BEZOUTIANS AND THE DISCRIMINANT OF A CERTAIN QUADRINOMIALS

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ABSTRACT. We give an explicit formula for the discriminant $\Delta_f(x)$ of the quadrinomials of the form

$$f(x) = x^n + t(x^2 + ax + b)$$

The proof uses Bezoutians of polynomials. This paper is an extended version of [31].

1. INTRODUCTION

In [34] Selmer studied polynomials $f(x) = x^n \pm x \pm 1$ and proved that $x^n - x - 1$ is irreducible for all ≥ 2, while $x^n + x + 1$ is irreducible for $n \not\equiv 2 \bmod 3$. In the process he proved that the discriminant of $f(x)$ is (up to a multiplication by -1) $\Delta = \pm\left(n^n \pm (n-1)^{n-1}\right)$, see [34] for precise formulas. He also noticed that polynomials $f(x)$ have very small discriminants. He checked his results for $n \leq 20$. Other mathematicians have considered trinomials $f(x) = x^n + ax + b$ for various reasons; see [28], [37]. In general, finding conditions on the coefficients of a polynomial to have minimal discriminant is a difficult problem. It is related to reduction of binary forms discussed in [5] and their heights [35]. More generally it is a special case of finding conditions on the coefficients of a binary form such that the corresponding point in the weighted moduli space of invariants is normalized; see [26].

This paper is the first of hopefully others to come to determine for what a, b, t the quadrinomial

$$f(x) = x^n + t(x^2 + ax + b) \tag{1}$$

has minimal discriminant, is reduced in the sense of [5], or has minimal naive height. There have been plenty of efforts to determine such formulas for certain classes of polynomials. In [17], [36] the authors focus on the computation of the discriminant of a trinomial which has been carried out in different ways. In this paper, we determine explicitly a formula for the discriminant of $f(x)$ by using the approach of Bezoutians.

We prove (see Thm. 2) that the discriminant $\Delta_f(x)$ of the polynomials in Eq. (1) is given by the formula

$$\Delta = (-1)^{m_1} t^{n-1} \left((n-2)^{n-2}(a^2 - 4b)t^2 + \gamma_{\boldsymbol{c}}\, t - n^n b^{n-1}\right), \tag{2}$$

such that

$$\gamma_{\boldsymbol{c}} = \sum_{k=0}^{m_0} (-1)^{n+k} n^k (n-1)^{n-2k-4} (n-2)^k a^{n-2k-4} b^k \cdot S_k,$$

and $m_0 = \lfloor (n-3)/2 \rfloor$, $m_1 = \lceil (n-3)/2 \rceil$ for $a \neq 0$ or $a = 0$ and n even, and $\gamma_{\boldsymbol{c}} = 0$ for $a = 0$ and n odd, and

$$S_k = (n-1)^3 \binom{n-k-3}{k} a^4 - \frac{n(n-1)\left\{5n^2 - (6k+23)n + 10k + 24\right\}}{n-k-3} \binom{n-k-3}{k} a^2 b + 4n^2(n-2) \binom{n-k-4}{k} b^2.$$

It is object of further investigation if such result could be generalized to polynomials $f(x) = x^n + t \cdot g(x)$, where $g(x)$ is a general cubic. As a quick application of Thm. 2 we get that for $b \neq 0$ and $\frac{(n-1)^2 a^2}{4n(n-2)} \leq b$ the polynomial $f(t, x)$ has no real roots, for any real number $t > 0$; Cor. 1. While there is an elementary proof of Cor. 1, it is interesting to check whether out approach would work for any polynomial of type $f(t, x) = x^n + t \cdot g(x)$, for $\deg g \geq 3$.

It is a quite open problem to determine for what integer values of a, b, and t the quadrinomial $f(x)$ is irreducible or the discriminant Δ_f has a minimal value. Moreover, it is our intention to study in the future for what conditions on a, b, t the quadrinomial is reduced in the sense of [5].

Another motivation of looking at the discriminant of such family of polynomials comes from our efforts to construct superelliptic Jacobians with large endomorphism rings. We have checked computationally that for small n and $f(t,x) \in \mathbb{Q}[t,x]$, for almost all t the Galois group Gal $_{\mathbb{Q}}(f,x)$ is isomorphic to S_n. Due to results of Zarhin this implies that the superelliptic curve $y^m = f(t,x)$ has large endomorphism ring; see [14] for related matters. Hence, curves $y^n = f(x)$, where $f(x)$ is as above, are smooth curves whose Jacobians are expected to have large endomorphism rings. This remains the focus of further investigation.

Notation: Throughout this paper n is a positive integer, Δ denotes the discriminant of a polynomial $f(x)$ with respect to the variable x. The symbol $\lfloor \cdot \rfloor$ is the floor function and $\lceil \cdot \rceil$ is the ceiling function. $B_n(f,g)$ denotes the level n Bezoutian of two polynomials f and g. For a polynomial $f \in \mathbb{R}[x]$ the symbol N_f denotes the number of real roots of $f(x)$. We denote by $\binom{n}{m}$ $(n,m \in \mathbb{Z}_{\geq 0})$ the binomial coefficient.

2. Preliminaries

Let F be a field of characteristic zero and $f_1(x)$, $f_2(x)$ be polynomials over F. Then, for any integer n such that $n \geq \max\{\deg f_1, \deg f_2\}$, we put

$$\begin{aligned} B_n(f_1,f_2) &: = \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x-y} = \sum_{i,j=1}^{n} \alpha_{ij} x^{n-i} y^{n-j} \in F[x,y], \\ M_n(f_1,f_2) &: = (\alpha_{ij})_{1 \leq i,j \leq n}. \end{aligned}$$

The $n \times n$ matrix $M_n(f_1,f_2)$ is called the Bezoutian of f_1 and f_2. When $f_2 = f_1'$, the formal derivative of f_1 with respect to the indeterminate x, we often write $B_n(f_1) := B_n(f_1, f_1')$, $M_n(f_1) := M_n(f_1, f_1')$ and the matrix $M_n(f_1)$ is called the Bezoutian of f_1. We denote by N_{f_1} the number of distinct real roots of $f_1(x)$ and $\Delta(f_1)$ the discriminant of $f_1(x)$. Moreover, for any real symmetric matrix M, we denote by $\sigma(M)$ the index of inertia of M. The following are the list of important properties of Bezoutians.

Lemma 1. *Notations as above, we have*

(i) $M_n(f_1,f_2)$ *is an* $n \times n$ *symmetric matrix over* F. $(M_n(f_1,f_2) \in Sym_n(F))$.
(ii) $B_n(f_1,f_2)$ $(M_n(f_1,f_2))$ *is linear in* f_1 *and* f_2*, separately.*
(iii) $B_n(f_1,f_2) = -B_n(f_2,f_1)$ $(M_n(f_1,f_2) = -M_n(f_2,f_1))$.
(iv) $N_{f_1} = \sigma(M_n(f_1))$.
(v) $\Delta(f_1) = \dfrac{1}{led(f_1)^2} \det M_n(f_1)$, *where* $led(f_1)$ *is the leading coefficient of* f_1. *In particular, if* f_1 *is monic, we have* $\Delta(f_1) = \det M_n(f_1)$.
(vi) Let λ, μ, ν *be integers such that* $\lambda \geq \mu > \nu \geq 0$. *Then* $M_\lambda(x^\mu, x^\nu) = (m_{ij})_{1 \leq i,j \leq \lambda}$, *where*

$$m_{ij} = \begin{cases} 1 & i+j = 2\lambda - (\mu+\nu) + 1 \quad (\lambda - \mu + 1 \leq i,j \leq \lambda - \nu), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. For (i)–(iv), see [16, Theorem 8.25, 9.2]. For (v), see equation (4) of [15, p.217]. For (vi), see [32, Lemma 2] ∎

For any vector $\boldsymbol{r} = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$, let us put

$$\begin{aligned} g_{\boldsymbol{r}}(x) = g(r_0, \cdots, r_s; x) &= \sum_{k=0}^{s} r_{s-k} x^{s-k} \in \mathbb{R}[x], \\ f_{\boldsymbol{r}}(t;x) = f^{(n)}(r_0, \cdots, r_s, t; x) &= x^n + t \cdot g_{\boldsymbol{r}}(x) \in \mathbb{R}(t)[x]. \end{aligned}$$

In the previous paper [32], by using the above properties of Bezoutians, we obtained the next theorem

Theorem 1. [32, Thm. 2] *Let* $\boldsymbol{r} = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ *be a vector such that* $g_{\boldsymbol{r}}(x)$ *is a degree* s *separable polynomial satisfying* $N_{g_{\boldsymbol{r}}(x)} = \gamma$ $(0 \leq \gamma \leq s)$. *Let us consider* $f_{\boldsymbol{r}}(t;x) = f^{(n)}(r_0, \cdots, r_s, t; x)$ *as a polynomial over* $\mathbb{R}(t)$ *in* x *and put*

$$P_{\boldsymbol{r}}(t) = \det M_n(f_{\boldsymbol{r}}(t;x)) = \det M_n(f_{\boldsymbol{r}}(t;x), f_{\boldsymbol{r}}'(t;x)),$$

where $f'_{\boldsymbol{r}}(t;x)$ *is a derivative of* $f_{\boldsymbol{r}}(t;x)$ *with respect to* x*. Then, for any real number* $t > \alpha_{\boldsymbol{r}} = \max\{\alpha \in \mathbb{R} \mid P_{\boldsymbol{r}}(\alpha) = 0\}$*, we have*

$$N_{f_{\boldsymbol{r}}(t;x)} = \begin{cases} \gamma + 1 & n - s \,:\, odd \\ \gamma & n - s \,:\, even,\ r_s > 0 \\ \gamma + 2 & n - s \,:\, even,\ r_s < 0. \end{cases}$$

Then, by using Thm. 1 and Thm. 2, we construct a certain family of totally complex polynomials of the form $f_{(b,a,1)}(t;x)$; see Cor. 1. Here, note that a real polynomial $f(x)$ to be called totally complex if it has no real roots, that is, $N_f = 0$.

Let $n \geq 3$ be an integer and a, b be real numbers. Let us put $\boldsymbol{c} = (b, a, 1) \in \mathbb{R}^3$ and

$$g_{\boldsymbol{c}}(x) = x^2 + ax + b \in \mathbb{R}[x],\ \ f_{\boldsymbol{c}}(t;x) = x^n + t g_{\boldsymbol{c}}(x) \in \mathbb{R}(t)[x].$$

We denote by $\binom{n}{m}$ $(n, m \in \mathbb{Z}_{\geq 0})$ the binomial coefficient. Note that $\binom{n}{0} = 1$ for any integer $n \geq 0$ and $\binom{n}{m} = 0$ if $n < m$. Moreover, we define $\binom{n}{m} = 0$ for any $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{<0}$.

Our main result is the following.

Theorem 2. *Put* $m_0 = \lfloor (n-3)/2 \rfloor$*,* $m_1 = \lceil (n-3)/2 \rceil$*. Moreover, for any integers* $n \geq 4$ *and* k $(0 \leq k \leq m_0)$*, put*

$$S_k = (n-1)^3 \binom{n-k-3}{k} a^4 - \frac{n(n-1)\left\{5n^2 - (6k+23)n + 10k + 24\right\}}{n-k-3} \binom{n-k-3}{k} a^2 b + 4n^2(n-2) \binom{n-k-4}{k} b^2.$$

Here, $\lfloor \cdot \rfloor$ *is the floor function and* $\lceil \cdot \rceil$ *is the ceiling function.*

(1) *Suppose* $n = 3$*. Then, we have*

$$\Delta\left(f_{\boldsymbol{c}}(t;x)\right) = t^2 \left\{ (a^2 - 4b)t^2 - (4a^3 - 18ab)t - 27b^2 \right\}.$$

(2) *Suppose* $n \geq 4$*. Then, we have*

$$\Delta\left(f_{\boldsymbol{c}}(t;x)\right) = (-1)^{m_1} t^{n-1} \left\{ (n-2)^{n-2}(a^2 - 4b)t^2 + \gamma_{\boldsymbol{c}} t - n^n b^{n-1} \right\},$$

where

$$\gamma_{\boldsymbol{c}} = \begin{cases} \displaystyle\sum_{k=0}^{m_0} (-1)^{n+k} n^k (n-1)^{n-2k-4} (n-2)^k a^{n-2k-4} b^k S_k & (a \neq 0 \text{ or } a = 0,\ n : even), \\ 0 & (a = 0,\ n : odd). \end{cases}$$

As a corollary of the above we get.

