SIX LINE CONFIGURATIONS AND STRING DUALITIES

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Abstract. We study the family of K3 surfaces of Picard rank sixteen associated with the double cover of the projective plane branched along the union of six lines, and the family of its Van Geemen-Sarti partners, i.e., K3 surfaces with special Nikulin involutions, such that quotienting by the involution and blowing up recovers the former. We prove that the family of Van Geemen-Sarti partners is a four-parameter family of K3 surfaces with $H \oplus E_7(-1) \oplus E_7(-1)$ lattice polarization. We describe explicit Weierstrass models on both families using even modular forms on the bounded symmetric domain of type IV. We also show that our construction provides a geometric interpretation, called geometric two-isogeny, for the F-theory/heterotic string duality in eight dimensions. As a result, we obtain novel F-theory models, dual to non-geometric heterotic string compactifications in eight dimensions with two non-vanishing Wilson line parameters.

1. Introduction

In this article, we consider configurations of six lines in general position on the projective plane. The double cover of the plane branched along their union is a K3 surface after resolving only ordinary double points. The moduli space of such K3 surfaces was described in [51]. Kloosterman classified all possible types of elliptic fibrations with a section on them in [36]. In [17], the authors consider K3 surfaces which are double covers of a blow-up of $\mathbb{P}^2$, branched along rational curves. They classified the elliptic fibrations on such surfaces and their van Geemen-Sarti involutions.

The assumption that the six lines are in general position implies that the Picard rank of the resulting K3 surface is sixteen. In the special case when the six lines are tangent to a conic, the Picard rank is, generically, seventeen and one obtains as K3 surface a Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ of the Jacobian $\mathrm{Jac}(\mathcal{C})$ of a generic genus-two curve $\mathcal{C}$. There is then, as shown in [11, 12, 38, 46], a closely related K3 surface, called the *Shioda-Inose surface* $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$, which carries a *Nikulin involution*, i.e., an automorphism of order two preserving the holomorphic two-form, such that quotienting by this involution and blowing up the fixed points recovers the Kummer surface. The Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ carries a canonical lattice polarization of type $H \oplus E_8(-1) \oplus E_7(-1)$ and is part of *a geometric two-isogeny*:

$$
\begin{tikzcd}
\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}) \arrow[r, dashed, shift right] & \mathrm{SI}(\mathrm{Jac}\,\mathcal{C}) \arrow[l, dashed, shift right]
\end{tikzcd}
\tag{1.1}
$$

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establishing a one-to-one correspondence between two different types of surfaces with the same Hodge-theoretic data: principally polarized abelian surfaces and algebraic K3 surfaces polarized the special lattice $H \oplus E_8(-1) \oplus E_7(-1)$. The key geometric ingredient in this construction is a normal form equation for an elliptically fibered K3 surface whose periods determine a point τ in the Siegel upper-half space $\mathbb{H}_2$, with the coefficients in the equation being Siegel modular forms. The normal form equation, as well as the two-isogeny construction, are due in different forms to Kumar [38] and to Clingher and Doran [12].

Compactifications of the type IIB string in which the axio-dilaton field varies over a base are generically referred to as F-theory. Eight-dimensional compactifications correspond to Jacobian elliptic fibrations on K3 surfaces. It is well-known that the moduli space of these F-theory models is isomorphic to the moduli space of the heterotic string compactified on an elliptic curve together with a principal G-bundle where G is the gauge group of the heterotic string with gauge algebra either $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_8$ or $\mathfrak{so}(32)$ [56, 65]. This is the basic form of the so called *F-theory/heterotic string duality* in eight dimensions. Geometric two-isogeny provides a more refined and geometric understanding for this string duality on a natural sub-space of the full eighteen dimensional moduli space [10, 46, 53]: by taking the K3 surface to be the Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$, the F-theory/heterotic string duality is manifested as the aforementioned geometric two-isogeny. In fact, the period lattice of $\mathrm{Jac}(\mathcal{C})$ describes a model dual to the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic string, with an unbroken gauge algebra of $\mathfrak{e}_8 \oplus \mathfrak{e}_7$ ensuring that a single Wilson line expectation value is non-zero; a similar result was established for the $\mathfrak{so}(32)$ heterotic string as well. By a result of Vinberg [68] and its interpretation in string theory in [46], the function field of the Narain moduli space of the heterotic compactifications turns out to be the ring of Siegel modular forms of *even weight.*

In this article, we extend the notion of geometric two-isogeny and its application to F-theory/heterotic string duality with K3 Picard rank sixteen. In this context, Kummer surfaces are replaced by what we shall refer to as *double sextic surfaces* - K3 surfaces $\mathcal{Y}$ obtained as minimal resolutions of double covers of the projective plane branched along a configuration of six distinct lines. The Shioda-Inose surfaces from above are then replaced, as shown by Clingher and Doran in [11] by K3 surfaces $\mathcal{X}$ polarized by the rank-sixteen lattice $H \oplus E_7(-1) \oplus E_7(-1)$. Similarly to the Shioda-Inose case, each of these K3 surfaces $\mathcal{X}$ carries a special Nikulin involution, $\jmath_{\mathcal{X}}$ called *Van Geemen-Sarti involution.* When quotienting by the involution $\jmath_{\mathcal{X}}$ and blowing up the fixed locus, one recovers the corresponding double-sextic surface $\mathcal{Y}$ together with a rational double cover map $\Phi\colon \mathcal{X} \dashrightarrow \mathcal{Y}$. However, the Van Geemen-Sarti involutions $\jmath_{\mathcal{X}}$ no longer determine Shioda-Inose structures. Instead, they appear as fiber-wise translation by two-torsion in a suitable Jacobian elliptic fibration $\pi^{\mathcal{X}}_{\text{alt}}$. The geometric two-isogeny picture is then given by the diagram below:

$$
\begin{tikzcd}
\mathcal{X} \arrow[loop left, "\jmath_{\mathcal{X}}"] \arrow[rr, dashed, bend right, "\hat{\Phi}"'] \arrow[dr, "\pi^{\mathcal{X}}_{\text{alt}}"'] & & \mathcal{Y} \arrow[loop right, "\jmath_{\mathcal{Y}}"'] \arrow[ll, dashed, bend right, "\Phi"'] \arrow[dl, "\pi^{\mathcal{Y}}_{\text{alt}}"] \\
& \mathbb{P}^1 &
\end{tikzcd}
\tag{1.2}
$$

We shall refer to the K3 surfaces $\mathcal{X}$ as the *Van Geemen-Sarti partners* of the double sextic surface $\mathcal{Y}$. From a physics point of view, the period lattice of the Van Geemen-Sarti partners describes a model dual to the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic string, with an unbroken gauge algebra $\mathfrak{e}_7 \oplus \mathfrak{e}_7$ ensuring that two Wilson line expectation values are non-zero. A similar result holds for the $\mathfrak{so}(32)$ heterotic string with an unbroken gauge algebra $\mathfrak{so}(24) \oplus \mathfrak{su}(2)^{\oplus 2}$. The function field of the Narain moduli space of these heterotic theories turns out to be the ring of modular forms of *even characteristic* on the bounded symmetric domain of type IV introduced by Matsumoto et al. [48].

This article is structured as follows: in Section 2 we review the work of Dolgachev and Ortland [18] and the moduli space associated with six-line configurations in the projective plane. We define new invariants of six-line configurations that generalize the Igusa invariants of binary sextics. We construct the function field of the moduli space explicitly, by determining a complete set of generators for the ring of modular forms of even characteristic. In Section 3 we construct explicit Weierstrass models for three Jacobian elliptic fibrations on the family of double-sextic surfaces $\mathcal{Y}$. One of them, which we call the *alternate fibration*, is of particular importance: the coefficients in its Weierstrass equation are the generators of the ring of modular forms derived before. In Section 4 we construct the family of Van Geemen-Sarti partners $\mathcal{X}$ of the double-sextic surfaces $\mathcal{Y}$ polarized by the lattice $H \oplus E_7(-1) \oplus E_7(-1)$. There are four non-isomorphic elliptic fibrations on $\mathcal{X}$; three will be important for the considerations in this article, and Weierstrass models will be constructed for them. Using the Van Geemen-Sarti involution, we will determine the coefficients of these Weierstrass models in terms of the modular forms found in Section 2. In Section 5 we discuss the specialization of six-line configurations tangent to a common conic and the associated K3 surfaces. We find perfect agreement in this case with the results in [11, 43, 46]. In Section 6 we discuss the construction of F-theory models, dual to the heterotic string with *two* non-vanishing Wilson line parameters that naturally follows from our geometric construction. We determine the function field of the Narain moduli space of these heterotic theories explicitly in terms of the modular forms on the bounded symmetric domain of type IV of even characteristic.

2. Invariants of six-line configurations in the projective plane

The Plücker embedding algebraically embeds the Grassmannian $\mathrm{Gr}(k, n; \mathbb{C})$ of all k-dimensional sub-spaces of an n-dimensional complex vector space V as a sub-variety of the projective space $\mathbb{P}(\wedge^k V)$. The homogeneous coordinates of the image under the Plücker embedding, with respect to the natural basis of the exterior space $\wedge^k V$

relative to a chosen basis in V, are called *Plücker coordinates*. The image of the Plücker embedding is an intersection of a number of quadrics defined by the so called *Plücker relations.*

We consider the situation $k = 3$ and $n = 6$ with $\dim \mathrm{Gr}(k, n; \mathbb{C}) = 9$. We start with the geometric setup of an *ordered* configuration of six lines in general position in the projective plane $\mathbb{P}^2$. We write each line in the form $\ell_i : a_i z_1 + b_i z_2 + c_i z_3 = 0$ for $i = 1, \dots, 6$ with $[z_1 : z_2 : z_3] \in \mathbb{P}^2$. The coefficients of the lines are assembled in vectors $\mathbf{v}_i = \langle a_i, b_i, c_i\rangle^t$ and form a matrix $\mathbf{A} \in \mathrm{Mat}(3, 6; \mathbb{C})$ given by $\mathbf{A} = [\mathbf{v}_1| \cdots |\mathbf{v}_6]$. Let $\mathbf{A}_{ijk} = [\mathbf{v}_i|\mathbf{v}_j|\mathbf{v}_k]$ and $D_{ijk} = \det \mathbf{A}_{ijk}$ be the Plücker coordinates derived from $\mathbf{A} \in \mathrm{Mat}(3, 6; \mathbb{C})$ considered as an element of the Grassmannian $\mathrm{Gr}(3, 6; \mathbb{C})$.

We consider the following cases of configurations of six lines in $\mathbb{P}^2$:

Definition 2.1. *We consider configurations of six lines in $\mathbb{P}^2$ that*

(0) *contain six lines in general position,*
(1) *are tangent to a common conic,*
(2) *contain three lines which are coincident in one point,*
(3) *contain one line which is coincident with two different pairs of lines in two different points,*
(4) *contain three lines pairwise coincident in three different points, and each of the three remaining lines is coincident in one intersection point,*
(5) *are combinations of case (1) and cases (2) through (4),*
(6a) *contain four lines which intersect in one point,*
(6b) *contain one double line.*

Configurations that include cases (0) through (6a) and (6b) are called *semi-stable* configurations. On configurations of six lines we have a right action of $(\mathbb{C}^*)^6$ given by rescaling each line separately, and the obvious left action of $\mathrm{GL}_3(\mathbb{C})$ by acting on $[z_1 : z_2 : z_3] \in \mathbb{P}^2$. Next, we want to describe the isomorphism classes of such configurations of six lines. We define the so called degree-one Dolgachev-Ortland coordinates [18] for configurations of six lines in $\mathbb{P}^2$ to be given by

$$\begin{array}{rclcrcl} t_1 & = & D_{135}D_{246}, & & t_2 & = & D_{145}D_{236}, \\ t_3 & = & D_{146}D_{235}, & & t_4 & = & D_{136}D_{245}, \\ t_5 & = & D_{125}D_{346}, & & t_6 & = & D_{126}D_{345}, \\ t_7 & = & D_{134}D_{256}, & & t_8 & = & D_{124}D_{356}, \\ t_9 & = & D_{156}D_{234}, & & t_{10} & = & D_{123}D_{456}. \end{array} \tag{2.1}$$

We have the following:

Lemma 2.2. *The degree-one coordinates $t_1, \dots, t_{10}$ satisfy the relations*

$$\begin{array}{lll} t_1 - t_2 - t_5 - t_9, & t_1 - t_2 - t_6 - t_7, & t_1 - t_3 - t_5 - t_{10}, \\ t_1 - t_3 - t_6 - t_8, & t_1 - t_4 - t_7 - t_{10}, & t_1 - t_4 - t_8 - t_9, \\ t_2 - t_3 + t_7 - t_8, & t_2 - t_3 + t_9 - t_{10}, & t_2 - t_4 + t_5 - t_8, \\ t_2 - t_4 + t_6 - t_{10}, & t_3 - t_4 + t_5 - t_7, & t_3 - t_4 + t_6 - t_9, \\ t_5 - t_6 - t_7 + t_9, & t_5 - t_6 - t_8 + t_{10}, & t_7 - t_8 - t_9 + t_{10}. \end{array} \tag{2.2}$$

In particular, only five relations among the fifteen relations are linearly independent.

Proof. The proof follows by explicit computation for any matrix $\mathbf{A} \in \mathrm{Mat}(3,6;\mathbb{C})$. □

One also introduces the degree-two Dolgachev-Ortland coordinate given by

$$R = D_{123}D_{145}D_{246}D_{356} - D_{124}D_{135}D_{236}D_{456}. \tag{2.3}$$

We have the following:

Lemma 2.3. *The degree-two coordinate R satisfies*

$$R^2 = \frac{1}{12}\left(\Big(\sum_{i=1}^{10} t_i^2\Big)^2 - 4\sum_{i=1}^{10} t_i^4\right). \tag{2.4}$$

Proof. The proof follows by explicit computation for any matrix $\mathbf{A} \in \mathrm{Mat}(3,6;\mathbb{C})$. □

The different strata in the moduli space can now be characterized as follows:

Lemma 2.4. *In Definition 2.1 we have the following:*

(0) $\Leftrightarrow$ *no element of $(t_i)_{i=1}^{10}$ vanishes and $R \neq 0$,*
(1) $\Leftrightarrow$ *no element of $(t_i)_{i=1}^{10}$ vanishes and $R = 0$,*
(2) $\Leftrightarrow$ *exactly one element of $(t_i)_{i=1}^{10}$ vanishes,*
(3) $\Leftrightarrow$ *exactly two elements of $(t_i)_{i=1}^{10}$ vanish,*
(4) $\Leftrightarrow$ *exactly three elements of $(t_i)_{i=1}^{10}$ vanish,*
(5) $\Leftrightarrow$ *up to three elements of $(t_i)_{i=1}^{10}$ vanish and $R = 0$,*
(6) $\Leftrightarrow$ *exactly four elements of $(t_i)_{i=1}^{10}$ vanish and $R = 0$.*

Proof. Configurations of six lines no three of which are concurrent have four homogeneous moduli which we denote by a, b, c, d. A general matrix $\mathbf{A} \in \mathrm{Mat}(3,6;\mathbb{C})$ is written in terms of only a, b, c, d using a $\mathrm{GL}_3(\mathbb{C})$ transformation. The lines are then in the form of Equations (3.2). We discuss the details in Section 3.1. Equations (3.4) determine the Dolgachev-Ortland coordinates in terms of these moduli. We can easily check necessary and sufficient conditions for cases (1) through (6). It follows from Equation (3.4) and [14, Prop. 5.13] that $R = 0$ in Equation (2.3) if and only if the six lines in general position are tangent to a common conic. □

We have the following:

Lemma 2.5. *For a configuration of six lines in $\mathbb{P}^2$ the point*

$$[t_1 : \cdots : t_{10} : R] \in \mathbb{P}(1, \ldots, 1, 2) \tag{2.5}$$

in complex weighted projective space, is well-defined and invariant under the right action of $(\mathbb{C}^*)^6$ and the left action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbf{A}$.

