SOME REMARKS ON THE NON-REAL ROOTS OF POLYNOMIALS

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Abstract. Let $f \in \mathbb{R}(t)[x]$ be given by $f(t,x) = x^n + t \cdot g(x)$ and $\beta_1 < \cdots < \beta_m$ the distinct real roots of the discriminant $\Delta_{(f,x)}(t)$ of $f(t,x)$ with respect to x. Let γ be the number of real roots of $g(x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}$. For any $\xi > |\beta_m|$, if $n-s$ is odd then the number of real roots of $f(\xi, x)$ is $\gamma + 1$, and if $n-s$ is even then the number of real roots of $f(\xi,x)$ is γ, $\gamma+2$ if $t_s > 0$ or $t_s < 0$ respectively. A special case of the above result is constructing a family of degree $n \geq 3$ irreducible polynomials over $\mathbb{Q}$ with many non-real roots and automorphism group S_n.

1. Introduction

Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 2$ and Gal (f) its Galois group over $\mathbb{Q}$. Let us assume that over $\mathbb{R}$, $f(x)$ is factored as

$$f(x) = a \prod_{j=1}^{r} (x - \alpha_j) \prod_{i=1}^{s} (x^2 + a_i x + b_i),$$

where $a_i^2 < 4b_i$, for all $i = 1, \ldots, s$. The pair (r,s) is called the *signature* of $f(x)$. Obviously deg $f = 2s + r$. If $s = 0$ then $f(x)$ is called *totally real* and if $r = 0$ it is called *totally complex*. Equivalently the above terminology can be defined for binary forms $f(x,z)$. By a reordering of the roots we may assume that if $f(x)$ has $2s$ non-real roots then

$$\alpha := (1,2)(3,4)\cdots(2s-1,2s) \in Gal(f).$$

In [4] it is proved that if deg $f = p$, for a prime p, and s satisfies

$$s\,(s \log s + 2 \log s + 3) \leq p$$

then $Gal(f) = A_p, S_p$. Moreover, a list of all possible groups for various values of r is given for $p \leq 29$; see [4, Thm. 2]. There are some follow up papers to [4].

In [1] the author proves that if $p \geq 4s + 1$, then the Galois group is either S_p or A_p. This improves the bound given in [4]. The author also studies when polynomials with non-real roots are solvable by radicals, which are consequences of Table 2 and Theorem 2 in [4]. In [13] the author uses Bezoutians of a polynomial and its derivative to construct polynomials with real coefficients where the number of real roots can be counted explicitly. Thereby, irreducible polynomials in $\mathbb{Q}[x]$ of prime degree p are constructed for which the Galois group is either S_p or A_p.

In this paper we study a family of polynomials with non-real roots whose degree is not necessarily prime. Given a polynomial $g(x) = \sum_{i=0}^{s} t_i x^i$ and with γ number of non-real roots we construct a polynomial $f(t,x) = x^n + t\, g(x)$ which has $\gamma, \gamma+1, \gamma+2$ non-real roots for certain values of $t \in \mathbb{R}$; see Theorem 2. The values of $t \in$

$\mathbb{R}$ are given in terms of the Bezoutian matrix of polynomials or equivalently the discriminant of $f(t,x)$ with respect to x. This is the focus of Section 3 in the paper.

While most of the efforts have been focusing on the case of irreducible polynomials over $\mathbb{Q}$ which have real roots, the case of polynomials with no real roots is equally interesting. How should an irreducible polynomial over $\mathbb{Q}$ with all non-real roots must look like? What can be said about the Galois group of such totally complex polynomials? In [5] is developed a reduction theory for such polynomials via the hyperbolic center of mass. A special case of Theorem 2 provides a class of totally complex polynomials.

Notation For any polynomial $f(x)$ we denote by $\Delta_{(f,x)}$ its discriminant with respect to x. If f is a univariate polynomial then Δ_f is used and the leading coefficient is denoted by $\text{led}(f)$. Throughout this paper the ground field is a field of characteristic zero.

2. Preliminaries

Let $f_1(x)$, $f_2(x)$ be polynomials over a field F of characteristic zero and, let n be an integer which is greater than or equal to $\max\{\deg f_1, \deg f_2\}$. Then, we put

$$B_n(f_1,f_2) := \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x-y} = \sum_{i,j=1}^{n} \alpha_{ij} x^{n-i} y^{n-j} \in F[x,y],$$
$$M_n(f_1,f_2) := (\alpha_{ij})_{1\le i,j\le n}.$$

The matrix $M_n(f_1,f_2)$ is called the *Bezoutian* of f_1 and f_2. Clearly, $B_n(f_1,f_1)=0$ and hence $M_n(f_1,f_1)$ is the zero matrix. The following properties hold true; see [6, Theorem 8.25] for details.

Proposition 1. *The following are true:*

1. $M_n(f_1,f_2)$ *is an* $n\times n$ *symmetric matrix over* F.
2. $B_n(f_1,f_2)$ *is linear in* f_1 *and* f_2*, separately.*
3. $B_n(f_1,f_2) = -B_n(f_2,f_1)$.

When $f_2 = f_1'$, the formal derivative of f_1 (with respect to the indeterminate x), we often write $B_n(f_1) := B_n(f_1, f_1')$. From now on, for any degree $n\ge 2$ polynomial $f(x)\in\mathbb{R}[x]$ we will denote by $M_n(f) := M_n(f,f')$ as above. The matrix $M_n(f)$ is called the **Bezoutian matrix** of f.

Remark 1. *It is often the case that the matrix* $M_n'(f_1,f_2) = (\alpha_{ij}')_{1\le i,j\le n}$ *defined by the generating function*

$$B_n'(f_1,f_2) := \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x-y} = \sum_{i,j=1}^{n} \alpha_{ij}' x^{i-1} y^{j-1} \in F[x,y]$$

is called the Bezoutian of f_1 *and* f_2*. But no difference can be seen between these two definitions as far as we consider the corresponding quadratic forms*

$$\sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \quad \textit{and} \quad \sum_{i,j=1}^{n} \alpha_{ij}' x_i x_j.$$

In fact, these two quadratic forms are equivalent over the prime field $\mathbb{Q}$ $(\subset F)$ *since we have* $M'_n(f_1, f_2) = {}^tJ_nM_n(f_1, f_2)J_n$*, where*

$$J_n = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{bmatrix}$$

is an $n \times n$ *anti-identity matrix. This implies that above two quadratic forms are equivalent over* $\mathbb{Q}$ *or more precisely, over the ring of rational integers* $\mathbb{Z}$*.*

Let $f(x) \in \mathbb{R}[x]$ be a degree $n \geq 2$ polynomial which is given by

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

Then over $\mathbb{R}$ this polynomial is factored as

$$f(x) = a\prod_{j=1}^{r}(x - \alpha_j)\prod_{i=1}^{s}(x^2 + a_ix + b_i)$$

for some $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ and $a_i, b_i, a \in \mathbb{R}$, where $a_i^2 < 4b_i$, for all $i = 1, \cdots, s$.

Throughout this paper, for a univariate polynomial f, its discriminant will be denoted by Δ_f. For any two polynomials $f_1(x)$, $f_2(x)$ the resultant with respect to x will be denoted by $\mathrm{Res}(f_1, f_2, x)$. We notice the following elementary fact, its proof is elementary and we skip the details.

Remark 2. *For any polynomial* $f(x)$*, the determinant of the Bezoutian is the same as the discriminant up to a multiplication by a constant. More precisely,*

$$\Delta_f = \frac{1}{led(f)^2}\det M_n(f),$$

where $led(f)$ *is the leading coefficient of* $f(x)$*.*

If $f(x) \in \mathbb{Q}[x]$ is irreducible and its degree is a prime number, say $\deg f = p$, then there is enough known for the Galois group of polynomials with some non-real roots; see [4], [1], [13] for details. If the number of non-real roots is "small" enough with respect to the prime degree $\deg f = p$ of the polynomial, then the Galois group is A_p or S_p. Furthermore, using the classification of finite simple groups one can provide a complete list of possible Galois groups for every polynomial of prime degree p which has non-real roots; see [4] for details.

On the other extreme are the polynomials which have all roots non-real. We called them above, totally complex polynomials. We have the following:

Lemma 1. *The followings are equivalent:*
i) $f(x) \in \mathbb{R}[x]$ *is totally complex*
ii) $f(x)$ *can be written as*

$$f(x) = a\prod_{i=1}^{n} f_i$$

where $f_i = x^2 + a_ix + b_i$*, for* $i = 1, \ldots, n$ *and* $a_i, b_i, a \in \mathbb{R}$*, where* $a_i^2 < 4b_i$*, for all* $i = 1, \ldots, n$*. Moreover, the determinant of the Bezoutian* $M_n(f)$ *is given by*

$$\Delta_f = \frac{1}{led(f)^2}\det M_n(f) = \prod_{i=1}^{n}\Delta_{f_i}\cdot\prod_{i,j,i\neq j}^{n}(Res(f_i, f_j, x))^2$$

where led(f) is the leading coefficient of $f(x)$.

ii) the index of inertia of Bezoutian $M(f)$ is 0

iii) if $\Delta_f \neq 0$ then the equivalence class of $M(f)$ in the Witt ring $W(R)$ is 0.

Proof. The equivalence between i), ii), and iii) can be found in [6]. □

It is not clear when such polynomials are irreducible over $\mathbb{Q}$. If that's the case, what is the Galois group Gal (f)? Clearly the group generated by the involution $(1,2)(3,4)\cdots(2n-1,2n)$ is embedded in Gal (f). Is Gal (f) larger in general?

3. On the number of real roots of polynomials

For any degree $n \geq 2$ polynomial $f(x) \in \mathbb{R}[x]$ and any symmetric matrix $M := M_n(f)$ with real entries, let N_f be the **number of distinct real roots** of f and $\sigma(M)$ be the index of inertia of M, respectively. The next result plays a fundamental role throughout this section ([6, Theorem 9.2]).

Proposition 2. *For any real polynomial $f \in \mathbb{R}[x]$, the number N_f of its distinct real roots is the index of inertia of the Bezoutian matrix $M_n(f)$. In other words,*

$$N_f = \sigma\left(M_n(f)\right).$$

Let us cite one more result which says that the roots of a polynomial depend continuously on its coefficients ([11, Theorem 1.4], [16, Theorem 1.3.1]).

