delta = 0$.

It has been claimed and used in [6] and [8] that this parametrization is not just a bijection between sets; the hyperbolic geometry of $\mathcal{H}_2$ represents faithfully the algebra of quadratic forms. This was probably known even before. In any case, here is the appropriate statement and a proof of it.

Proposition 3. *Let $\overline{\mathcal{H}}_2 = \mathcal{H}_2 \cup \partial\mathcal{H}_2 = \mathcal{H}_2 \cup \mathbb{RP}^1$ and $\omega_1, \omega_2 \in \overline{\mathcal{H}}_2$. The quadratics of the form*

$$sQ_{\omega_1} + tQ_{\omega_2}, s \geq 0, t \geq 0, s + t = 1$$

parametrize the hyperbolic segment that joins ω_1 and ω_2.

Proof. We will show only the case when the hyperbolic segment is part of a semicircle. The vertical geodesic is similar. Let $a < b$ be two real numbers such that $A(a, 0), B(b, 0)$ are the ideal points of the geodesic $C_{a,b}$ that passes through ω_1, ω_2. We first show that $C_{a,b}$ parametrizes quadratics of the form

$$\lambda Q_a + \mu Q_b, \lambda \geq 0, \mu \geq 0, \lambda + \mu = 1,$$

i.e. $\xi(\lambda Q_a + \mu Q_b) \in C_{a,b}$. The center of $C_{a,b}$ is on the real axis at $\dfrac{a+b}{2}$ and its radius is $\dfrac{b-a}{2}$. Let $\lambda \geq 0, \mu \geq 0, \lambda + \mu = 1$. Then

$$\lambda Q_a + \mu Q_b = \lambda(x - a)^2 + \mu(x - b)^2 = x^2 - 2(\lambda a + \mu b)x + \lambda a^2 + \mu b^2.$$

The root of $\lambda Q_a + \mu Q_b$ in $\mathcal{H}_2$ is

$$(\lambda a + \mu b) + \mathbf{i}(b - a)\sqrt{\lambda\mu},$$

and its distance from $((a + b)/2, 0)$ is easily computed to be $(b - a)/2$.

The proposition now follows easily. Let

$$Q_{\omega_1} = \lambda_1 Q_a + \mu_1 Q_b \text{ and } Q_{\omega_2} = \lambda_2 Q_a + \mu_2 Q_b \text{ with } \lambda_i + \mu_i = 1, \text{ for } i = 1, 2.$$

Then, for $s \geq 0, t \geq 0, s + t = 1$ we have

$$sQ_{\omega_1} + tQ_{\omega_2} = (s\lambda_1 + t\lambda_2)Q_a + (s\mu_1 + t\mu_2)Q_b, \text{ with } (s\lambda_1 + t\lambda_2) + (s\mu_1 + t\mu_2) = 1,$$

hence $\xi(sQ_{\omega_1} + tQ_{\omega_2}) \in C_{a,b}$. It is obvious that $\xi(sQ_{\omega_1} + tQ_{\omega_2})$ lives in fact in the hyperbolic segment that joins ω_1 and ω_2. □

This proposition can be generalized by induction as follows.

Proposition 4. *Let $\omega_1, \omega_2, ..., \omega_n \in \overline{\mathcal{H}}_2$ such that for all i, ω_i is not in the hyperbolic convex hull of $\omega_1, \omega_2, ..., \omega_{i-1}$. Then the convex hull of $\omega_1, \omega_2, \ldots, \omega_n$ parametrizes the linear combinations $\sum_{i=1}^n \lambda_i Q_{\omega_i}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$.*

Proof. We proceed by induction. For $n = 2$ the statement is true due to the previous proposition. Consider $\sum_{i=1}^n \lambda_i Q_{\omega_i}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Then

$$\sum_{i=1}^{n} \lambda_i Q_{\omega_i} = \left(\sum_{i=1}^{n-1} \lambda_i \right) \sum_{i=1}^{n-1} \left(\frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} \right) Q_{\omega_i} + \lambda_n Q_{\omega_n}.$$

By induction hypothesis, there exists ω_0 in the convex hull of $\omega_1, \omega_2, ..., \omega_{n-1}$ such that

$$\sum_{i=1}^{n-1} \left(\frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} \right) Q_{\omega_i} = Q_{\omega_0}$$

It follows that

$$\sum_{i=1}^{n} \lambda_i Q_{\omega_i} = \left(\sum_{i=0}^{n-1} \lambda_i \right) Q_{\omega_0} + \lambda_n Q_{\omega_n}$$

represents a point ω in the hyperbolic segment that joins ω_0 and ω_n. Clearly ω is also in the convex hull of $\alpha_1, \alpha_2, ..., \alpha_n$.