Proof. The point in weighted projective space is well-defined because of Lemma 2.4. The invariance under the right action of $(\mathbb{C}^*)^6$ on $\mathbf{A}$ is immediate. The invariance under the left action of $\mathrm{GL}_3(\mathbb{C})$ follows from a computation showing that the coordinates t_i for $1 \leq i \leq 10$ and R rescale by the determinant with weight two and four, respectively, and the point in weighted projective space remains invariant. □

For more details we refer to [18, 66]. The following is a corollary of Lemma 2.5:

Corollary 2.6 ([18]). *The moduli space of configurations of six lines in $\mathbb{P}^2$ is isomorphic to the algebraic variety in $\mathbb{P}(1,\dots,1,2)$ with the coordinates $[t_1 : \cdots : t_{10} : R]$ given by Equations (2.2) and (2.4), and $R \neq 0$ and $t_i \neq 0$ for all $i \in \{1,\dots,10\}$.*

We define the moduli space $\mathfrak{M}(2)^+$ to be the moduli space of *ordered* configurations of six lines in $\mathbb{P}^2$ that fall into cases (0) through (5) in Definition 2.1, i.e.,

$$\mathfrak{M}(2)^+ = \left\{ [t_1 : \cdots : t_{10} : R] \;\middle|\; \begin{array}{l} t_i = 0 \text{ for at most three } i \in \{1,\dots,10\}, \\ \text{Eqns. (2.2) and (2.4) hold.} \end{array} \right\}. \tag{2.6}$$

The notation $\mathfrak{M}(2)^+$ indicates (i) the existence of a level-two structure obtained by splitting up six indices into two pairs of three, and (ii) the fact that we include all cases (1) through (5) in Definition 2.1 in addition to case (0).

For a given ordered configuration of lines $\{\ell_1,\dots,\ell_6\}$ in general position, let us fix six out of fifteen points of intersection, namely the points

$$\begin{array}{lll} p_1 = \ell_2 \cap \ell_3, & p_2 = \ell_1 \cap \ell_3, & p_3 = \ell_1 \cap \ell_2, \\ p_4 = \ell_5 \cap \ell_6, & p_5 = \ell_4 \cap \ell_6, & p_6 = \ell_4 \cap \ell_5. \end{array} \tag{2.7}$$

Given any non-singular conic $C \subset \mathbb{P}^2$, we define the dual of a point $p_i \notin C$ to be the line ℓ'_i that joins the two points of C on the two tangent lines of C passing through p_i; if $p_i \in C$ we define ℓ'_i to be the tangent line of C at p_i. Changing the conic C to another non-singular conic C' in this construction simply transforms the lines ℓ'_i by a projective automorphism of $\mathbb{P}^2$. We then say that the two configurations $\{\ell'_1,\dots,\ell'_6\}$ and $\{\ell_1,\dots,\ell_6\}$ are *in association*. It was proved in [66] that $\{\ell'_1,\dots,\ell'_6\}$ and $\{\ell_1,\dots,\ell_6\}$ are associated if and only if their respective matrices $\mathbf{A}'$ and $\mathbf{A}$ satisfy $\mathbf{A}' \cdot D \cdot \mathbf{A}^t = 0$ for some diagonal matrix D with $\det D \neq 0$.

Mapping an ordered configuration of six lines to an associated ordered configuration defines an involution $\imath$ on $\mathfrak{M}^+(2)$ with a fixed point set that consists of configurations of six lines tangent to a common conic, and in terms of the Dolgachev-Ortland coordinates it is given by

$$\imath : [\, t_1 : \cdots : t_{10} : R \,] \to [\, t_1 : \cdots : t_{10} : -R \,]. \tag{2.8}$$

We define a four-dimensional sub-space $\mathfrak{M}(2)$ of $\mathbb{P}^9$ by setting

$$\mathfrak{M}(2) = \left\{ [t_1 : \cdots : t_{10}] \in \mathbb{P}^9 \;\middle|\; \begin{array}{l} t_i = 0 \text{ for at most three } i \in \{1,\dots,10\}, \\ \text{and Eqns. (2.2) hold.} \end{array} \right\}. \tag{2.9}$$

We also set

$$\overline{\mathfrak{M}(2)} = \left\{ [t_1 : \cdots : t_{10}] \in \mathbb{P}^9 \;\middle|\; \text{Eqns. (2.2) hold.} \right\}. \tag{2.10}$$

Notice that, apart from the six-line configurations listed in Definition 2.1, there are more degenerate configurations: there are configurations such that exactly six elements of $(t_i)_{i=1}^{10}$ vanish; there are also configurations such that exactly four elements of $(t_i)_{i=1}^{10}$ vanish, $R \neq 0$, and all non-vanishing t_i's equal ± 1. Since $\mathfrak{M}(2)$ is a four-dimensional linear sub-space of $\mathbb{P}^9$, it is easy to show [48, Sec. 3.2] that $\overline{\mathfrak{M}(2)}$ is in fact isomorphic to $\mathbb{P}^4$.

We take the map pr to be the projection from $\mathbb{P}(1,\dots,1,2)\backslash\{[0:\dots:0:1]\} \to \mathbb{P}^9$ given by $[t_1:\dots:t_{10}:R] \mapsto [t_1:\dots:t_{10}]$. We have the following:

Lemma 2.7. *We have* $\mathrm{pr} = \mathrm{pr} \circ \imath : \mathfrak{M}(2)^+ \to \mathfrak{M}(2)$ *and* $\mathrm{pr}(\mathfrak{M}(2)^+) \cong \mathfrak{M}(2)$.

2.1. **The modular description.** The moduli spaces $\mathfrak{M}(2)$ and $\mathfrak{M}(2)^+$ have modular descriptions based on the seminal work in [48]. By $\mathbf{H}_2$ we denote the set of all complex two-by-two matrices ϖ over $\mathbb{C}$ such that the hermitian matrix $(\varpi - \varpi^\dagger)/(2i)$ is positive definite, i.e.,

$$\mathbf{H}_2 = \left\{ \left(\begin{array}{cc} \tau_1 & z_1 \\ z_2 & \tau_2 \end{array} \right) \in \mathrm{Mat}(2,2;\mathbb{C}) \;\middle|\; 4\,\mathrm{Im}\tau_1\,\mathrm{Im}\tau_2 > |z_1 - \bar{z}_2|^2,\ \mathrm{Im}\tau_2 > 0 \right\}, \tag{2.11}$$

and the modular group $\Gamma \subset \mathrm{U}(2,2)$ given by

$$\Gamma = \left\{ G \in \mathrm{GL}_4\left(\mathbb{Z}[i]\right) \;\middle|\; G^\dagger \cdot \left(\begin{array}{cc} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{array} \right) \cdot G = \left(\begin{array}{cc} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{array} \right) \right\}. \tag{2.12}$$

The modular group acts on $\varpi \in \mathbf{H}_2$ by

$$\forall\, G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma : \quad G \cdot \varpi = (A \cdot \varpi + B)(C \cdot \varpi + D)^{-1}.$$

It was shown in [48, Prop. 1.5.1] that Γ is generated by the five elements G_1, G_2, G_3, G_4, G_5 given by

$$\begin{pmatrix} i & & & \\ & 1 & & \\ & & i & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}, \tag{2.13}$$

with determinants $\det(G_1) = -1$ and $\det(G_k) = 1$ for $k = 2, \dots, 5$. We also introduce the principal modular sub-group of *complex level* $1+i$ (over the Gaussian integers) given by

$$\Gamma(1+i) = \left\{ G \in \Gamma \;\middle|\; G \equiv \mathbb{I}_4 \mod 1+i \right\}. \tag{2.14}$$

There is an additional involution $\mathcal{T}$ acting on elements of $\mathbf{H}_2$ by transposition, i.e., $\varpi \mapsto \mathcal{T} \cdot \varpi = \varpi^t$, yielding extended groups obtained from the semi-direct products

$$\Gamma_{\mathcal{T}} = \Gamma \rtimes \langle \mathcal{T} \rangle, \qquad \Gamma_{\mathcal{T}}(1+i) = \Gamma(1+i) \rtimes \langle \mathcal{T} \rangle, \tag{2.15}$$

where $\langle \mathcal{T} \rangle$ is the sub-group generated by $\mathcal{T}$. We will always write elements $g \in \Gamma_{\mathcal{T}}$ in the form $g = G\,\mathcal{T}^n$ with $G \in \Gamma$ and $n \in \{0, 1\}$. A *modular form* f *of weight* $2k$ *relative to a finite-index sub-group* $\Gamma' \subset \Gamma_{\mathcal{T}}$ *with character* χ_f is a holomorphic function on $\mathbf{H}_2$ such that

$$\forall\, \varpi \in \mathbf{H}_2,\ \forall\, g = G\mathcal{T}^n \in \Gamma' : \ f\big(g \cdot \varpi\big) = \chi_f(g)\ \det(C\varpi + D)^{2k}\ f(\varpi). \tag{2.16}$$

There is a well-known isomorphism $\Gamma/\Gamma(1+i) \cong \mathrm{S}_6$ – since both groups are in fact isomorphic to $\mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z})$ – where S_6 is the permutation group of six elements. By S_G we denote the image of $G \in \Gamma$ under the natural quotient map $\Gamma \to \mathrm{S}_6$ and by $\mathrm{sign}(S_G)$ the sign of this permutation S_G. The following was proven in [48]:

Theorem 2.8 (Props. 3.1.1, 3.1.3, 3.1.5 in [48])**.**

(1) *There are ten theta functions $\theta_i^2(\varpi)$ for $1 \le i \le 10$ which are non-zero modular forms of weight two relative to $\Gamma_{\mathcal{T}}(1+i)$ and for each $g = G\,\mathcal{T}^n \in \Gamma_{\mathcal{T}}(1+i)$ with $n \in \{0,1\}$ the modular forms $\theta_i^2(\varpi)$ transform with $\chi_{\theta_i}(g) = \det(G)$.*

(2) *Any five of the ten functions $\theta_i^2(\varpi)$ for $1 \le i \le 10$ generate the ring of modular forms of level $1+i$ and character $\chi(g) = \det(G)$ for all $g \in \Gamma_{\mathcal{T}}(1+i)$.*

(3) *There is a unique function $\Theta(\varpi)$ which is a non-zero modular form of weight four relative to $\Gamma_{\mathcal{T}}$ such that for each $g = G\mathcal{T}^n \in \Gamma_{\mathcal{T}}$ with $n \in \{0,1\}$ the modular form $\Theta(\varpi)$ transforms with character $\chi_\Theta(g) = (-1)^n \det(G)\,\mathrm{sign}(S_G)$ and satisfies*

$$\Theta(\varpi)^2 = 2^{-6} \cdot 3^5 \cdot 5^2 \left(\sum_{i=1}^{10} \theta_i(\varpi)^2 - 4 \sum_{i=1}^{10} \theta_i(\varpi)^4 \right). \tag{2.17}$$

In the interest of keeping this section short, we do not give explicit formulas for $\theta_i^2(\varpi)$ with $1 \le i \le 10$. However, just as there are simple sum formulas for theta functions of even and odd characteristic in genus two and genus one, the same holds for the theta functions $\theta_i^2(\varpi)$ in Theorem 2.8: they are simply theta functions of complex characteristic. All quadratic relations among the even theta functions $\theta_i^2(\varpi)$ for $1 \le i \le 10$ can then be derived explicitly. We refer to [48, Sec. 2] for details.

Remark 2.9. *The space $\mathbf{H}_2$ is a generalization of the Siegel upper-half space $\mathbb{H}_2$. In fact, elements invariant under the involution $\mathcal{T}$ are precisely the two-by-two symmetric matrices over $\mathbb{C}$ whose imaginary part is positive definite, i.e.,*

$$\mathbb{H}_2 = \left\{ \varpi \in \mathbf{H}_2 \,\middle|\, \varpi^t = \varpi \right\}. \tag{2.18}$$

It was proven in [48, Lemma 2.1.1(vi)] *that for $\varpi = \tau \in \mathbb{H}_2$ we have $\theta_i(\varpi) = \vartheta_i(\tau)^2$ where $\vartheta_i(\tau)$ for $1 \le i \le 10$ are the even theta functions of genus two. We provide a geometric cross-check for (the squares of) these reduction formulas in Proposition 5.6.*

The following describes the action of the *full* modular group on the theta functions:

Lemma 2.10. *The action of the generators $\mathcal{T}, G_1, \ldots, G_5 \in \Gamma_{\mathcal{T}}$ in Equation (2.13) on $\theta_i(\varpi)$ with $1 \le i \le 10$ and $\rho = -\det(\varpi)$ is given in the following table:*

(2.19)

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
$\mathcal{T}$	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
$G_1^{\pm1}$	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	$-\theta_{10}$
$G_2^{\pm1}$	θ_1	θ_4	θ_3	θ_2	θ_8	θ_{10}	θ_7	θ_5	θ_9	θ_6
$G_3^{\pm1}$	θ_1	θ_2	θ_4	θ_3	θ_7	θ_9	θ_5	θ_8	θ_6	θ_{10}
$G_4^{\pm1}$	θ_3	θ_4	θ_1	θ_2	$\pm i\,\theta_5$	$\pm i\,\theta_6$	θ_9	$\pm i\,\theta_8$	θ_7	$\pm i\,\theta_{10}$
$G_5^{\pm1}$	$\rho^{\pm1}\theta_1$	$\rho^{\pm1}\theta_8$	$\rho^{\pm1}\theta_5$	$\rho^{\pm1}\theta_7$	$\rho^{\pm1}\theta_3$	$\rho^{\pm1}\theta_9$	$\rho^{\pm1}\theta_4$	$\rho^{\pm1}\theta_2$	$\rho^{\pm1}\theta_6$	$\rho^{\pm1}\theta_{10}$

Proof. The proof follows from an explicit computation applying the formulas in Lemmas 2.1.1(ii) and Lemma 2.1.2(viii)-(x) in [48]. $\square$

Lemma 2.11. *Under the action $\varpi \mapsto \mathcal{T} \cdot \varpi = \varpi^t$ we have*

$$\Big(\theta_1(\varpi), \dots, \theta_{10}(\varpi), \Theta(\varpi)\Big) \;\mapsto\; \Big(\theta_1(\varpi), \dots, \theta_{10}(\varpi), -\Theta(\varpi)\Big). \tag{2.20}$$

Proof. The transformation for $\Theta(\varpi)$ was proven in [48, Cor. 3.1.4]. $\square$

Lemma 2.12. *Under the action $\varpi \mapsto M_i \cdot G_1 \cdot M_i^{-1} \cdot \varpi$ we have*

$$\Big[\theta_1(\varpi) : \cdots : \theta_{10}(\varpi)\Big] \mapsto \Big[(-1)^{\delta_{i,1}}\theta_1(\varpi) : \cdots : (-1)^{\delta_{i,10}}\theta_{10}(\varpi)\Big], \tag{2.21}$$

where $M_i \in \Gamma$ with $\det(M_i) = 1$ and $1 \leq i \leq 10$, $\delta_{\mu,\nu}$ is the Kronecker delta function, and the matrices M_i are given in the following table:

(2.22)

i	M_i	$[(-1)^{\delta_{i,1}}, \dots, (-1)^{\delta_{i,10}}]$
1	$G_4G_3G_5G_4G_3G_2$	$[-1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$
2	$G_2G_5G_4G_3G_2$	$[1, -1, 1, 1, 1, 1, 1, 1, 1, 1]$
3	$G_3G_5G_4G_3G_2$	$[1, 1, -1, 1, 1, 1, 1, 1, 1, 1]$
4	$G_5G_4G_3G_2$	$[1, 1, 1, -1, 1, 1, 1, 1, 1, 1]$
5	$G_3G_4G_3G_2$	$[1, 1, 1, 1, -1, 1, 1, 1, 1, 1]$
6	G_2	$[1, 1, 1, 1, 1, -1, 1, 1, 1, 1]$
7	$G_4G_3G_2$	$[1, 1, 1, 1, 1, 1, -1, 1, 1, 1]$
8	$G_2G_3G_4G_3G_2$	$[1, 1, 1, 1, 1, 1, 1, -1, 1, 1]$
9	G_3G_2	$[1, 1, 1, 1, 1, 1, 1, 1, -1, 1]$
10	$\mathbb{I}_2$	$[1, 1, 1, 1, 1, 1, 1, 1, 1, -1]$

In particular, $\theta_i(\varpi)$ has simple zeros exactly on the $\Gamma_{\mathcal{T}}(1+i)$-orbit of the fixed locus of $M_i \cdot G_1 \cdot M_i^{-1}$ for $1 \leq i \leq 10$.