Corollary 1. *Suppose* $n \geq 4$ *is an even integer. Moreover, let us suppose* $b \neq 0$ *and* $(n-1)^2 a^2 / 4n(n-2) \leq b$*. Then we have* $N_{f_{\boldsymbol{c}}(t;x)} = 0$ *for any positive real number* t*.*

By a direct computation, we have

$$\begin{aligned} \Delta\left(x^3 + t(x^2 + ax + b)\right) &= t^2 \left\{ (a^2 - 4b)t^2 - (4a^3 - 18ab)t - 27b^2 \right\}, \\ \Delta\left(x^4 + t(x^2 + ax + b)\right) &= -t^3 \left\{ (4a^2 - 16b)t^2 + (27a^4 - 144a^2 b + 128b^2)t - 256b^3 \right\} \end{aligned}$$

and we get Thm. 2 for $n = 3$, 4. Moreover, if $n = 4$ and $(n-1)^2 a^2 / 4n(n-2) = b$, we have

$$P_{\boldsymbol{c}}(t) = \Delta\left(x^4 + t(x^2 + ax + b)\right) = \frac{1}{2} a^2 \left(t + \frac{27}{8} a^2 \right)^2 t^3$$

and hence $\alpha_{\boldsymbol{c}} = 0$, which implies $N_{f_{\boldsymbol{c}}(t;x)} = 0$ for any $t > 0$ by Thm. 1 since $x^2 + ax + b$ has no real root in this case. Then, by considering the graph of the function $y = x^4 + t(x^2 + ax + b)$, we have $N_{f_{\boldsymbol{c}}(t;x)} = 0$ whenever $(n-1)^2 a^2 / 4n(n-2) \leq b$ and $t > 0$, which is the claim of Cor. 1 for $n = 4$. Therefore, let us assume $n \geq 5$ hereafter.

In the following, we put

$$A_{\boldsymbol{c}}(t) = (a_{ij}^{(\boldsymbol{c})}(t))_{1\le i,j\le n} = M_n(f_{\boldsymbol{c}}(t;x)) \in \mathrm{Sym}_n(\mathbb{R}(t)),$$
$$B_{\boldsymbol{c}} = (b_{ij}^{(\boldsymbol{c})})_{1\le i,j\le 2} = M_2(g_{\boldsymbol{c}}(x)) \in \mathrm{Sym}_2(\mathbb{R}),$$
$$P_{\boldsymbol{c}}(t) = \det A_{\boldsymbol{c}}(t) = \Delta(f_{\boldsymbol{c}}(t;x)).$$

Then, by Thm. 1, we have the next lemma.

Corollary 2. *Suppose n is even and $a^2/4 < b$. Put $\alpha_{\boldsymbol{c}} = \max\{\alpha \in \mathbb{R} \mid P_{\boldsymbol{c}}(\alpha) = 0\}$. Then, for any real number $t > \alpha_{\boldsymbol{c}}$, we have $N_{f_{\boldsymbol{c}}(t,x)} = 0$.*

To prove Thm. 2, we will use the equality $\Delta(f_{\boldsymbol{c}}(t;x)) = \det A_{\boldsymbol{c}}(t)$. Also, to prove Cor. 1, it is enough to prove $\alpha_{\boldsymbol{c}} = 0$ when $(n-1)^2a^2/4n(n-2) \le b$ and $(a,b) \ne (0,0)$, which implies we need to compute the polynomial $P_{\boldsymbol{c}}(t) = \det A_{\boldsymbol{c}}(t)$ precisely. Here, let $Q_m(k;c) = (q_{ij})_{1\le i,j\le m}$, $R_m(k,l;c) = (r_{ij})_{1\le i,j\le m}$ be $m\times m$ elementary matrices such that

$$Q_m(k;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},\ R_m(k,l;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & c & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},$$

where $q_{kk} = c$, $r_{kl} = c$, respectively and, for any $m\times m$ matrices $M_1, M_2, \cdots, M_l$, put $\prod_{k=1}^{l} M_k = M_1M_2\cdots M_l$. Moreover, put $l_k = n+k$. Then, as is the case in [32], we inductively define the matrix $A_{\boldsymbol{c}}(t)_k$ $(1 \le k \le n-2)$ as follows:

(1) $A_{\boldsymbol{c}}(t)_1 = (a_{ij}^{(\boldsymbol{c})}(t)_1)_{1\le i,j\le n} = {}^tS_{\boldsymbol{c}}(t)_1 A_{\boldsymbol{c}}(t) S_{\boldsymbol{c}}(t)_1$, where

$$S_{\boldsymbol{c}}(t)_1 = Q_n(1; 1/\sqrt{n}) \prod_{k=0}^{1} R_n(1, l_k - 1; -a_{1,l_k-1}^{(\boldsymbol{c})}(t)/\sqrt{n}).$$

(2) Put

$$n_0 = \begin{cases} (n-1)/2, & n:\text{odd}, \\ n/2, & n:\text{even}. \end{cases}$$

Then, $A_{\boldsymbol{c}}(t)_k = (a_{ij}^{(\boldsymbol{c})}(t)_k)_{1\le i,j\le n} = {}^tS_{\boldsymbol{c}}(t)_k A_{\boldsymbol{c}}(t)_{k-1} S_{\boldsymbol{c}}(t)_k$, where

$$S_{\boldsymbol{c}}(t)_k = \begin{cases} \displaystyle\prod_{m=l_0-k+1}^{n} R_n\left(l_0-k, m; -\frac{a_{km}^{(\boldsymbol{c})}(t)_{k-1}}{-(n-2)t}\right), & (2 \le k \le n_0) \\ \displaystyle R_n\left(l_0-k, k; -\frac{a_{kk}^{(\boldsymbol{c})}(t)_{k-1}}{-2(n-2)t}\right) \prod_{m=k+1}^{n} R_n\left(l_0-k, m; -\frac{a_{km}^{(\boldsymbol{c})}(t)_{k-1}}{-(n-2)t}\right), & (n_0 < k \le n-2). \end{cases}$$

By the definition of $A_{\boldsymbol{c}}(t)_k$ $(1 \le k \le n-2)$, we have $\det A_{\boldsymbol{c}}(t) = n \det A_{\boldsymbol{c}}(t)_{n-2}$. Therefore, to compute the polynomial $P_{\boldsymbol{c}}(t) = \det A_{\boldsymbol{c}}(t)$, let us first compute the matrix $A_{\boldsymbol{c}}(t)_{n-2}$ concretely.

3. Computation of the matrix $A_{\boldsymbol{c}}(t)_{n-2}$

By Lem. 1, we have

$$B_{\boldsymbol{c}} = M_2(x^2+ax+b, 2x+a) = 2M_2(x^2,x) + aM_2(x^2,1) + (a^2-2b)M_2(x,1) = \begin{bmatrix} 2 & a \\ a & a^2-2b \end{bmatrix}.$$

Here, let us give some examples of the matrices $A_{\boldsymbol{c}}(t)$ and $A_{\boldsymbol{c}}(t)_1$ for some small n.

Example 1. (1) *Let $n=5$. Then,*

$$A_{\boldsymbol{c}}(t)=\begin{bmatrix} 5 & 0 & 0 & 2t & at \\ 0 & 0 & -3t & -4at & -5bt \\ 0 & -3t & -4at & -5bt & 0 \\ 2t & -4at & -5bt & 2t^2 & at^2 \\ at & -5bt & 0 & at^2 & (a^2-2b)t^2 \end{bmatrix}, \quad A_{\boldsymbol{c}}(t)_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3t & -4at & -5bt \\ 0 & -3t & -4at & -5bt & 0 \\ 0 & -4at & -5bt & (6/5)t^2 & (3/5)at^2 \\ 0 & -5bt & 0 & (3/5)at^2 & \left\{(4/5)a^2-2b\right\}t^2 \end{bmatrix}.$$

(2) *Let $n=6$. Then,*

$$A_{\boldsymbol{c}}(t)=\begin{bmatrix} 6 & 0 & 0 & 0 & 2t & at \\ 0 & 0 & 0 & -4t & -5at & -6bt \\ 0 & 0 & -4t & -5at & -6bt & 0 \\ 0 & -4t & -5at & -6bt & 0 & 0 \\ 2t & -5at & -6bt & 0 & 2t^2 & at^2 \\ at & -6bt & 0 & 0 & at^2 & (a^2-2b)t^2 \end{bmatrix}, \ A_{\boldsymbol{c}}(t)_1=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4t & -5at & -6bt \\ 0 & 0 & -4t & -5at & -6bt & 0 \\ 0 & -4t & -5at & -6bt & 0 & 0 \\ 0 & -5at & -6bt & 0 & \frac{4}{3}t^2 & \frac{2}{3}at^2 \\ 0 & -6bt & 0 & 0 & \frac{2}{3}at^2 & \left(\frac{5}{6}a^2-2b\right)t^2 \end{bmatrix}.$$

□

Let's continue our discussion for $n \geq 5$. Then, by [32, Prop 4] and [32, Eq. (5)], we have

$$A_{\boldsymbol{c}}(t)=\begin{bmatrix} n & 0 & \cdots & \cdots & \cdots & 0 & 2t & at \\ 0 & & & & & -(n-2)t & -(n-1)at & -nbt \\ \vdots & & & & \iddots & -(n-1)at & -nbt & 0 \\ \vdots & & & \iddots & \iddots & -nbt & 0 & \vdots \\ \vdots & & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots \\ 0 & -(n-2)t & -(n-1)at & -nbt & \iddots & & 0 & 0 \\ 2t & -(n-1)at & -nbt & 0 & \cdots & 0 & 2t^2 & at^2 \\ at & -nbt & 0 & \cdots & \cdots & 0 & at^2 & (a^2-2b)t^2 \end{bmatrix} \tag{3}$$

and hence

$$A_{\boldsymbol{c}}(t)_1=\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & & & & & -(n-2)t & -(n-1)at & -nbt \\ \vdots & & & & \iddots & -(n-1)at & -nbt & 0 \\ \vdots & & & \iddots & \iddots & -nbt & 0 & \vdots \\ \vdots & & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots \\ 0 & -(n-2)t & -(n-1)at & -nbt & \iddots & & 0 & 0 \\ 0 & -(n-1)at & -nbt & 0 & \cdots & 0 & 2\left(1-\frac{2}{n}\right)t^2 & \left(1-\frac{2}{n}\right)at^2 \\ 0 & -nbt & 0 & \cdots & \cdots & 0 & \left(1-\frac{2}{n}\right)at^2 & \left\{\left(1-\frac{1}{n}\right)a^2-2b\right\}t^2 \end{bmatrix}. \tag{4}$$

Here, similar to $A_{\boldsymbol{c}}(t)_k$ $(2 \leq k \leq n-2)$, we inductively define the matrix $W(t)_k = (w_{ij}(t)_k)_{1\leq i,j\leq n}$ $(2 \leq k \leq n-2)$ as follows;