□

2.3. **The hyperbolic three dimensional space $\mathcal{H}_3$.** As a set, $\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}^+$. Points of $\mathcal{H}_3$ will be written in the form $z + t\mathbf{j}$ where $z \in \mathbb{C}$ and $t > 0$. The equation $t = 0$ represents the floor $\mathbb{C}$ of $\mathcal{H}_3$. The hyperbolic space $\mathcal{H}_3$ is foliated via horospheres

$$H_t := \{z + t\mathbf{j} : \ z \in \mathbb{C}\}$$

which are centered at ∞ and indexed by the height t above $\partial\mathcal{H}_3 = \mathbb{CP}^1$. The algebra of $\mathcal{H}_3$ is not commutative. The following identities are essential to computations:

$$\mathbf{j}^2 = -1, \ \mathbf{ij} = -\mathbf{ji}, \ \mathbf{j}z = \bar{z}\mathbf{j} \text{ (see the lemma below for a proof of this).}$$

The notion of complex modulus extends to $\mathcal{H}_3$: $|z+t\mathbf{j}| = |z|^2+t^2$. There is a natural isometrical inclusion map $\mathcal{H}_2 \to \mathcal{H}_3$ via $x + \mathbf{i}t \to x + \mathbf{j}t$, the upper half-plane $\mathcal{H}_2$ thus, sits as a vertical cross-section inside $\mathcal{H}_3$. The invariant elements of $\mathcal{H}_3$ under the partial conjugation

$$z + \mathbf{j}t \mapsto \bar{z} + \mathbf{j}t$$

are precisely the elements of $\mathcal{H}_2$. The hyperbolic metric of $\mathcal{H}_3$ is

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2}.$$

The geodesics are either semicircles centered on the floor $\mathbb{C}$ and perpendicular to $\mathbb{C}$, or rays $\{z_0 + \mathbf{j}t\}$ perpendicular to $\mathbb{C}$.

For $\omega = z + t\mathbf{j} \in \mathcal{H}_3$ and $w + 0\mathbf{j} \in \mathbb{C}$ on the floor, define

$$d_H(\omega, w) := \frac{|z - w|^2 + t^2}{y}. \tag{1}$$

The following proposition and its proof are straightforward generalizations from $\mathcal{H}_2$.

Proposition 5. *If one of the ideal points of the geodesic through ω_1, ω_2 is at w, then*

$$d_H(\omega_1, \omega_2) = |d_H(\omega_1, w) - d_H(\omega_2, w)|.$$

There is a right action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{H}_3$. If $M \in \mathrm{SL}_2(\mathbb{C})$ and $M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its action is described as follows

$$(z + \mathbf{j}t) \cdot M = M^{-1}(z + \mathbf{j}t) = [a(z + \mathbf{j}t) + b][(c(z + \mathbf{j}t) + d]^{-1},$$

where the inverse indicates the right inverse in the non commutative structure of $\mathcal{H}_3$. Note that for $t = 0$ we get the standard $\mathrm{SL}_2(\mathbb{C})$-action on the boundary $\mathbb{CP}^1$ of $\mathcal{H}_3$.

Lemma 1. *The action of* $\mathrm{SL}_2(\mathbb{C})$ *on* $\mathcal{H}_3$ *can be written in the form*

$$(z + \mathbf{j}t) \cdot M = \frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2 + \mathbf{j}t}{|cz + d|^2 + |c|^2t^2}.$$

Proof. First, with $z = x + y\mathbf{i}$ we have

$$\mathbf{j}\bar{z} = \mathbf{j}(x - y\mathbf{i}) = x\mathbf{j} - y\mathbf{j}\mathbf{i} = x\mathbf{j} + y\mathbf{i}\mathbf{j} = (x + y\mathbf{i})\mathbf{j} = z\mathbf{j}.$$

Using this identity, it is straightforward to show that

$$[c(z + t\mathbf{j}) + d]\left[(\bar{z} - t\mathbf{j})\bar{c} + \bar{d}\right] = |cz + d|^2 + t^2|c|^2.$$