Proposition 3. *Let be given a polynomial*

$$f(x) = \sum_{l=0}^{n} a_l x^l \in \mathbb{C}[x],$$

with distinct roots $\alpha_1, \ldots, \alpha_k$ of multiplicities $m_1, \ldots, m_k$ respectively. Then, for any given a positive

$$\varepsilon < \min_{1\leq i<j\leq k} \left\{ \frac{|\alpha_i - \alpha_j|}{2} \right\},$$

there exists a real number $\delta > 0$ such that any monic polynomial $g(x) = \sum_{l=0}^{n} b_l x^l \in \mathbb{C}[x]$ whose coefficients satisfy

$$|b_l - a_l| < \delta,$$

for $l = 0, \cdots, n-1$, has exactly m_j roots in the disk

$$\mathcal{D}(\alpha_j; \varepsilon) = \{z \in \mathbb{C} \mid |z - \alpha_j| < \epsilon\} \ (j = 1, \cdots, k).$$

Let n, s be positive integers such that $n > s$ and let

$$g(t_0, \cdots, t_s; x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}, \tag{1}$$
$$f^{(n)}(t_0, \cdots, t_s, t; x) = x^n + t \cdot g(t_0, \cdots, t_s; x)$$

be polynomials in x over $E_1 = \mathbb{R}(t_0, \cdots, t_s)$, $E_2 = \mathbb{R}(t_0, \cdots, t_s, t)$, respectively. Here, E_1 (resp., E_2) is a rational function field with $s+1$ (*resp.*, $(s+2)$) variables $t_0, \cdots, t_s$ (*resp.*, $(t_0, \cdots, t_s, t)$). To ease notation, let us put

$$g(x) = g(t_0, \cdots, t_s; x), \ \ f(t; x) = f^{(n)}(t_0, \cdots, t_s, t; x)$$

and for any real vector $\boldsymbol{v} = (v_0, \cdots, v_s) \in \mathbb{R}^{s+1}$, we put

$$g_{\boldsymbol{v}}(x) = g(v_0, \cdots, v_s; x), \ \ f_{\boldsymbol{v}}(t; x) = f^{(n)}(v_0, \cdots, v_s, t; x). \tag{2}$$

By using Proposition 2, we can prove the next theorem ([13, Main Theorem 1.3]).

Theorem 1. *Let $\boldsymbol{r} = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $N_{g_{\boldsymbol{r}}} = s$. Let us consider $f_{\boldsymbol{r}}(t;x) = f^{(n)}(r_0, \cdots, r_s, t; x)$ as a polynomial over $\mathbb{R}(t)$ in x and put*

$$P_{\boldsymbol{r}}(t) = \det M_n(f_{\boldsymbol{r}}(t;x)) = \det M_n(f_{\boldsymbol{r}}(t;x), f'_{\boldsymbol{r}}(t;x)),$$

where $f'_{\boldsymbol{r}}(t;x)$ is a derivative of $f_{\boldsymbol{r}}(t;x)$ with respect to x. Then, for any real number $\xi > \alpha_{\boldsymbol{r}} = \max\{\alpha \in \mathbb{R} \mid P_{\boldsymbol{r}}(\alpha) = 0\}$, we have

$$N_{f_{\boldsymbol{r}}(\xi;x)} = \begin{cases} s+1 & \text{if } n-s : \text{odd} \\ s & \text{if } n-s : \text{even}, \ r_s > 0 \\ s+2 & \text{if } n-s : \text{even}, \ r_s < 0. \end{cases}$$

By this theorem and a theorem of Oz Ben-Shimol [1, Theorem 2.6], we can obtain an algorithm to construct prime degree p polynomials with given number of real roots, and whose Galois groups are isomorphic to the symmetric group S_p or the alternating group A_p ([13, Corollary 1.6]).

In this section, we extend this theorem as follows;

Theorem 2. *Let $\boldsymbol{r} = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $g_{\boldsymbol{r}}(x)$ is a degree s separable polynomial satisfying $N_{g_{\boldsymbol{r}}(x)} = \gamma$ $(0 \leq \gamma \leq s)$. Let us consider $f_{\boldsymbol{r}}(t;x) = f^{(n)}(r_0, \cdots, r_s, t; x)$ as a polynomial over $\mathbb{R}(t)$ in x and put*

$$P_{\boldsymbol{r}}(t) = \det M_n(f_{\boldsymbol{r}}(t;x)) = \det M_n(f_{\boldsymbol{r}}(t;x), f'_{\boldsymbol{r}}(t;x)),$$

where $f'_{\boldsymbol{r}}(t;x)$ is a derivative of $f_{\boldsymbol{r}}(t;x)$ with respect to x. Then, for any real number $\xi > \alpha_{\boldsymbol{r}} = \max\{\alpha \in \mathbb{R} \mid P_{\boldsymbol{r}}(\alpha) = 0\}$, we have

$$N_{f_{\boldsymbol{r}}(\xi;x)} = \begin{cases} \gamma+1 & \text{if } n-s : \text{odd} \\ \gamma & \text{if } n-s : \text{even}, \ r_s > 0 \\ \gamma+2 & \text{if } n-s : \text{even}, \ r_s < 0. \end{cases} \tag{3}$$

The above theorem can be restated as follows:

Corollary 1. *Let $f \in \mathbb{R}(t)[x]$ be given by*

$$f(t,x) = x^n + t \cdot \sum_{k=0}^{s} t_{s-k} x^{s-k}$$

and $\beta_1 < \cdots < \beta_m$ the distinct real roots of the degree s polynomial

$$P(t) := \frac{1}{t^{n-1}} \Delta_{(f,x)}(t).$$

For any $\xi > |\beta_m|$, the number of real roots of $f(\xi, x)$ is

$$N_{f(\xi,x)} = \begin{cases} \gamma+1 & \text{if } n-s : \text{odd} \\ \gamma & \text{if } n-s : \text{even}, \ t_s > 0 \\ \gamma+2 & \text{if } n-s : \text{even}, \ t_s < 0. \end{cases}$$

where γ is the number or real roots of $g(x) = \frac{f(x)-x^n}{t} \in \mathbb{R}[x]$.

The rest of the section is concerned with proving Thm. 2.

3.1. **The Bezoutian of $f(t;x)$.** First, let us put

$$A(t_0,\cdots,t_s,t)=(a_{ij}(t_0,\cdots,t_s,t))_{1\le i,j\le n}=M_n(f(t;x))\in \mathrm{Sym}_n(E_2),$$
$$B(t_0,\cdots,t_s)=(b_{ij}(t_0,\cdots,t_s))_{1\le i,j\le s}=M_s(g(x))\in \mathrm{Sym}_s(E_1).$$

For ease of notation, we also write

$$A(t_0,\cdots,t_s,t)=A(t)=(a_{ij}(t))_{1\le i,j\le n},\ B(t_0,\cdots,t_s)=B=(b_{ij})_{1\le i,j\le s}$$

and we put $B(t)=(b_{ij}(t))_{1\le i,j\le s}=t^2B$. Then, by Proposition 1, we have

$$\begin{aligned}A(t)&=M_n(x^n+tg(x),nx^{n-1}+tg'(x))\\&=nM_n(x^n,x^{n-1})-ntM_n(x^{n-1},g(x))+tM_n(x^n,g'(x))+t^2M_n(g(x),g'(x))\\&=nM_n(x^n,x^{n-1})-nt\sum_{k=0}^{s}t_{s-k}M_n(x^{n-1},x^{s-k})\\&\qquad+t\sum_{k=0}^{s-1}(s-k)t_{s-k}M_n(x^n,x^{s-k-1})+t^2M_n(g(x),g'(x)).\end{aligned}$$

Lemma 2. *Let λ,μ,ν be integers such that $\lambda\ge\mu>\nu\ge 0$. Then $M_\lambda(x^\mu,x^\nu)=(m_{ij})_{1\le i,j\le\lambda}$, where*

$$m_{ij}=\begin{cases}1 & i+j=2\lambda-(\mu+\nu)+1\quad(\lambda-\mu+1\le i,j\le\lambda-\nu),\\0 & \textit{otherwise.}\end{cases}$$

Proof. By definition, we have

$$\begin{aligned}B_\lambda(x^\mu,x^\nu)&=\frac{x^\mu y^\nu-x^\nu y^\mu}{x-y}\\&=\sum_{k=1}^{\mu-\nu}x^{\mu-k}y^{\nu+k-1}=\sum_{k=1}^{\mu-\nu}x^{\lambda-(\lambda-\mu+k)}y^{\lambda-(\lambda-\nu-k+1)},\end{aligned}$$

which implies

$$\begin{aligned}m_{ij}&=\begin{cases}1 & (i,j)=(\lambda-\mu+k,\lambda-\nu-k+1)\quad(1\le k\le\mu-\nu)\\0 & \text{otherwise}\end{cases}\\&=\begin{cases}1 & i+j=2\lambda-(\mu+\nu)+1\quad(\lambda-\mu+1\le i,j\le\lambda-\nu),\\0 & \text{otherwise.}\end{cases}\end{aligned}$$

This completes the proof. □

Here, let us divide $A(t)$ into two parts $\hat{A}(t)$ and $\tilde{A}(t)$, where

$$\begin{aligned}\hat{A}(t)=(\hat{a}_{ij}(t))_{1\le i,j\le n}&=nM_n(x^n,x^{n-1})-nt\sum_{k=0}^{s}t_{s-k}M_n(x^{n-1},x^{s-k})\\&\qquad+t\sum_{k=0}^{s-1}(s-k)t_{s-k}M_n(x^n,x^{s-k-1}),\\\tilde{A}(t)=(\tilde{a}_{ij}(t))_{1\le i,j\le n}&=t^2M_n(g(x),g'(x))\end{aligned}$$

and put $l_k = n - s + k + 2$ $(= 2n - (n + s - k - 1) + 1)$. Then, by lemma 2, we have

$$\begin{cases} \hat{a}_{11}(t) = n \\ \hat{a}_{1,l_k-1}(t) = \hat{a}_{l_k-1,1}(t) = (s-k)t_{s-k}t \ (0 \leq k \leq s-1). \end{cases}$$

Moreover, when $i + j = l_k$, we have

$$\hat{a}_{ij}(t) = -ntt_{s-k} + t(s-k)t_{s-k} = -(l_k - 2)t_{s-k}t \quad (2 \leq i, j \leq l_k - 2,\ 0 \leq k \leq s). \tag{4}$$

Remark 3. *Note that, if $s = n - 1$, we have*

$$-nt \sum_{k=0}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k}) = -nt \sum_{k=1}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k}),$$

Thus, when $i + j = l_k$, equation (4) should be modified by

$$\hat{a}_{ij}(t) = -ntt_{s-k} + t(s-k)t_{s-k} = -(l_k - 2)t_{s-k}t \quad (2 \leq i, j \leq l_k - 2,\ 1 \leq k \leq s).$$

We avoid this minor defect by considering that there is no entries satisfying $2 \leq i, j \leq l_0 - 2$ when $s = n - 1$ since $l_0 - 2 = n - s = 1$.