(1) $W(t)_1 = (w_{ij}(t)_1)_{1\leq i,j\leq n}$, where

$$w_{ij}(t)_1 = \begin{cases} qt, & i+j=n,\ 2\leq i,j\leq n-2, \\ rt, & i+j=n+1,\ 2\leq i,j\leq n-1, \\ st, & i+j=n+2,\ 2\leq i,j\leq n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) $W(t)_k = (w_{ij}(t)_k)_{1\leq i,j\leq n}$ $(2\leq k\leq n-2) = {}^tS(t)_k W(t)_{k-1} S(t)_k$, where

$$S(t)_k = \begin{cases} \displaystyle\prod_{m=n-k+1}^{n} R_n\left(n-k, m; -\frac{w_{km}(t)_{k-1}}{qt}\right) & (2\leq k\leq n_0), \\ \displaystyle R_n\left(n-k, k; -\frac{w_{kk}(t)_{k-1}}{-2qt}\right)\prod_{m=k+1}^{n} R_n\left(n-k, m; -\frac{w_{km}(t)_{k-1}}{qt}\right), & (n_0<k\leq n-2). \end{cases}$$

Example 2. (1) *Let $n = 5$. Then,*

$$W(t)_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & qt & rt & st \\ 0 & qt & rt & st & 0 \\ 0 & rt & st & 0 & 0 \\ 0 & st & 0 & 0 & 0 \end{bmatrix},\ W(t)_{n_0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & qt & 0 & 0 \\ 0 & qt & rt & (qs-r^2)t/q & -rst/q \\ 0 & 0 & (qs-r^2)t/q & -(2qs-r^2)rt/q^2 & -(qs-r^2)st/q^2 \\ 0 & 0 & -rst/q & -(qs-r^2)st/q^2 & rs^2t/q^2 \end{bmatrix},$$

$$W(t)_{n-2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & qt & 0 & 0 \\ 0 & qt & 0 & 0 & 0 \\ 0 & 0 & 0 & -(2qs-r^2)rt/q^2 & -(qs-r^2)st/q^2 \\ 0 & 0 & 0 & -(qs-r^2)st/q^2 & rs^2t/q^2 \end{bmatrix}.$$

(2) *Let $n = 6$. Then,*

$$W(t)_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & qt & rt & st \\ 0 & 0 & qt & rt & st & 0 \\ 0 & qt & rt & st & 0 & 0 \\ 0 & rt & st & 0 & 0 & 0 \\ 0 & st & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W(t)_{n_0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & qt & 0 & 0 \\ 0 & 0 & qt & 0 & 0 & 0 \\ 0 & qt & 0 & (qs-r^2)t/q & -(2qs-r^2)rt/q^2 & -(qs-r^2)st/q^2 \\ 0 & 0 & 0 & -(2qs-r^2)rt/q^2 & -(q^2s^2-3qr^2s+r^4)t/q^3 & (2qs-r^2)rst/q^3 \\ 0 & 0 & 0 & -(qs-r^2)st/q^2 & (2qs-r^2)rst/q^3 & (qs-r^2)s^2t/q^3 \end{bmatrix},$$

$$W(t)_{n-2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & qt & 0 & 0 \\ 0 & 0 & qt & 0 & 0 & 0 \\ 0 & qt & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(q^2s^2-3qr^2s+r^4)t/q^3 & (2qs-r^2)rst/q^3 \\ 0 & 0 & 0 & 0 & (2qs-r^2)rst/q^3 & (qs-r^2)s^2t/q^3 \end{bmatrix}.$$

In the same way, let us compute the matrix $W(t)_{n-2}$ for any integer $n \geq 7$.

Lemma 2. *Suppose $n \geq 7$. Let $\{x_m\}$, $\{y_m\}$ be sequences defined by the next recurrence relations;*

$$x_0 = 0,\ x_1 = -qt,\ x_2 = rt,\ x_{m+2} = -\frac{r}{q}x_{m+1} - \frac{s}{q}x_m\ (m \geq 1),$$
$$y_0 = y_1 = 0,\ y_{m+1} = -\frac{s}{q}x_m\ (m \geq 1)$$

and put

$$n_1 = \begin{cases} n_0 - 1 & n : \textit{odd}, \\ n_0 - 2 & n : \textit{even}. \end{cases}$$

Then, for any integer k such that $2 \leq k \leq n_1$, we have

$$w_{ij}(t)_k = \begin{cases} 0 & (i,j) = (k,\ell) \text{ or } (\ell,k)\ (n-k+1 \leq \ell \leq n), \\ x_{\ell-n+k+2} & (i,j) = (k+1,\ell) \text{ or } (\ell,k+1)\ (n-k \leq \ell \leq n-1), \\ y_{\ell-n+k+2} & (i,j) = (k+2,\ell) \text{ or } (\ell,k+2)\ (n-k \leq \ell \leq n-1), \\ -\frac{sx_k}{q} & (i,j) = (k+1,n) \text{ or } (n,k+1), \\ -\frac{sy_k}{q} & (i,j) = (k+2,n) \text{ or } (n,k+2), \\ w_{ij}(t)_{k-1} & \textit{otherwise}. \end{cases}$$

Proof. Let us prove this lemma by induction on k. First, suppose $k = 2$. Then, by the definition of $W(t)_2$, we have

$$W(t)_2 = \begin{bmatrix} 0 & \cdots & \cdots & \dots & \dots & \dots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & qt & 0 & 0 \\ \vdots & & & & & & qt & rt & \frac{(qs-r^2)t}{q} & -\frac{rst}{q} \\ \vdots & & & & & \iddots & rt & st & -\frac{rst}{q} & -\frac{s^2t}{q} \\ \vdots & & & & \iddots & \iddots & st & 0 & 0 & 0 \\ \vdots & & & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 0 & qt & rt & st & \iddots & \iddots & \iddots & & \vdots \\ 0 & qt & rt & st & 0 & \iddots & \iddots & & & \vdots \\ 0 & 0 & (qs-r^2)\frac{t}{q} & -\frac{rst}{q} & 0 & \iddots & & & & \vdots \\ 0 & 0 & -\frac{rst}{q} & -\frac{s^2t}{q} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

which implies the claim of Lem. 2 for $k = 2$ since

$$x_2 = rt,\ x_3 = \frac{(qs-r^2)t}{q},\ y_2 = st,\ y_3 = -\frac{rst}{q}.$$

Next, suppose Lem. 2 is true for $k = 2, \cdots, m-1$ $(m-1 < n_1)$. Then, since

$$W(t)_{m-1} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & & & & & \iddots & \iddots & \vdots & \vdots \\ \vdots & & & & & & & & qt & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & & qt & x_2 & x_3 & \cdots & x_m & -sx_{m-1}/q \\ \vdots & & & & & & \iddots & rt & y_2 & y_3 & \cdots & y_m & -sy_{m-1}/q \\ \vdots & & & & & \iddots & \iddots & st & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \iddots & \iddots & \iddots & \iddots & \iddots & & & \iddots & \vdots \\ \vdots & & & qt & rt & st & \iddots & \iddots & & & \iddots & & \vdots \\ \vdots & & qt & x_2 & y_2 & 0 & \iddots & & & \iddots & & & \vdots \\ \vdots & \iddots & 0 & x_3 & y_3 & 0 & & & \iddots & & & & \vdots \\ \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & & \iddots & & & & & \vdots \\ 0 & \cdots & 0 & x_m & y_m & 0 & \iddots & & & & & & \vdots \\ 0 & \cdots & 0 & -sx_{m-1}/q & -sy_{m-1}/q & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

we have

$$W(t)_m = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & & & & & \iddots & \iddots & \vdots & \vdots \\ \vdots & & & & & & & & qt & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & & qt & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \iddots & rt & x'_2 & x'_3 & \cdots & x'_m & x'_{m+1} \\ \vdots & & & & & \iddots & \iddots & st & y'_2 & y'_3 & \cdots & y'_m & y'_{m+1} \\ \vdots & & & & \iddots & \iddots & \iddots & \iddots & \iddots & & & \iddots & \vdots \\ \vdots & & & qt & rt & st & \iddots & \iddots & & & \iddots & & \vdots \\ \vdots & & qt & 0 & x'_2 & y'_2 & \iddots & & & \iddots & & & \vdots \\ \vdots & \iddots & 0 & 0 & x'_3 & y'_3 & & & \iddots & & & & \vdots \\ \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & & \iddots & & & & & \vdots \\ 0 & \cdots & 0 & 0 & x'_m & y'_m & \iddots & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & x'_{m+1} & y'_{m+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

where

$$
\begin{aligned}
x'_\ell &= y_\ell - \frac{x_\ell}{qt}\cdot rt = -\frac{r}{q}x_\ell - \frac{s}{q}x_{\ell-1} = x_{\ell+1}\ (2\le \ell\le m),\\
y'_\ell &= -\frac{x_\ell}{qt}\cdot st = -\frac{s}{q}x_\ell = y_{\ell+1}\ (2\le \ell\le m),\\
x'_{m+1} &= -\frac{s}{q}y_{m-1} + \left(-\frac{r}{q}\right)\cdot\left(-\frac{s}{q}x_{m-1}\right) = -\frac{s}{q}\left(-\frac{r}{q}x_{m-1} - \frac{s}{q}x_{m-2}\right) = -\frac{s}{q}x_m\\
y'_{m+1} &= \left(-\frac{s}{q}\right)\cdot\left(-\frac{s}{q}x_{m-1}\right) = -\frac{s}{q}y_m.
\end{aligned}
$$

This completes the proof of Lem. 2. □

Proposition 1. *For any integer $n\ge 5$, we have*

$$
w_{ij}(t)_{n-2} = \begin{cases} qt & i+j=n\ (2\le i,j\le n-2),\\ x_{n-1} & (i,j)=(n-1,n-1),\\ y_{n-1} & (i,j)=(n-1,n)\ \text{or}\ (n,n-1),\\ (-s/q)y_{n-2} & (i,j)=(n,n),\\ 0 & \text{otherwise.}\end{cases}
$$

Proof. Prop. 1 has been proved for $n=5,6$ in Example 2. Thus, we suppose $n\ge 7$. First, by solving the recurrence relation given in Lem. 2, we have

$$
x_m = \begin{cases} \frac{(-1)^m m r^{m-1}}{2^{m-1}q^{m-2}}t & (r^2-4qs=0),\\[2mm] \frac{(-1)^m\left\{\left(r+\sqrt{r^2-4qs}\right)^m - \left(r-\sqrt{r^2-4qs}\right)^m\right\}}{2^m q^{m-2}\sqrt{r^2-4qs}}t & (r^2-4qs\ne 0).\end{cases} \tag{5}
$$

Note that

$$
\frac{(-1)^m\left\{\left(r+\sqrt{r^2-4qs}\right)^m - \left(r-\sqrt{r^2-4qs}\right)^m\right\}}{2^m q^{m-2}\sqrt{r^2-4qs}}t = \frac{(-1)^m t}{2^m q^{m-2}}\cdot 2\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}r^{m-2k-1}(r^2-4qs)^k
$$

and hence, by putting $r^2-4qs=0$, we have

$$
\frac{(-1)^m\left\{\left(r+\sqrt{r^2-4qs}\right)^m - \left(r-\sqrt{r^2-4qs}\right)^m\right\}}{2^m q^{m-2}\sqrt{r^2-4qs}}t = \frac{(-1)^m m r^{m-1}}{2^{m-1}q^{m-2}}t,
$$

the solution for the case of $r^2-4qs=0$. Here, suppose n is odd. Then, we have $n_1=\frac{n-3}{2}$ and hence by Lem. 2,

$$
w_{ij}(t)_{n_1} = \begin{cases} 0 & (i,j)=((n-3)/2,\ell)\ \text{or}\ (\ell,(n-3)/2)\ ((n+5)/2\le\ell\le n),\\ x_{\ell-(n-1)/2} & (i,j)=((n-1)/2,\ell)\ \text{or}\ (\ell,(n-1)/2)\ ((n+3)/2\le\ell\le n-1),\\ y_{\ell-(n-1)/2} & (i,j)=((n+1)/2,\ell)\ \text{or}\ (\ell,(n+1)/2)\ ((n+3)/2\le\ell\le n-1),\\ -sx_{(n-3)/2}/q & (i,j)=((n-1)/2,n)\ \text{or}\ (n,(n-1)/2),\\ -sy_{(n-3)/2}/q & (i,j)=((n+1)/2,n)\ \text{or}\ (n,(n+1)/2),\\ w_{ij}(t)_{(n-5)/2} & \text{otherwise.}\end{cases}
$$