Real numbers commute with both $\mathbf{i}$ and $\mathbf{j}$ in $\mathcal{H}_3$, hence they have a well-defined inverse. We obtain the right inverse as follows:

$$[c(z + t\mathbf{j}) + d]^{-1} = \frac{(\bar{z} - t\mathbf{j})c + \bar{d}}{|cz + d|^2 + t^2|c|^2}.$$

The lemma follows from the straightforward calculation

$$[a(z + \mathbf{j}t) + b][(\bar{z} - t\mathbf{j}) + \bar{d}] = (az + b)(\overline{cz + d}) + a\bar{c}t^2 + t\mathbf{j}.$$

∎

2.4. **The upper half-space $\mathcal{H}_3$ as a parameter space for positive definite Hermitian quadratics.** Let

$$H(X, Z) = a|X|^2 - bX\bar{Z} - \bar{b}\bar{X}Z + c|Z|^2, a, c \in \mathbb{R}$$

be a Hermitian quadratic form with homogeneous variables $[X, Z] \in \mathbb{CP}^1$. Notice that the values of $H(X, Z)$ are always real. Let $\Delta = ac - |b|^2$ be its discriminant. Then

$$H(X, Z) = a\left|X - (\bar{b}/a)Z\right|^2 + (\Delta/a)Z^2,$$

hence $H(X, Z) > 0$ for all (X, Z) when $\Delta > 0, a > 0$. Such a form is called *positive definite.* Denote the set of all positive definite Hermitian forms by $V^+_{2,\mathbb{C}}$. There is an $\mathrm{SL}_2(\mathbb{C})$ action on $V^+_{2,\mathbb{C}}$ similar to the real case. The natural $\mathrm{SL}_2(\mathbb{R})$ equivariant inclusion $\psi : V^+_{2,\mathbb{R}} \to V^+_{2,\mathbb{C}}$ via

$$\psi(aX^2 - 2bXZ + cZ^2) = a|X|^2 - bX\bar{Z} - \bar{b}\bar{X}Z + c|Z|^2,$$

gives rise to an extension of the zero map.

Definition 2. *The zero map* $\xi : V^+_{2,\mathbb{C}} \to \mathcal{H}_3$ *is defined via*

$$\xi(a|X|^2 - bX\bar{Z} - \bar{b}\bar{X}Z + c|Z|^2) = \frac{\bar{b}}{a} + \mathbf{j}\frac{\sqrt{\Delta}}{a} \tag{2}$$

Proposition 6. *The map* ξ *is* $\mathrm{SL}_2(\mathbb{C})$ *equivariant.*

Proof. The generators of $\mathrm{SL}_2(\mathbb{C})$ are matrices of the form $\begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$, for $a \in \mathbb{C}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is easy to show that for any generator matrix M:

$$\xi(H^M) = M^{-1}\xi(H).$$

□

The hyperbolic space $\mathcal{H}_3$ is a parameter space for positive definite $(\Delta > 0, a > 0)$ Hermitian forms via the inverse map

$$\xi^{-1}(\omega) = \xi^{-1}(z + \mathbf{j}t) = |X|^2 - \bar{z}\bar{X}Z - zX\bar{Z} + (|z|^2 + t^2)|Z|^2 = H_\omega.$$

The boundary $\mathbb{CP}^1 = \mathbb{C} \cup \infty$ of $\mathcal{H}_3$ is a parameter space for the decomposable $(\Delta = 0)$ Hermitian forms

$$H_\beta = (X - \bar{\beta}Z)(\bar{X} - \beta\bar{Z}) = |X - \beta Z|^2 \text{ for } \beta \in \mathbb{C}, \;\; H_\infty = |Z|^2,$$

Just as in the case of the upper half-plane $\mathcal{H}_2$, we have the following proposition:

Proposition 7. *Let* $\overline{\mathcal{H}}_3 = \mathcal{H}_3 \cup \partial\mathcal{H}_3 = \mathcal{H}_3 \cup \mathbb{CP}^1$. *The hyperbolic convex hull of* $\omega_1, \omega_2, ..., \omega_n \in \overline{\mathcal{H}}_3$ *parametrizes Hermitian forms* $\sum_{i=1}^n \lambda_i H_{\omega_i}$ *with* $\lambda_i \geq 0$ *for* $i = 1, 2, ..., n$ *and* $\sum_{i=1}^n \lambda_i = 1$.