Proof. The first part of the proof follows from Lemma 2.10. The fact that it is a simple zero must only be proven for one theta function, say θ_{10}. This was done in [48, Lemma 2.3.1]. $\square$

The groups $\Gamma_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}(1+i)$ have the index-two subgroups given by

$$\begin{aligned} \Gamma^+_{\mathcal{T}} &= \Big\{ g = G\,\mathcal{T}^n \in \Gamma_{\mathcal{T}} \;\Big|\; n \in \{0,1\},\ (-1)^n \det G = 1 \Big\}, \\ \Gamma^+_{\mathcal{T}}(1+i) &= \Gamma^+_{\mathcal{T}} \cap \Gamma_{\mathcal{T}}(1+i). \end{aligned} \tag{2.23}$$

Obviously, we have the following:

Lemma 2.13. *The group $\Gamma^+_{\mathcal{T}}$ is generated by elements $G_1\mathcal{T}$ and $G_2, \dots, G_5$ where G_k for $k = 1, \dots, 5$ were given in Equation (2.13).*

We now consider the quotient spaces $\mathbf{H}_2/\Gamma_{\mathcal{T}}(1+i)$ and $\mathbf{H}_2/\Gamma^+_{\mathcal{T}}(1+i)$, and the Satake compactification

$$\overline{\mathbf{H}_2/\Gamma_{\mathcal{T}}(1+i)}\ . \tag{2.24}$$

For details on the construction of the Satake-Baily-Borel compactification we refer to [6,62]. We define a holomorphic map

$$\mathcal{F}': \quad \mathbf{H}_2 \to \mathbb{P}^9, \quad \varpi \mapsto \Big[\theta_1^2(\varpi) : \cdots : \theta_{10}^2(\varpi)\Big]. \tag{2.25}$$

It follows immediately from Theorem 2.8 that the map $\mathcal{F}'$ descends to a holomorphic map on the quotient space $\mathbf{H}_2/\Gamma_\mathcal{T}(1+i)$. Equations (2.2) coincide with the quadratic relations between the even theta functions in Theorem 2.8 under the identification given by the map $\mathcal{F}'$. An analysis of the simple zeros of the theta functions in [51, Lemma 2.3.1] shows that the image is in fact contained in $\mathfrak{M}(2)$. Thus, we obtain a holomorphic map

$$\begin{aligned} \mathcal{F}: \ \mathbf{H}_2/\Gamma_\mathcal{T}(1+i) \ &\longrightarrow \ \mathfrak{M}(2) \subset \mathbb{P}^9, \\ \varpi \ &\mapsto \ \Big[t_1 : \cdots : t_{10}\Big] = \Big[\theta_1^2(\varpi) : \cdots : \theta_{10}^2(\varpi)\Big]. \end{aligned} \tag{2.26}$$

We have the following:

Theorem 2.14 ([48, Thm. 3.2.1])**.** *The map $\mathcal{F}$ in Equation (2.26) extends to an isomorphism between the Satake compactification of $\mathbf{H}_2/\Gamma_\mathcal{T}(1+i)$ and $\overline{\mathfrak{M}(2)}$ given by*

$$\mathcal{F}: \ \overline{\mathbf{H}_2/\Gamma_\mathcal{T}(1+i)} \ \stackrel{\cong}{\longrightarrow} \ \overline{\mathfrak{M}(2)} \subset \mathbb{P}^9.$$

We also define a holomorphic map

$$\mathcal{G}': \quad \mathbf{H}_2 \to \mathbb{P}(1,\dots,1,2), \ \varpi \mapsto \Big[\theta_1^2(\varpi) : \cdots : \theta_{10}^2(\varpi) : 2^2 3^{-3} 5^{-2} \Theta(\varpi)\Big]. \tag{2.27}$$

We have the following:

Proposition 2.15. *The map $\mathcal{G}'$ descends to a holomorphic map*

$$\begin{aligned} \mathcal{G}: \ \mathbf{H}_2/\Gamma^+_\mathcal{T}(1+i) \ &\longrightarrow \ \mathfrak{M}(2)^+ \subset \mathbb{P}(1,\dots,1,2), \\ \varpi \ &\mapsto \ \Big[t_1 : \cdots : t_{10} : R\Big] = \Big[\theta_1^2(\varpi) : \cdots : \theta_{10}^2(\varpi) : 2^2 3^{-3} 5^{-2} \Theta(\varpi)\Big]. \end{aligned} \tag{2.28}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd} \mathbf{H}_2/\Gamma^+_\mathcal{T}(1+i) \arrow[r, "\mathcal{G}"] \arrow[d, two heads] & \mathfrak{M}(2)^+ \arrow[d, two heads, "\mathrm{pr}"] \\ \mathbf{H}_2/\Gamma_\mathcal{T}(1+i) \arrow[r, "\mathcal{F}"] & \mathfrak{M}(2) \end{tikzcd} \tag{2.29}$$

with $\mathcal{G} \circ \mathcal{T} = \imath \circ \mathcal{G}$ and $\mathrm{pr} \circ \imath = \mathrm{pr}$ and $\Gamma^+_\mathcal{T}(1+i)$ the index-two sub-group of $\Gamma_\mathcal{T}(1+i)$ defined in Equation (2.23).

Proof. For elements $g = G\,\mathcal{T}^n \in \Gamma^+_\mathcal{T}(1+i)$ we have by definition $(-1)^n \det G = 1$ and $\operatorname{sign}(S_G) = 1$. It then follows from Theorem 2.8 that the map $\mathcal{G}'$ descends to a holomorphic map on the quotient space $\mathbf{H}_2/\Gamma^+_\mathcal{T}(1+i)$. Combined with the results in Theorem 2.14 this shows that the image is contained in $\mathfrak{M}(2)^+$. The branching locus of the covering $\mathbf{H}_2/\Gamma^+_\mathcal{T}(1+i) \to \mathbf{H}_2/\Gamma_\mathcal{T}(1+i)$ is given by the simple zeros of the modular form $\Theta(\varpi)$ of weight four which is unique up to constant; see Theorem 2.8. Moreover, the ratio of the right hand sides of Equation (2.17) and Equation (2.4) yields $\Theta(\varpi)^2/R^2 = 2^{-4}\cdot 3^6 \cdot 5^2$ under the identification given by the holomorphic map $\mathcal{F}$. Thus, the branching locus is identical. The equivariance follows from Equation (2.8) and Equation (2.20) and the fact that their branching locus is identical. $\square$

2.2. **The Satake sextic and ring of modular forms.** We introduce the following linear combinations of the degree-one invariants t_i which we call the generalized *level-two Satake coordinate functions* $x_1, \dots, x_6$. We set $x_1 + \dots + x_6 = 0$. Choosing three Satake roots out of the five roots $x_1, \dots, x_5$, we want to obtain all invariants $t_1, \dots, t_{10}$ by setting

$$
\begin{array}{rlllll}
-3\,t_9 & = x_1 & +x_2 & +x_3, & & \\
3\,t_8 & = x_1 & +x_2 & & +x_4, & \\
-3\,t_6 & = x_1 & +x_2 & & & +x_5, \\
3\,t_5 & = x_1 & & +x_3 & +x_4, & \\
-3\,t_{10} & = x_1 & & +x_3 & & +x_5, \\
3\,t_7 & = x_1 & & & +x_4 & +x_5, \\
-3\,t_3 & = & +x_2 & +x_3 & +x_4, & \\
-3\,t_1 & = & +x_2 & +x_3 & & +x_5, \\
-3\,t_4 & = & +x_2 & & +x_4 & +x_5, \\
-3\,t_2 & = & & +x_3 & +x_4 & +x_5.
\end{array}
\tag{2.30}
$$

The j-th power sums s_j are defined by $s_j = \sum_{k=1}^{6} x_k^j$ for $j = 1, \dots, 6$. It can be easily checked using Equation (2.30) that $s_1 = \sum_k x_k = 0$. We combine the level-two Satake functions in a sextic curve, called the *Satake sextic*, given by

$$\mathcal{S}(x) = \prod_{k=1}^{6} (x - x_k) \; .$$

The coefficients of the Satake sextic are polynomials in $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, s_2, s_3, s_4, s_5, s_6\right]$. In fact, we obtain

$$\mathcal{S}(x) = x^6 + \sum_{k=1}^{6} \frac{(-1)^k}{k!} b_k \, x^{6-k}$$

where b_k is the k-th Bell polynomials in the variables $\{s_1, -s_2, 2!s_3, -3!s_4, 4!s_5, -5!s_6\}$. The following holds:

Lemma 2.16. *The generalized level-two Satake coordinate functions* $x_1, \dots, x_6$ *are the roots of the Satake sextic* $\mathcal{S} \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, s_2, s_3, s_4, s_5, s_6\right][x]$ *given by*

$$\begin{aligned} &\mathcal{S}(x) = \mathcal{B}(x)^2 - 4\,\mathcal{A}(x), \\ \text{with} \quad &\mathcal{B}(x) = x^3 - \frac{s_2}{4}\,x - \frac{s_3}{6}, \\ &\mathcal{A}(x) = \frac{4s_4 - s_2^2}{64} x^2 - \frac{5s_2s_3 - 12s_5}{240} x + \frac{3s_2^3 - 4s_3^2 - 18s_2s_4 + 24s_6}{576}. \end{aligned} \tag{2.31}$$

Proof. The proof follows from the explicit computation of the Bell polynomials using the relation $s_1 = 0$. □

We define new quantities J_2, J_3, J_4, J_5, J_6 by setting

$$\begin{aligned} J_2 = \frac{s_2}{12}, \qquad J_3 = \frac{s_3}{12}, \qquad J_4 = \frac{4s_4 - s_2^2}{64} \\ J_5 = \frac{5s_2s_3 - 20s_5}{240}, \qquad s_6 = \frac{3s_2^3 - 4s_3^2 - 18s_2s_4 + 24s_6}{576}, \end{aligned} \tag{2.32}$$

such that

$$\begin{aligned} s_2 = 12\,J_2, \qquad s_3 = 12\,J_3, \qquad s_4 = 36\,J_2^2 + 16\,J_4, \\ s_5 = 60\,J_2J_3 - 20\,J_5, \qquad s_6 = 108\,J_2^3 + 144\,J_2J_4 + 24\,J_3^2 + 24\,J_6. \end{aligned} \tag{2.33}$$

We have the following:

Lemma 2.17. J_2, J_3, J_4, J_5, J_6 *are polynomials over* $\mathbb{Q}$ *in* t_i *for* $1 \leq i \leq 10$ *that are invariant under the action of the permutation group* S_6 *on the variables* t_i *induced by permuting the lines of a six-line configuration.*

Proof. Using Equations (2.2) we can solve for $x_1, \dots, x_6$ using any five of the then t_i's and obtain

$$\begin{array}{rrrrrr} x_1 = & 2\,t_1 & +2\,t_5 & -3\,t_6 & -t_7 & -t_8, \\ x_2 = & -t_1 & -t_5 & & -t_7 & +2\,t_8, \\ x_3 = & -t_1 & +2\,t_5 & & -t_7 & -t_8, \\ x_4 = & -t_1 & -t_5 & +3\,t_6 & +2\,t_7 & +2\,t_8, \\ x_5 = & -t_1 & -t_5 & & +2\,t_7 & -t_8, \\ x_6 = & 2t_1 & -t_5 & & -t_7 & -t_8. \end{array} \tag{2.34}$$

Plugging Equations (2.34) into the j-th power sums s_j and, in turn, into Equations (2.32) proves that J_k for $2 \leq k \leq 6$ are polynomials in t_i with $1 \leq i \leq 10$ and rational coefficients. One checks that for a set of generators of the permutation group of six elements S_6, acting on the variables t_i with $1 \leq i \leq 10$ as defined by permuting lines in Equations (2.1), the polynomials J_k for $2 \leq k \leq 6$ remain invariant. □

Notice that there are many notations in the literature for invariants of binary equations. We will show in Section 5 how our invariants J_k for $2 \leq k \leq 6$, when restricted to a six-line configuration tangent to a common conic, are related to the Igusa invariants of a binary sextic which we denote by I_2, I_4, I_6, I_{10}; see Equation (5.10). We have the following:

Lemma 2.18. *The moduli space $\mathfrak{M}(2)^+$ of six lines in $\mathbb{P}^2$ embeds into the variety in $\mathbb{P}(1,1,1,1,1,1,2)$ with coordinates $[x_1 : \cdots : x_6 : X]$ given by the equations $s_1 = 0$ and $X^2 = 4s_4 - s_2^2$.*

Proof. Given a point in the image, setting $x_6 = -(x_1 + \cdots + x_5)$, Equations (2.30) for t_1, t_5, t_6, t_7, t_8 constitute the inverse transformation to Equations (2.34). Moreover, one checks that $4s_4 - s_2^2 = (18\,R)^2$ in Equation (2.3). □

Remark 2.19. *The sub-variety defined by $X = 0$ comprises an algebraic variety in $\mathbb{P}^4$ given by $s_1 = 0$ and $s_2^2 = 4s_4$ known as Igusa's quartic. It corresponds to the moduli space of configurations of six-lines tangent to a conic and is closely related to the moduli space of genus-two curves with level-two structure. We will discuss the details in Section 5.*

In terms of the invariants J_k with $2 \leq k \leq 6$, the Satake sextic is given by

$$\begin{gathered} \mathcal{S}(x) = \mathcal{B}(x)^2 - 4\,\mathcal{A}(x), \\ \text{with} \qquad \mathcal{B}(x) = x^3 - 3\,J_2 x - 2\,J_3, \qquad \mathcal{A}(x) = J_4\,x^2 - J_5\,x + J_6. \end{gathered} \tag{2.35}$$

One also checks that the square of the degree-two Dolgachev-Ortland invariant in Equation (2.4) is given by J_4, i.e.,

$$R^2 = 2^4 3^{-4} J_4\,. \tag{2.36}$$

We introduce three more invariants: the discriminants of the Satake sextic $\mathcal{S}(x)$ and the quadratic polynomial $\mathcal{A}(x)$ which have degrees 30 and 10, respectively, as well as the resultant of the polynomials $\mathcal{A}(x)$ and $\mathcal{B}(x)$ which has degree 18. By construction, they are all homogeneous polynomials in the invariants $\mathcal{J}_k$ for $2 \leq k \leq 6$ with integer coefficients. One checks that

$$\begin{aligned} \operatorname{Disc}(\mathcal{A}) &= J_5^2 - 4\,J_4 J_6 = 2^{-4} 3^{10} \prod_{i=1}^{10} t_i, \\ \operatorname{Res}(\mathcal{A},\mathcal{B}) &= 9\,J_2^2 J_4^2 J_6 + 6\,J_2\,J_3\,J_4^2 J_5 + 4\,J_3^2 J_4^3 + 6\,J_2\,J_4\,J_6^2 \\ &\quad - 3\,J_2\,J_5^2\,J_6 + 6\,J_3\,J_4\,J_5\,J_6 - 2\,J_3\,J_5^3 + J_6^3. \end{aligned} \tag{2.37}$$

For the discriminant of the Satake sextic, i.e., $\mathcal{S}(x) = \prod_{i<j}(x_i - x_j)^2$, we suppress the lengthy polynomial expression in terms of the J_k for $k = 2, \ldots, 6$. We rather give the following formula in terms of modular forms on $\mathfrak{M}(2)$, namely

$$\begin{gathered} \operatorname{Disc}(\mathcal{S}) = 3^{30} \prod_{j=2}^{10} (t_1 - t_j)^2 \\ \times (t_2 - t_3)^2 (t_3 - t_4)^2 (t_4 - t_5)^2 (t_5 - t_6)^2 (t_2 - t_4)^2 (t_4 - t_6)^2. \end{gathered} \tag{2.38}$$

We have the following:

Corollary 2.20. *A configuration of six lines in $\mathbb{P}^2$ falls into cases (0) through (5) in Definition 2.1 if and only if the invariants in Equation (2.32) satisfy*

$$(J_3,\ J_4,\ J_5) \neq (0,0,0). \tag{2.39}$$

For cases (1) through (5) in Definition 2.1, we find the following equivalences (where all invariants not specified remain generic):

$$\begin{aligned} (1) &\Leftrightarrow\ J_4 = 0, \\ (2) &\Leftrightarrow\ \mathrm{Disc}(\mathcal{A}) = 0, \\ (3,\,4) &\Leftrightarrow\ \mathrm{Disc}(\mathcal{A}) = \mathrm{Res}(\mathcal{A},\mathcal{B}) = 0, \\ (5) &\Leftrightarrow\ J_4 = J_5 = 0, \end{aligned}$$

For cases (6a) and (6b), we find $(J_3, J_4, J_5) = (0,0,0)$.