Therefore, we have the next expression of the matrix $W(t)_{n_1}$;

$$W(t)_{n_1} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & & \iddots & & \vdots & \vdots \\ \vdots & & & & & qt & 0 & \cdots & 0 & 0 \\ \vdots & & & & qt & x_2 & x_3 & \cdots & x_{(n-1)/2} & -sx_{(n-3)/2}/q \\ \vdots & & & qt & rt & y_2 & y_3 & \cdots & y_{(n-1)/2} & -sy_{(n-3)/2}/q \\ \vdots & & qt & x_2 & y_2 & 0 & & & & \vdots \\ \vdots & \iddots & 0 & x_3 & y_3 & & & & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & \cdots & 0 & x_{(n-1)/2} & y_{(n-1)/2} & & & & & \vdots \\ 0 & \cdots & 0 & -sx_{(n-3)/2}/q & -sy_{(n-3)/2}/q & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

Then, since $W(t)_{n_1+1} = {}^tS(t)_{n_1+1}W(t)_{n_1}S(t)_{n_1+1}$, where

$$S(t)_{n_1+1} = \prod_{m=(n+3)/2}^{n} R_n\left(\frac{n+1}{2}, m; -\frac{w_{(n-1)/2,m}(t)_{n_1}}{qt}\right)$$

and $w_{n-1,(n+1)/2}(t)_{n_1+1} = w_{(n+1)/2,n-1}(t)_{n_1+1} = x_{(n+1)/2}$, we have

$$\begin{aligned} w_{n-1,n-1}(t)_{n_1+1} &= -\frac{x_{(n-1)/2}}{qt}\cdot y_{(n-1)/2} - \frac{x_{(n-1)/2}}{qt}\cdot x_{(n+1)/2} \\ &= -\frac{x_{(n-1)/2}}{qt}\cdot\left(-\frac{s}{q}x_{(n-3)/2}\right) - \frac{x_{(n-1)/2}}{qt}\cdot x_{(n+1)/2} \\ &= -\frac{1}{qt}\cdot\frac{(-1)^{(n-1)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n-1)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n-1)/2}\right\}}{2^{(n-1)/2}q^{(n-5)/2}\sqrt{r^2-4qs}}t \\ &\quad\times\left(-\frac{s}{q}\right)\cdot\frac{(-1)^{(n-3)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n-3)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n-3)/2}\right\}}{2^{(n-3)/2}q^{(n-7)/2}\sqrt{r^2-4qs}}t \\ &\quad-\frac{1}{qt}\cdot\frac{(-1)^{(n-1)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n-1)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n-1)/2}\right\}}{2^{(n-1)/2}q^{(n-5)/2}\sqrt{r^2-4qs}}t \\ &\quad\times\frac{(-1)^{(n+1)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n+1)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n+1)/2}\right\}}{2^{(n+1)/2}q^{(n-3)/2}\sqrt{r^2-4qs}}t \\ &= -s\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-2} - (4qs)^{(n-3)/2}\left(r+\sqrt{r^2-4qs}\right)\right. \\ &\quad\left.-(4qs)^{(n-3)/2}\left(r-\sqrt{r^2-4qs}\right) + \left(r-\sqrt{r^2-4qs}\right)^{n-2}\right\}t\Big/\left\{2^{n-2}q^{n-4}(r^2-4qs)\right\} \\ &\quad+\left\{\left(r+\sqrt{r^2-4qs}\right)^{n} - (4qs)^{(n-1)/2}\left(r+\sqrt{r^2-4qs}\right)\right. \\ &\quad\left.-(4qs)^{(n-1)/2}\left(r-\sqrt{r^2-4qs}\right) + \left(r-\sqrt{r^2-4qs}\right)^{n}\right\}t\Big/\left\{2^{n}q^{n-3}(r^2-4qs)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \left\{ \left(r+\sqrt{r^2-4qs}\right)^{n-2} \left\{ \left(r+\sqrt{r^2-4qs}\right)^2 - 4qs \right\} \right. \\
&\qquad \left. + \left(r-\sqrt{r^2-4qs}\right)^{n-2} \left\{ \left(r-\sqrt{r^2-4qs}\right)^2 - 4qs \right\} \right\} t \Big/ \left\{ 2^n q^{n-3} (r^2-4qs) \right\} \\
&= \frac{2\sqrt{r^2-4qs} \left\{ \left(r+\sqrt{r^2-4qs}\right)^{n-1} - \left(r-\sqrt{r^2-4qs}\right)^{n-1} \right\}}{2^n q^{n-3}(r^2-4qs)} t \\
&= \frac{\left\{ \left(r+\sqrt{r^2-4qs}\right)^{n-1} - \left(r-\sqrt{r^2-4qs}\right)^{n-1} \right\}}{2^{n-1} q^{n-3}\sqrt{r^2-4qs}} t = x_{n-1}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
w_{n-1,n}(t)_{n_1+1} &= w_{n,n-1}(t)_{n_1+1} = -\frac{x_{(n-1)/2}}{qt} \cdot \frac{-sy_{(n-3)/2}}{q} - \frac{-sx_{(n-3)/2}}{q^2 t} x_{(n+1)/2} \\
&= -\frac{s^2}{q^3 t} \cdot \frac{(-1)^{(n-1)/2} \left\{ \left(r+\sqrt{r^2-4qs}\right)^{(n-1)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n-1)/2} \right\}}{2^{(n-1)/2} q^{(n-5)/2} \sqrt{r^2-4qs}} t \\
&\qquad \times \frac{(-1)^{(n-5)/2} \left\{ \left(r+\sqrt{r^2-4qs}\right)^{(n-5)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n-5)/2} \right\}}{2^{(n-5)/2} q^{(n-9)/2} \sqrt{r^2-4qs}} t \\
&\quad + \frac{s}{q^2 t} \cdot \frac{(-1)^{(n-3)/2} \left\{ \left(r+\sqrt{r^2-4qs}\right)^{(n-3)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n-3)/2} \right\}}{2^{(n-3)/2} q^{(n-7)/2} \sqrt{r^2-4qs}} t \\
&\qquad \times \frac{(-1)^{(n+1)/2} \left\{ \left(r+\sqrt{r^2-4qs}\right)^{(n+1)/2} - \left(r-\sqrt{r^2-4qs}\right)^{(n+1)/2} \right\}}{2^{(n+1)/2} q^{(n-3)/2} \sqrt{r^2-4qs}} t \\
&= -\frac{s^2}{q} \cdot \left\{ (r+\sqrt{r^2-4qs})^{n-3} - (4qs)^{(n-5)/2} (r+\sqrt{r^2-4qs})^2 \right. \\
&\qquad \left. - (4qs)^{(n-5)/2} (r-\sqrt{r^2-4qs})^2 + (r-\sqrt{r^2-4qs})^{n-3} \right\} t \Big/ \left\{ 2^{n-3} q^{n-5} (r^2-4qs) \right\} \\
&\quad + \frac{s}{q} \cdot \left\{ (r+\sqrt{r^2-4qs})^{n-1} - (4qs)^{(n-3)/2} (r+\sqrt{r^2-4qs})^2 \right. \\
&\qquad \left. - (4qs)^{(n-3)/2} (r-\sqrt{r^2-4qs})^2 + (r-\sqrt{r^2-4qs})^{n-1} \right\} t \Big/ \left\{ 2^{n-1} q^{n-4} (r^2-4qs) \right\} \\
&= \frac{s}{q} \cdot \frac{(r+\sqrt{r^2-4qs})^{n-2} - (r-\sqrt{r^2-4qs})^{n-2}}{2^{n-2} q^{n-4} \sqrt{r^2-4qs}} t \\
&= -\frac{s}{q} x_{n-2} = y_{n-1}.
\end{aligned}
$$

Moreover, since $w_{n,(n+1)/2}(t)_{n_1+1} = w_{(n+1)/2,n}(t)_{n_1+1} = -sx_{(n-1)/2}/q$, we have

$$
\begin{aligned}
w_{n,n}(t)_{n_1+1} &= -\frac{-sx_{(n-3)/2}}{q^2t}\cdot\frac{-sy_{(n-3)/2}}{q} - \frac{-sx_{(n-3)/2}}{q^2t}\cdot\frac{-sx_{(n-1)/2}}{q} \\
&= \frac{s^3}{q^4t}\cdot\frac{(-1)^{(n-3)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n-3)/2}-\left(r-\sqrt{r^2-4qs}\right)^{(n-3)/2}\right\}}{2^{(n-3)/2}q^{(n-7)/2}\sqrt{r^2-4qs}}t \\
&\qquad\times\frac{(-1)^{(n-5)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n-5)/2}-\left(r-\sqrt{r^2-4qs}\right)^{(n-5)/2}\right\}}{2^{(n-5)/2}q^{(n-9)/2}\sqrt{r^2-4qs}}t \\
&\quad-\frac{s^2}{q^3t}\cdot\frac{(-1)^{(n-3)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n-3)/2}-\left(r-\sqrt{r^2-4qs}\right)^{(n-3)/2}\right\}}{2^{(n-3)/2}q^{(n-7)/2}\sqrt{r^2-4qs}}t \\
&\qquad\times\frac{(-1)^{(n-1)/2}\left\{\left(r+\sqrt{r^2-4qs}\right)^{(n-1)/2}-\left(r-\sqrt{r^2-4qs}\right)^{(n-1)/2}\right\}}{2^{(n-1)/2}q^{(n-5)/2}\sqrt{r^2-4qs}}t \\
&= -\frac{s^3}{q^2}\cdot\Big\{(r+\sqrt{r^2-4sq})^{n-4}-(4qs)^{(n-5)/2}(r+\sqrt{r^2-4qs}) \\
&\quad-(4qs)^{(n-5)/2}(r-\sqrt{r^2-4qs})+(r-\sqrt{r^2-4qs})^{n-4}\Big\}t\Big/\left\{2^{n-4}q^{n-6}(r^2-4qs)\right\} \\
&\quad+\frac{s^2}{q^2}\cdot\Big\{(r+\sqrt{r^2-4sq})^{n-2}-(4qs)^{(n-3)/2}(r+\sqrt{r^2-4qs}) \\
&\quad-(4qs)^{(n-3)/2}(r-\sqrt{r^2-4qs})+(r-\sqrt{r^2-4qs})^{n-2}\Big\}t\Big/\left\{2^{n-2}q^{n-5}(r^2-4qs)\right\} \\
&= \frac{s^2}{q^2}\cdot\frac{(r+\sqrt{r^2-4sq})^{n-3}-(r-\sqrt{r^2-4qs})^{n-3}}{2^{n-3}q^{n-5}\sqrt{r^2-4qs}}t \\
&= -\frac{s}{q}\left(-\frac{s}{q}x_{n-3}\right) = -\frac{s}{q}y_{n-2}.
\end{aligned}
$$

Therefore, we can express the matrix $W(t)_{n_0}$ $(= W(t)_{n_1+1})$ as follows;

$$
W(t)_{n_0} = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & & & & & & \iddots & & \vdots & \vdots \\
\vdots & & & & & qt & 0 & \cdots & 0 & 0 \\
\vdots & & & & qt & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & qt & * & * & * & \cdots & * & * \\
\vdots & & qt & 0 & * & * & * & \cdots & * & * \\
\vdots & \iddots & 0 & 0 & * & * & * & \cdots & * & * \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & * & * & * & \cdots & x_{n-1} & y_{n-1} \\
0 & \cdots & 0 & 0 & * & * & * & \cdots & y_{n-1} & (-s/q)y_{n-2}
\end{bmatrix}.
$$

Then, by the definition of the matrix W_k $(n_0 < k \le n-2)$, we have

$$W(t)_{n-2} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & \iddots & & \vdots & \vdots \\ \vdots & & & & qt & 0 & \cdots & 0 & 0 \\ \vdots & & & qt & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & qt & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \iddots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & 0 & 0 & 0 & 0 & \cdots & x_{n-1} & y_{n-1} \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & y_{n-1} & (-s/q)y_{n-2} \end{bmatrix},$$

which completes the proof of Prop. 1 for odd n.