The equivariant connection between the geometry of hyperbolic spaces and the algebra of positive definite forms, which extends to the boundary as well, can be expressed in the following equivariant commutative diagram:

$$\begin{array}{ccc} V^+_{2,\mathbb{R}} & \xrightarrow{\xi} & \mathcal{H}_2 \\ \downarrow & & \downarrow \\ V^+_{2,\mathbb{C}} & \xrightarrow{\xi} & \mathcal{H}_3 \end{array}$$

Next, we will see how to use the equivariance of the zero map to construct a reduction method.

3. Reduction of binary forms via the Julia quadratic

In this section we summarize the reduction of binary forms via the zero map obtained in [6] and [8]. We will focus especially on the geometric features of the theory which are of particular interest to us.

Let $V_n(\mathbb{C})$ denote the space of complex binary forms of degree n. If $F \in V_n(\mathbb{C})$ then

$$F(X, Z) = a_0 \prod_{i=1}^n (X - \alpha_i Z)$$

for some complex numbers α_j and $a_0 \neq 0$. For $t_1, t_2, ..., t_n \geq 0$ define

$$Q_F(t_1, t_2, ..., t_n) = \sum_{i=1}^n t_i |X - \alpha_i Z|^2 = \sum_{i=1}^n t_i H_{\alpha_i}(X, Z).$$

From Prop. 7, the positive definite Hermitian forms $Q_F(t_1, t_2, ..., t_n)$ parametrize the hyperbolic convex hull of $\alpha_1, \alpha_2, ..., \alpha_n \in \overline{\mathcal{H}}_3$. Let $(t_1^0, t_2^0, ..., t_n^0)$ be the values that minimize

$$\theta_0 := \frac{a_0^2 (\mathrm{disc}(Q_F))^{n/2}}{n^n t_1 t_2 ... t_n}.$$

Definition 3. *The form $\mathcal{J}_F := Q_F(t_1^0, t_2^0, ..., t_n^0) \in V_{2,\mathbb{C}}^+$ is called the Julia quadratic of F. The zero map extends to $\xi : V_n(\mathbb{C}) \to \mathcal{H}_3$ via $\xi(F) = \xi(J_F) \in \mathcal{H}_3$, a point in the hyperbolic convex hull of the roots of F. The form F is called reduced if $\xi(F)$ is in the fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{C})$.*

To reduce a real binary form $F(X, Z)$, we first compute its zero map value $\xi(F)$ in $\mathcal{H}_2$. If $\xi(F)$ is in the fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{R})$, then $F(X, Z)$ is already reduced. If not, choose $M \in \mathrm{SL}_2(\mathbb{R})$ such that $M^{-1}\xi(F) \in \mathcal{F}$. The form $F(X, Z)$ reduces to $F^M(X, Z)$, which is expected to have smaller coefficients. Similar procedure holds for complex binary forms in $\mathcal{H}_3$.

In [8], the authors provide a geometric construction of the zero map. The roots $\alpha_i,\ i = 1, 2, ..., n$ of $F(X, Z)$ are placed in the floor $t = 0$ of $\mathcal{H}_3$.

Proposition 8. *(Proposition 5.3 in [8]) Let d_H be as in Eqn.(1). The zero map value $\xi(F)$ is the unique point $w_0 \in \mathcal{H}_3$ that minimizes the sum of distances*

$$\tilde{F}(w) := \sum_{i=1}^{n} d_H(w, \alpha_i).$$

Remark 1. *This minimizing solution w_0 is $\mathrm{SL}_2(\mathbb{C})$-invariant. Furthermore, when $F(X, Z)$ has real coefficients, w_0 is also invariant with respect to the partial conjugation $w_0 = z_0 + t_0\mathbf{j} \mapsto \bar{z}_0 + t_0\mathbf{j}$. Hence, z_0 is real number, i.e. $w_0 \in \mathcal{H}_2$.*

4. The reduction of real forms via the hyperbolic centroid

In this section we introduce an alternative zero map for binary forms with real coefficients and no real roots. It is based on the notion of *hyperbolic centroid* in hyperbolic spaces. We focus in $\mathcal{H}_2$ which is the case of interest for us, but the general case is straightforward. Our treatment follows closely that of [5].