Proof. One first checks that $\mathrm{Disc}(\mathcal{A}) = \mathrm{Res}(\mathcal{A},\mathcal{B}) = 0$ implies $\mathrm{Disc}(\mathcal{S}) = 0$. The proof follows the same strategy as the one applied in the proof of Lemma 2.4. We explicitly compute the invariants $J_3,\ J_4,\ J_5$ in terms of the moduli a, b, c, d and then restrict to cases (1) through (5) in Definition 2.1. Moreover, $(J_3, J_4, J_5) = (0,0,0)$ implies $\mathrm{Disc}(\mathcal{A}) = \mathrm{Disc}(\mathcal{S}) = \mathrm{Res}(\mathcal{A},\mathcal{B}) = 0$. □

In light of Lemma 2.17 and Corollary 2.20 we can now define a moduli space of *unordered* configurations of six lines in $\mathbb{P}^2$. We define $\mathfrak{M}$ to be the four-dimensional open complex variety given by

$$\mathfrak{M} = \Big\{ [J_2 :\ J_3 :\ J_4 :\ J_5 :\ J_6] \in \mathbb{P}(2,3,4,5,6) \ \Big|\ (J_3,\ J_4,\ J_5) \neq (0,0,0) \Big\}, \tag{2.40}$$

and we also set $\overline{\mathfrak{M}} = \mathbb{P}(2,3,4,5,6)$.

By construction, the points in projective space $\mathbb{P}^9$ arising as image of the map $\mathcal{F}' : \mathbf{H}_2 \to \mathbb{P}^9$ in Equation (2.25), i.e., the points given by

$$\mathcal{F}' :\quad \mathbf{H}_2 \to \mathbb{P}^9, \quad \varpi \mapsto \Big[\theta_1^2(\varpi) : \cdots : \theta_{10}^2(\varpi)\Big], \tag{2.41}$$

are invariant under the action of the sub-group $\Gamma(1+i)$ of level $(1+i)$; see Theorem 2.8. As explained, there is a natural action of the permutation group of six elements S_6 on the variables t_i with $1 \le i \le 10$ induced by permuting the six lines. This action coincides with the action of $\Gamma/\Gamma(1+i)$.

Equation (2.38) provides a geometric characterization of the locus $\mathrm{Disc}(\mathcal{S}) = 0$ in $\mathfrak{M}(2)$. It turns out that the fifteen components of the vanishing locus are in one-to one correspondence with permutations of the from $\sigma_\alpha = (ij)(kl)(mn)$ where $(ij) = i \leftrightarrow j$ is the permutation of the i-th and j-th line. We have the following:

Lemma 2.21. *The vanishing locus $\mathrm{Disc}(\mathcal{S}) = 0$ is the union of the $\Gamma_{\mathcal{T}}(1+i)$-orbits of the fixed loci of $\varpi \mapsto S_j \cdot G_2\mathcal{T} \cdot S_j^{-1} \cdot \varpi$ in $\mathfrak{M}(2)$ where $S_j \in \Gamma$ with $\det(S_j) = 1$ and $1 \le j \le 15$. The fixed loci, the elements S_j, and corresponding permutations σ_j*

are given in the following table:

(2.42)

j	S_j	σ_j	fixed locus
1	$G_5G_4G_5G_3G_5G_2G_5G_4$	$(15)(26)(34)$	$t_1 = t_2,\ t_5 = -t_9,\ t_6 = -t_7$
2	$G_5G_2G_5G_4$	$(12)(35)(46)$	$t_1 = t_3,\ t_5 = -t_{10},\ t_6 = -t_8$
3	$G_5G_4G_3G_5G_2G_5G_4$	$(13)(24)(56)$	$t_1 = t_4,\ t_7 = -t_{10},\ t_8 = -t_9$
4	$G_3G_5G_2G_5G_4$	$(15)(23)(46)$	$t_1 = t_5,\ t_2 = -t_9,\ t_3 = -t_{10}$
5	$G_4G_2G_5G_4$	$(14)(26)(35)$	$t_1 = t_6,\ t_2 = -t_7,\ t_3 = -t_8$
6	$G_5G_3G_5G_2G_5G_4$	$(13)(26)(45)$	$t_1 = t_7,\ t_2 = -t_6,\ t_4 = -t_{10}$
7	$G_4G_3G_5G_2G_5G_4$	$(16)(24)(35)$	$t_1 = t_8,\ t_3 = -t_6,\ t_4 = -t_9$
8	$G_4G_5G_4G_5G_4$	$(15)(24)(36)$	$t_1 = t_9,\ t_2 = -t_5,\ t_4 = -t_8$
9	$G_4G_5G_4$	$(13)(25)(46)$	$t_1 = t_{10},\ t_3 = -t_5,\ t_4 = -t_7$
10	$G_2G_5G_4$	$(14)(23)(56)$	$t_2 = t_3,\ t_7 = t_8,\ t_9 = t_{10}$
11	$G_5G_4G_5G_4$	$(12)(36)(45)$	$t_2 = t_4,\ t_5 = t_8,\ t_6 = t_{10}$
12	G_4	$(14)(25)(36)$	$t_2 = t_8,\ t_3 = t_7,\ t_4 = t_5$
13	$G_4G_5G_3G_5G_2G_5G_4$	$(16)(23)(45)$	$t_2 = t_{10},\ t_3 = t_9,\ t_4 = t_6$
14	G_5G_4	$(16)(25)(34)$	$t_3 = t_4,\ t_5 = t_7,\ t_6 = t_9$
15	$\mathbb{I}_4$	$(12)(34)(56)$	$t_5 = t_6,\ t_7 = t_9,\ t_8 = t_{10}$

Proof. The relation between the components of vanishing locus and the fixed loci of the transformations $\varpi \mapsto S_j \cdot G_2\mathcal{T} \cdot S_j^{-1} \cdot \varpi$ with $1 \le j \le 15$ follow from Equation (2.38) and Lemma 2.19. The relation between permutations acting on t_i with $1 \le i \le 10$ and the listed fixed loci is checked directly using Equations (2.1) □

We denote the six-line configuration discussed in Lemma 2.21 by (2b), adding to cases (0) through (5) in Definition 2.1. Equation (2.42) provides the explicit from of the isomorphism $\Gamma/\Gamma(1+i) \cong \mathrm{S}_6$. We have the following:

Corollary 2.22. *The following vanishing loci are fixed loci of elements in $\Gamma_{\mathcal{T}} \backslash \Gamma^+_{\mathcal{T}}$:*

(1) *The locus $J_4 = 0$ is the fixed locus of $\mathcal{T} \in \Gamma_{\mathcal{T}}$.*
(2) *The locus $\mathrm{Disc}(\mathcal{A}) = 0$ is the union of the fixed loci of $M_i \cdot G_1 \cdot M_i^{-1} \in \Gamma_{\mathcal{T}}$ with $M_i \in \Gamma^+_{\mathcal{T}}$ and $1 \le i \le 10$ given in Lemma 2.12.*
(2b) *The locus $\mathrm{Disc}(\mathcal{S}) = 0$ is the union of the fixed loci of $S_j \cdot G_2\mathcal{T} \cdot S_j^{-1} \in \Gamma_{\mathcal{T}}$ with $S_j \in \Gamma^+_{\mathcal{T}}$ and $1 \le j \le 15$ given in Lemma 2.21.*

Proof. Parts (1), (2) follow from Equations (2.36) and (2.37) when using Lemma 2.11 and Lemma 2.12. Part (3) follows from Lemma 2.21. □

If we plug into the expressions for J_k in Equation (2.32) for $k = 2, 3, 4, 5, 6$, the theta functions $t_i = \theta_i^2(\varpi)$ for $1 \le i \le 10$ of Theorem 2.8, we obtain modular forms which we will denote by $J_2(\varpi), J_3(\varpi), J_4(\varpi), J_5(\varpi), J_6(\varpi)$. We have the following:

Lemma 2.23. *The functions $J_k(\varpi)$ are modular forms relative to $\Gamma_{\mathcal{T}}$ of even characteristic, i.e., with character $\chi_{2k}(g) = \det(G)^k$ for all $g \in \Gamma_{\mathcal{T}}$ in weight $2k$.*

Proof. It follows from Lemma 2.17 that $J_k(\varpi)$ are homogeneous polynomials of degree k in $t_i = \theta_i^2(\varpi)$ for $1 \le i \le 10$. Using Theorem 2.8, we conclude that $J_k(\varpi)$ are modular forms of weight $2k$ relative to $\Gamma_{\mathcal{T}}(1+i)$ with character $\chi_{2k}(g) = \det(G)^k$.

The isomorphism $\Gamma/\Gamma(1+i) \cong \mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z})$ extends the group homomorphism obtained from the projection

$$\mathbb{Z}[i] \mapsto \mathbb{Z}[i]/(1+i)\mathbb{Z}[i] \cong \mathbb{Z}/2\mathbb{Z}. \tag{2.43}$$

On the other hand, there is a group isomorphism $\mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{S}_6$. We showed in Lemma 2.10 that this action coincides with the natural action of the permutation group of six elements S_6 on the variables t_i with $1 \leq i \leq 10$ due to Equations (2.1). Since the invariants J_k are polynomials in the symmetric polynomials s_k with $2 \leq k \leq 6$ given in Equation (2.32), hence invariant under the action of S_6, $J_k(\varpi)$ are modular forms of weight $2k$ relative to the full modular group $\Gamma_{\mathcal{T}}$. □

We have the following:

Theorem 2.24. *The graded ring of modular forms relative to $\Gamma_{\mathcal{T}}$ of even characteristic is generated over $\mathbb{C}$ by the five algebraically independent modular forms $J_k(\varpi)$ of weight $2k$ with $k = 2, \dots, 6$.*

Proof. It follows from [67, Thm. 1] and Section 6.2 that the ring of modular forms relative to $\Gamma_{\mathcal{T}}$ is generated by five modular forms of weights $4, 6, 8, 10, 12$. For arguments $\varpi = \tau$ invariant under $\mathcal{T}$, the functions $J_k(\varpi)$ for $k = 2, 3, 4, 5, 6$ descend to Siegel modular forms of even weight. In fact, we will check in Equation (5.10) that under the restriction from $\mathbf{H}_2/\Gamma_{\mathcal{T}}$ to $\mathbb{H}_2/\mathrm{Sp}_4(\mathbb{Z})$, we obtain

$$\Big[J_2(\varpi) : J_3(\varpi) : J_4(\varpi) : J_5(\varpi) : J_6(\varpi)\Big] = \Big[\psi_4(\tau) : \psi_6(\tau) : 0 : 2^{12}3^5\chi_{10}(\tau) : 2^{12}3^6\chi_{12}(\tau)\Big]. \tag{2.44}$$

Here, ψ_4, ψ_6, χ_{10}, χ_{12} are Siegel modular forms of respective weights $4, 6, 10, 12$ that, as Igusa proved in [30, 31], generate the ring of Siegel modular forms of even weight. Thus, the forms $J_k(\varpi)$ for $k = 2, 3, 5, 6$ must be independent. After looking at some Fourier coefficients to ensure that J_4 is not identically zero, we adjoin J_4 as a fifth form to the ring generated by J_k for $k = 2, 3, 5, 6$. The fundamental theorem of symmetric polynomials establishes the power sums as an algebraic basis for the space of symmetric polynomials. Therefore, J_k for $k = 2, 3, 4, 5, 6$ are algebraically independent. Moreover, we check that $J_4(\varpi) = (\Theta(\varpi)/15)^2$ using Theorem 2.8. Θ does *not* have the character $\chi(g) = \det(G)$ for all $g \in \Gamma_{\mathcal{T}}$. It follows that J_4 cannot be decomposed further as a modular form relative to $\Gamma_{\mathcal{T}}$ with even characteristic. □

We define the holomorphic map

$$\mathcal{H}' : \mathbf{H}_2 \to \mathbb{P}^9, \quad \varpi \mapsto \Big[J_2(\varpi) : \; J_3(\varpi) : \; J_4(\varpi) : \; J_5(\varpi) : \; J_6(\varpi)\Big]. \tag{2.45}$$

We have the following:

Corollary 2.25. *The map $\mathcal{H}'$ descends to a holomorphic map*

$$\begin{aligned} \mathcal{H} : \; \mathbf{H}_2/\Gamma_{\mathcal{T}} &\longrightarrow \mathfrak{M} \subset \mathbb{P}(2,3,4,5,6), \\ \varpi &\mapsto \Big[J_2(\varpi) : \; J_3(\varpi) : \; J_4(\varpi) : \; J_5(\varpi) : \; J_6(\varpi)\Big]. \end{aligned} \tag{2.46}$$

The map $\mathcal{H}$ in Equation (2.46) extends to an isomorphism between the Satake compactification of $\mathbf{H}_2/\Gamma_{\mathcal{T}}$ and $\overline{\mathfrak{M}}$ given by

$$\mathcal{H}: \ \overline{\mathbf{H}_2/\Gamma_{\mathcal{T}}} \ \xrightarrow{\cong} \ \overline{\mathfrak{M}} \subset \mathbb{P}(2,3,4,5,6).$$

Proof. Using Theorem 2.24 and Corollary 2.20 it follows that $\mathcal{H}'$ descends to the holomorphic map $\mathcal{H}$ as stated. By construction the Satake compactification of $\mathbf{H}_2/\Gamma_{\mathcal{T}}$ is given by

$$\operatorname{Proj}\mathbb{C}\Big[J_2, J_3, J_4, J_5, J_6\Big] \cong \mathbb{P}(2,3,4,5,6). \tag{2.47}$$

□

3. K3 surfaces associated with double covers of six lines

In this section, we discuss several Jacobian elliptic fibrations on the K3 surface associated with configurations of six lines ℓ_i in $\mathbb{P}^2$ with $i = 1, \dots, 6$, no three of which are concurrent. The double cover branched along six lines ℓ_i with $i = 1, \dots, 6$ given in terms of variables z_1, z_2, z_3 of $\mathbb{P}^2$ is the solution of

$$z_4^2 = \ell_1\ell_2\ell_3\ell_4\ell_5\ell_6 \tag{3.1}$$

with $[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}(1,1,1,3)$. It is well known that the minimal resolution is a K3 surface of Picard-rank 16 which we will always denote by $\mathcal{Y}$. In [36] Kloosterman classified all Jacobian elliptic fibrations on $\mathcal{Y}$, i.e., elliptic fibrations $\pi^{\mathcal{Y}} : \mathcal{Y} \to \mathbb{P}^1$ together with a section $\sigma : \mathbb{P}^1 \to \mathcal{Y}$ such that $\pi^{\mathcal{Y}} \circ \sigma = 1$. We will construct explicit Weierstrass models for three of these Jacobian elliptic fibrations which we call the *natural* fibration, the *base-fiber-dual* of the natural fibration, and the *alternate* fibration.