Next, suppose n is even. Then, we have $n_1 = (n-4)/2$ and hence by Lem. 2, we have

$$w_{ij}(t)_{n_1} = \begin{cases} 0 & (i,j) = ((n-4)/2, \ell) \text{ or } (\ell, (n-4)/2) \ ((n+6)/2 \le \ell \le n), \\ x_{\ell - n/2} & (i,j) = ((n-2)/2, \ell) \text{ or } (\ell, (n-2)/2) \ ((n+4)/2 \le \ell \le n-1), \\ y_{\ell - n/2} & (i,j) = (n/2, \ell) \text{ or } (\ell, n/2) \ ((n+4)/2 \le \ell \le n-1), \\ -sx_{(n-4)/2}/q & (i,j) = ((n-2)/2, n) \text{ or } (n, (n-2)/2), \\ -sy_{(n-4)/2}/q & (i,j) = (n/2, n) \text{ or } (n, n/2), \\ w_{ij}(t)_{(n-6)/2} & \text{otherwise.} \end{cases}$$

Thus, we have the next expression of the matrix $W(t)_{n_1}$;

$$W(t)_{n_1} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & & & \iddots & & \vdots & \vdots \\ \vdots & & & & & & qt & 0 & \cdots & 0 & 0 \\ \vdots & & & & & qt & x_2 & x_3 & \cdots & x_{(n-2)/2} & -sx_{(n-4)/2}/q \\ \vdots & & & & qt & rt & y_2 & y_3 & \cdots & y_{(n-2)/2} & -sy_{(n-4)/2}/q \\ \vdots & & & qt & rt & st & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & qt & x_2 & y_2 & 0 & 0 & & & & \vdots \\ \vdots & \iddots & 0 & x_3 & y_3 & 0 & & & & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & \cdots & 0 & x_{(n-2)/2} & y_{(n-2)/2} & 0 & & & & & \vdots \\ 0 & \cdots & 0 & -sx_{(n-4)/2}/q & -sy_{(n-4)/2}/q & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

Then, for any integer ℓ $((n+4)/2 \le \ell \le n)$, let us put

$$V(t)_\ell = (v_{ij}(t)_\ell)_{1 \le i,j \le n} = {}^t R(t)_\ell W(t)_{n_1} R(t)_\ell,$$

where

$$R(t)_\ell = \prod_{m=(n+4)/2}^{\ell} R_n\left((n+2)/2, m; -\frac{w_{(n-2)/2,m}(t)_{n_1}}{qt}\right).$$

Note that we have $V(t)_n = W(t)_{n_1+1}$.

Claim . *For any integer* ℓ' $((n+4)/2 \leq \ell' \leq n-1)$*, we have*

$$v_{ij}(t)_{\ell'} = \begin{cases} 0 & (i,j) = ((n-2)/2, \ell) \text{ or } (\ell, (n-2)/2) \ ((n+4)/2 \leq \ell \leq \ell'), \\ x_{\ell-(n-2)/2} & (i,j) = (n/2, \ell) \text{ or } (\ell, n/2) \ ((n+2)/2 \leq \ell \leq \ell'), \\ y_{\ell-(n-2)/2} & (i,j) = ((n+2)/2, \ell) \text{ or } (\ell, (n+2)/2) \ ((n+2)/2 \leq \ell \leq \ell'), \\ \left(-x_{\ell-n/2}/qt\right)^2 y_2 & (i,j) = (\ell,\ell) \ ((n+4)/2 \leq \ell \leq \ell'), \\ (-x_{m-n/2}/qt) & \\ \qquad \times y_{\ell-(n-2)/2} & (i,j) = (\ell, m) \text{ or } (m, \ell) \ ((n+4)/2 \leq \ell < m \leq \ell'), \\ w_{ij}(t)_{(n-4)/2} & \text{otherwise.} \end{cases}$$

Proof. To ease notation, let us put $k = (n-2)/2$. Then, by definition,

$$V(t)_{(n+4)/2} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & & & \iddots & & \vdots & \vdots \\ \vdots & & & & & & qt & 0 & \cdots & 0 & 0 \\ \vdots & & & & & qt & 0 & x_3 & \cdots & x_k & -sx_{k-1}/q \\ \vdots & & & & qt & x_2 & x_3 & y_3 & \cdots & y_k & -sy_{k-1}/q \\ \vdots & & & qt & x_2 & y_2 & y_3 & 0 & \cdots & 0 & 0 \\ \vdots & & qt & 0 & x_3 & y_3 & \left(-x_2/qt\right)^2 y_2 & & & & \vdots \\ \vdots & \iddots & 0 & x_3 & y_3 & 0 & & & & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & \cdots & 0 & x_k & y_k & 0 & & & & & \vdots \\ 0 & \cdots & 0 & -sx_{k-1}/q & -sy_{k-1}/q & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

and we get Claim for $V(t)_{(n+4)/2}$. Here, suppose Claim is true for $V(t)_{\ell'-1}$ and hence

$$V(t)_{\ell'-1} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots \\ \vdots & & & & & \iddots & & & \vdots & \vdots & \\ \vdots & & & & qt & 0 & 0 & \cdots & 0 & x_{\ell'-k-1} & \cdots \\ \vdots & & & qt & x_2 & x_3 & x_4 & \cdots & x_{\ell'-k-1} & y_{\ell'-k-1} & \cdots \\ \vdots & & qt & x_2 & y_2 & y_3 & y_4 & \cdots & y_{\ell'-k-1} & 0 & \cdots \\ \vdots & \iddots & 0 & x_3 & y_3 & \left(-x_2/qt\right)^2 y_2 & (-x_3/qt)\, y_3 & \cdots & (-x_{\ell'-k-2}/qt) y_3 & 0 & \cdots \\ \vdots & & 0 & x_4 & y_4 & (-x_3/qt)\, y_3 & \left(-x_3/qt\right)^2 y_2 & \cdots & (-x_{\ell'-k-2}/qt) y_4 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & & 0 & x_{\ell'-k-1} & y_{\ell'-k-1} & (-x_{\ell'-k-2}/qt) y_3 & (-x_{\ell'-k-2}/qt) y_4 & \cdots & (-x_{\ell'-k-2}/qt)^2 y_2 & 0 & \cdots \\ 0 & \cdots & x_{\ell'-k-1} & y_{\ell'-k-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}.$$

Thus, by a direct computation, we have

$$V(t)_{\ell'} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots \\ \vdots & & & & & \iddots & & & \vdots & \vdots & \\ \vdots & & & & qt & 0 & 0 & \cdots & 0 & x_{\ell'-k} & \cdots \\ \vdots & & & qt & x_2 & x_3 & x_4 & \cdots & x_{\ell'-k} & y_{\ell'-k} & \cdots \\ \vdots & & qt & x_2 & y_2 & y_3 & y_4 & \cdots & y_{\ell'-k} & 0 & \cdots \\ \vdots & \iddots & 0 & x_3 & y_3 & (-x_2/qt)^2\, y_2 & (-x_3/qt)\, y_3 & \cdots & (-x_{\ell'-k-1}/qt)y_3 & 0 & \cdots \\ \vdots & & 0 & x_4 & y_4 & (-x_3/qt)\, y_3 & (-x_3/qt)^2\, y_2 & \cdots & (-x_{\ell'-k-1}/qt)y_4 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & & 0 & x_{\ell'-k} & y_{\ell'-k} & (-x_{\ell'-k-1}/qt)y_3 & (-x_{\ell'-k-1}/qt)y_4 & \cdots & (-x_{\ell'-k-1}/qt)^2 y_2 & 0 & \cdots \\ 0 & \cdots & x_{\ell'-k} & y_{\ell'-k} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}$$

and we get Claim by induction on ℓ'. $\square$

By above Claim,

$$V(t)_{n-1} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & \iddots & & & \vdots & \vdots \\ \vdots & & & & qt & 0 & 0 & \cdots & 0 & -sx_{k-1}/q \\ \vdots & & & qt & x_2 & x_3 & x_4 & \cdots & x_{k+1} & -sy_{k-1}/q \\ \vdots & & qt & x_2 & y_2 & y_3 & y_4 & \cdots & y_{k+1} & 0 \\ \vdots & \iddots & 0 & x_3 & y_3 & (-x_2/qt)^2\, y_2 & (-x_3/qt)\, y_3 & \cdots & (-x_k/qt)y_3 & 0 \\ \vdots & & 0 & x_4 & y_4 & (-x_3/qt)\, y_3 & (-x_3/qt)^2\, y_2 & \cdots & (-x_k/qt)y_4 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & 0 & x_{k+1} & y_{k+1} & (-x_k/qt)y_3 & (-x_k/qt)y_4 & \cdots & (-x_k/qt)^2 y_2 & 0 \\ 0 & \cdots & -sx_{k-1}/q & -sy_{k-1}/q & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and hence, by the definition of $V(t)_n$, we have

$$W(t)_{n_1+1} = V(t)_n =$$
$$\begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & \iddots & & & \vdots & \vdots \\ \vdots & & & & qt & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & qt & x_2 & x_3 & x_4 & \cdots & x_{k+1} & -sx_k/q \\ \vdots & & qt & x_2 & y_2 & y_3 & y_4 & \cdots & y_{k+1} & -sy_k/q \\ \vdots & \iddots & 0 & x_3 & y_3 & (-x_2/qt)^2\, y_2 & (-x_3/qt)\, y_3 & \cdots & (-x_k/qt)y_3 & (x_{k-1}/q^2t^2)y_2y_3 \\ \vdots & & 0 & x_4 & y_4 & (-x_3/qt)\, y_3 & (-x_3/qt)^2\, y_2 & \cdots & (-x_k/qt)y_4 & (x_{k-1}/q^2t^2)y_2y_4 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & 0 & x_{k+1} & y_{k+1} & (-x_k/qt)y_3 & (-x_k/qt)y_4 & \cdots & (-x_k/qt)^2 y_2 & (x_{k-1}/q^2t^2)y_2y_{k+1} \\ 0 & \cdots & 0 & -sx_k/q & -sy_k/q & (x_{k-1}/q^2t^2)y_2y_3 & (x_{k-1}/q^2t^2)y_2y_4 & \cdots & (x_{k-1}/q^2t^2)y_2y_{k+1} & (x_{k-1}/q^2t^2)^2 y_2^3 \end{bmatrix}.$$