4.1. The centroid via the hyperboloid model of the hyperbolic plane.

Let M be the Minkowski pairing in $\mathbb{R}^3$: for $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3)$

$$M(\mathbf{x}, \mathbf{y}) = -x_1 y_1 - x_2 y_2 + x_3 y_3.$$

It comes with a corresponding norm: $||\mathbf{x}||^2 = M(\mathbf{x}, \mathbf{x}) = -x_1^2 - x_2^2 + x_3^2$. Let $\mathcal{H}$ be the upper sheet of the hyperboloid

$$\mathcal{H} := \{\mathbf{x} \ :\ ||\mathbf{x}|| = 1,\ x_3 > 0\}.$$

Its equation is $-x_1^2 - x_2^2 + x_3^2 = 1$ and its metric is given by $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$. If $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, the hyperbolic distance $d_H(\mathbf{x}, \mathbf{y})$ in this model can be found via

$$\cosh d_H(\mathbf{x}, \mathbf{y}) = M(\mathbf{x}, \mathbf{y}).$$

Definition 4. *The centroid of $\mathbf{x}_j \in \mathcal{H},\ j = 1, 2, ..., r$ is defined as*

$$\mathcal{C} = \mathcal{C}_{\mathcal{H}}(x_1, x_2, ..., x_r) := \frac{\sum_{j=1}^{r} \mathbf{x}_j}{||\sum_{j=1}^{r} \mathbf{x}_j||}.$$

Notice that

$$\sum_{i=1}^{r} \cosh(d_H(\mathcal{C}, \mathbf{x}_i)) = \sum_{i=1}^{r} M(\mathcal{C}, \mathbf{x}_i) = M(\mathcal{C}, \sum_{i=1}^{r} \mathbf{x}_i) = || \sum_{i=1}^{r} \mathbf{x}_i || M(\mathcal{C}, \mathcal{C}) = || \sum_{i=1}^{r} \mathbf{x}_i ||$$

Proposition 9. *The centroid* $\mathcal{C}_{\mathcal{H}}(x_1, x_2, ..., x_r)$ *is* $\mathrm{SL}_2(\mathbb{R})$ *invariant. It is the unique point* $\mathbf{x} \in \mathcal{H}$ *that minimizes* $\sum_{j=1}^{r} \cosh(d_H(\mathbf{x}, \mathbf{x}_j))$.

Proof. Recall that $\mathrm{SL}_2(\mathbb{R})$ action on $\mathcal{H}$ preserves hyperbolic distances, hence the center of mass is $\mathrm{SL}_2(\mathbb{R})$ invariant. The proof of the second part follows easily by solving the minimizing problem

$$\text{minimize } M(\mathbf{x}, \sum_{j=1}^{r} \mathbf{x}_j), \text{ subject to } \mathbf{x} \in \mathcal{H}$$

using the Lagrange Multipliers method and the inequality $|| \sum_{j=1}^{r} \mathbf{x}_j || > 1$. ∎

We use the minimizing property to transfer the notion of centroid in $\mathcal{H}_2$. There is an isometry $\mathcal{H}_2 \to \mathcal{H}$ given by

$$u + \mathbf{i}v \to \left(\frac{1 - u^2 - v^2}{2u}, \frac{u}{v}, \frac{1 + u^2 + v^2}{2v} \right).$$

The following identity holds in $\mathcal{H}_2$:

$$\cosh d_H(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2y_1 y_2}$$

for $z_1 = x_1 + \mathbf{i}y_1 \in \mathcal{H}_2$, $z_2 = x_2 + \mathbf{i}y_2 \in \mathcal{H}_2$. It follows that if $\alpha_j = x_j + \mathbf{i}y_j \in \mathcal{H}_2$, $j = 1, 2, ..., n$, their centroid is the complex number $t + \mathbf{i}u \in \mathcal{H}_2$ such that

$$\sum_{j=1}^{n} \left[1 + \frac{(t - x_j)^2 + (u - y_j)^2}{2uy_j} \right]$$

is minimal. By excluding the constant summands, we obtain the following:

Definition 5. *The hyperbolic centroid, or simply centroid,* $\mathcal{C}_{\mathcal{H}}(\alpha_1, \alpha_2, ..., \alpha_n)$ *of the collection* $\{\alpha_j \in \mathcal{H}_2 \mid j = 1, 2, ..., n\}$ *is the unique point* $t + \mathbf{i}u \in \mathcal{H}_2$ *that minimizes*

$$\sum_{j=1}^{n} \frac{(t - x_j)^2 + (u - y_j)^2}{uy_j}.$$

Setting the partials equal to zero, we obtain a system of equations for t and u:

$$\begin{cases} \sum_{j=1}^{n} \frac{t - x_j}{y_j} = 0 \\ \sum_{j=1}^{n} \frac{u^2 - (t^2 - 2x_j t + |\alpha_j|^2)}{y_j} = 0. \end{cases} \tag{3}$$

Let the $(n-1)$-st elementary symmetric polynomial be

$$\mathfrak{s}_{n-1}(y_1, y_2, ..., y_n) = \sum_{i=1}^{n} y_1 y_2 \cdots y_{i-1} y_{i+1} \cdots y_n.$$

Straightforward algebraic computations yield the solution to the system (3).