Proposition 4. *Put $l_k = n - s + k + 2$. Then*

$$\hat{a}_{ij}(t) = \begin{cases} n & (i,j) = (1,1) \\ (s-k)t_{s-k}t & (i,j) = (1, l_k - 1) \text{ or } (l_k - 1, 1) \quad (0 \leq k \leq s-1) \\ -(l_k - 2)t_{s-k}t & i + j = l_k,\ 2 \leq i, j \leq l_k - 2,\ (0 \leq k \leq s) \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{a}_{ij}(t) = \begin{cases} b_{i-(n-s),j-(n-s)} t^2 & n - s + 1 \leq i, j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The statement for $\hat{a}_{ij}(t)$ has just been proved. For $\tilde{a}_{ij}(t)$, it is enough to see that we can denote

$$M_s(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} mt_\ell t_m M_s(x^\ell, x^{m-1}),$$

$$M_n(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} mt_\ell t_m M_n(x^\ell, x^{m-1}),$$

that is, we can obtain $M_n(g(x))$ from $M_s(g(x))$ by just replacing s with n for all $M_s(x^\ell, x^m)$, which, by Lemma 2, means that $s \times s$ matrix $M_s(g(x))$ occupies the part $\{b^\dagger_{ij} \mid n - s + 1 \leq i, j \leq n\}$ of the matrix $M_n(g(x)) = (b^\dagger_{ij})_{1 \leq i,j \leq n}$. ☐

By Proposition 4, we can express the matrix $A(t)$ as follows;

$$A(t) = \left[\begin{array}{cccc|cccc} n & 0 & \cdots & 0 & st_st & (s-1)t_{s-1}t & \cdots & t_1t \\ 0 & & & -(n-s)t_st & -(n-s+1)t_{s-1}t & \cdots & -(n-1)t_1t & -nt_0t \\ \vdots & & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)t_st & \iddots & & & \iddots & 0 & 0 \\ \hline st_st & -(n-s+1)t_{s-1}t & & & & & & \\ (s-1)t_{s-1}t & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)t_1t & \iddots & 0 & \multicolumn{4}{c}{C(t)} \\ t_1t & -nt_0t & 0 & 0 & & & & \end{array}\right]. \tag{5}$$

Here, $C(t) = (c_{ij}(t))_{1\le i,j\le s} = C(t_0, \cdots, t_s, t) = (c_{ij}(t_0, \cdots, t_s, t))_{1\le i,j\le s}$ is an $s\times s$ symmetric matrix whose entries are of the form

$$\begin{aligned} c_{ij}(t_0, \cdots, t_s, t) &= b_{ij}t^2 + \lambda_{ij}t \\ &= b_{ij}(t_0, \cdots, t_s)t^2 + \lambda_{ij}(t_0, \cdots, t_s)t \quad (\lambda_{ij} = \lambda_{ij}(t_0, \cdots, t_s) \in E_1). \end{aligned}$$

Next, let $A(t)_1 = (a_{ij}(t)_1)_{1\le i,j\le n} = A(t_0, \cdots, t_s, t)_1 = (a_{ij}(t_0, \cdots, t_s, t)_1)_{1\le i,j\le n}$ be the $n \times n$ symmetric matrix obtained from $A(t)$ by multiplying the first row and the first column by $1/\sqrt{n}$ and then sweeping out the entries of the first row and the first column by the $(1, 1)$ entry 1. Here, let $Q_m(k; c) = (q_{ij})_{1\le i,j\le m}$ and $R_m(k, l; c) = (r_{ij})_{1\le i,j\le m}$ be $m \times m$ elementary matrices such that

$$Q_m(k;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},\ R_m(k,l;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & c & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},$$

where $q_{kk} = c$ and $r_{kl} = c$. Moreover, for any $m \times m$ matrices $M_1, M_2, \cdots, M_l$, put $\prod_{k=1}^{l} M_k = M_1M_2\cdots M_l$. Then, we have $A(t)_1 = {}^tS(t)_1A(t)S(t)_1$, where

$$S(t)_1 = Q_n(1; 1/\sqrt{n}) \prod_{k=0}^{s-1} R_n(1, l_k - 1; -a_{1,l_k-1}(t)/\sqrt{n}).$$

The matrix $A(t)_1$ can be expressed as follows;

(6)

$$A(t)_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -(n-s)t_st & -(n-s+1)t_{s-1}t & \cdots & -(n-1)t_1t & -nt_0t \\ \vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)t_st & \iddots & & & \iddots & 0 & 0 \\ \hline 0 & -(n-s+1)t_{s-1}t & & & & & & \\ 0 & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)t_1t & \iddots & 0 & \multicolumn{4}{c}{C(t)_1} \\ 0 & -nt_0t & 0 & 0 & & & & \end{array}\right].$$

Here, $C(t)_1 = (c_{ij}(t)_1)_{1\le i,j\le s} = C(t_0, \cdots, t_s, t)_1 = (c_{ij}(t_0, \cdots, t_s, t)_1)_{1\le i,j\le s}$ is an $s \times s$ symmetric matrix whose entries are of the form

$$c_{ij}(t_0, \cdots, t_s, t)_1 = \bar{b}_{ij}(t_0, \cdots, t_s)t^2 + \lambda_{ij}(t_0, \cdots, t_s)t \quad (\bar{b}_{ij}(t_0, \cdots, t_s) \in E_1),$$

where

$$\bar{b}_{ij}(t_0, \cdots, t_s) = b_{ij}(t_0, \cdots, t_s) - \frac{(s-i+1)(s-j+1)}{n} t_{s-i+1}t_{s-j+1} \tag{7}$$

for any i, j $(1 \le i, j \le s)$. We put $\bar{b}_{ij}(t_0, \cdots, t_s) = \bar{b}_{ij}$ and $\bar{B} = (\bar{b}_{ij})_{1\le i,j\le s}$.

3.2. **Some results for the Bezoutian of $f_{\boldsymbol{r}}(t;x)$.** Let $\boldsymbol{r} = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector as in Theorem 2. We put

$$\begin{aligned} A_{\boldsymbol{r}}(t) &= (a_{ij}^{(\boldsymbol{r})}(t))_{1\le i,j\le n} = A(r_0, \cdots, r_s, t) \in \mathrm{Sym}_n(\mathbb{R}(t)), \\ B_{\boldsymbol{r}} &= (b_{ij}^{(\boldsymbol{r})})_{1\le i,j\le s} = B(r_0, \cdots, r_s) \in \mathrm{Sym}_s(\mathbb{R}) \end{aligned}$$

and $B_{\boldsymbol{r}}(t) = t^2 B_{\boldsymbol{r}}$. Let us also put $A_{\boldsymbol{r}}(t)_1 = A(r_0, \cdots, r_s, t)_1$. By equation (6), the matrix $A_{\boldsymbol{r}}(t)_1$ can be expressed as follows;

$$A_{\boldsymbol{r}}(t)_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -(n-s)r_s t & -(n-s+1)r_{s-1}t & \cdots & -(n-1)r_1 t & -nr_0 t \\ \vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)r_s t & \iddots & & & \iddots & 0 & 0 \\ \hline 0 & -(n-s+1)r_{s-1}t & & & & & & \\ 0 & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)r_1 t & \iddots & 0 & & C_{\boldsymbol{r}}(t)_1 & & \\ 0 & -nr_0 t & 0 & 0 & & & & \end{array}\right].$$

Here, $C_{\boldsymbol{r}}(t)_1 = (c_{ij}^{(\boldsymbol{r})}(t)_1)_{1\le i,j\le s} = C(r_0, \cdots, r_s, t)_1$ and

$$c_{ij}^{(\boldsymbol{r})}(t)_1 = \bar{b}_{ij}(r_0, \cdots, r_s)t^2 + \lambda_{ij}(r_0, \cdots, r_s)t \quad (\bar{b}_{ij}(r_0, \cdots, r_s), \lambda_{ij}(r_0, \cdots, r_s) \in \mathbb{R}).$$

Note that, by equation (7), we have

$$\bar{b}_{ij}(r_0, \cdots, r_s) = b_{ij}^{(\boldsymbol{r})} - \frac{(s-i+1)(s-j+1)}{n} r_{s-i+1} r_{s-j+1} \quad (1 \le i,j \le s).$$

To ease notation, we put $\bar{b}_{ij}(r_0, \cdots, r_s) = \bar{b}_{ij}^{(\boldsymbol{r})}$ and $\bar{B}_{\boldsymbol{r}} = (\bar{b}_{ij}^{(\boldsymbol{r})})_{1\le i,j\le s}$.

In particular, since

$$\begin{aligned} M_s(g_{\boldsymbol{r}}) &= M_s\left(r_s x^s, \sum_{k=0}^{s-1}(s-k)r_{s-k}x^{s-k-1}\right) + M_s\left(\sum_{k=1}^{s} r_{s-k}x^{s-k}, g_{\boldsymbol{r}}'\right) \\ &= \sum_{k=0}^{s-1}(s-k)r_s r_{s-k} M_s(x^s, x^{s-k-1}) + M_s\left(\sum_{k=1}^{s} r_{s-k}x^{s-k}, g_{\boldsymbol{r}}'\right), \end{aligned}$$

we have

$$b_{1,k+1}^{(\boldsymbol{r})} = b_{k+1,1}^{(\boldsymbol{r})} = (s-k)r_s r_{s-k} \ (0 \le k \le s-1) \tag{8}$$

by Lemma 2 and hence

$$\begin{aligned} \bar{b}_{1j}^{(\boldsymbol{r})} &= (s-j+1)r_s r_{s-j+1} - \frac{s(s-j+1)}{n} r_s r_{s-j+1} \\ &= (s-j+1)\left(1 - \frac{s}{n}\right) r_s r_{s-j+1} \ (1 \le j \le s). \end{aligned} \tag{9}$$

Lemma 3. *Put $\bar{B}_{\boldsymbol{r}}(t) = t^2 \bar{B}_{\boldsymbol{r}}$. Then, $B_{\boldsymbol{r}}(\xi)$ and $\bar{B}_{\boldsymbol{r}}(\xi)$ are equivalent over $\mathbb{R}$ for any real number ξ and we have $\sigma(\bar{B}_{\boldsymbol{r}}(\xi)) = N_{g_{\boldsymbol{r}}}$ for any non-zero real number ξ.*