Therefore, by the definition of the matrix $W(t)_{n_1+2} = W(t)_{n_0}$, we have

$$
\begin{aligned}
w_{n-1,n-1}(t)_{n_0} &= \left(-\frac{x_{(n-2)/2}}{qt}\right)^2 y_2 - \frac{\left(x_{n/2}\right)^2}{qt} \\
&= \frac{s\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-2}+\left(r-\sqrt{r^2-4qs}\right)^{n-2}-2(4qs)^{(n-2)/2}\right\}}{2^{n-2}q^{n-4}\left(r^2-4qs\right)}t \\
&\qquad - \frac{\left(r+\sqrt{r^2-4qs}\right)^{n}+\left(r-\sqrt{r^2-4qs}\right)^{n}-2(4qs)^{n/2}}{2^{n}q^{n-3}\left(r^2-4qs\right)}t \\
&= \frac{\left(r+\sqrt{r^2-4qs}\right)^{n-2}\left\{4qs-\left(r+\sqrt{r^2-4qs}\right)^2\right\}}{2^{n}q^{n-3}\left(r^2-4qs\right)}t \\
&\qquad + \frac{\left(r-\sqrt{r^2-4qs}\right)^{n-2}\left\{4qs-\left(r-\sqrt{r^2-4qs}\right)^2\right\}}{2^{n}q^{n-3}\left(r^2-4qs\right)}t \\
&= \frac{\left(r+\sqrt{r^2-4qs}\right)^{n-2}\left(-2\sqrt{r^2-4qs}\right)\left(r+\sqrt{r^2-4qs}\right)}{2^{n}q^{n-3}\left(r^2-4qs\right)}t \\
&\qquad + \frac{\left(r-\sqrt{r^2-4qs}\right)^{n-2}\left(2\sqrt{r^2-4qs}\right)\left(r-\sqrt{r^2-4qs}\right)}{2^{n}q^{n-3}\left(r^2-4qs\right)}t = x_{n-1}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&w_{n-1,n}(t)_{n_0} = w_{n,n-1}(t)_{n_0} = \frac{x_{(n-4)/2}}{q^2t^2}y_2y_{n/2} + \frac{sx_{(n-2)/2}x_{n/2}}{q^2t} \\
&= \frac{x_{(n-4)/2}}{q^2t^2}y_2\left(-\frac{s}{q}x_{(n-2)/2}\right) + \frac{sx_{(n-2)/2}x_{n/2}}{q^2t} \\
&= -\frac{s^2}{q^3}\cdot\frac{-\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-3}+\left(r-\sqrt{r^2-4qs}\right)^{n-3}-2r\left(4qs\right)^{(n-4)/2}\right\}}{2^{n-3}q^{n-7}\left(r^2-4qs\right)}t \\
&\qquad + \frac{s}{q^2}\cdot\frac{-\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-1}+\left(r-\sqrt{r^2-4qs}\right)^{n-1}-2r\left(4qs\right)^{(n-2)/2}\right\}}{2^{n-1}q^{n-5}\left(r^2-4qs\right)}t \\
&= \frac{s\left\{4qs\left(r+\sqrt{r^2-4qs}\right)^{n-3}+4qs\left(r-\sqrt{r^2-4qs}\right)^{n-3}-2r\left(4qs\right)^{(n-2)/2}\right\}}{2^{n-1}q^{n-3}\left(r^2-4qs\right)}t \\
&\qquad - \frac{s\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-1}+\left(r-\sqrt{r^2-4qs}\right)^{n-1}-2r\left(4qs\right)^{(n-2)/2}\right\}}{2^{n-1}q^{n-3}\left(r^2-4qs\right)}t \\
&= \frac{s\left(r+\sqrt{r^2-4qs}\right)^{n-3}\left(-2\sqrt{r^2-4qs}\right)\left(r+\sqrt{r^2-4qs}\right)}{2^{n-1}q^{n-3}\left(r^2-4qs\right)}t \\
&\qquad + \frac{s\left(r-\sqrt{r^2-4qs}\right)^{n-3}\left(2\sqrt{r^2-4qs}\right)\left(r-\sqrt{r^2-4qs}\right)}{2^{n-1}q^{n-3}\left(r^2-4qs\right)}t \\
&= -\frac{s}{q}\cdot\frac{\left(r+\sqrt{r^2-4qs}\right)^{n-2}-\left(r-\sqrt{r^2-4qs}\right)^{n-2}}{2^{n-2}q^{n-4}\sqrt{r^2-4qs}}t = y_{n-1},
\end{aligned}
$$

$$\begin{aligned}
w_{n,n}(t)_{n_0} &= \left(\frac{x_{(n-4)/2}}{q^2t^2}\right)^2 y_2^3 - \frac{s^2\left(x_{(n-2)/2}\right)^2}{q^3t} \\
&= \frac{s^3\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-4}+\left(r-\sqrt{r^2-4qs}\right)^{n-4}-2(4qs)^{(n-4)/2}\right\}}{2^{n-4}q^{n-4}(r^2-4qs)}t \\
&\qquad - \frac{s^2\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-2}+\left(r-\sqrt{r^2-4qs}\right)^{n-2}-2(4qs)^{(n-2)/2}\right\}}{2^{n-2}q^{n-3}(r^2-4qs)}t \\
&= \frac{s^2\left\{4qs\left(r+\sqrt{r^2-4qs}\right)^{n-4}+4qs\left(r-\sqrt{r^2-4qs}\right)^{n-4}-2(4qs)^{(n-2)/2}\right\}}{2^{n-2}q^{n-3}(r^2-4qs)}t \\
&\qquad - \frac{s^2\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-2}+\left(r-\sqrt{r^2-4qs}\right)^{n-2}-2(4qs)^{(n-2)/2}\right\}}{2^{n-2}q^{n-3}(r^2-4qs)}t \\
&= \frac{s^2\left(r+\sqrt{r^2-4qs}\right)^{n-4}\left(-2\sqrt{r^2-4qs}\right)\left(r+\sqrt{r^2-4qs}\right)}{2^{n-2}q^{n-3}\left(r^2-4qs\right)}t \\
&\qquad + \frac{s^2\left(r-\sqrt{r^2-4qs}\right)^{n-4}\left(2\sqrt{r^2-4qs}\right)\left(r-\sqrt{r^2-4qs}\right)}{2^{n-2}q^{n-3}\left(r^2-4qs\right)}t \\
&= \left(-\frac{s}{q}\right)\cdot\left(-\frac{s}{q}\right)\cdot\frac{-\left\{\left(r+\sqrt{r^2-4qs}\right)^{n-3}-\left(r-\sqrt{r^2-4qs}\right)^{n-3}\right\}}{2^{n-3}q^{n-5}\sqrt{r^2-4qs}}t = -\frac{s}{q}y_{n-2}.
\end{aligned}$$

Therefore, we can express the matrix $W(t)_{n_0}$ as follows;

$$W(t)_{n_0} = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & & & & & \iddots & & \vdots & \vdots \\
\vdots & & & & qt & 0 & \cdots & 0 & 0 \\
\vdots & & & qt & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & qt & 0 & * & * & \cdots & * & * \\
\vdots & \iddots & 0 & 0 & * & * & \cdots & * & * \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & & 0 & 0 & * & * & \cdots & x_{n-1} & y_{n-1} \\
0 & \cdots & 0 & 0 & * & * & \cdots & y_{n-1} & (-s/q)y_{n-2}
\end{bmatrix}.$$

Then, by the definition of the matrix W_k $(n_0 < k \leq n-2)$, we have

$$W(t)_{n-2} = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & & & & & \iddots & & \vdots & \vdots \\
\vdots & & & & qt & 0 & \cdots & 0 & 0 \\
\vdots & & & qt & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & qt & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \iddots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & & 0 & 0 & 0 & 0 & \cdots & x_{n-1} & y_{n-1} \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & y_{n-1} & (-s/q)y_{n-2}
\end{bmatrix},$$

which completes the proof of Prop. 1 for even n. $\square$

In the end, let us apply Prop. 1 to the matrix $A_{\boldsymbol{c}}(t)_1$ $(n \geq 5)$. Then, by the definition of the matrix $A_{\boldsymbol{c}}(t)_k$ $(2 \leq k \leq n-2)$ and Prop. 1, we have

$$A_{\boldsymbol{c}}(t)_{n-2} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & & & & & -(n-2)t & 0 & 0 \\ \vdots & & & & \iddots & \iddots & 0 & 0 \\ \vdots & & & \iddots & \iddots & \iddots & \vdots & \vdots \\ \vdots & & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots \\ 0 & -(n-2)t & \iddots & \iddots & \iddots & & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & a^{(\boldsymbol{c})}_{n-1,n-1}(t)_{n-2} & a^{(\boldsymbol{c})}_{n-1,n}(t)_{n-2} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & a^{(\boldsymbol{c})}_{n,n-1}(t)_{n-2} & a^{(\boldsymbol{c})}_{n,n}(t)_{n-2} \end{bmatrix}, \tag{6}$$

where

$$\begin{cases} a^{(\boldsymbol{c})}_{n-1,n-1}(t)_{n-2} = 2\left(1-2/n\right)t^2 + \bar{x}_{n-1} \\ a^{(\boldsymbol{c})}_{n-1,n}(t)_{n-2} = a^{(\boldsymbol{c})}_{n,n-1}(t)_{n-2} = \left(1-2/n\right)at^2 - (nb\bar{x}_{n-2})/(n-2) \\ a^{(\boldsymbol{c})}_{n,n}(t)_{n-2} = \left\{\left(1-1/n\right)a^2 - 2b\right\}t^2 + (n^2b^2\bar{x}_{n-3})/(n-2)^2. \end{cases}$$

Here, for any integer $m \geq 0$, we denote

$$\begin{aligned} \bar{x}_m &= \frac{(-1)^m\left(A^m - B^m\right)}{2^m\{-(n-2)\}^{m-2}\sqrt{(n-1)^2a^2 - 4n(n-2)b}}t, \\ A &= -(n-1)a + \sqrt{(n-1)^2a^2 - 4n(n-2)b}, \\ B &= -(n-1)a - \sqrt{(n-1)^2a^2 - 4n(n-2)b}. \end{aligned}$$

4. Proof of Theorems

By a direct computation, we have

$$\begin{aligned} &\det\begin{bmatrix} a^{(\boldsymbol{c})}_{n-1,n-1}(t)_{n-2} & a^{(\boldsymbol{c})}_{n-1,n}(t)_{n-2} \\ a^{(\boldsymbol{c})}_{n,n-1}(t)_{n-2} & a^{(\boldsymbol{c})}_{n,n}(t)_{n-2} \end{bmatrix} \\ &= \left(1 - \frac{2}{n}\right)(a^2-4b)t^4 \\ &\quad + \frac{2(n-2)}{n}t^2 \cdot \frac{n^2b^2\bar{x}_{n-3}}{(n-2)^2} + \left\{\left(\frac{n-1}{n}a^2 - 2b\right)\right\}t^2 \cdot \bar{x}_{n-1} + \frac{2(n-2)}{n}at^2 \cdot \frac{nb\bar{x}_{n-2}}{n-2} \\ &\qquad + \frac{n^2b^2}{(n-2)^2}\bar{x}_{n-1}\bar{x}_{n-3} - \frac{n^2b^2}{(n-2)^2}\bar{x}^2_{n-2} \\ &= \frac{(n-2)(a^2-4b)}{n}t^4 + \left\{\frac{2nb^2\bar{x}_{n-3}}{n-2} + \frac{\left\{(n-1)a^2 - 2nb\right\}\bar{x}_{n-1}}{n} + 2ab\bar{x}_{n-2}\right\}t^2 \\ &\qquad + \frac{n^2b^2}{(n-2)^2}\left(\bar{x}_{n-1}\bar{x}_{n-3} - \bar{x}^2_{n-2}\right). \end{aligned} \tag{7}$$

Here, let us put

$$\begin{aligned} \alpha(t) &= \left\{\frac{2nb^2\bar{x}_{n-3}}{n-2} + \frac{\left\{(n-1)a^2 - 2nb\right\}\bar{x}_{n-1}}{n} + 2ab\bar{x}_{n-2}\right\}t^2, \\ \beta(t) &= \frac{n^2b^2}{(n-2)^2}\left(\bar{x}_{n-1}\bar{x}_{n-3} - \bar{x}^2_{n-2}\right). \end{aligned}$$

Lemma 3. *Suppose $n \geq 5$ and put*

$$\begin{aligned} S_k =&(n-1)^3 \binom{n-k-3}{k} a^4 - \frac{n(n-1)\left\{5n^2-(6k+23)n+10k+24\right\}}{n-k-3} \binom{n-k-3}{k} a^2 b \\ &+ 4n^2(n-2)\binom{n-k-4}{k} b^2. \end{aligned}$$

Then, we have

$$\alpha(t) = \begin{cases} \displaystyle\sum_{k=0}^{m_0} \frac{(-1)^{n+k} n^k (n-1)^{n-2k-4}(n-2)^k a^{n-2k-4} b^k S_k}{n(n-2)^{n-3}} t^3 & (a \neq 0, \text{ or } a = 0,\ n: \text{even}), \\ 0 & (a = 0,\ n: \text{odd}). \end{cases}$$

$$\beta(t) = -\frac{n^{n-1} b^{n-1}}{(n-2)^{n-3}} t^2.$$

To carry out these computations, we need some combinatorial identities.