Proposition 10. *The centroid $\mathcal{C}_\mathcal{H} = t + \mathbf{i}u \in \mathcal{H}_2$ of $\alpha_1, \alpha_2, ..., \alpha_n$ satisfies*

$$
\begin{aligned}
t &= \sum_{i=1}^{n} \left(\frac{y_1 y_2 \cdots y_{i-1} y_{i+1} \cdots y_n}{\mathfrak{s}_{n-1}(y_1, y_2, ..., y_n)} \right) x_i \\
|\mathcal{C}_\mathcal{H}|^2 &= \sum_{i=1}^{n} \left(\frac{y_1 y_2 \cdots y_{i-1} y_{i+1} \cdots y_n}{\mathfrak{s}_{n-1}(y_1, y_2, ..., y_n)} \right) |\alpha_i|^2 \\
Q_{\mathcal{C}_\mathcal{H}}(X,Z) &= \sum_{i=1}^{n} \left(\frac{y_1 y_2 \cdots y_{i-1} y_{i+1} \cdots y_n}{\mathfrak{s}_{n-1}(y_1, y_2, ..., y_n)} \right) Q_{\alpha_i}(X,Z).
\end{aligned} \tag{4}
$$

Remark 2. *All equations in Eq. (4) are described in terms of the function $\psi : \mathbb{R}^n \times \mathbb{R}^n_{>0} \mapsto \mathbb{R}$ defined by*

$$
\psi\left((x_1, \ldots, x_n), (y_1, \ldots, y_n)\right) = \sum_{i=1}^{n} \left(\frac{y_1 y_2 \cdots y_{i-1} y_{i+1} \cdots y_n}{\mathfrak{s}_{n-1}(y_1, y_2, ..., y_n)} \right) x_i. \tag{5}
$$

The function ψ has symmetries and is a convex linear combination of x_i's with weights that depend only on $y_1, ..., y_n$. It is probably a well-known and standard function in areas where symmetries and group actions are relevant.

4.2. **Reduction based on the notion of centroid.** Let $V^+_{2n,\mathbb{R}}(0,n)$ denote binary forms of degree $2n$ with real coefficients and no real roots. Every $F(X,Z) \in V^+_{2n,\mathbb{R}}(0,n)$ can be factored

$$
F(X,Z) = \prod_{j=1}^{n} Q_{\alpha_j}(X,Z)
$$

where

$$
\alpha_j = x_j + \mathbf{i} y_j, \; Q_{\alpha_j}(X,Z) = (X - \alpha_j Z)(X - \overline{\alpha_j} Z)
$$

Definition 6. *The <u>centroid zero map</u> $\xi_\mathcal{C} : V^+_{2n,\mathbb{R}}(0,n) \to \mathcal{H}_2$ is defined via*

$$
\xi_\mathcal{C}(F) := \mathcal{C}_\mathcal{H} = \mathcal{C}_\mathcal{H}(\alpha_1, \alpha_2, ..., \alpha_n).
$$

The form

$$
\mathcal{J}^\mathcal{C}_F := (X - \mathcal{C}_\mathcal{H} Z)(X - \overline{\mathcal{C}_\mathcal{H}} Z) = \sum_{j=1}^{n} \left(\frac{y_1 y_2 \cdots y_{j-1} y_{j+1} \cdots y_n}{\mathfrak{s}_{n-1}(y_1, y_2, ..., y_n)} \right) Q_{\alpha_j}(X,Z)
$$

is called the <u>centroid quadratic</u> of F.

Remark 3. *The reduction theory based on the centroid proceeds as before. Let $F(X,Z)$ be a real binary form with no real roots. If $\xi_\mathcal{C}(F) \in \mathcal{F}$ then F is reduced. Otherwise, let $M \in \mathrm{SL}_2(\mathbb{R})$ such that $M^{-1}\xi_\mathcal{C}(F) \in \mathcal{F}$. The form F reduces to $F^M(X,Z)$.*

Here is a comparison between the reduction of [6] [8] and the one via the hyperbolic centroid.