Proof. Let us denote by $B_{\boldsymbol{r}}^* = (b_{ij}^{(\boldsymbol{r},*)})_{1\le i,j\le s}$ $(\bar{B}_{\boldsymbol{r}}^* = (\bar{b}_{ij}^{(\boldsymbol{r},*)})_{1\le i,j\le s})$ the matrix obtained from $B_{\boldsymbol{r}}$ $(\bar{B}_{\boldsymbol{r}})$ by multiplying the first row and the first column by $1\Big/\pm\sqrt{b_{11}^{(\boldsymbol{r})}}$ $\left(1\Big/\pm\sqrt{\bar{b}_{11}^{(\boldsymbol{r})}}\right)$ (the sign before $\sqrt{b_{11}^{(\boldsymbol{r})}}$ $\left(\sqrt{\bar{b}_{11}^{(\boldsymbol{r})}}\right)$ are the same as the sign of r_s; see the definition of d $(\bar{d})$ below) and then sweeping out the entries of the first row and the first column by the $(1,1)$ entry 1. Since $b_{11} = sr_s^2$ (>0) and $\bar{b}_{11} = s(1-s/n)r_s^2$ (>0) by (8) and (9), we have

$$B^*_{\boldsymbol{r}} = {}^tTB_{\boldsymbol{r}}T, \ \bar{B}^*_{\boldsymbol{r}} = {}^t\bar{T}\bar{B}_{\boldsymbol{r}}\bar{T}, \tag{10}$$

where

$$T = Q_s(1;1/d)\prod_{k=2}^{s} R_s(1,k;-b^{(\boldsymbol{r})}_{1k}/d) \ (d = \sqrt{s}\cdot r_s),$$
$$\bar{T} = Q_s(1;1/\bar{d})\prod_{k=2}^{s} R_s(1,k;-\bar{b}^{(\boldsymbol{r})}_{1k}/\bar{d}) \ (\bar{d} = \sqrt{s(1-s/n)}\cdot r_s).$$

Note that in [13, Lemma 3.3], we have proved $b^{(\boldsymbol{r},*)}_{ij} = \bar{b}^{(\boldsymbol{r},*)}_{ij}$ $(1 \le i,j \le s)$ and hence $t^2B^*_{\boldsymbol{r}} = t^2\bar{B}^*_{\boldsymbol{r}}$, which, by (10), implies that symmetric matrices $B_{\boldsymbol{r}}(\xi)$ and $\bar{B}_{\boldsymbol{r}}(\xi)$ are equivalent over $\mathbb{R}$ for any real number ξ. Then, since $N_{g_{\boldsymbol{r}}} = \sigma(B_{\boldsymbol{r}}) = \sigma(B_{\boldsymbol{r}}(\xi))$ for any $\xi \in \mathbb{R}\setminus\{0\}$, the latter half of the statement have also been proved. □

3.3. **Nonvanishingness of some coefficients.** In this subsection, we prove the next lemma.

Lemma 4. *Let*

$$\Phi(x) = \Phi(t_0,\cdots,t_s;x) = \sum_{k=0}^{s} h_{s-k}(t_0,\cdots,t_s)x^{s-k} \in E_1[x] \tag{11}$$

be the characteristic polynomial of $\bar{B}$. Then, $h_{s-k}(t_0,\cdots,t_s)$ is a non-zero polynomial in E_1 for any k $(1 \le k \le s)$.

Proof. Lemma 4 is clear for $s = 1$, since we have

$$B = M_1(t_1x + t_0) = \left[\ t_1^2\ \right]$$

and hence, by equation (7),

$$\bar{B} = \left[\ t_1^2 - \frac{1}{n}t_1^2\ \right] = \left[\ \frac{n-1}{n}t_1^2\ \right].$$

Next, suppose $s \ge 2$. Then, by equation (7) and the definition of the Bezoutian, we have $h_{s-k}(t_0,\cdots,t_s) \in \mathbb{R}[t_0,\cdots,t_s]$ for any k $(1 \le k \le s)$. Thus, we have only to prove that $h_{s-k}(t_0,\cdots,t_s) \ne 0$ for any k $(1 \le k \le s)$, which is clear from the next Lemma 5. □

Lemma 5. *Suppose $s \ge 2$ and put $u_0 = u_s = 1$, $u_1 = t_1$ and $u_k = 0$ $(2 \le k \le s-1)$. Then, $h_{s-k}(u_0,\cdots,u_s)$ is a non-constant polynomial in $\mathbb{R}(t_1)$ for any k $(1 \le k \le s)$, i.e., $h_{s-k}(u_0,\cdots,u_s) \in \mathbb{R}[t_1]\setminus\mathbb{R}$ $(1 \le k \le s)$.*

To prove lemma 5, let us put $\boldsymbol{u} = (u_0,\cdots,u_s)$ and

$$g_{\boldsymbol{u}}(x) = g(u_0,\cdots,u_s;x) = x^s + t_1x + 1 \in \mathbb{R}(t_1)[x],$$
$$f_{\boldsymbol{u}}(t;x) = x^n + tg_{\boldsymbol{u}}(x) \in \mathbb{R}(t_1,t)[x] \ (n > s),$$
$$A_{\boldsymbol{u}}(t) = (a^{(\boldsymbol{u})}_{ij}(t))_{1\le i,j\le n} = A(u_0,\cdots,u_s,t) \in \mathrm{Sym}_n(\mathbb{R}(t_1,t)),$$
$$B_{\boldsymbol{u}} = (b^{(\boldsymbol{u})}_{ij})_{1\le i,j\le s} = B(t_0,\cdots,u_s) \in \mathrm{Sym}_s(\mathbb{R}(t_1)),\ B_{\boldsymbol{u}}(t) = t^2B_{\boldsymbol{u}}.$$

Then, by equation (5), we have

$$A_{\boldsymbol{u}}(t) = \left[\begin{array}{cccc|cccc} n & 0 & \dots & 0 & st & 0 & \dots & t_1 t \\ 0 & & & -(n-s)t & 0 & \dots & -(n-1)t_1 t & -nt \\ \vdots & & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)t & \iddots & & & \iddots & 0 & 0 \\ \hline st & 0 & & & & & & \\ 0 & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)t_1 t & \iddots & 0 & & & C_{\boldsymbol{u}}(t) & \\ t_1 t & -nt & 0 & 0 & & & & \end{array}\right],$$

where $C_{\boldsymbol{u}}(t) = (c_{ij}^{(\boldsymbol{u})}(t))_{1\le i,j\le s} = C(u_0, \cdots, u_s, t)$ and

$$c_{ij}^{(\boldsymbol{u})}(t) = b_{ij}(u_0, \cdots, u_s)t^2 + \lambda_{ij}(u_0, \cdots, u_s)t \quad (\lambda_{ij}(u_0, \cdots, u_s) \in \mathbb{R}(t_1)).$$

Moreover, by equation (6), we also have

$$A_{\boldsymbol{u}}(t)_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & -(n-s)t & 0 & \dots & -(n-1)t_1 t & -nt \\ \vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)t & \iddots & & & \iddots & 0 & 0 \\ \hline 0 & 0 & & & & & & \\ 0 & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)t_1 t & \iddots & 0 & & & C_{\boldsymbol{u}}(t)_1 & \\ 0 & -nt & 0 & 0 & & & & \end{array}\right].$$

Here, $C_{\boldsymbol{u}}(t)_1 = (c_{ij}^{(\boldsymbol{u})}(t)_1)_{1\le i,j\le s} = C(u_0, \cdots, u_s, t)_1$ and

$$c_{ij}^{(\boldsymbol{u})}(t)_1 = \bar{b}_{ij}(u_0, \cdots, u_s)t^2 + \lambda_{ij}(u_0, \cdots, u_s)t \quad (\bar{b}_{ij}(u_0, \cdots, u_s) \in \mathbb{R}).$$

Note that, by equation (7), we have

$$\bar{b}_{ij}^{(\boldsymbol{u})} = \begin{cases} b_{11}^{(\boldsymbol{u})} - (s^2/n) & (i,j) = (1,1) \\ b_{1s}^{(\boldsymbol{u})} - (s/n)t_1 & (i,j) = (1,s) \text{ or } (s,1) \\ b_{ss}^{(\boldsymbol{u})} - (1/n)t_1^2 & (i,j) = (s,s) \\ b_{ij}^{(\boldsymbol{u})} & \text{otherwise.} \end{cases} \tag{12}$$

Let us put $\bar{B}_{\boldsymbol{u}} = (\bar{b}_{ij}^{(\boldsymbol{u})})_{1\le i,j\le s}$ and $\bar{B}_{\boldsymbol{u}}(t) = t^2 \bar{B}_{\boldsymbol{u}}$. Then, since

$$\begin{aligned} M_s(g_{\boldsymbol{u}}) &= M_s(x^s + t_1 x + 1, sx^{s-1} + t_1) \\ &= sM_s(x^s, x^{s-1}) + t_1 M_s(x^s, 1) - st_1 M_s(x^{s-1}, x) - sM_s(x^{s-1}, 1) \\ &\qquad + t_1^2 M_s(x, 1) + t_1 M_s(1, 1), \end{aligned}$$

we have

(a) if $s = 2$,

$$B_{\boldsymbol{u}} = \begin{bmatrix} 2 & t_1 \\ t_1 & t_1^2 - 2 \end{bmatrix},$$

(b) if $s \geq 3$,

$$b_{ij}^{(\boldsymbol{u})} = \begin{cases} s & (i,j) = (1,1) \\ t_1 & (i,j) = (1,s) \text{ or } (s,1) \\ (1-s)t_1 & i+j = s+1,\ 2 \leq i,j \leq s-1 \\ -s & i+j = s+2 \\ t_1^2 & (i,j) = (s,s), \\ 0 & \text{otherwise,} \end{cases}$$

which, by equation (12), implies

(a') if $s = 2$,

$$\bar{B}_{\boldsymbol{u}} = \begin{bmatrix} 2(n-2)/n & (n-2)t_1/n \\ (n-2)t_1/n & (n-1)t_1^2/n - 2 \end{bmatrix},$$

(b') if $s \geq 3$,

$$\bar{b}_{ij}^{(\boldsymbol{u})} = \begin{cases} s(n-s)/n & (i,j) = (1,1) \\ (n-s)t_1/n & (i,j) = (1,s) \text{ or } (s,1) \\ (1-s)t_1 & i+j = s+1,\ 2 \leq i,j \leq s-1 \\ -s & i+j = s+2 \\ (n-1)t_1^2/n & (i,j) = (s,s), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $s \geq 3$, the matrix $\bar{B}_{\boldsymbol{u}} = (\bar{b}_{ij}^{(\boldsymbol{u})})_{1 \leq i,j \leq s}$ has the expression of the form

$$\begin{bmatrix} s(n-s)/n & 0 & 0 & 0 & \cdots & 0 & (n-s)t_1/n \\ 0 & 0 & 0 & \cdots & 0 & (1-s)t_1 & -s \\ 0 & 0 & & \iddots & (1-s)t_1 & -s & 0 \\ 0 & \vdots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & (1-s)t_1 & \iddots & \iddots & & 0 \\ 0 & (1-s)t_1 & -s & \iddots & & & 0 \\ (n-s)t_1/n & -s & 0 & \cdots & 0 & 0 & (n-1)t_1^2/n \end{bmatrix}.$$

Here, let us denote by

$$\Phi_{\boldsymbol{u}}(x) = \sum_{k=0}^{s} h_{s-k}^{(\boldsymbol{u})} x^{s-k} = \Phi(u_0, \cdots, u_s; x) \quad \left(= \sum_{k=0}^{s} h_{s-k}(u_0, \cdots, u_s) x^{s-k} \right)$$

the characteristic polynomial of $\bar{B}_{\boldsymbol{u}}$. Note that since we have $h_{s-k}^{(\boldsymbol{u})} \in \mathbb{R}[t_1]$ by the proof of Lemma 4, we have only to prove $h_{s-k}^{(\boldsymbol{u})}$ is non-constant for any k $(1 \leq k \leq s)$.