Lemma 4. [33, Chapter 2, Problem 18 (c), Chapter 6, Problem 18 (a)] *Let N (≥ 0) be an integer and put $\underline{m} = \lfloor N/2 \rfloor$, $\overline{m} = \lceil N/2 \rceil = \lfloor (N+1)/2 \rfloor$. Then, we have*

$$\sum_{j=k}^{\underline{m}} \binom{N+1}{2j+1}\binom{j}{k} = 2^{N-2k}\binom{N-k}{k} \qquad (k \in \mathbb{Z},\ 0 \leq k \leq \underline{m}),$$

$$\begin{aligned} \sum_{j=k}^{\overline{m}} \binom{N+1}{2j}\binom{j}{k} &= 2^{N-2k}\left[2\binom{N+1-k}{k} - \binom{N-k}{k}\right] \\ &= 2^{N-2k}\left[\binom{N+1-k}{k} + \binom{N-k}{k-1}\right] \qquad (k \in \mathbb{Z},\ 0 \leq k \leq \overline{m}). \end{aligned}$$

Proof. We omit the proof of the first identity and let us prove the second one. By using the convention

$$\binom{n}{n+m} = 0 \quad (n \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}_{\geq 1}),$$

we have

$$\sum_{j=k}^{\underline{m}} \binom{N+1}{2j+1}\binom{j}{k} = \sum_{j=k}^{\overline{m}} \binom{N+1}{2j+1}\binom{j}{k}$$

and hence

$$\begin{aligned} \sum_{j=k}^{\overline{m}} \binom{N+1}{2j}\binom{j}{k} + \sum_{j=k}^{\underline{m}} \binom{N+1}{2j+1}\binom{j}{k} &= \sum_{j=k}^{\overline{m}} \binom{N+1}{2j}\binom{j}{k} + \sum_{j=k}^{\overline{m}} \binom{N+1}{2j+1}\binom{j}{k} \\ &= \sum_{j=k}^{\overline{m}} \left\{\binom{N+1}{2j} + \binom{N+1}{2j+1}\right\}\binom{j}{k} \\ &= \sum_{j=k}^{\overline{m}} \binom{N+2}{2j+1}\binom{j}{k} \end{aligned}$$

for any k $(0 \leq k \leq \overline{m})$. Thus, by using the first identity, we get the second one. $\square$

Proof of Lem. 3. We first prove the second equality. Then, by the definition of $\bar{x}_m$, we have

$$\begin{aligned}
\beta(t) &= \frac{n^2b^2}{(n-2)^2}\cdot\frac{-A^{n-1}B^{n-3}-A^{n-3}B^{n-1}+2A^{n-2}B^{n-2}}{2^{2n-4}(n-2)^{2n-8}\left\{(n-1)^2a^2-4n(n-2)b\right\}}t^2\\
&= -\frac{n^2b^2\left\{(4n(n-2)b)^{n-3}\left(A^2+B^2\right)-2\left(4n(n-2)b\right)^{n-2}\right\}}{2^{2n-4}(n-2)^{2n-6}\left\{(n-1)^2a^2-4n(n-2)b\right\}}t^2\\
&= -\frac{2^{2n-6}n^{n-1}(n-2)^{n-3}b^{n-1}\cdot 4\left\{(n-1)^2a^2-4n(n-2)b\right\}}{2^{2n-4}(n-2)^{2n-6}\left\{(n-1)^2a^2-4n(n-2)b\right\}}t^2\\
&= -\frac{n^{n-1}b^{n-1}}{(n-2)^{n-3}}t^2,
\end{aligned}$$

which is the second equality. For the first equality, let us first suppose $a=0$. Then, we have

$$\begin{aligned}
\bar{x}_m &= \frac{(-1)^m\left\{\left\{-4n(n-2)b\right\}^{m/2}-(-1)^m\left\{-4n(n-2)b\right\}^{m/2}\right\}}{2^m\left\{-(n-2)\right\}^{m-2}\left\{-4n(n-2)b\right\}^{1/2}}t\\
&= \begin{cases}\dfrac{2\left\{-4n(n-2)b\right\}^{(m-1)/2}}{2^m(n-2)^{m-2}}t & (m:\text{odd}),\\ 0 & (m:\text{even})\end{cases}\\
&= \begin{cases}\dfrac{(-nb)^{(m-1)/2}}{(n-2)^{(m-3)/2}}t & (m:\text{odd}),\\ 0 & (m:\text{even})\end{cases}
\end{aligned}$$

and hence we have

$$\begin{aligned}
\alpha(t) &= \left\{\frac{2nb^2\bar{x}_{n-3}}{n-2}-2b\bar{x}_{n-1}\right\}t^2\\
&= \begin{cases}\left\{\dfrac{2nb^2}{n-2}\cdot\dfrac{(-nb)^{(n-4)/2}}{(n-2)^{(n-6)/2}}t-\dfrac{2b(-nb)^{(n-2)/2}}{(n-2)^{(n-4)/2}}t\right\}t^2 & (n:\text{even}),\\ 0 & (n:\text{odd})\end{cases}\\
&= \begin{cases}\dfrac{(-1)^{n/2}4n^{(n-2)/2}b^{n/2}}{(n-2)^{(n-4)/2}}t^3 & (n:\text{even}),\\ 0 & (n:\text{odd}),\end{cases}
\end{aligned}$$

which is the claim of Lem. 3 for the case $a=0$ since for even n $(n\geq 6)$,

$$\begin{aligned}
&\frac{\sum_{k=0}^{m_0}(-1)^{n+k}n^k(n-1)^{n-2k-4}(n-2)^ka^{n-2k-4}b^kS_k}{n(n-2)^{n-3}}t^3\\
&= \frac{(-1)^{(3n-4)/2}n^{(n-4)/2}(n-2)^{(n-4)/2}b^{(n-4)/2}\cdot 4n^2(n-2)b^2}{n(n-2)^{n-3}}t^3\\
&= \frac{(-1)^{n/2}4n^{(n-2)/2}b^{n/2}}{(n-2)^{(n-4)/2}}t^3.
\end{aligned}$$

Next, we suppose $a\neq 0$ and let us put

$$\begin{aligned}
&F=(n-1)^2a^3-2n(2n-3)ab,\ G=(n-1)a^2-2nb,\\
&H=(n-1)^2a^2-4n(n-2)b.
\end{aligned}$$

Then, we have

$$
\begin{aligned}
&\frac{2nb^2\bar{x}_{n-3}}{n-2}+\frac{\left\{(n-1)a^2-2nb\right\}\bar{x}_{n-1}}{n}+2ab\bar{x}_{n-2}\\
&=\frac{2nb^2}{n-2}\frac{\left(A^{n-3}-B^{n-3}\right)}{2^{n-3}(n-2)^{n-5}\sqrt{H}}t+\frac{(n-1)a^2-2nb}{n}\frac{\left(A^{n-1}-B^{n-1}\right)}{2^{n-1}(n-2)^{n-3}\sqrt{H}}t\\
&\qquad\qquad+\frac{2ab(A^{n-2}-B^{n-2})}{2^{n-2}(n-2)^{n-4}\sqrt{H}}t\\
&=\frac{8n^2(n-2)b^2(A^{n-3}-B^{n-3})}{2^{n-1}n(n-2)^{n-3}\sqrt{H}}t+\frac{\left\{(n-1)a^2-2nb\right\}(A^{n-1}-B^{n-1})}{2^{n-1}n(n-2)^{n-3}\sqrt{H}}t\\
&\qquad\qquad+\frac{4n(n-2)ab(A^{n-2}-B^{n-2})}{2^{n-1}n(n-2)^{n-3}\sqrt{H}}t\\
&=\frac{8n^2(n-2)b^2+\left\{(n-1)a^2-2nb\right\}A^2+4n(n-2)abA}{2^{n-1}n(n-2)^{n-3}\sqrt{H}}A^{n-3}t\\
&\qquad\qquad-\frac{8n^2(n-2)b^2+\left\{(n-1)a^2-2nb\right\}B^2+4n(n-2)abB}{2^{n-1}n(n-2)^{n-3}\sqrt{H}}B^{n-3}t
\end{aligned}
$$

and

$$
\begin{aligned}
&8n^2(n-2)b^2+\left\{(n-1)a^2-2nb\right\}A^2+4n(n-2)abA\\
&=8n^2(n-2)b^2+\left\{(n-1)a^2-2nb\right\}\left\{2(n-1)^2a^2-4n(n-2)b-2(n-1)a\sqrt{H}\right\}\\
&\qquad\qquad+4n(n-2)ab\left\{-(n-1)a+\sqrt{H}\right\}\\
&=-2\left\{(n-1)^2a^3-2n(2n-3)ab\right\}\sqrt{H}+2\left\{(n-1)a^2-2nb\right\}H\\
&=-2F\sqrt{H}+2GH,
\end{aligned}
$$

$$
\begin{aligned}
&8n^2(n-2)b^2+\left\{(n-1)a^2-2nb\right\}B^2+4n(n-2)abB\\
&=8n^2(n-2)b^2+\left\{(n-1)a^2-2nb\right\}\left\{2(n-1)^2a^2-4n(n-2)b+2(n-1)a\sqrt{H}\right\}\\
&\qquad\qquad+4n(n-2)ab\left\{-(n-1)a-\sqrt{H}\right\}\\
&=2\left\{(n-1)^2a^3-2n(2n-3)ab\right\}\sqrt{H}+2\left\{(n-1)a^2-2nb\right\}H,\\
&=2F\sqrt{H}+2GH.
\end{aligned}
$$

Hence, by putting

$$A^{n-3}=\left\{-(n-1)a+\sqrt{(n-1)^2a^2-4n(n-2)b}\right\}^{n-3}=I+J\sqrt{H},$$

we have

$$
\begin{aligned}
&\frac{2nb^2\bar{x}_{n-3}}{n-2}+\frac{\left\{(n-1)a^2-2nb\right\}\bar{x}_{n-1}}{n}+2ab\bar{x}_{n-2}\\
&=\frac{\left\{-2F\sqrt{H}+2GH\right\}\left(I+J\sqrt{H}\right)-\left\{2F\sqrt{H}+2GH\right\}\left(I-J\sqrt{H}\right)}{2^{n-1}n(n-2)^{n-3}\sqrt{H}}t\\
&=\frac{-FI+GHJ}{2^{n-3}n(n-2)^{n-3}}t.
\end{aligned}
$$