3.1. The natural fibration.

Configurations of six lines no three of which are concurrent have four homogeneous moduli which we will denote by a, b, c, d. These moduli can be constructed as follows: the six lines ℓ_i for $1 \leq i \leq 6$ can always be brought into the form

$$\ell_1 = z_1, \ell_2 = z_2, \ell_3 = z_3, \ell_4 = z_1+z_2+z_3, \ell_5 = z_1+az_2+bz_3, \ell_6 = z_1+cz_2+dz_3. \tag{3.2}$$

The matrix $\mathbf{A}$ defined in Section 2 for this six-line configuration is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & a & c \\ 0 & 0 & 1 & 1 & b & d \end{pmatrix}, \tag{3.3}$$

with Dolgachev-Ortland coordinates given by

$$\begin{gathered} t_1 = a(d-1),\ t_2 = b-a,\ t_3 = d-c,\ t_4 = c(b-1),\ t_5 = b(c-1), \\ t_6 = d(a-1),\ t_7 = d-b,\ t_8 = c-a,\ t_9 = ad-bc,\ t_{10} = ad-bc-a+b+c-d, \\ R = -abc+abd+acd-bcd-ad+bc. \end{gathered} \tag{3.4}$$

The six lines intersect as follows:
(3.5)

	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	–	$[0:0:1]$	$[0:1:0]$	$[0:1:-1]$	$[0:b:-a]$	$[0:d:-c]$
ℓ_2	$[0:0:1]$	–	$[1:0:0]$	$[1:0:-1]$	$[-b:0:1]$	$[-d:0:1]$
ℓ_3	$[0:1:0]$	$[1:0:0]$	–	$[1:-1:0]$	$[-a:1:0]$	$[-c:1:0]$
ℓ_4	$[0:1:-1]$	$[1:0:-1]$	$[1:-1:0]$	–	$[a-b:b-1:1-a]$	$[c-d:d-1:1-c]$
ℓ_5	$[0:b:-a]$	$[-b:0:1]$	$[-a:1:0]$	$[a-b:b-1:1-a]$	–	$[ad-bc:b-d:c-a]$
ℓ_6	$[0:d:-c]$	$[-d:0:1]$	$[-c:1:0]$	$[c-d:d-1:1-c]$	$[ad-bc:b-d:c-a]$	–

Setting

$$z_1 = \frac{u(u+1)(au+b)(cu+d)}{X-u(au+b)(cu+d)}, \quad z_2 = u, \quad z_3 = 1 \tag{3.6}$$

in Equation (3.1), it is transformed into the Weierstrass equation

$$Y^2 = X\Big(X - 2\,u\big(\mu(u)-\nu(u)\big)\Big)\Big(X - 2\,u\big(\mu(u)+\nu(u)\big)\Big), \tag{3.7}$$

with discriminant

$$\Delta_{\mathcal{Y}}(u) = 2^8 u^6 \mu(u)^2 \left(\mu(u)^2 - \nu(u)^2\right)^2 \tag{3.8}$$

and

$$\begin{aligned} 2(\mu-\nu) &= \Big(au+b\Big)\Big((c-1)\,u+(d-1)\Big), \\ 2(\mu+\nu) &= \Big(cu+d\Big)\Big((a-1)\,u+(b-1)\Big). \end{aligned} \tag{3.9}$$

In this way, the K3 surfaces associated with the double cover of $\mathbb{P}^2$ branched along the union of six lines, no three of which are concurrent, are equipped with an elliptical fibration $\pi^{\mathcal{Y}}_{\text{nat}} : \mathcal{Y} \to \mathbb{P}^1$ with section σ given by the point at infinity and with a fiber $\mathcal{Y}_u$ given by the minimal Weierstrass equation (3.7). We call this fibration the *natural fibration*. Three two-torsion sections are obvious from the explicit Weierstrass points in Equation (3.7). The following is immediate:

Lemma 3.1. *Equation (3.7) defines a Jacobian elliptic fibration $\pi^{\mathcal{Y}}_{\text{nat}} : \mathcal{Y} \to \mathbb{P}^1$ with six singular fibers of type I_2, two singular fibers of type I_0^*, and the Mordell-Weil group of sections* $\mathrm{MW}(\pi^{\mathcal{Y}}_{\text{nat}}) = (\mathbb{Z}/2\mathbb{Z})^2$.

One checks that the Weierstrass model in Equation (3.7) is minimal if and only if the configuration of six lines falls into cases (0) through (5) in Definition 2.1, but not into case (6). In the latter case, the singularities of Equation (3.7) are not canonical singularities. The following proposition was given in [28]:

Proposition 3.2 ([28])**.** *For generic parameters a, b, c, d, the K3 surface $\mathcal{Y}$ has the transcendental lattice $T_{\mathcal{Y}} \cong H(2) \oplus H(2) \oplus \langle -2 \rangle^{\oplus 2}$.*

Remark 3.3. *We choose $z_1 = 1$ as an affine chart for Equation (3.1) with lines given by Equations (3.2), and the holomorphic two-form $dz_2 \wedge dz_3/z_4$. After relabelling variables, the period of the holomorphic two-form for the family of K3 surfaces $\mathcal{Y}(a,b,c,d)$ over a transcendental two-cycle $\Sigma \in T_{\mathcal{Y}}$ is given by*

$$\iint_\Sigma \frac{dz_1}{\sqrt{z_1}} \, \frac{dz_2}{\sqrt{z_2}} \, \frac{1}{\sqrt{(1+z_1+z_2)(1+az_1+bz_2)(1+cz_1+dz_2)}} \, . \tag{3.10}$$

We set

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), \quad \beta = (\beta_1, \beta_2) = \left(-\frac{1}{2}, -\frac{1}{2}\right) \tag{3.11}$$

and

$$P_1 = 1 + z_1 + z_2, \quad P_2 = 1 + az_1 + bz_2, \quad P_3 = 1 + cz_1 + dz_2. \tag{3.12}$$

We observe that $\forall\, i : \alpha_i \notin \mathbb{Z}$, $\forall\, j : \beta_j \notin \mathbb{Z}$, and $\sum_i \alpha_i + \sum_j \beta_j \notin \mathbb{Z}$, and write the periods in the form

$$F_\Sigma\Big(\alpha, \beta, \{P_i\} \Big| \, a, b, c, d\Big) = \iint_\Sigma \frac{dz_1}{z_1^{-\beta_1}} \, \frac{dz_2}{z_2^{-\beta_2}} \, \prod_{i=1}^{3} P_i^{\alpha_i} \, . \tag{3.13}$$

This identifies the periods as so called $\mathcal{A}$-hypergeometric functions that satisfy a system of linear differential equations known as non-resonant GKZ system [24]. *The particular GKZ system satisfied by the periods in Equation (3.10) is a system of differential equations of rank six in four variables known as Aomoto-Gel'fand Hypergeometric System of Type* $(3,6)$. *It was studied in great detail in* [48–52].

A basis of the solutions $F_1, \dots F_6$ defines a map from the Grassmanian $\mathrm{Gr}(3,6;\mathbb{C})$ *into the projective space $\mathbb{P}^5$, more precisely into the domain in $\mathbb{P}^5$ cut out by the Hodge-Riemann relations. The period map is equivariant with respect to the action of $\Gamma_{\mathcal{T}}(1+i)$ on the domain and the monodromy group on the image. In fact, the map $[F_1 : \dots : F_6] \in \mathbb{P}^5$ is a multi-valued vector function from the moduli space* $\mathfrak{M}(2)$ *to the period domain acted upon by a monodromy group moving the branching locus of six lines around. The map $\mathcal{F}$ in Equation (2.26) is the inverse of this period mapping.*

We obtain a new map by quotienting further by the permutation group S_6. *The monodromy group is then generated by reflections and transformations caused by permuting the lines in a six-line configuration. The resulting map is the period map for a family of K3 surfaces $\mathcal{X}$ closely related to the K3 surfaces $\mathcal{Y}$ discussed in Section 4.*

We have the following:

Corollary 3.4. *Switching the roles of base and fiber in Equation (3.7) defines a second Jacobian elliptic fibration $\check{\pi}^{\mathcal{Y}}_{\mathrm{nat}} : \mathcal{Y} \to \mathbb{P}^1$ with 12 singular fibers of type I_1, a fiber of type I_4, a fiber of type I_8, and* $\mathrm{MW}(\check{\pi}^{\mathcal{Y}}_{\mathrm{nat}})_{\mathrm{tor}} = \{\mathbb{I}\}$ *and* $\mathrm{rk}\,\mathrm{MW}(\check{\pi}^{\mathcal{Y}}_{\mathrm{nat}}) = 4$ *and* $\det \mathrm{discr}\, \mathrm{MW}(\check{\pi}^{\mathcal{Y}}_{\mathrm{nat}}) = 1/2$.

The elliptic fibration appears in the list of all Jacobian elliptic fibrations in [36]. We call this fibration the *base-fiber-dual* of the natural fibration.

Proof. In [14] it was proved that a Weierstrass model of the form given by Equation (3.7) is equivalent to the genus-one fibration

$$\eta^2 = \nu(u)^2\xi^4 + 2\,u\,\mu(u)\,\xi^2 + u^2, \tag{3.14}$$

with one apparent rational point. Since μ, ν are given as polynomials in u in Equation (3.9), the equation can be rewritten as

$$\eta^2 = A(\xi)\,u^4 + B(\xi)\,u^3 + C(\xi)\,u^2 + D(\xi)\,u + E(\xi)^2, \tag{3.15}$$

where A, B, C, D, E are polynomials in ξ. Because there always is the rational point $(u, \eta) = (0, E(\xi))$, it can be brought into the Weierstrass form

$$y^2 = 4\,x^3 - g_2(\xi)\,x - g_3(\xi), \tag{3.16}$$

with

$$\begin{aligned} g_2 &= \frac{16}{3}\,C(\xi)^2 + 64\,A(\xi)\,E(\xi)^2 - 16\,B(\xi)\,D(\xi),\\ g_3 &= -\frac{64}{27}\,C(\xi)^3 + \frac{256}{3}A(\xi)\,C(\xi)\,E(\xi)^2 + \frac{32}{3}B(\xi)\,C(\xi)\,D(\xi)\\ &\quad - 32\;A(\xi)\,D(\xi)^2 - 32\,B(\xi)^2\,E(\xi)^2. \end{aligned} \tag{3.17}$$

This is a Weierstrass model with 12 singular fibers of type I_1, a fiber of type I_4, a fiber of type I_8, and $\mathrm{MW}(\check{\pi}^{\mathcal{Y}}_{\mathrm{nat}})_{\mathrm{tor}} = \{\mathbb{I}\}$. In Proposition 3.2 we showed that the transcendental lattice of the K3 surfaces $\mathcal{Y}$ has rank six and is given by

$$T_{\mathcal{Y}} \cong H(2) \oplus H(2) \oplus \langle -2 \rangle^{\oplus 2}. \tag{3.18}$$

Therefore, the determinant of the discriminant group for the rank-six lattice $T_{\mathcal{Y}}$ is $\det(\operatorname{discr} T_{\mathcal{Y}}) = 2^6$. The root lattice associated with the singular fibers in Equation (3.16) has rank ten and determinant 2^5. The claim follows. □

3.2. **The alternate fibration.** In the list of all possible fibrations on the K3 surface $\mathcal{Y}$ associated with the double cover branched along the union of six lines given in [36] we find the following fibration which we call the *alternate* fibration:

Corollary 3.5 ([36])**.** *On the K3 surface $\mathcal{Y}$ there is a Jacobian elliptic fibration $\pi^{\mathcal{Y}}_{\mathrm{alt}} : \mathcal{Y} \to \mathbb{P}^1$ with six singular fibers of type I_2, two singular fibers of type I_1, one singular fiber of type I_4^*, and the Mordell-Weil group of sections* $\mathrm{MW}(\pi^{\mathcal{Y}}_{\mathrm{alt}}) = \mathbb{Z}/2\mathbb{Z}$.

The alternate fibration on the K3 surface $\mathcal{Y}$ can be obtained explicitly from Equation (3.1). In fact, we obtained the Weierstrass model for this fibration using a 2-neighbor-step procedure applied twice, a method explained in [39], starting with the natural fibration. The details of this computation will be published in another forthcoming article [13]. We have the following:

Theorem 3.6. *On the K3 surface $\mathcal{Y}$ associated with the double cover branched along six lines, the Jacobian elliptic fibration $\pi^{\mathcal{Y}}_{\mathrm{alt}} : \mathcal{Y} \to \mathbb{P}^1$ has the Weierstrass equation*

$$Y^2 = X\Big(X^2 - 2\,\mathcal{B}(t)X + \mathcal{B}(t)^2 - 4\,\mathcal{A}(t)\Big), \tag{3.19}$$

with discriminant

$$\Delta_{\mathcal{Y}}(t) = 16\,\mathcal{A}(t)\left(\mathcal{B}(t)^2 - 4\,\mathcal{A}(t)\right)^2, \tag{3.20}$$

and

$$\mathcal{B}(t) = t^3 - 3\,J_2\,t - 2\,J_3, \qquad \mathcal{A}(t) = J_4\,t^2 - J_5\,t + J_6, \tag{3.21}$$

where J_k for $k = 2, \dots, 6$ are the invariants of the configuration of six lines defined in Equations (2.33).

Proof. The invariants J_2, J_3, J_4, J_5, J_6 defined by Equations (2.33) are determined by the symmetric polynomials in terms of the five degree-one invariants t_1, t_5, t_6, t_7, t_8 which in turn are given in terms of the affine moduli a, b, c, d using Equations (3.4). These affine moduli a, b, c, d were defined by arranging the six lines to be in the form of Equation (3.2). On the other hand, the 2-neighbor-step procedure, when applied twice, gives the Weierstrass model in Equation (3.19) with

$$\mathcal{B}(t) = t^3 - 3\,J_2'\,t - 2\,J_3', \qquad \mathcal{A}(t) = J_4'\,t^2 - J_5'\,t + J_6'. \tag{3.22}$$

We computed the coefficients J_i' following the same procedure as the one outlined in [39] for a general Kummer surface using a computer algebra system. At the end of the computation, the coefficients J_i' for $2 \leq 6$ are obtained directly in terms of the affine moduli a, b, c, d. The resulting expressions for the coefficients are then given in Equations (A.2). One easily checks that $J_i = J_i'$ for $2 \leq i \leq 6$. □

4. The Van Geemen-Sarti partner

To extend the notion of geometric two-isogeny to Picard rank 16, we replaced the Kummer surfaces by the K3 surface $\mathcal{Y}$ associated with the double cover branched along the union of six lines discussed in Section 3. The Shioda-Inose surface is now replaced by a K3 surface $\mathcal{X}$ introduced by Clingher and Doran in [11]. The K3 surface occurs as the general member of a six-parameter family of K3 surfaces polarized by the lattice $H \oplus E_7(-1) \oplus E_7(-1)$. Each K3 surface in the family carries a special Nikulin involution, called *Van Geemen-Sarti involution*, such that quotienting by this involution and blowing up fixed points recovers a double-sextic surface.