By the above expression of $\bar{B}_{\boldsymbol{u}}$, we have

(a'') if $s = 2$,

$$\Phi_{\boldsymbol{u}}(x) = x^2 - \frac{(n-1)t_1^2 - 4}{n} x + \frac{(n-2)t_1^2 - 4n + 8}{n},$$

(b'') if $s \geq 3$,

$$
\Phi_{\boldsymbol{u}}(x) = \begin{cases}
\begin{vmatrix}
x-s(n-s)/n & & & & & & & & -(n-s)t_1/n \\
& x & & & & & & (s-1)t_1 & s \\
& & \ddots & & & & \iddots & s & \\
& & & \ddots & & \iddots & \iddots & & \\
& & & & x+(s-1)t_1 & s & & & \\
& & & \iddots & s & x & & & \\
& & \iddots & \iddots & & & \ddots & & \\
& (s-1)t_1 & s & & & & & x & \\
-(n-s)t_1/n & s & & & & & & & x-(n-1)t_1^2/n
\end{vmatrix} & (s \text{ is odd}), \\[2ex]
\begin{vmatrix}
x-s(n-s)/n & & & & & & & -(n-s)t_1/n \\
& x & & & & & (s-1)t_1 & s \\
& & \ddots & & & \iddots & s & \\
& & & x & (s-1)t_1 & \iddots & & \\
& & & (s-1)t_1 & x+s & & & \\
& & \iddots & \iddots & & \ddots & & \\
& (s-1)t_1 & s & & & & x & \\
-(n-s)t_1/n & s & & & & & & x-(n-1)t_1^2/n
\end{vmatrix} & (s \text{ is even}).
\end{cases}
$$

Example 1. (1) *Put* $s = 7$ *and* $n = 10$. *Then, we have*

$$
g_{\boldsymbol{u}}(x) = x^7 + t_1 x + 1, \quad f_{\boldsymbol{u}}(t; x) = x^{10} + t(x^7 + t_1 x + 1),
$$

$$
\begin{aligned}
\Phi_{\boldsymbol{u}}(x) = &\begin{vmatrix}
x-21/10 & 0 & 0 & 0 & 0 & 0 & -3t_1/10 \\
0 & x & 0 & 0 & 0 & 6t_1 & 7 \\
0 & 0 & x & 0 & 6t_1 & 7 & 0 \\
0 & 0 & 0 & x+6t_1 & 7 & 0 & 0 \\
0 & 0 & 6t_1 & 7 & x & 0 & 0 \\
0 & 6t_1 & 7 & 0 & 0 & x & 0 \\
-3t_1/10 & 7 & 0 & 0 & 0 & 0 & x-9t_1^2/10
\end{vmatrix} \\
=& x^7 + \left(-\frac{9}{10}t_1^2 + 6t_1 - \frac{21}{10}\right)x^6 + \left(-\frac{27}{5}t_1^3 - \frac{351}{5}t_1^2 - \frac{63}{5}t_1 - 147\right)x^5 \\
&+ \left(\frac{324}{5}t_1^4 - \frac{2106}{5}t_1^3 + \frac{1197}{5}t_1^2 - 588t_1 + \frac{3087}{10}\right)x^4 \\
&+ \left(\frac{1944}{5}t_1^5 + \frac{5832}{5}t_1^4 + \frac{5859}{5}t_1^3 + \frac{16758}{5}t_1^2 + \frac{6174}{5}t_1 + 7203\right)x^3 \\
&+ \left(-\frac{5832}{5}t_1^6 + \frac{34992}{5}t_1^5 - \frac{21546}{5}t_1^4 + \frac{50274}{5}t_1^3 - \frac{95697}{10}t_1^2 + 14406t_1 - \frac{151263}{10}\right)x^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(-\frac{34992}{5}t_1^7 + \frac{11664}{5}t_1^6 - \frac{81648}{5}t_1^5 + \frac{15876}{5}t_1^4 - \frac{111132}{5}t_1^3 + \frac{21609}{5}t_1^2 - \frac{151263}{5}t_1 \right. \\
&\left. - 117649\right)x + \frac{69984}{5}t_1^7 + \frac{2470629}{10}.
\end{aligned}
$$

(2) *Put $s=8$ and $n=12$. Then, we have*

$$g_{\boldsymbol{u}}(x)=x^8+t_1x+1,\ f_{\boldsymbol{u}}(t;x)=x^{12}+t(x^8+t_1x+1)$$

and

$$\Phi_{\boldsymbol{u}}(x)=\begin{vmatrix} x-8/3 & 0 & 0 & 0 & 0 & 0 & 0 & -t_1/3 \\ 0 & x & 0 & 0 & 0 & 0 & 7t_1 & 8 \\ 0 & 0 & x & 0 & 0 & 7t_1 & 8 & 0 \\ 0 & 0 & 0 & x & 7t_1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 7t_1 & x+8 & 0 & 0 & 0 \\ 0 & 0 & 7t_1 & 8 & 0 & x & 0 & 0 \\ 0 & 7t_1 & 8 & 0 & 0 & 0 & x & 0 \\ -t_1/3 & 8 & 0 & 0 & 0 & 0 & 0 & x-11t_1^2/12 \end{vmatrix}$$

$$\begin{aligned}=&x^8+\left(-\frac{11}{12}t_1^2+\frac{16}{3}\right)x^7+\left(-152t_1^2-\frac{640}{3}\right)x^6+\left(\frac{539}{4}t_1^4-256t_1^2-1024\right)x^5\\ &+\left(\frac{22736}{3}t_1^4+\frac{45824}{3}t_1^2+16384\right)x^4+\left(-\frac{26411}{4}t_1^6-\frac{22736}{3}t_1^4+\frac{31744}{3}t_1^2+65536\right)x^3\\ &+\left(-\frac{355348}{3}t_1^6-213248t_1^4-\frac{1064960}{3}t_1^2-524288\right)x^2+\left(\frac{1294139}{12}t_1^8+\frac{1075648}{3}t_1^6\right.\\ &\left.+\frac{1404928}{3}t_1^4+\frac{1835008}{3}t_1^2-\frac{4194304}{3}\right)x-\frac{823543}{3}t_1^8+\frac{16777216}{3}.\end{aligned}$$

Proof of Lemma 5. To prove Lemma 5, it is enough to prove deg $h_{s-k}^{(\boldsymbol{u})}\geq 1$ for any k $(1\leq k\leq s)$. This is clear for $s=2$ by (a'') and we suppose $s\geq 3$ hereafter. To prove deg $h_{s-k}^{(\boldsymbol{u})}\geq 1$ $(1\leq k\leq s)$, let us compute the leading term of $h_{s-k}^{(\boldsymbol{u})}$ $(\in\mathbb{R}[t_1])$. Then, since $h_{s-k}^{(\boldsymbol{u})}$ is the coefficient of the term $h_{s-k}^{(\boldsymbol{u})}x^{s-k}$ of the characteristic polynomial $\Phi_{\boldsymbol{u}}(x)$, we need to maximize the degree in t_1 when we take '$s-k$' x and the remaining k elements from $\mathbb{R}[t_1]$.

(a) Suppose s is odd. Let us divide the case into three other sub-cases.

$(a1)$ Suppose k is odd and $1\leq k\leq s-2$.
In this case, the degree of the leading term of $h_{s-k}^{(\boldsymbol{u})}$ is $k+1$. In fact, it is obtained by taking

$(a11)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,
$(a12)$ '$k-1$' $(s-1)t_1$ from entries of the form $(i,s+1-i)$ $(2\leq i\leq s-1)$.

First, suppose we take the (s,s) entry $x-(n-1)t_1^2/n$ from the s-th row. Then we must take the $(1,1)$ entry from the first row. Next, let us proceed to the $(s-1)$-th row. If we take the $(s-1,s-1)$ entry x from the $(s-1)$-th row, then we must also take x from the second row, while if we take $(s-1)t_1$ from the $(s-1)$-th row, then we must also take $(s-1)t_1$ from the second row. The situation is the same for the $(s-2)$-th row, the $(s-3)$-th row ... and so on, which implies that $(s-1)t_1$ must occur in pair.

Hence, the leading term of $h_{s-k}^{(\boldsymbol{u})}$ is

$$-\frac{n-1}{n}t_1^2\cdot\binom{(s-3)/2}{(k-1)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-1)/2}\quad\left(\binom{n}{0}=1\ (n\geq 0)\right)$$

and the degree of this term is $k+1$ (≥ 2).

$(a2)$ Suppose k is odd and $k=s$.

If $k = s$, $h_{s-k}^{(\boldsymbol{u})} = h_0^{(\boldsymbol{u})}$ is the constant term of $\Phi_{\boldsymbol{u}}(x)$. In this case, the degree of the leading term of $h_0^{(\boldsymbol{u})}$ is s. In fact, it is obtained by taking

($a21$) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

($a22$) If $s \geq 5$ ($\Leftrightarrow (s,k) \neq (3,3)$), '$(s-3)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq (s-1)/2,\ (s+3)/2 \leq i \leq s-1)$,

($a23$) $(s-1)t_1$ from the $((s+1)/2, (s+1)/2)$ entry $x+(s-1)t_1$,

($a24$) $-s(n-s)/n$ from the $(1,1)$ entry $x - s(n-s)/n$

or by taking

($a25$) all anti-diagonal entries.

Therefore, the leading term of $h_0^{(\boldsymbol{u})}$ is

$$\begin{aligned} &-\frac{n-1}{n}t_1^2 \cdot \{(-1)\cdot(s-1)^2t_1^2\}^{(s-3)/2}\cdot(s-1)t_1\cdot\left(-\frac{s(n-s)}{n}\right) \\ &\qquad\qquad + (-1)\cdot\left(-\frac{n-s}{n}t_1\right)^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-3)/2}\cdot(s-1)t_1 \\ &= \frac{(n-s)(s-1)}{n}\cdot(-1)^{(s-3)/2}(s-1)^{s-2}t_1^s \\ &= (-1)^{(s-3)/2}\frac{(n-s)(s-1)^{s-1}}{n}t_1^s \end{aligned}$$

for any s $(s \geq 3)$ and the degree of this term is s.