Example 1. *Let $F(X,Z)$ be the binary sextic with roots $\alpha_1 = 2 + 3\mathbf{i}$, $\alpha_2 = 6 + 4\mathbf{i}$, $\alpha_3 = 4 + 7\mathbf{i}$ and their conjugates. Then*

$$
F(X,Z) = (X^2 - 4XZ + 13Z^2)(X^2 - 12XZ + 52Z^2)(X^2 - 8XZ + 65Z^2).
$$

Consider the genus 2 curve

$$
Y^2 = F(X,1) = X^6 - 24X^5 + 306X^4 - 2308X^3 + 10933X^2 - 29068X + 43940
$$

with height $\mathfrak{h} = 43940$. *Reducing it via* [8] *yields a curve* C' *with equation*

$$Y^2 + (X^3 + X)Y = 16X^4 + 7X^3 + 273X^2 + 343X + 3185$$

which is isomorphic to

$$Y^2 = (X^3 + X)^2 + 4\left(16X^4 + 7X^3 + 273X^2 + 343X + 3185\right)$$
$$X^6 + 66X^4 + 28X^3 + 1093X^2 + 1372X + 12740.$$

This last curve has height $\mathfrak{h} = 12740$, *which is smaller than the original height.*

The reduction via the centroid is as follows. The centroid zero map $\xi_{\mathcal{C}}(F)$ *is*

$$\xi_{\mathcal{C}}(F) = \frac{230}{61} + i\,\frac{14}{61}\sqrt{2 \cdot 3 \cdot 71} \approx 3.77 + i\,4.73$$

To bring this point to the fundamental domain we have to shift it to the left by 4 units. Hence, we must compute

$$f(X+4) = F(X+4, 1) = X^6 + 66X^4 + 28X^3 + 1093X^2 + 1372X + 12740.$$

which has height $\mathfrak{h} = 12740$, *the same as in the Julia case.*

We generalize the case of totally complex sextics. Recall that a binary form is **totally complex** when all its roots are non-real (complex) numbers.

Proposition 11. *Let* $F(X,Z) \in \mathbb{Z}[X,Z]$ *be a totally complex sextic factored over* $\mathbb{R}$ *as*

$$F(X,Z) = (X^2 + a_1XZ + b_1Z^2)(X^2 + a_2XZ + b_2Z^2)(X^2 + a_3XZ + b_3Z^2).$$

Let $d_j = \sqrt{4b_j - a_j^2}$, $\mathbf{d} = (d_1, d_2, d_3)$, $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$. *The centroid zero map* $\xi_{\mathcal{C}}(F) = t + \mathbf{i}u \in H_2$ *of* F *is determined by*

$$t = -\frac{1}{2}\left(\frac{d_2d_3}{\mathfrak{s}_2(d_1,d_2,d_3)}a_1 + \frac{d_1d_3}{\mathfrak{s}_2(d_1,d_2,d_3)}a_2 + \frac{d_1d_2}{\mathfrak{s}_2(d_1,d_2,d_3)}a_3\right) = -\frac{1}{2}\psi(\mathbf{d}, \mathbf{a}),$$

$$|\xi_{\mathcal{C}}(F)|^2 = \frac{d_2d_3}{\mathfrak{s}_2(d_1,d_2,d_3)}b_1 + \frac{d_1d_3}{\mathfrak{s}_2(d_1,d-2,d_3)}b_2 + \frac{d_1d_2}{\mathfrak{s}_2(d_1,d_2,d_3)}b_3 = \psi(\mathbf{d}, \mathbf{b}).$$

The centroid quadratic of F *is given by*

$$\frac{d_2d_3(X^2 + a_1XZ + b_1Z^2) + d_1d_3(X^2 + a_2XZ + b_2Z^2) + d_1d_2(X^2 + a_3XZ + b_3Z^2)}{\mathfrak{s}_2(d_1,d_2,d_3)}.$$

The reduction is defined over $\mathbb{Q}(d_1, d_2, d_3)$.

Proof. Let $\alpha_j = x_j + \mathbf{i}y_j$, $i = 1,2,3$ be the roots of $F(X,Z)$. The lemma follows after substituting $x_j = -\frac{a_j}{2}$, $y_j = \frac{1}{2}d_j$ and $Q_{\alpha_j} = X^2 + a_jXZ + b_j$ into the equations of Prop. 10.