4.1. **A Six-Parameter Family of K3 Surfaces.**

Let $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \in \mathbb{C}^6$. We consider the projective quartic surface $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \subset \mathbb{P}^3(x, y, z, w)$ defined by the homogeneous equation:

$$\mathbf{Y}^2\mathbf{Z}\mathbf{W} - 4\mathbf{X}^3\mathbf{Z} + 3\alpha\mathbf{X}\mathbf{Z}\mathbf{W}^2 + \beta\mathbf{Z}\mathbf{W}^3 + \gamma\mathbf{X}\mathbf{Z}^2\mathbf{W} - \tfrac{1}{2}\left(\delta\mathbf{Z}^2\mathbf{W}^2 + \zeta\mathbf{W}^4\right) + \varepsilon\mathbf{X}\mathbf{W}^3 = 0. \tag{4.1}$$

The family in Equation (4.1) was first introduced by Clingher and Doran in [11] as a generalization of the Inose quartic introduced in [32]. We denote by $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ the smooth complex surface obtained as the minimal resolution of $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$. We have the following:

Theorem 4.1. *Assume that* $(\gamma, \delta) \neq (0,0)$ *and* $(\varepsilon, \zeta) \neq (0,0)$. *Then, the surface* $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ *obtained as the minimal resolution of* $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ *is a K3 surface endowed with a canonical* $H \oplus E_7(-1) \oplus E_7(-1)$ *lattice polarization.*

Proof. The conditions imposed on the pairs (γ, δ) and (ε, ζ) ensure that singularities of $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ are rational double points. This fact, in connection with the degree of Equation (4.1) being four, guarantees that the minimal resolution $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ is a K3 surface.

Note that the quartic $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ has two special singularities at the following points:

$$\mathrm{P}_1 = [0,1,0,0], \qquad \mathrm{P}_2 = [0,0,1,0].$$

One verifies that the singularity at P_1 is a rational double point of type A_9 if $\varepsilon \neq 0$, and of type A_{11} if $\varepsilon = 0$. The singularity at P_2 is of type A_5 if $\gamma \neq 0$, and of type E_6 if $\gamma = 0$. For a generic sextuple $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$, the points P_1 and P_2 are the only singularities of Equation (4.1).

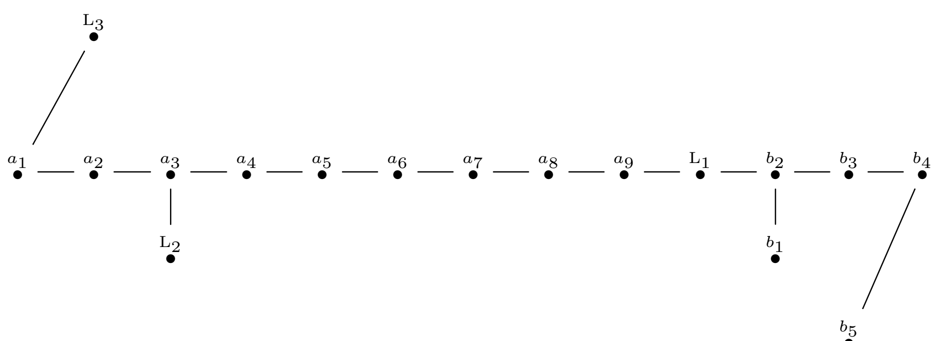
As a first step in uncovering the claimed lattice polarization, we introduce the following three special lines, denoted L_1, L_2, L_3:

$$\mathbf{X} = \mathbf{W} = 0, \quad \mathbf{Z} = \mathbf{W} = 0, \quad 2\varepsilon\mathbf{X} - \zeta\mathbf{W} = \mathbf{Z} = 0 \ .$$

Note that L_1, L_2, L_3 lie on the quartic in Equation (4.1).

Assume the case $\gamma\varepsilon \neq 0$. Then L_1, L_2, L_3 are distinct and concurrent, meeting at P_1. When taking the minimal resolution $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$, one obtains a configuration of smooth rational curves intersecting according to the dual diagram below.

(4.2)

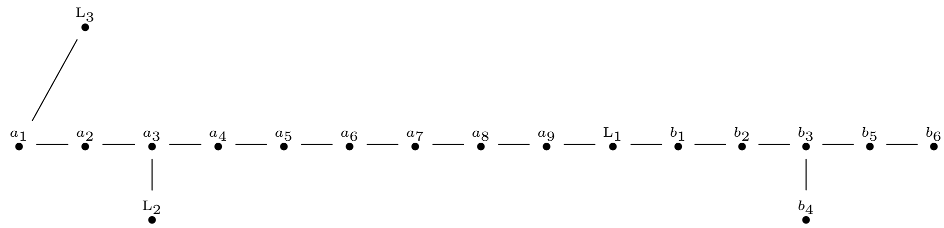


The two sets $a_1, a_2, \dots, a_9$ and $b_1, b_2, \dots, b_5$ denote the curves appearing from resolving the rational double point singularities at P_1 and P_2, respectively. In the context of diagram (4.2), the lattice polarization $H \oplus E_7(-1) \oplus E_7(-1)$ is generated by:

$$\begin{aligned}
\mathrm{H} &= \langle a_7,\ \mathrm{L}_3 + 2a_1 + 3a_2 + 4a_3 + 2\mathrm{L}_2 + 3a_4 + 2a_5 + a_6 \rangle \\
\mathrm{E}_7 &= \langle\ \mathrm{L}_3,\ a_1,\ a_2,\ a_3,\ \mathrm{L}_2,\ a_4,\ a_5\ \rangle \\
\mathrm{E}_7 &= \langle\ b_5,\ b_4,\ b_3,\ b_2,\ b_1,\ \mathrm{L}_1,\ a_9\ \rangle \ .
\end{aligned}$$

In the case $\gamma = 0$, $\varepsilon \neq 0$, the diagram (4.2) changes to:

(4.3)

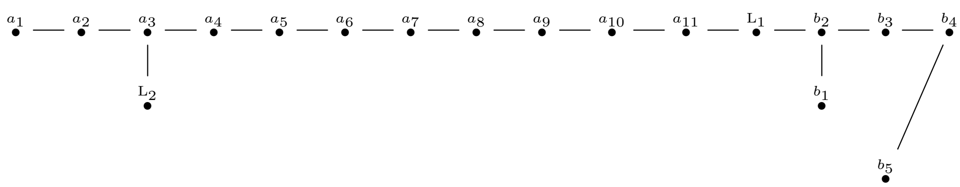


One obtains an enhanced lattice polarization of type $H \oplus E_8(-1) \oplus E_7(-1)$ with:

$$\begin{aligned} \mathrm{H} &= \langle a_7,\ \mathrm{L}_3 + 2a_1 + 3a_2 + 4a_3 + 2\mathrm{L}_2 + 3a_4 + 2a_5 + a_6 \rangle \\ \mathrm{E}_7 &= \langle\ \mathrm{L}_3,\ a_1,\ a_2,\ a_3,\ \mathrm{L}_2,\ a_4,\ a_5\ \rangle \\ \mathrm{E}_8 &= \langle\ b_6,\ b_5,\ b_4,\ b_3,\ b_2,\ b_1,\ \mathrm{L}_1,\ a_9\ \rangle\ . \end{aligned}$$

In the case $\gamma \neq 0$, $\varepsilon = 0$, the lines L_2 and L_3 coincide but the rational double point type at P_1 changes from A_9 to A_{11}. One obtains rational curves on $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ as follows:

(4.4)

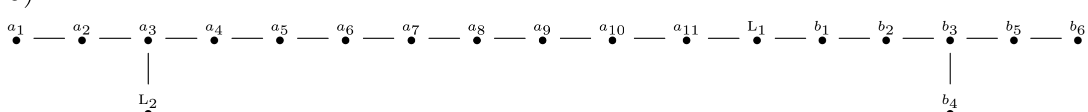


This provides a $H \oplus E_8(-1) \oplus E_7(-1)$ polarization as follows:

$$\begin{aligned} \mathrm{H} &= \langle a_9,\ 2a_1 + 4a_2 + 6a_3 + 3\mathrm{L}_2 + 5a_4 + 4a_5 + 3a_6 + 2a_7 + a_8 \rangle \\ \mathrm{E}_8 &= \langle\ a_1,\ a_2,\ a_3,\ \mathrm{L}_2,\ a_4,\ a_5,\ a_6,\ a_7\ \rangle \\ \mathrm{E}_7 &= \langle\ b_5,\ b_4,\ b_3,\ b_2,\ b_1,\ \mathrm{L}_1,\ a_{11}\ \rangle\ . \end{aligned}$$

Finally, in the case $\gamma = \varepsilon = 0$, the lines L_2, L_3 coincide and the rational double point types at P_1 and P_2 are A_{11} and E_6, respectively. This determines the following diagram of smooth rational curves on the resolution $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$.

(4.5)



This determines a lattice polarization of type $H \oplus E_8(-1) \oplus E_8(-1)$ polarization as follows:

$$\begin{aligned} \mathrm{H} &= \langle a_9,\ 2a_1 + 4a_2 + 6a_3 + 3\mathrm{L}_2 + 5a_4 + 4a_5 + 3a_6 + 2a_7 + a_8 \rangle \\ \mathrm{E}_8 &= \langle\ a_1,\ a_2,\ a_3,\ \mathrm{L}_2,\ a_4,\ a_5,\ a_6,\ a_7\ \rangle \\ \mathrm{E}_8 &= \langle\ b_6,\ b_5,\ b_4,\ b_3,\ b_2,\ b_1,\ \mathrm{L}_1,\ a_{11}\ \rangle\ . \end{aligned}$$

□

Remark 4.2. *The degree-four polarization determined on* $\mathrm{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ *by its quartic definition is described explicitly in the context of diagrams* (4.2)-(4.5). *For instance, assuming the case* $\gamma\varepsilon \neq 0$, *one can write a polarizing divisor as:*

$$\mathcal{L} \;=\; \mathrm{L}_2 + (a_1 + 2a_2 + 3a_3 + 3a_4 + 3a_5 + \cdots 3a_9) + 3\mathrm{L}_1 + (2b_1 + 4b_2 + 3b_3 + 2b_4 + b_5) \tag{4.6}$$

Similar formulas hold in the other three cases.

Diagrams (4.2), (4.3) and (4.4) from the above proof can be nicely augmented. Consider the following complete intersections:

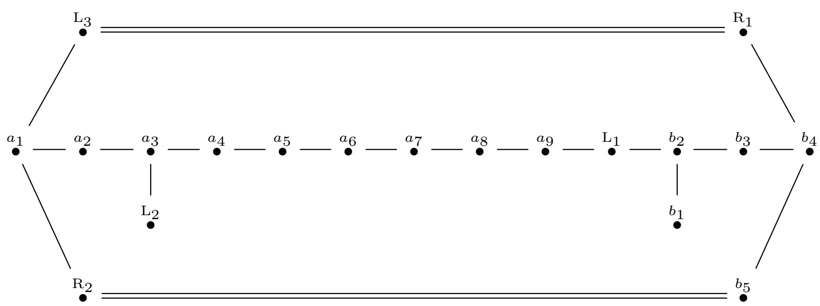
$$2\varepsilon\mathbf{X} - \zeta\mathbf{W} \;=\; \left(3\alpha\varepsilon^2\zeta + 2\beta\varepsilon^3 - \zeta^3\right)\mathbf{W}^2 - \varepsilon^2\left(\delta\varepsilon - \gamma\zeta\right)\mathbf{Z}\mathbf{W} + 2\varepsilon^3\mathbf{Y}^2 \;=\; 0$$

$$2\gamma\mathbf{X} - \delta\mathbf{W} \;=\; \left(3\alpha\gamma^2\delta + 2\beta\gamma^3 - \delta^3\right)\mathbf{Z}\mathbf{W}^2 - \gamma^2\left(\gamma\zeta - \delta\varepsilon\right)\mathbf{W}^3 + 2\gamma^3\mathbf{Y}^2\mathbf{Z} \;=\; 0\;.$$

Assuming appropriate generic conditions, the above equations determine two projective curves R_1, R_2, of degrees two and three, respectively. The conic R_1 is a (generically smooth) rational curve tangent to L_1 at P_2. The cubic R_2 has a double point at P_2, passes through P_1 and is generically irreducible. When resolving the quartic surface (4.1), these two curves lift to smooth rational curves on $\mathrm{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$, which by a slight abuse of notation we shall denote by the same symbol. One obtains the following dual diagrams of rational curves.

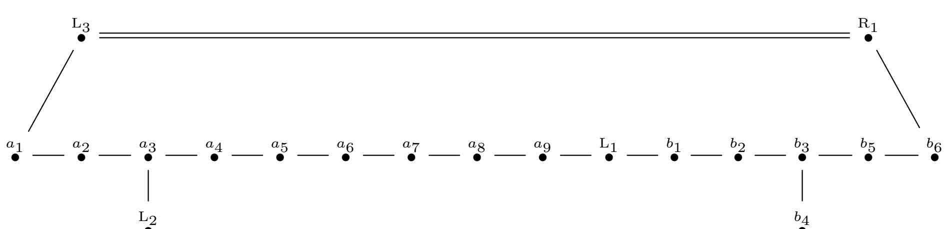
Case $\gamma \neq 0$, $\varepsilon \neq 0$:

(4.7)



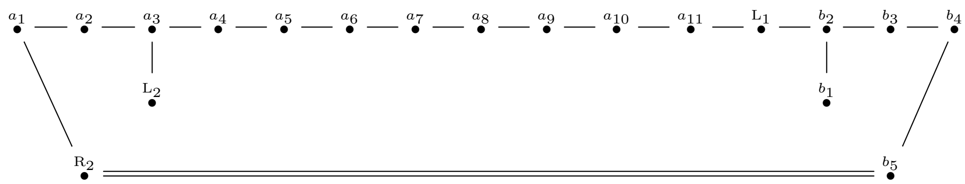
Case $\gamma = 0$, $\varepsilon \neq 0$:

(4.8)



Case $\gamma \neq 0$, $\varepsilon = 0$:

(4.9)



Note that diagrams (4.8) and (4.9) are similar in nature. This is not a fortuitous fact, as we shall see next.

Proposition 4.3. *Let* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \in \mathbb{C}^6$ *with* $(\gamma, \delta) \neq (0, 0)$ *and* $(\varepsilon, \zeta) \neq (0, 0)$. *Then, one has the following isomorphisms of* $H \oplus E_7(-1) \oplus E_7(-1)$ *lattice polarized K3 surfaces:*

(a) $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \; \simeq \; \mathcal{X}(t^2\alpha,\ t^3\beta,\ t^5\gamma,\ t^6\delta,\ t^{-1}\varepsilon,\ \zeta)$*, for any* $t \in \mathbb{C}^*$.
(b) $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \; \simeq \; \mathcal{X}(\alpha, \beta, \varepsilon, \zeta, \gamma, \delta)$.

Proof. Let q be a square root of t. Then, the projective automorphism given by

$$\mathbb{P}^3 \longrightarrow \mathbb{P}^3, \quad [\mathbf{X} : \mathbf{Y} : \mathbf{Z} : \mathbf{W}] \mapsto [\ q^8\mathbf{X} :\ q^9\mathbf{Y} : \mathbf{Z} :\ q^6\mathbf{W}\ ] \tag{4.10}$$

extends to an isomorphism $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \simeq \mathcal{X}(t^2\alpha, t^3\beta, t^5\gamma, t^6\delta, t^{-1}\varepsilon, \zeta)$ preserving the lattice polarization. Similarly, the birational involution:

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^3, \quad [\mathbf{X} : \mathbf{Y} : \mathbf{Z} : \mathbf{W}] \mapsto [\ \mathbf{XZ} :\ \mathbf{YZ}, \mathbf{W}^2,\ \mathbf{ZW}\ ]\,. \tag{4.11}$$

extends to an isomorphism between $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ and $\mathcal{X}(\alpha, \beta, \gamma, \delta, \zeta, \varepsilon)$. $\square$

4.2. **Elliptic Fibrations on $\mathcal{X}$.** By the nature of the $H \oplus E_7(-1) \oplus E_7(-1)$ lattice polarizations, K3 surfaces $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ carry interesting elliptic fibrations with sections. As discussed in [11], there are four non-isomorphic elliptic fibrations with section; three will be important for the considerations of this article.