($a3$) Suppose k is even.

In this case, we have $2 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{(\boldsymbol{u})}$ is $k+1$. In fact, it is obtained by taking

($a31$) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

($a32$) If $s \geq 5$ ($\Leftrightarrow (s,k) \neq (3,2)$), '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq (s-1)/2,\ (s+3)/2 \leq i \leq s-1)$,

($a33$) $(s-1)t_1$ from the $((s+1)/2, (s+1)/2)$ entry $x+(s-1)t_1$.

Therefore, the leading term of $h_{s-k}^{(\boldsymbol{u})}$ is

$$-\frac{n-1}{n}t_1^2\cdot\binom{(s-3)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\cdot(s-1)t_1$$

for any s $(s \geq 3)$ and the degree of this term is $k+1$ (≥ 3).

(b) Suppose s is even $(s \geq 4)$. We also divide this case into three other sub-cases.

($b1$) Suppose k is odd.

In this case, we have $1 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{(\boldsymbol{u})}$ is $k+1$. In fact, it is obtained by taking

($b11$) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

($b12$) '$(k-1)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq s-1)$.

Therefore, the leading term of $h_{s-k}^{(\boldsymbol{u})}$ is

$$-\frac{n-1}{n}t_1^2\cdot\binom{(s-2)/2}{(k-1)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-1)/2}$$

and the degree of this term is $k+1$ (≥ 2).

($b2$) Suppose k is even and $2 \le k \le s-2$.
In this case, the degree of the leading term of $h^{(\boldsymbol{u})}_{s-k}$ is k. In fact, it is obtained by taking

($b21$) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,
($b22$) '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$,
($b23$) $-s(n-s)/n$ from the $(1,1)$ entry $x-s(n-s)/n$

or by taking

($b24$) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,
($b25$) I f $s \ge 6$ $(\Leftrightarrow (s,k) \ne (4.2))$, '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le (s-2)/2,\ (s+4)/2 \le i \le s-1)$,
($b26$) s from the $((s+2)/2, (s+2)/2)$ entry $x+s$

or by taking

($b27$) '$k/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$

or by taking

($b28$) One pair of $-(n-s)t_1/n$ from the $(1,s)$ and the $(s,1)$ entry,
($b29$) '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$.

Here, note that if we take the $(s,1)$ entry $-(n-s)t_1/n$ from the s-th row, we must also take the $(1,s)$ entry $-(n-s)t_1/n$ from the first row.

Therefore, the leading term of $h^{(\boldsymbol{u})}_{s-k}$ is

$$
\begin{aligned}
&-\frac{n-1}{n}t_1^2 \cdot \binom{(s-2)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\cdot\left(-\frac{s(n-s)}{n}\right)\\
&\quad-\frac{n-1}{n}t_1^2 \cdot \binom{(s-4)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\cdot s+\binom{(s-2)/2}{k/2}\{(-1)\cdot(s-1)^2t_1^2\}^{k/2}\\
&\quad+\left((-1)\cdot\frac{\{-(n-s)\}^2}{n^2}t_1^2\right)\cdot\binom{(s-2)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\\
&=\left(\frac{s(n-s)(n-1)}{n^2}\binom{(s-2)/2}{(k-2)/2}-\frac{s(n-1)}{n}\binom{(s-4)/2}{(k-2)/2}\right.\\
&\qquad\left.-(s-1)^2\binom{(s-2)/2}{k/2}-\frac{(n-s)^2}{n^2}\binom{(s-2)/2}{(k-2)/2}\right)\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}t_1^2.
\end{aligned}
$$

for any s $(s \ge 4)$. Then, since

$$
\binom{(s-4)/2}{(k-2)/2}=\frac{s-k}{s-2}\binom{(s-2)/2}{(k-2)/2},\quad \binom{(s-2)/2}{k/2}=\frac{s-k}{k}\binom{(s-2)/2}{(k-2)/2},
$$

we have

$$
\begin{aligned}
&\frac{s(n-s)(n-1)}{n^2}\binom{(s-2)/2}{(k-2)/2}-\frac{s(n-1)}{n}\binom{(s-4)/2}{(k-2)/2}\\
&\qquad-(s-1)^2\binom{(s-2)/2}{k/2}-\frac{(n-s)^2}{n^2}\binom{(s-2)/2}{(k-2)/2}\\
&=\left(\frac{s(n-s)(n-1)}{n^2}-\frac{s(s-k)(n-1)}{n(s-2)}-\frac{(s-1)^2(s-k)}{k}-\frac{(n-s)^2}{n^2}\right)\binom{(s-2)/2}{(k-2)/2}\\
&=\frac{s\{\big(k(k+s^2-4s+2)-s^3+4s^2-5s+2\big)n-k(k+s^2-4s+2)\}}{nk(s-2)}\binom{(s-2)/2}{(k-2)/2}.
\end{aligned}
\tag{13}
$$

Hence, if the above value becomes zero, we have

$$\big(k(k+s^2-4s+2)-s^3+4s^2-5s+2\big)\,n-k(k+s^2-4s+2)=0,$$

which implies

$$k(k+s^2-4s+2)=0,\ \ -s^3+4s^2-5s+2=0 \tag{14}$$

or

$$n=\frac{k(k+s^2-4s+2)}{k(k+s^2-4s+2)-s^3+4s^2-5s+2}. \tag{15}$$

Here, (14) is impossible since $-s^3+4s^2-5s+2=-(s-1)^2(s-2)$ and $s\geq 4$. Also, (15) is impossible since, for any $s\geq 4$ and $2\leq k\leq s-2$, we have

$$k(k+s^2-4s+2)\geq 2(2+s^2-4s+2)\geq 2(s-2)^2>0$$

and

$$\begin{aligned}&k(k+s^2-4s+2)-s^3+4s^2-5s+2\\&\leq (s-2)\{(s-2)+s^2-4s+2\}-s^3+4s^2-5s+2\\&=-s^2+s+2\\&=-(s+1)(s-2)<0,\end{aligned}$$

which implies $n<0$, a contradiction. Thus, the above value (13) is non-zero and the degree of the leading term of $h_{s-k}^{(\boldsymbol{u})}$ is k.

$(b3)$ Suppose k is even and $k=s$.

If $k=s$, $h_{s-k}^{(\boldsymbol{u})}=h_0^{(\boldsymbol{u})}$ is the constant term of $\Phi_{\boldsymbol{u}}(x)$. In this case, the degree of the leading term of $h_0^{(\boldsymbol{u})}$ is s. In fact, it is obtained by taking

- $(b31)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,
- $(b32)$ '$(s-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i,s+1-i)$ $(2\leq i\leq s-1)$,
- $(b33)$ $-s(n-s)/n$ from the $(1,1)$ entry $x-s(n-s)/n$

or by taking

- $(b34)$ all anti-diagonal entries.

Therefore, the leading term of $h_0^{(\boldsymbol{u})}$ is

$$\begin{aligned}&-\frac{n-1}{n}t_1^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-2)/2}\cdot\left(-\frac{s(n-s)}{n}\right)\\&\qquad\qquad+(-1)\cdot\left(-\frac{n-s}{n}t_1\right)^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-2)/2}\\&=(-1)^{(s-2)/2}\frac{(n-s)(s-1)^{s-1}}{n}t_1^s\end{aligned}$$

and the degree of this term is s $(s\geq 4)$. □

Lemma 6. *Let $\boldsymbol{v}=(v_0,\cdots,v_s)\in\mathbb{R}^{s+1}$ be a real vector and n $(>s)$ be an integer. Put*

$$P_{\boldsymbol{v}}(t)=\det M_n(f_{\boldsymbol{v}}(t;x))=\det M_n(f^{(n)}(v_0,\cdots,v_s,t;x))$$

and $\alpha_{\boldsymbol{v}}=\max\{\alpha\in\mathbb{R}\mid P_{\boldsymbol{v}}(\alpha)=0\}$. If there exists a real number ρ_0 $(>\alpha_{\boldsymbol{v}})$ such that $N_{f_{\boldsymbol{v}}(\xi;x)}=\gamma_0$ for any $\xi>\rho_0$, we have $N_{f_{\boldsymbol{v}}(\xi;x)}=\gamma_0$ for any $\xi>\alpha_{\boldsymbol{v}}$.

Proof. Put $A_{\boldsymbol{v}}(t) = M_n(f_{\boldsymbol{v}}(t;x))$. Then, by Proposition 2, we have $\gamma_0 = \sigma(A_{\boldsymbol{v}}(\xi))$ for any $\xi > \rho_0$. Let us also put

$$R = \{\rho \in \mathbb{R} \mid \rho > \alpha_{\boldsymbol{v}},\ \sigma(A_{\boldsymbol{v}}(\xi)) = \gamma_0 \text{ for any } \xi > \rho\}.$$

Since R is a nonempty set ($\rho_0 \in R$) having a lower bound $\alpha_{\boldsymbol{v}}$, R has the infimum $\rho_{\boldsymbol{v}}$; $\rho_{\boldsymbol{v}} = \inf R$. Then, it is enough to prove $\rho_{\boldsymbol{v}} = \alpha_{\boldsymbol{v}}$. Here, suppose to the contrary that $\rho_{\boldsymbol{v}} > \alpha_{\boldsymbol{v}}$ and we denote by

$$\Omega_{\boldsymbol{v}}(t;x) = \sum_{k=0}^{n} \omega_k(t) x^k \in \mathbb{R}(t)[x]$$

the characteristic polynomial of $A_{\boldsymbol{v}}(t)$. Note that $\omega_k(t) \in \mathbb{R}[t]$ ($0 \le k \le n$) and for any $\xi > \alpha_{\boldsymbol{v}}$, $\Omega_{\boldsymbol{v}}(\xi;x)$ has n non-zero real roots (counted with multiplicity) since $A_{\boldsymbol{v}}(\xi)$ is symmetric and $\det A_{\boldsymbol{v}}(\xi) \neq 0$. Then, by Proposition 3, there exists a positive real number δ such that $\rho_{\boldsymbol{v}} - \delta > \alpha_{\boldsymbol{v}}$ and for any $\xi \in [\rho_{\boldsymbol{v}} - \delta, \rho_{\boldsymbol{v}} + \delta]$, $\Omega_{\boldsymbol{v}}(\xi;x)$ has the same number of positive and hence negative real roots with $\Omega_{\boldsymbol{v}}(\rho_{\boldsymbol{v}};x)$. On the other hand, since $\rho_{\boldsymbol{v}} = \inf R$, there exist real numbers ξ_+ ($\rho_{\boldsymbol{v}} < \xi_+ < \rho_{\boldsymbol{v}} + \delta$) and ξ_- ($\rho_{\boldsymbol{v}} - \delta < \xi_- < \rho_{\boldsymbol{v}}$) such that $\sigma(A_{\boldsymbol{v}}(\xi_+)) \neq \sigma(A_{\boldsymbol{v}}(\xi_-))$, which implies $\Omega_{\boldsymbol{v}}(\xi_+;x)$ and $\Omega_{\boldsymbol{v}}(\xi_-;x)$ have different number of positive and hence negative real roots. This is a contradiction and we have $\rho_{\boldsymbol{v}} = \alpha_{\boldsymbol{v}}$. □