□

Example 2. *We consider a totally complex sextic treated in* [1]. *Let*

$$F(X,Z) = X^6 - 12X^5Z + 96X^4Z^2 - 458X^3Z^3 + 1489X^2Z^4 - 3014XZ^5 + 3770Z^6$$

whose roots in $\mathcal{H}_2$ *are* $\alpha_1 = 1 + 3\mathbf{i}, \alpha_2 = 2 + 5\mathbf{i}, \alpha_3 = 3 + 2\mathbf{i}$. *Hence*

$$F(X,Z) = (X^2 - 2XZ + 10Z^2)(X^2 - 4XZ + 29Z^2)(X^2 - 6XZ + 13Z^2).$$

Numerical computations lead to the following Julia zero map:

$$\xi(F) = 2.12067657 + 3.26692991\mathbf{i}.$$

The formulas for the centroid zero map on the other yield

$$Re(\xi_{\mathcal{C}}(F)) = 67/31 \approx 2.16129032, \ |\xi_{\mathcal{C}}(F)|^2 = 202/13.$$

It is clear that the zero maps ξ and $\xi_{\mathcal{C}}$ are different. Neither of them is in the fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{Z})$, a left horizontal shift by 2 is needed in either case. The polynomial

$$f(x) = F(x,1) = x^6 - 12x^5 + 96x^4 - 458x^3 + 1489x^2 - 3014x + 3770$$

with height $\mathfrak{h} = 3770$ reduces to

$$f(x+2) = x^6 + 36x^4 - 10x^3 + 325x^2 - 250x + 1250$$

with a smaller height $\mathfrak{h} = 1250$. We should mention however, that in [1], the author finds another form with much smaller height which it is not in the same $\mathrm{SL}_2(\mathbb{Z})$-orbit.

It is straightforward to generalize and prove these results to any degree.

Proposition 12. *Let $F(X,Z)$ be a totally complex form factored over $\mathbb{R}$ as below*

$$F(X,Z) = \prod_{i=1}^{n} (X^2 + a_i XZ + b_i Z^2)$$

Denote by $d_i = \sqrt{4b_i - a_i^2}$, for $i = 1, \dots, n$ the discriminants for each factor of $F(X,Z)$. Let $\mathfrak{s}_{n-1} = \sum_{i=1}^{r} d_1 \cdots d_{i-1} \hat{d}_i d_{i+1} \cdots d_r$ where $\hat{x}$ denote a missing x, and

$$\mathbf{a} = (a_1, ..., a_n), \ \ \mathbf{b} = (b_1, ..., b_n), \ \ \mathbf{d} = (d_1, ..., d_n).$$

The centroid quadratic of $F(X,Z)$ is given by

$$\mathcal{J}_F^{\mathcal{C}} = \sum_{i=1}^{n} \left(\frac{d_1 d_2 \cdots d_{i-1} d_{i+1} \cdots d_n}{\mathfrak{s}_{n-1}} \right) (X^2 + a_i XZ + b_i Z^2).$$

The centroid zero map $\xi_{\mathcal{C}}(F) = t + \mathbf{i}u \in \mathcal{H}_2$ is given by

$$\begin{aligned} t &= -\frac{1}{2} \sum_{i=1}^{n} \frac{d_1 \cdots d_{i-1} d_{i+1} \cdots d_n}{\mathfrak{s}_{n-1}} a_i = \psi(\mathbf{d}, \mathbf{a}), \\ u^2 &= \frac{1}{4\mathfrak{s}_{n-1}^2} \prod_{i=1}^{n} d_i \left(\mathfrak{s}_{n-1} \sum_{i=1}^{n} d_i + \sum_{i}^{n} d_1 \cdots \hat{d}_i \cdots \hat{d}_j \cdots d_n \, (a_i - a_j)^2 \right) \qquad (6) \\ |\xi_{\mathcal{C}}(F)|^2 &= \sum_{i=1}^{n} \frac{d_1 \cdots d_{i-1} d_{i+1} \cdots d_n}{\mathfrak{s}_{n-1}} b_i = \psi(\mathbf{d}, \mathbf{b}). \end{aligned}$$

The reduction is defined over $\mathbb{Q}(d_1, d_2, ..., d_n)$

It would be interesting to express $\xi_{\mathcal{C}}(F)$ in terms of invariants of F or symmetries of the roots of F, and as a more overarching goal, to incorporate the real roots of the binary form F in this approach. We will continue to explore these issues.

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