4.2.1. *The standard fibration.* The first elliptic fibration with section is canonically associated with the lattice polarization, as the classes of its fiber and section span the hyperbolic factor in $H \oplus E_7(-1) \oplus E_7(-1)$. Following the terminology of [10], we shall refer to this elliptic fibration as *standard* and denote it by

$$\pi^{\mathcal{X}}_{\text{std}} \colon \mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \to \mathbb{P}^1\ .$$

One obtains the fibers of $\pi^{\mathcal{X}}_{\text{std}}$ by considering the pencil of planes in $\mathbb{P}^3$ that contain the line L_1, where L is the degree-four canonical hyperplane polarization of $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$.

It is obtained from residual intersections with the pencil of planes containing the line $\mathbf{Z} = \mathbf{W} = 0$. Setting

$$\mathbf{X} = sx, \quad \mathbf{Y} = y, \quad \mathbf{W} = 4\,s^3, \quad \mathbf{Z} = 4\,s^4, \tag{4.12}$$

in Equation (4.1), we obtain the Weierstrass equation

$$\mathcal{X}_s : \quad y^2 = x^3 + f(s)\,x + g(s), \tag{4.13}$$

with discriminant

$$\Delta_{\mathcal{X}}(s) = 4\,f(s)^3 + 27\,g(s)^2 = -64\,s^9 P(s) \tag{4.14}$$

and

$$f(s) = 4\,s^3\Big(\gamma\,s^2 - 3\,\alpha\,s + \varepsilon\Big), \quad g(s) = -8\,s^5\Big(\delta\,s^2 + 2\,\beta\,s + \zeta\Big), \tag{4.15}$$

and

$$\begin{aligned} P(s) = &\; 4\,\gamma^3 s^6 - 9\,(4\,\alpha\gamma^2 - 3\,\delta^2)\,s^5 + 12\,(9\,\alpha^2\gamma + 9\,\beta\delta + \gamma^2\varepsilon)\,s^4 \\ - &\; 18\,(6\,\alpha^3 + 4\,\alpha\gamma\varepsilon - 6\,\beta^2 - 3\,\delta\zeta)\,s^3 + 12\,(9\,\alpha^2\varepsilon + 9\,\beta\zeta + \gamma\varepsilon^2)\,s^2 \\ - &\; 9\,(4\,\alpha\varepsilon^2 - 3\,\zeta^2)\,s + 4\,\varepsilon^3. \end{aligned} \tag{4.16}$$

In this way, we obtain an elliptically fibered K3 surface $\pi^{\mathcal{X}}_{\text{std}}\colon \mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \to \mathbb{P}^1$ with section given by the point at infinity in each fiber $\mathcal{X}_s$ and minimal Weierstrass equation (4.13). The fibration has singularities of Kodaira type III^* over $s=0$ and $s=\infty$. The following lemma is immediate:

Lemma 4.4. *Equation (4.13) is a Jacobian elliptic fibration $\pi^{\mathcal{X}}_{\text{std}}\colon \mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \to \mathbb{P}^1$ with six singular fibers of type I_1, two singular fibers of type III^*, and the Mordell-Weil group of sections $\mathrm{MW}(\pi^{\mathcal{X}}_{\text{std}}) = \{\mathbb{I}\}$.*

Application of Tate's algorithm shows immediately:

Lemma 4.5. *We have the following:*

- *If $\varepsilon \neq 0$, there is a singular fiber of Kodaira type III^* at $s=0$. Otherwise, it is a singular fiber of Kodaira type II^*.*
- *If $\gamma \neq 0$, there is a singular fiber of Kodaira type III^* at $s=\infty$. Otherwise, it is a singular fiber of Kodaira type II^*.*

4.2.2. *The alternate fibration.* Another elliptic fibration with section is obtained from residual intersections with the pencil of planes containing the line $\mathbf{X} = \mathbf{W} = 0$. Setting

$$\mathbf{X} = t\,x^3, \quad \mathbf{Y} = \sqrt{2}\,x^2 y, \quad \mathbf{W} = 2\,x^3, \quad \mathbf{Z} = 2\,x^2(-\varepsilon t + \zeta), \tag{4.17}$$

in Equation (4.1), determines a second Jacobian elliptic fibration $\pi^{\mathcal{X}}_{\text{alt}}\colon \mathcal{X} \to \mathbb{P}^1$ with fiber $\mathcal{X}_t$ given by the Weierstrass equation

$$\mathcal{X}_t: \quad y^2 = x\Big(x^2 + B(t)\,x + A(t)\Big), \tag{4.18}$$

with discriminant

$$\Delta_{\mathcal{X}}(t) = A(t)^2\,\Big(B(t)^2 - 4\,A(t)\Big) \tag{4.19}$$

and

$$\begin{aligned} A(t) &= (\gamma t - \delta)(\varepsilon t - \zeta) = \gamma\varepsilon t^2 - (\gamma\zeta + \delta\varepsilon)t + \delta\zeta, \\ B(t) &= t^3 - 3\alpha t - 2\beta. \end{aligned} \tag{4.20}$$

Thus, we obtain an elliptically fibered K3 surface $\pi^{\mathcal{X}}_{\text{alt}}\colon \mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \to \mathbb{P}^1$ which we call the *alternate fibration*, with section given by the point at infinity in each fiber $\mathcal{X}_t$ and minimal Weierstrass equation (4.18). It has a two-torsion section $(x,y) = (0,0)$, two singularities of Kodaira type I_2 over $A(t)=0$, and a singularity of Kodaira type I_8^* over $t=\infty$. Therefore, the following is immediate:

Lemma 4.6. *Equation (4.18) defines a Jacobian elliptic fibration* $\pi^{\mathcal{X}}_{\text{alt}}\colon \mathcal{X} \to \mathbb{P}^1$ *with six singular fibers of type* I_1*, two singular fibers of type* I_2*, one singular fibers of type* I_8^**, and the Mordell-Weil group of sections* $\mathrm{MW}(\pi^{\mathcal{X}}_{\text{alt}}) = \mathbb{Z}/2\mathbb{Z}$*.*

Setting

$$x = T, \quad y = \frac{Y}{T^2}, \quad t = \frac{X - \frac{1}{3}\gamma\varepsilon T}{T^2}, \tag{4.21}$$

in Equation (4.18) determines another Jacobian elliptic fibration $\check{\pi}^{\mathcal{X}}_{\text{alt}} : \mathcal{X} \to \mathbb{P}^1$ with fiber $\mathcal{X}_T$ given by the minimal Weierstrass equation

$$\mathcal{X}_T : \quad Y^2 = X^3 + \check{f}(T)\, X + \check{g}(T), \tag{4.22}$$

with discriminant

$$\Delta_{\mathcal{X}}(T) = 4\, \check{f}(T)^3 + 27\, \check{g}(T)^2 \tag{4.23}$$

and

$$\begin{aligned} \check{f}(T) &= -\frac{1}{3}\, T^2 \Big(9\,\alpha\, T^2 + 3(\gamma\zeta + \delta\varepsilon)\, T + (\gamma\varepsilon)^2 \Big), \\ \check{g}(T) &= \frac{1}{27}\, T^3 \Big(27\, T^4 - 54\,\beta\, T^3 + 27\,(\alpha\gamma\varepsilon + \delta\zeta)\, T^2 \\ &\qquad + 9\,\gamma\varepsilon\,(\delta\varepsilon + \gamma\zeta)\, T + 2(\gamma\varepsilon)^3 \Big). \end{aligned}$$

Thus, we obtain a Jacobian elliptic fibration $\check{\pi}^{\mathcal{X}}_{\text{alt}}\colon \mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \to \mathbb{P}^1$ which we call the *base-fiber-dual* of the alternate fibration. We have the following:

Lemma 4.7. *Equation (4.22) defines a Jacobian elliptic fibration* $\check{\pi}^{\mathcal{X}}_{\text{alt}}\colon \mathcal{X} \to \mathbb{P}^1$ *with 6 singular fibers of Kodaira type* I_1*, a singular fibers of Kodaira type* I_2^**, and a singular fiber of Kodaira type* II^**, and the Mordell-Weil group of sections* $\mathrm{MW}(\check{\pi}^{\mathcal{X}}_{\text{alt}}) = \{\mathbb{I}\}$*.*

4.3. **Van Geemen-Sarti involutions and moduli.** Equation (4.18) is a minimal Weierstrass model for the Jacobian elliptic fibration $\pi^{\mathcal{X}}_{\text{alt}} : \mathcal{X} \to \mathbb{P}^1$ with fiber $\mathcal{X}_t$ given by

$$\mathcal{X}_t : \quad y^2 = x\Big(x^2 + B(t)\, x + A(t)\Big). \tag{4.24}$$

The singular fibers of $\mathcal{X}$ are located over the support of $\Delta_{\mathcal{X}} = A(t)^2(B(t)^2 - 4\,A(t))$. A smooth section σ is given by the point at infinity in each fiber. A two-torsion section τ is given by $\tau : t \mapsto (x,y) = (0,0)$ such that $2\tau = \sigma$. Thus, we have $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{MW}(\pi^{\mathcal{X}}_{\text{alt}})$. The holomorphic two-form is given by $\omega_{\mathcal{X}} = dt \wedge dx/y$.

A *Nikulin involution* on a K3 surface $\mathcal{X}$ is a symplectic involution $\jmath_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$, i.e., an involution with $\jmath_{\mathcal{X}}^*(\omega) = \omega$. If a Nikulin involution exists on a K3 surface $\mathcal{X}$, then it necessarily has eight fixed points, and the minimal resolution of the quotient surface is another K3 surface $\mathcal{Y} = \widehat{\mathcal{X}/\{1, \jmath_{\mathcal{X}}\}}$ [59]. Special Nikulin involution are obtained in our situation: the fiberwise translation by the two-torsion section acting by $p \mapsto p+\tau$ for all $p \in \mathcal{X}_t$ extends to a Nikulin involution $\jmath_{\mathcal{X}}$ on $\mathcal{X}$, called *Van Geemen-Sarti* involution. A computation shows that the involution is, on each fiber $\mathcal{X}_t$, given by

$$\jmath_{\mathcal{X}_t} : (x, y) \mapsto (x, y) + (0, 0) = \left(\frac{A(t)}{x}, - \frac{A(t)\, y}{x^2} \right) \tag{4.25}$$

for $p \notin \{\sigma, \tau\}$ and interchanges σ and τ. It is also easy to check that $\jmath_{\mathcal{X}}$ leaves the holomorphic two-form $\omega_{\mathcal{X}}$ invariant. Using the smooth two-isogeneous elliptic curve $\mathcal{X}_t/\{\sigma, \tau\}$ for each smooth fiber, we obtain the new K3 surface $\mathcal{Y}$ equipped with an elliptic fibration $\pi^{\mathcal{Y}}_{\text{alt}} : \mathcal{Y} \to \mathbb{P}^1$ with section Σ as the Weierstrass model with fiber $\mathcal{Y}_t$ given by

$$\mathcal{Y}_t : \quad Y^2 = X \Big(X^2 - 2\, B(t)\, X + B(t)^2 - 4\, A(t) \Big). \tag{4.26}$$

The singular fibers of $\mathcal{Y}$ are located over the support of $\Delta_{\mathcal{Y}} = 16\, A(t)\, (B(t)^2 - 4A(t))^2$. The holomorphic two-form on $\mathcal{Y}$ is $\omega_{\mathcal{Y}} = dt \wedge dX/Y$. The fiberwise isogeny given by

$$\hat{\Phi}|_{\mathcal{X}_t} : (x, y) \mapsto (X, Y) = \left(\frac{y^2}{x^2}, \frac{(x^2 - A(t))y}{x^2} \right) \tag{4.27}$$

extends to a degree-two rational map $\hat{\Phi} : \mathcal{X} \dashrightarrow \mathcal{Y}$. We observe that the K3 surface $\mathcal{Y}$ satisfies $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{MW}(\pi^{\mathcal{Y}}_{\text{alt}})$ with a two-torsion section T given by $T : t \mapsto (X, Y) = (0, 0)$. Therefore, the surface $\mathcal{Y}$ is itself equipped with a Van Geemen-Sarti involution $\jmath_{\mathcal{Y}}$, namely

$$\jmath_{\mathcal{Y}_t} : (X, Y) \mapsto (X, Y) + (0, 0) = \left(\frac{B(t)^2 - 4\, A(t)}{X}, - \frac{(B(t)^2 - 4\, A(t))Y}{X^2} \right). \tag{4.28}$$

The involution $\jmath_{\mathcal{Y}}$ leaves the holomorphic two-form $\omega_{\mathcal{Y}}$ invariant and covers the map Φ extending the fiberwise dual isogeny $P \mapsto P + T$ for all $P \in \mathcal{Y}_t$ given by

$$\Phi|_{\mathcal{Y}_t} : (X, Y) \mapsto (x, y) = \left(\frac{Y^2}{4X^2}, \frac{Y(X^2 - B(t)^2 + 4\, A(t))}{8\, X^2} \right). \tag{4.29}$$

The situation is summarized in the following diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[loop left, "\jmath_{\mathcal{X}}"] \arrow[rr, dashed, bend right, "\hat{\Phi}"'] \arrow[dr, "\pi^{\mathcal{X}}_{\text{alt}}"'] & & \mathcal{Y} \arrow[loop right, "\jmath_{\mathcal{Y}}"'] \arrow[ll, dashed, bend right, "\Phi"'] \arrow[dl, "\pi^{\mathcal{Y}}_{\text{alt}}"] \\
& \mathbb{P}^1 &
\end{tikzcd} \tag{4.30}$$

We refer to such K3 surfaces $\mathcal{X}$ and $\mathcal{Y}$ as *Van Geemen-Sarti partners*. Therefore, the family of K3 surfaces $\mathcal{Y}$ associated with the double cover of the projective plane branched along the union of six lines equipped with the alternate fibration in Equation (3.19) and the Clingher-Doran family of K3 surfaces equipped with the alternate fibration in Equation (4.18) constitute such Van Geemen-Sarti partners; see [14, 15]. The notion of Van Geemen-Sarti partners is more general than the one of a Shioda-Inose structure. We make the following:

Remark 4.8. *In Picard rank* 17, $\mathcal{X}$ *carries a Shioda-Inose structure* [10, 11]. *The quotient map* $\hat{\Phi} : \mathcal{X} \dashrightarrow \mathcal{Y} = \mathrm{Kum}(A)$ *induces a Hodge isometry* $T_{\mathcal{X}}(2) \cong T_{\mathrm{Kum}(A)}$. *In Picard rank 16, the map* $\hat{\Phi} : \mathcal{X} \dashrightarrow \mathcal{Y}$ *in Equation (4.30) does NOT induce a Hodge isometry. In Proposition 3.2 the transcendental lattice* $T_{\mathcal{Y}}$ *of the family of K3 surfaces* $\mathcal{Y}$*, and in Proposition 4.3 the lattice polarization of the family of K3 surfaces* $\mathcal{X}$ *were determined. For generic parameters, we have*

$$\begin{aligned} T_{\mathcal{X}} &= H \oplus H \oplus \langle -2 \rangle^{\oplus 2}, \\ T_{\mathcal{Y}} &= H(2) \oplus H(2) \oplus \langle -2 \rangle^{\oplus 2}. \end{aligned} \tag{4.31}$$

Hence, it is no longer the case that $T_{\mathcal{X}}(2) \cong T_{\mathcal{Y}}$.