3.4. **Proof of Theorem 2.** Let $\boldsymbol{r} = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be the vector as in Theorem 2 and put

$$n_0 = \begin{cases} (n-s+1)/2, & n-s-1 : \text{even} \\ (n-s+2)/2, & n-s-1 : \text{odd}. \end{cases}$$

When $n - s \ge 2$, we inductively define the matrix $A_{\boldsymbol{r}}(t)_k = (a_{ij}^{(\boldsymbol{r})}(t)_k)_{1 \le i,j \le n}$ ($2 \le k \le n-s$) as the matrix obtained from $A_{\boldsymbol{r}}(t)_{k-1}$ by sweeping out the entries of the k-th row (k-th column) by the $(k, l_0 - k)$ entry $-(n-s)r_s t$ ($(l_0 - k, k)$ entry $-(n-s)r_s t$). That is, we define $A_{\boldsymbol{r}}(t)_k = {}^t S_{\boldsymbol{r}}(t)_k A_{\boldsymbol{r}}(t)_{k-1} S_{\boldsymbol{r}}(t)_k$, where

$$S_{\boldsymbol{r}}(t)_k = \begin{cases} \displaystyle\prod_{m=l_0-k+1}^{n} R_n\left(l_0 - k, m; -\frac{a_{km}^{(\boldsymbol{r})}(t)_{k-1}}{-(n-s)r_s t}\right) & (2 \le k \le n_0) \\ \displaystyle R_n\left(l_0 - k, k; -\frac{a_{kk}^{(\boldsymbol{r})}(t)_{k-1}}{-2(n-s)r_s t}\right) \prod_{m=k+1}^{n} R_n\left(l_0 - k, m; -\frac{a_{km}^{(\boldsymbol{r})}(t)_{k-1}}{-(n-s)r_s t}\right) & \\ & (n_0 < k \le n-s). \end{cases}$$

Then, if $n - s \ge 1$, we can express the matrix $A_{\boldsymbol{r}}(t)_{n-s}$ as follows;

$$A_{\boldsymbol{r}}(t)_{n-s} = \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & 0 & \dots & -(n-s)r_s t & \\ \vdots & \vdots & \iddots & 0 & O \\ 0 & -(n-s)r_s t & 0 & 0 & \\ \hline & & O & & C_{\boldsymbol{r}}(t)_{n-s} \end{array}\right].$$

Note that $a_{km}^{(\boldsymbol{r})}(t)_{k-1}$ and $a_{kk}^{(\boldsymbol{r})}(t)_{k-1}$ appearing in $S_{\boldsymbol{r}}(t)_k$ are degree 1 monomials in t and hence the numbers $-a_{km}^{(\boldsymbol{r})}(t)_{k-1}/(-(n-s)r_s t)$, $-a_{kk}^{(\boldsymbol{r})}(t)_{k-1}/(-2(n-s)r_s t)$

appearing in $S_{\boldsymbol{r}}(t)_k$ are just real numbers. Therefore, the entries of the $s \times s$ symmetric matrix $C_{\boldsymbol{r}}(t)_{n-s} = (c_{ij}^{(\boldsymbol{r})}(t)_{n-s})_{1 \leq i,j \leq s}$ $(n-s \geq 1)$ are of the form

$$c_{ij}^{(\boldsymbol{r})}(t)_{n-s} = \bar{b}_{ij}^{(\boldsymbol{r})} t^2 + \bar{\lambda}_{ij}^{(\boldsymbol{r})} t \quad (\bar{\lambda}_{ij}^{(\boldsymbol{r})} \in \mathbb{R}). \tag{16}$$

Moreover, since the matrix

$$D_{\boldsymbol{r}}(t)_{n-s} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & -(n-s)r_s t \\ \vdots & \vdots & \iddots & 0 \\ 0 & -(n-s)r_s t & 0 & 0 \end{bmatrix}$$

is equivalent to the matrix

$$\bar{D}_{\boldsymbol{r}}(t)_{n-s} = \begin{cases} \begin{bmatrix} 1 & & & \\ & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} \end{bmatrix} & (n-s : \text{odd}) \\ \begin{bmatrix} 1 & & & & \\ & -(n-s)r_s t & & & \\ & & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} & & \\ & & & \ddots & \\ & & & & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} \end{bmatrix} & (n-s : \text{even}) \end{cases}$$

over $\mathbb{R}$, we have

$$\sigma(D_{\boldsymbol{r}}(\xi)_{n-s}) = \sigma(\bar{D}_{\boldsymbol{r}}(\xi)_{n-s}) = \begin{cases} 1 & n-s : \text{odd} \\ 0 & n-s : \text{even}, \ r_s > 0 \\ 2 & n-s : \text{even}, \ r_s < 0 \end{cases} \tag{17}$$

for any real number $\xi > \alpha_{\boldsymbol{r}}$ (≥ 0). Here, note that since $P_{\boldsymbol{r}}(0) = 0$, we have $\alpha_{\boldsymbol{r}} \geq 0$.

Next, let $\Phi_{\boldsymbol{r}}(t;x)$, $\Psi_{\boldsymbol{r}}(t;x)$ be characteristic polynomials of $\bar{B}_{\boldsymbol{r}}(t)$, $C_{\boldsymbol{r}}(t)_{n-s}$, respectively. Then, by equations (11) and (16), we have

$$\Phi_{\boldsymbol{r}}(t;x) = x^s + h_{s-1}^{(\boldsymbol{r})} t^2 x^{s-1} + \cdots + h_1^{(\boldsymbol{r})} t^{2s-2} x + h_0^{(\boldsymbol{r})} t^{2s}$$
$$\left(h_{s-k}^{(\boldsymbol{r})} = h_{s-k}(r_0, \cdots, r_s) \in \mathbb{R} \ (1 \leq k \leq s)\right),$$

$$\Psi_{\boldsymbol{r}}(t;x) = x^s + \left(h_{s-1}^{(\boldsymbol{r})} t^2 + \psi_{s-1}(t)\right) x^{s-1} + \cdots$$
$$+ \left(h_1^{(\boldsymbol{r})} t^{2s-2} + \psi_1(t)\right) x + \left(h_0^{(\boldsymbol{r})} t^{2s} + \psi_0(t)\right)$$
$$(\psi_0(t), \cdots, \psi_{s-1}(t) \in \mathbb{R}[t], \deg \psi_{s-k}(t) < 2k \ (1 \leq k \leq s)).$$

Here, let us divide the proof into next two cases.

(i) The case $h_0^{(\boldsymbol{r})} h_1^{(\boldsymbol{r})} \cdots h_{s-1}^{(\boldsymbol{r})} \neq 0$.

In this case, we have

$$\Psi_{\boldsymbol{r}}(t;x) = x^s + h^{(\boldsymbol{r})}_{s-1}t^2\left(1+\frac{\psi_{s-1}(t)}{h^{(\boldsymbol{r})}_{s-1}t^2}\right)x^{s-1}+\cdots + h^{(\boldsymbol{r})}_{1}t^{2s-2}\left(1+\frac{\psi_{1}(t)}{h^{(\boldsymbol{r})}_{1}t^{2s-2}}\right)x + h^{(\boldsymbol{r})}_{0}t^{2s}\left(1+\frac{\psi_{0}(t)}{h^{(\boldsymbol{r})}_{0}t^{2s}}\right)$$

and $1+\psi_{s-k}(t)/h^{(\boldsymbol{r})}_{s-k}t^{2k} \to 1$ $(t\to\infty)$ for any k $(1\le k\le s)$. Moreover, since $h^{(\boldsymbol{r})}_0 h^{(\boldsymbol{r})}_1 \cdots h^{(\boldsymbol{r})}_{s-1} \neq 0$, we have $h^{(\boldsymbol{r})}_0 \neq 0$, which implies that for any non-zero real number ξ, $\Phi_{\boldsymbol{r}}(\xi;x)$ have s non-zero real roots (counted with multiplicity). Thus, there exists a real number ρ_0 $(>\alpha_{\boldsymbol{r}})$ such that for any real number $\xi>\rho_0$, $\Psi_{\boldsymbol{r}}(\xi;x)$ have the same number of positive (hence also negative) real roots with $\Phi_{\boldsymbol{r}}(\xi;x)$ by Proposition 3, which implies $\sigma(C_{\boldsymbol{r}}(\xi)_{n-s}) = \sigma(\bar{B}_{\boldsymbol{r}}(\xi))$ and hence $\sigma(C_{\boldsymbol{r}}(\xi)_{n-s}) = N_{g_{\boldsymbol{r}}} = \gamma$ $(\xi>\rho_0)$ by Lemma 3. Then, by the equation (17), we have

$$\sigma(A_{\boldsymbol{r}}(\xi)_{n-s}) = \begin{cases} \gamma+1 & n-s:\text{odd} \\ \gamma & n-s:\text{even},\ r_s>0 \\ \gamma+2 & n-s:\text{even},\ r_s<0 \end{cases}$$

for any $\xi>\rho_0$, which implies

$$N_{f_{\boldsymbol{r}}(\xi;x)} = \sigma(A_{\boldsymbol{r}}(\xi)) = \begin{cases} \gamma+1 & n-s:\text{odd} \\ \gamma & n-s:\text{even},\ r_s>0 \\ \gamma+2 & n-s:\text{even},\ r_s<0 \end{cases}$$

for any $\xi>\rho_0$ since $A_{\boldsymbol{r}}(\xi)$ and $A_{\boldsymbol{r}}(\xi)_{n-s}$ are equivalent over $\mathbb{R}$. Hence, by Lemma 6, we have

$$N_{f_{\boldsymbol{r}}(\xi;x)} = \begin{cases} \gamma+1 & n-s:\text{odd} \\ \gamma & n-s:\text{even},\ r_s>0 \\ \gamma+2 & n-s:\text{even},\ r_s<0 \end{cases}$$

for any $\xi>\alpha_{\boldsymbol{r}}$.