Then, by using Lem. 4, we have

$$\begin{aligned}
I &= \sum_{\ell=0}^{m_0} \binom{n-3}{2\ell} \{-(n-1)a\}^{n-2\ell-3} \left\{(n-1)^2a^2 - 4n(n-2)b\right\}^{\ell} \\
&= \sum_{\ell=0}^{m_0} \binom{n-3}{2\ell} \{-(n-1)a\}^{n-2\ell-3} \left\{\sum_{k=0}^{\ell} \binom{\ell}{k} \left\{(n-1)^2a^2\right\}^{\ell-k} \{-4n(n-2)b\}^{k}\right\} \\
&= \sum_{k=0}^{m_0}\sum_{\ell=k}^{m_0} \binom{n-3}{2\ell}\binom{\ell}{k}(-1)^{n+k+1}2^{2k}n^k(n-1)^{n-2k-3}(n-2)^k a^{n-2k-3}b^k \\
&= \sum_{k=0}^{m_0} 2^{n-2k-4}\left\{\binom{n-k-3}{k} + \binom{n-k-4}{k-1}\right\} \\
&\qquad\qquad\qquad \times (-1)^{n+k+1}2^{2k}n^k(n-1)^{n-2k-3}(n-2)^k a^{n-2k-3}b^k \\
&= \sum_{k=0}^{m_0}(-1)^{n+k+1}2^{n-4}n^k(n-1)^{n-2k-3}(n-2)^k \frac{n-3}{n-k-3}\binom{n-k-3}{k}a^{n-2k-3}b^k
\end{aligned}$$

and

$$\begin{aligned}
J &= \sum_{\ell=0}^{m_0} \binom{n-3}{2\ell+1} \{-(n-1)a\}^{n-2\ell-4} \left\{(n-1)^2a^2 - 4n(n-2)b\right\}^{\ell} \\
&= \sum_{\ell=0}^{m_0} \binom{n-3}{2\ell+1} \{-(n-1)a\}^{n-2\ell-4} \left\{\sum_{k=0}^{\ell} \binom{\ell}{k}\{(n-1)^2a^2\}^{\ell-k} \{-4n(n-2)b\}^{k}\right\} \\
&= \sum_{k=0}^{m_0}\sum_{\ell=k}^{m_0} \binom{n-3}{2\ell+1}\binom{\ell}{k}(-1)^{n+k}2^{2k}n^k(n-1)^{n-2k-4}(n-2)^k a^{n-2k-4}b^k \\
&= \sum_{k=0}^{m_0} 2^{n-2k-4}\binom{n-k-4}{k}(-1)^{n+k}2^{2k}n^k(n-1)^{n-2k-4}(n-2)^k a^{n-2k-4}b^k \\
&= \sum_{k=0}^{m_0}(-1)^{n+k}2^{n-4}n^k(n-1)^{n-2k-4}(n-2)^k\binom{n-k-4}{k}a^{n-2k-4}b^k,
\end{aligned}$$

which implies,

$$\begin{aligned}
&-FI + GHJ \\
&= -\{(n-1)^2a^3 - 2n(2n-3)ab\}I + \{(n-1)^3a^4 - 2n(n-1)(3n-5)a^2b + 8n^2(n-2)b^2\}J \\
&= \sum_{k=0}^{m_0}(-1)^{n+k}2^{n-4}n^k(n-1)^{n-2k-4}(n-2)^k a^{n-2k-4}b^k \\
&\quad \times\Bigg\{(n-1)^3\frac{n-3}{n-k-3}\binom{n-k-3}{k}a^4 - 2n(n-1)(2n-3)\frac{n-3}{n-k-3}\binom{n-k-3}{k}a^2b \\
&\qquad + (n-1)^3\binom{n-k-4}{k}a^4 - 2n(n-1)(3n-5)\binom{n-k-4}{k}a^2b + 8n^2(n-2)\binom{n-k-4}{k}b^2\Bigg\} \\
&= \sum_{k=0}^{m_0}(-1)^{n+k}2^{n-3}n^k(n-1)^{n-2k-4}(n-2)^k a^{n-2k-4}b^k \\
&\quad \times\Bigg\{(n-1)^3\binom{n-k-3}{k}a^4 - \frac{n(n-1)\left\{5n^2-(6k+23)n+10k+24\right\}}{n-k-3}\binom{n-k-3}{k}a^2b \\
&\qquad\qquad\qquad\qquad + 4n^2(n-2)\binom{n-k-4}{k}b^2\Bigg\}
\end{aligned}$$

Therefore, we finally obtain

$$\begin{aligned}\alpha(t) &= \left\{ \frac{2nb^2\bar{x}_{n-3}}{n-2} + \frac{\left\{(n-1)a^2 - 2nb\right\}\bar{x}_{n-1}}{n} + 2ab\bar{x}_{n-2} \right\} t^2 = \frac{-FI+GHJ}{2^{n-3}n(n-2)^{n-3}} t^3 \\ &= \sum_{k=0}^{m_0} \frac{(-1)^{n+k} n^k (n-1)^{n-2k-4}(n-2)^k a^{n-2k-4} b^k S_k}{n(n-2)^{n-3}} t^3,\end{aligned}$$

where

$$\begin{aligned}S_k = (n-1)^3 \binom{n-k-3}{k} a^4 &- \frac{n(n-1)\left\{5n^2-(6k+23)n+10k+24\right\}}{n-k-3} \binom{n-k-3}{k} a^2 b \\ &+ 4n^2(n-2)\binom{n-k-4}{k} b^2,\end{aligned}$$

which completes the proof of Lem. 3. □

Proof of Thm. 2. Theorem 3 has been proved for $n = 3, 4$ and hence let us assume $n \geq 5$. Then, by the definition of $A_{\boldsymbol{c}}(t)_{n-2}$ and equations (6), (7), we have

$$\begin{aligned}\Delta\left(f_{\boldsymbol{c}}(t;x)\right) &= \det A_{\boldsymbol{c}}(t) = n \cdot \det A_{\boldsymbol{c}}(t)_{n-2} \\ &= n \cdot (-1)^{\lceil (n-3)/2 \rceil}(n-2)^{n-3} t^{n-3} \cdot \det \begin{bmatrix} a^{(\boldsymbol{c})}_{n-1,n-1}(t)_{n-2} & a^{(\boldsymbol{c})}_{n-1,n}(t)_{n-2} \\ a^{(\boldsymbol{c})}_{n,n-1}(t)_{n-2} & a^{(\boldsymbol{c})}_{n,n}(t)_{n-2} \end{bmatrix}.\end{aligned}$$

Therefore, by Lem. 3, we have

$$\Delta\left(f_{\boldsymbol{c}}(t;x)\right) = (-1)^{m_1} t^{n-1} \left\{ (n-2)^{n-2}(a^2-4b)t^2 + \gamma_{\boldsymbol{c}} t - n^n b^{n-1} \right\},$$

where

$$\gamma_{\boldsymbol{c}} = \begin{cases} \displaystyle\sum_{k=0}^{m_0} (-1)^{n+k} n^k (n-1)^{n-2k-4}(n-2)^k a^{n-2k-4} b^k S_k & (a \neq 0, \text{ or } a = 0,\ n : \text{even}), \\ 0 & (a = 0,\ n : \text{odd}), \end{cases}$$

which completes the proof. □

Now we are ready to prove Cor. 1.

Proof of Cor. 1. Put $Q(t) = (n-2)^{n-2}(a^2-4b)t^2 + \gamma_{\boldsymbol{c}} t - n^n b^{n-1}$ and suppose $b = (n-1)^2 a^2/4n(n-2)$. Then, by equation (7) and Lem. 3, we have

$$\begin{aligned}
&Q(t) \\
&= n(n-2)^{n-3}\left\{\frac{(n-2)(a^2-4b)}{n}t^2 + \left\{\frac{2nb^2\bar{x}_{n-3}}{n-2} + \frac{\left\{(n-1)a^2-2nb\right\}\bar{x}_{n-1}}{n} + 2ab\bar{x}_{n-2}\right\}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - \frac{n^{n-1}b^{n-1}}{(n-2)^{n-3}}\right\} \\
&= (n-2)^{n-2}\left\{a^2 - \frac{(n-1)^2a^2}{n(n-2)}\right\}t^2 \\
&\qquad + n(n-2)^{n-3}\left\{\frac{2nb^2\bar{x}_{n-3}}{n-2} + \frac{\left\{(n-1)a^2-2nb\right\}\bar{x}_{n-1}}{n} + 2ab\bar{x}_{n-2}\right\} - n^n b^{n-1} \\
&= -\frac{(n-2)^{n-3}a^2}{n}t^2 \\
&\qquad + n(n-2)^{n-3}\left\{\frac{2nb^2\bar{x}_{n-3}}{n-2} + \frac{\left\{(n-1)a^2-2nb\right\}\bar{x}_{n-1}}{n} + 2ab\bar{x}_{n-2}\right\} - n^n b^{n-1}.
\end{aligned}$$

Here, by equation (5), we have

$$\begin{aligned}
&n(n-2)^{n-3}\left\{\frac{2nb^2\bar{x}_{n-3}}{n-2} + \frac{\left\{(n-1)a^2-2nb\right\}\bar{x}_{n-1}}{n} + 2ab\bar{x}_{n-2}\right\} \\
&= n(n-2)^{n-3}\left\{\frac{2n}{n-2}\cdot\frac{(n-1)^4a^4}{16n^2(n-2)^2}\cdot\frac{(-1)^{n-3}(n-3)\{-(n-1)a\}^{n-4}}{2^{n-4}\left\{-(n-2)\right\}^{n-5}}t\right. \\
&\qquad\qquad + \left\{\frac{(n-1)a^2}{n} - \frac{(n-1)^2a^2}{2n(n-2)}\right\}\cdot\frac{(-1)^{n-1}(n-1)\{-(n-1)a\}^{n-2}}{2^{n-2}\left\{-(n-2)\right\}^{n-3}}t \\
&\qquad\qquad\qquad\qquad \left. + \frac{(n-1)^2a^3}{2n(n-2)}\cdot\frac{(-1)^{n-2}(n-2)\{-(n-1)a\}^{n-3}}{2^{n-3}\left\{-(n-2)\right\}^{n-4}}t\right\} \\
&= n(n-2)^{n-3}\left\{\frac{(-1)^n(n-1)^n(n-3)a^n}{2^{n-1}n(n-2)^{n-2}}t + \frac{(-1)^n(n-1)^n(n-3)a^n}{2^{n-1}n(n-2)^{n-2}}t - \frac{(-1)^n(n-1)^{n-1}a^n}{2^{n-2}n(n-2)^{n-4}}t\right\} \\
&= -\frac{(-1)^n(n-1)^{n-1}a^n}{2^{n-2}(n-2)}t,
\end{aligned}$$

which implies

$$\begin{aligned}
Q(t) &= -\frac{(n-2)^{n-3}a^2}{n}t^2 - \frac{(-1)^n(n-1)^{n-1}a^n}{2^{n-2}(n-2)}t - n^n b^{n-1}, \\
\Delta\left(Q(t)\right) &= \left\{-\frac{(-1)^n(n-1)^{n-1}a^n}{2^{n-2}(n-2)}\right\}^2 - 4\cdot\left\{-\frac{(n-2)^{n-3}a^2}{n}\right\}\cdot\left(-n^n b^{n-1}\right) \\
&= \frac{(n-1)^{2n-2}a^{2n}}{2^{2n-4}(n-2)^2} - \frac{4(n-2)^{n-3}a^2}{n}\cdot n^n\cdot\left\{\frac{(n-1)^2a^2}{4n(n-2)}\right\}^{n-1} = 0
\end{aligned}$$

and hence, by completing the square, we have

$$Q(t) = -\frac{(n-2)^{n-3}a^2}{n}\left\{t + \frac{(-1)^n n(n-1)^{n-1}a^{n-2}}{2^{n-1}(n-2)^{n-2}}\right\}^2.$$

Thus, if n is even, we have $\alpha_{\boldsymbol{c}} = 0$ and hence by Thm. 1, we have $N_{f_{\boldsymbol{c}}(t;x)} = 0$ for any positive real number t since $x^2 + ax + b$ has no real root in this case. Then, by considering the graph of the function $y = x^n + t(x^2 + ax + b)$ $(t > 0)$, we can conclude that $N_{f_{\boldsymbol{c}}(t;x)} = 0$ whenever $(n-1)^2 a^2 / 4n(n-2) \leq b$. $\square$

Remark 1. *It was pointed out to us by J. Gutierrez that the proof of the Cor. 1, it is elementary and all the machinery of Thm. 2 is not needed. However, we decided to present it here with the intention that perhaps such techniques can be generalized to polynomials* $f(x) = x^n + t \cdot g(x)$*, deg* $g \geq 3$.

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