(ii) General case.

Let ε_0 be a positive real number and for any vector $\boldsymbol{v}\in\mathbb{R}^{s+1}$, set

$$\alpha'_{\boldsymbol{v}} = \max\{|\alpha| \mid \alpha\in\mathbb{C}, P_{\boldsymbol{v}}(\alpha)=0\}.$$

Clearly, we have $\alpha'_{\boldsymbol{v}} \ge \alpha_{\boldsymbol{v}}$ for any $\boldsymbol{v}\in\mathbb{R}^{s+1}$. Here, let us put $\rho'_0 = \alpha'_{\boldsymbol{r}} + \varepsilon_0$. Then, by Lemma 6, it is enough to prove the next claim.

Claim 1. *For any real number $\xi>\rho'_0$, we have*

$$N_{f_{\boldsymbol{r}}(\xi;x)} = \begin{cases} \gamma+1 & n-s:\textit{odd} \\ \gamma & n-s:\textit{even},\ r_s>0 \\ \gamma+2 & n-s:\textit{even},\ r_s<0. \end{cases}$$

Proof. By the assumption that $g_{\boldsymbol{r}}(x)$ is a separable polynomial of degree s and the fact that the non-real roots must occur in pair with its complex conjugate, there exists a real number δ_0 such that for any vector $\boldsymbol{v}=(v_0,\cdots,v_s)\in\mathbb{R}^{s+1}$ satisfying

$|\boldsymbol{r}-\boldsymbol{v}|_0 = \max_{0\le k\le s}\{|r_k - v_k|\} < \delta_0$, $g_{\boldsymbol{v}}(x)$ is also a degree s separable polynomial satisfying $N_{g_{\boldsymbol{v}}} = N_{g_{\boldsymbol{r}}} = \gamma$ by Proposition 3.

(S1) If a vector $\boldsymbol{v}\in\mathbb{R}^{s+1}$ satisfies $|\boldsymbol{r}-\boldsymbol{v}|_0 < \delta_0$, then $g_{\boldsymbol{v}}(x)$ is also a degree s separable polynomial satisfying $N_{g_{\boldsymbol{v}}} = N_{g_{\boldsymbol{r}}} = \gamma$.

Next, we put

$$P(t) = \sum_{k\ge 0} x_k(t_0,\cdots,t_s)t^k = \det A(t)\ (A(t) = A(t_0,\cdots,t_s,t))$$

and let us consider $P(t)$ as a polynomial over $E_1 = \mathbb{R}(t_0,\cdots,t_s)$ in t. Then, since $x_k(t_0,\cdots,t_s)\in\mathbb{R}[t_0,\cdots,t_s]$ for any $k\ge 0$, there exists a real number $\delta_1 > 0$ such that for any vector $\boldsymbol{v}\in\mathbb{R}^{s+1}$ satisfying $|\boldsymbol{r}-\boldsymbol{v}|_0 < \delta_1$, we have $|\alpha'_{\boldsymbol{r}} - \alpha'_{\boldsymbol{v}}| < \varepsilon_0$ by Proposition 3;

(S2) If a vector $\boldsymbol{v}\in\mathbb{R}^{s+1}$ satisfies $|\boldsymbol{r}-\boldsymbol{v}|_0 < \delta_1$, we have $|\alpha'_{\boldsymbol{r}} - \alpha'_{\boldsymbol{v}}| < \varepsilon_0$.

Here, let ξ be any real number such that $\xi > \rho'_0 = \alpha'_{\boldsymbol{r}} + \varepsilon_0$ and let

$$\Omega(t_0,\cdots,t_s,\xi;x) = \sum_{k=0}^{n} y_k(t_0,\cdots,t_s)x^k \in E_1[x]$$

be the characteristic polynomial of the Bezoutian

$$A(t_0,\cdots,t_s,\xi;x) = M_n(f^{(n)}(t_0,\cdots,t_s,\xi;x), f^{(n)}(t_0,\cdots,t_s,\xi;x)').$$

Here, $f^{(n)}(t_0,\cdots,t_s,\xi;x)'$ is the derivative of

$$f^{(n)}(t_0,\cdots,t_s,\xi;x) = \sum_{k=0}^{n} z_k(t_0,\cdots,t_s)x^k \in E_1[x]$$

with respect to x. Then, since $z_k(t_0,\cdots,t_s)\in\mathbb{R}[t_0,\cdots,t_s]$ $(0\le k\le n)$, we also have $y_k(t_0,\cdots,t_s)\in\mathbb{R}[t_0,\cdots,t_s]$ $(0\le k\le n)$. Moreover, since $\xi > \rho'_0 > \alpha_{\boldsymbol{r}}$, we have $\det A_{\boldsymbol{r}}(\xi) = \det A(r_0,\cdots,r_s,\xi) \ne 0$.

By these arguments, we can also deduce that there exists a positive real number δ_2 such that for any vector $\boldsymbol{v}\in\mathbb{R}^{s+1}$ satisfying $|\boldsymbol{r}-\boldsymbol{v}|_0 < \delta_2$, the characteristic polynomial $\Omega_{\boldsymbol{v}}(\xi;x)$ have the same number of positive and hence negative real roots with $\Omega_{\boldsymbol{r}}(\xi;x)$ (counted with multiplicity), which implies $N_{f_{\boldsymbol{r}}(\xi;x)} = \sigma(A_{\boldsymbol{r}}(\xi)) = \sigma(A_{\boldsymbol{v}}(\xi)) = N_{f_{\boldsymbol{v}}(\xi;x)}$.

(S3) If a vector $\boldsymbol{v}\in\mathbb{R}^{s+1}$ satisfies $|\boldsymbol{r}-\boldsymbol{v}|_0 < \delta_2$, we have $N_{f_{\boldsymbol{r}}(\xi;x)} = N_{f_{\boldsymbol{v}}(\xi;x)}$.

Put $\delta = \min\{\delta_0,\delta_1,\delta_2\} > 0$. Then, there exists a vector $\boldsymbol{w} = (w_0,\cdots,w_s)\in\mathbb{R}^{s+1}$ such that

$$(a)\ |\boldsymbol{r}-\boldsymbol{w}|_0 < \delta,\ (b)\ h_0^{(\boldsymbol{w})}h_1^{(\boldsymbol{w})}\cdots h_{s-1}^{(\boldsymbol{w})} \ne 0.$$

Here, we put $h_{s-k}^{(\boldsymbol{w})} = h_{s-k}(w_0,\cdots,w_s)$ for any k $(1\le k\le s)$. In fact, since $h_{s-k}(t_0,\cdots,t_s)$ is a non-zero polynomial for any k $(1\le k\le s)$ by Lemma 4, the product $\prod_{k=1}^{s} h_{s-k}(t_0,\cdots,t_s)$ is also non-zero, which implies that there exists a vector $\boldsymbol{w}\in\mathbb{R}^{s+1}$ satisfying (a) and (b).

Let $\boldsymbol{w} \in \mathbb{R}^{s+1}$ be the vector as above. Then, since $|\boldsymbol{r} - \boldsymbol{w}|_0 < \delta \leq \delta_0$, $g_{\boldsymbol{w}}(x)$ is a degree s separable polynomial satisfying $N_{g_{\boldsymbol{w}}} = \gamma$ by (S1) and also, by (S2), we have $\alpha_{\boldsymbol{w}} \leq \alpha'_{\boldsymbol{w}} < \alpha'_{\boldsymbol{r}} + \varepsilon_0 = \rho'_0 < \xi$. Thus, by (b) and the case (i), we have

$$N_{f_{\boldsymbol{w}}(\xi;x)} = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even}, \, r_s > 0 \\ \gamma + 2 & n - s : \text{even}, \, r_s < 0, \end{cases}$$

which, by (S3), implies

$$N_{f_{\boldsymbol{r}}(\xi;x)} = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even}, \, r_s > 0 \\ \gamma + 2 & n - s : \text{even}, \, r_s < 0. \end{cases}$$

Since ξ is any real number such that $\xi > \rho'_0$, this completes the proof of Claim and hence the proof of Theorem 2. ☐

Proposition 5. *Let* $g(x) = \sum_{i=0}^{s} a_i x^i$ *be a polynomial in* $\mathbb{R}[x]$ *such that* $\Delta_g \neq 0$ *and*

$$f(t, x) = x^n + t \cdot g(x) \tag{18}$$

If $g(x)$ *is totally complex,* $(n-s)$ *is even, and* $a_s > 0$ *then* $f(\beta, x)$ *is totally complex for all* $\beta > \max\{\alpha \,|\, \Delta_{(f,x)}(\alpha) = 0\}$.

Proof. We have to show that $f(\beta, x)$ has no real roots. Since $g(x)$ is totally complex we have that $\gamma = 0$. $N_{f(\beta,x)} = \gamma$ as $\beta > \max\{\alpha \,|\, \Delta_{(f,x)}(\alpha) = 0\}$ and $a_s > 0$, so $N_{f(\beta,x)} = \gamma = 0$. Hence, $f(\beta, x)$ is totally complex. ☐

Let $K := \mathbb{Q}(t, a_0, \dots, a_s)$ be the field of transcendental degree $s + 1$ and $g(x) = \sum_{i=0}^{s} a_i x^i$. Then we have the following.

Corollary 2. *Let* $K := \mathbb{Q}(t, a_0, \dots, a_s)$ *be the field of transcendental degree* $s + 1$, $g(x) = \sum_{i=0}^{s} a_i x^i$ *and*

$$f(t, x) = x^n + t \cdot g(x)$$

For any value of $(\lambda_0, \dots, \lambda_s) \in \mathbb{Z}^{s+1}$, *if* $g(\lambda_0, \dots, \lambda_s, x) \in \mathbb{Z}[x]$ *is irreducible and satisfies the conditions of the Eisenstein criteria, then* $f(x)$ *is irreducible, over* $\mathbb{Q}$.

We also note:

Remark 4. *It can be verified computationally by Maple that if* $n \leq 9$ *and* $1 \leq s < n$ *then the Galois group Gal* $_K(f, x)$ *is isomorphic to* S_n.

Remark 5. *Polynomials in Eq.* (18) *for* $s = 1$ *and* $t = 1$ *has been treated by Y. Zarhin in* [18] *while studying Mori trinomials. It is shown there that the Galois group of* $f(x)$ *over* $\mathbb{Q}$ *is isomorphic to* S_n; *see* [18, Cor. 3.5] *for details.*

In general, if we let $K := \mathbb{Q}(t, a_0, \dots, a_s)$ be the field of transcendental degree $s + 1$, for $1 \leq s < n$, then we expect that Gal $_K(f) \cong S_n$ for all $n \geq 1$. If true, this would generalize Zarhin's result to a more general class of polynomials.

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