In the context of the above results, we have the following:

Lemma 4.9. *Any* $H \oplus E_7(-1) \oplus E_7(-1)$*-polarized K3 surface* $\mathcal{X}$ *given by Equation (4.1) is the Van Geemen-Sarti partner of a K3 surface* $\mathcal{Y}$ *given in Theorem 3.6 associated with a six-line configuration in* $\mathbb{P}^2$ *with invariants* J_k *for* $k = 2, \ldots, 6$. *In particular, we have*

$$[J_2 : \; J_3 : \; J_4 : \; J_5 : \; J_6] = [\, \alpha : \; \beta : \; \gamma \cdot \varepsilon : \; \gamma \cdot \zeta + \delta \cdot \varepsilon : \; \delta \cdot \zeta] \tag{4.32}$$

as points in the four-dimensional weighted projective space $\mathbb{P}(2, 3, 4, 5, 6)$.

Proof. The proof follows directly by comparing Equation (4.26) – obtained by fiber-wise two-isogeny from Equation (4.18) – with Equation (3.19). It then follows that $\mathcal{A}(t) = A(t)$ and $\mathcal{B}(t) = B(t)$, and the claim follows. □

We also have the following:

Lemma 4.10. *The isomorphism classes in the family of K3 surfaces* $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ *in Equation (4.1) are parametrized by the four-dimensional open complex variety* $\mathfrak{M}$ *defined in Equation (2.40).*

Proof. As a consequence of Theorem 4.1, one has an isomorphism of polarized K3 surfaces

$$\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \;\simeq\; \mathcal{X}(\alpha, \; \beta, \; t\gamma, \; t\delta, \; t^{-1}\varepsilon, \; t^{-1}\zeta) \tag{4.33}$$

for any $t \in \mathbb{C}^*$. The conditions imposed on the pairs $(J_3, \; J_4, \; J_5) \neq (0, 0, 0)$ ensure that singularities of $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ are rational double points. □

Combing the results of Lemma 4.9 and Lemma 4.10 we obtain the following:

Corollary 4.11. *The moduli space* $\mathfrak{M}$ *in Equation (2.40) is the coarse moduli space of K3 surfaces endowed with* $H \oplus E_7(-1) \oplus E_7(-1)$ *lattice polarization.*

We have the immediate consequence:

Corollary 4.12. *The loci of the singular fibers in the alternate fibration on the K3 surface $\mathcal{X}$ are determined by the Satake sextic in Section 2.2. That is, if $\varpi \in \mathfrak{M}$ is the point in the moduli space associated with the six-line configuration defining $\mathcal{Y}$ and $\mathcal{X}$, the loci of the fibers of Kodaira type I_1 and I_2 are given by $\mathcal{S} = \mathcal{B}^2 - 4\,\mathcal{A} = 0$ and $\mathcal{A} = 0$, respectively, with*

$$\mathcal{B} = t^3 - 3\,J_2(\varpi)\,t - 2\,J_3(\varpi), \quad \mathcal{A} = J_4(\varpi)\,t^2 - J_5(\varpi)\,t + J_6(\varpi), \tag{4.34}$$

where $J_k(\varpi)$ are the modular forms of weights $2k$ for $k = 2, \dots, 6$ in Theorem 2.24 generating the ring of modular forms relative to $\Gamma_{\mathcal{T}}$.

Next, we describe what confluences of singular fibers appear in the three Jacobian elliptic fibrations determined above. We discuss several cases for each fibration where the labelling corresponds to the one used to characterize six-line configurations in Definition 2.1, Lemma 2.20, and Corollary 2.22. We have the following:

Lemma 4.13. *The Weierstrass model in Equation (4.18) associated with a six-line configuration in $\mathbb{P}^2$ with invariants J_k for $k = 2, \dots, 6$ satisfies the following:*

- (0) *In the generic case, there are singular fibers $I_8^* + 2\,I_2 + 6\,I_1$.*
- (0b) *If* $\operatorname{Res}(\mathcal{A}, \mathcal{B}) = 0$*, one I_1 and one I_2 fiber coalesce to a III fiber.*
- (1) *If $J_4 = 0$, one I_2 and the I_8^* fiber coalesce to an I_{10}^* fiber.*
- (2) *If* $\operatorname{Disc}(\mathcal{A}) = 0$*, two I_2 fibers coalesce to an I_4 fiber.*
- (2b) *If* $\operatorname{Disc}(\mathcal{S}) = 0$*, two I_1 fibers coalesce to an I_2 fiber.*
- (3+4) *If* $\operatorname{Disc}(\mathcal{A}) = \operatorname{Res}(\mathcal{A}, \mathcal{B}) = 0$*, two I_1 and two I_2 fibers coalesce to an I_0^* fiber.*
- (5) *If $J_4 = J_5 = 0$, two I_2 fibers and the I_8^* fiber coalesce to an I_{12}^* fiber.*

Proof. The coefficients of the Weierstrass model in Equation (4.18) can be written in terms of modular forms; this is Equation (6.17). The proof follows from the application of Tate's algorithm. Notice that $J_4 = \operatorname{Disc}(\mathcal{A}) = 0$ is equivalent to $J_4 = J_5 = 0$; and $\operatorname{Disc}(\mathcal{A}) = \operatorname{Res}(\mathcal{A}, \mathcal{B}) = 0$ implies $\operatorname{Disc}(\mathcal{S}) = 0$. □

We also have the following:

Lemma 4.14. *The Weierstrass model in Equation (4.22) associated with a six-line configuration in $\mathbb{P}^2$ with invariants J_k for $k = 2, \dots, 6$ satisfies the following:*

- (0) *In the generic case, there are singular fibers $II^* + 6\,I_1 + I_2^*$.*
- (0b) *If* $\operatorname{Res}(\check{f}\,T^{-2}, \check{g}\,T^{-3}) = 0$*, two I_1 fibers coalesce to a II fiber.*
- (1) *If $J_4 = 0$, one I_1 and the I_2^* fiber coalesce to a III^* fiber.*
- (2) *If* $\operatorname{Disc}(\mathcal{A}) = 0$*, one I_1 and the I_2^* fiber coalesce to an I_3^* fiber.*
- (2b) *If* $\operatorname{Disc}(\mathcal{S}) = 0$*, two I_1 fibers coalesce to an I_2 fiber.*
- (3+4) *If* $\operatorname{Disc}(\mathcal{A}) = \operatorname{Res}(\mathcal{A}, \mathcal{B}) = 0$*, two I_1 and the I_2^* fiber coalesce to an I_4^* fiber.*
- (5) *If $J_4 = J_5 = 0$, two I_1 fibers and the I_2^* fiber coalesce to a II^* fiber.*

Proof. The coefficients of the Weierstrass model in Equation (4.22) can be written in terms of modular forms; this is Equation (6.8). The proof then follows from the application of Tate's algorithm. ☐

Similarly, one proves the following:

Lemma 4.15. *The Weierstrass model in Equation (4.13) associated with a six-line configuration in $\mathbb{P}^2$ with invariants J_k for $k = 2, \dots, 6$ satisfies the following:*

(0) *In the generic case, there are singular fibers $III^* + 6\, I_1 + III^*$.*
(0b) *If $\operatorname{Res}(fs^{-2}, gs^{-5}) = 0$, two I_1 fibers coalesce to a II fiber.*
(1) *If $J_4 = 0$, one I_1 and one III^* fiber coalesce to a II^* fiber.*
(2b) *If $\operatorname{Disc}(\mathcal{S}) = 0$, two I_1 fibers coalesce to an I_2 fiber.*
(5) *If $J_4 = J_5 = 0$, two pairs of I_1 and III^* fiber coalesce each to a II^* fiber.*

Notice that cases in which two I_1's coalesce to form a fiber of type II or one I_1 fiber and one I_2 fiber coalesce to a fiber of type III – a case we labelled (0b), adding to cases (0) through (5) in Definition 2.1 – do not affect the lattice polarization. An immediate consequence is the following:

Corollary 4.16. *The family of K3 surfaces in Equation (4.1) satisfies the following:*

(0) *For a generic point in $\mathfrak{M}$, there is a $H \oplus E_7(-1) \oplus E_7(-1)$ polarization.*
(1) *If $J_4 = 0$, the polarization extends to $H \oplus E_8(-1) \oplus E_7(-1)$.*
(2) *if $\operatorname{Disc}(\mathcal{A}) = 0$, the polarization extends to $H \oplus E_8(-1) \oplus D_7(-1)$.*
(2b) *If $\operatorname{Disc}(\mathcal{S}) = 0$, the polarization extends to $H \oplus E_7(-1) \oplus E_7(-1) \oplus \langle -2 \rangle$.*
(3+4) *If $\operatorname{Disc}(\mathcal{A}) = \operatorname{Res}(\mathcal{A}, \mathcal{B}) = 0$, the polarization extends to $H \oplus E_8(-1) \oplus D_8(-1)$.*
(5) *If $J_4 = J_5 = 0$, the polarization extends to $H \oplus E_8(-1) \oplus E_8(-1)$.*

Proof. The presence of a singular fiber of Kodaira type II^* in the fibration given by Equation (4.22) implies that we have, in all cases, a Mordell-Weil group of sections $\mathrm{MW}(\check{\pi}^{\mathcal{X}}_{\text{alt}}) = \{\mathbb{I}\}$ [63, Lemma 7.3]. Therefore, the lattice polarization coincides with the trivial lattice generated by the singular fibers extended by H generated by the classes of the smooth fiber and the section of the elliptic fibration. ☐

5. Specialization to six lines tangent to a conic

In this section we consider the specialization of the generic six-line configuration when the six lines are tangent to a common conic. Such a configuration has three moduli which we will denote by $\lambda_1, \lambda_2, \lambda_3$. It follows from [14, Prop. 5.13] that the lines can be brought into the form

$$
\begin{array}{rrcl}
\ell_1 : & z_1 & = & 0, \\
\ell_2 : & z_2 & = & 0, \\
\ell_3 : & z_1 + z_2 - z_3 & = & 0, \\
\ell_4 : & \lambda_1^2 \, z_1 + z_2 - \lambda_1 z_3 & = & 0, \\
\ell_5 : & \lambda_2^2 \, z_1 + z_2 - \lambda_2 z_3 & = & 0, \\
\ell_6 : & \lambda_3^2 \, z_1 + z_2 - \lambda_3 z_3 & = & 0,
\end{array}
\tag{5.1}
$$

where $\lambda_i \neq 0, 1, \infty$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$. We have the following:

Lemma 5.1. *The six lines in Equation (5.1) are tangent to* $C : z_3^2 - 4z_1z_2 = 0$.

Proof. It is easy to prove that the intersection of the conic $C : z_3^2 - 4z_1z_2$ with any of the six lines ℓ_i for $1 \leq i \leq 6$ in Equation (5.1) yields a root of order two, that is, a point of tangency. □

The following lemma is immediate:

Lemma 5.2. *For a configuration of six lines tangent to a conic, the K3 surface* $\mathcal{Y}$ *satisfies the following:*

(1) *Equation (3.7) is a Jacobian elliptic fibration* $\pi^{\mathcal{Y}}_{\text{nat}} : \mathcal{Y} \to \mathbb{P}^1$ *with 6 singular fibers of type* I_2*, two singular fibers of type* I_0^**, and the Mordell-Weil group of sections* $\mathrm{MW}(\pi^{\mathcal{Y}}_{\text{nat}}) = (\mathbb{Z}/2\mathbb{Z})^2 + \langle 1 \rangle$.
(2) *Equation (3.19) is a Jacobian elliptic fibration* $\pi^{\mathcal{Y}}_{\text{alt}} : \mathcal{Y} \to \mathbb{P}^1$ *with 6 singular fibers of type* I_2*, one fiber of type* I_5^**, one fiber of type* I_1*, and a Mordell-Weil group of sections* $\mathrm{MW}(\pi^{\mathcal{Y}}_{\text{alt}}) = \mathbb{Z}/2\mathbb{Z}$.

Proof. The proof of (1) was given in [14, Prop. 5.13]. The proof of (2) was given in [43, Prop. 9]. □

Lemma 5.3. *For a configuration of six lines tangent to a conic, the K3 surface* $\mathcal{Y}$ *is the Kummer surface* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ *of the principally polarized abelian surface* $\mathrm{Jac}(\mathcal{C})$*, i.e., the Jacobian variety of a generic genus-two curve* $\mathcal{C}$*. In particular, the curve* $\mathcal{C}$ *is given in Rosenhain normal form as*

$$\mathcal{C}: \quad Y^2 = F(X) = X(X-1)(X-\lambda_1)(X-\lambda_2)(X-\lambda_3). \tag{5.2}$$

Proof. All inequivalent elliptic fibrations on a generic Kummer surface where determined explicitly by Kumar in [39]. In fact, Kumar computed elliptic parameters and Weierstrass equations for all twenty five different fibrations that appear, and analyzed the reducible fibers and Mordell-Weil lattices. Equation (3.7) is the Weierstrass model of the elliptic fibration `(7)` in the list of all possible elliptic fibrations in [39, Thm. 2]. □

The ordered tuple $(\lambda_1, \lambda_2, \lambda_3)$ determines a point in the moduli space of genus-two curves together with a level-two structure, and, in turn, a level-two structure on the corresponding Jacobian variety, i.e., a point in the moduli space of principally polarized abelian surfaces with level-two structure

$$\mathfrak{A}_2(2) = \mathbb{H}_2/\Gamma_2(2), \tag{5.3}$$

where $\Gamma_2(2)$ is the principal congruence sub-group of level two of the Siegel modular group $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z})$. In turn, the Rosenhain invariants generate the function field $\mathbb{C}(\lambda_1, \lambda_2, \lambda_3)$ of $\mathfrak{A}_2(2)$. For a Jacobian variety with level-two structure corresponding to $\tau \in \mathfrak{A}_2(2)$, we have six odd theta characteristics and ten even theta characteristics; see [8, 61] for details. We denote the even theta characteristics by

$$\vartheta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \vartheta_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \vartheta_3 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \vartheta_4 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \vartheta_5 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\vartheta_6 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \vartheta_7 = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \vartheta_8 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \vartheta_9 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \vartheta_{10} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

and write

$$\vartheta_i(\tau) \text{ instead of } \vartheta \begin{bmatrix} a^{(i)} \\ b^{(i)} \end{bmatrix} (\tau) \text{ with } i = 1, \dots, 10, \tag{5.4}$$

and $\vartheta_i = \vartheta_i(0)$. Fourth powers of theta constants are modular forms of $\mathfrak{A}_2(2)$ and define the Satake compactification of $\mathfrak{A}_2(2)$ given by $\operatorname{Proj}[\vartheta_1^4 : \cdots : \vartheta_{10}^4]$.

The three λ-parameters in the Rosenhain normal (5.2) can be expressed as ratios of even theta constants by Picard's lemma. There are 720 choices for such expressions since the forgetful map, i.e., forgetting the level-two structure, is a Galois covering of degree $720 = |\mathrm{S}_6|$ since S_6 acts on the roots of $\mathcal{C}$ by permutations. Any of the 720 choices may be used, we chose the one from [29]:

Lemma 5.4. *If $\mathcal{C}$ is a non-singular genus-two curve with period matrix τ for $\operatorname{Jac}(\mathcal{C})$, then $\mathcal{C}$ is equivalent to the curve (5.2) with Rosenhain parameters $\lambda_1, \lambda_2, \lambda_3$ given by*

$$\lambda_1 = \frac{\vartheta_1^2 \vartheta_4^2}{\vartheta_2^2 \vartheta_3^2}, \quad \lambda_2 = \frac{\vartheta_4^2 \vartheta_7^2}{\vartheta_2^2 \vartheta_9^2}, \quad \lambda_3 = \frac{\vartheta_1^2 \vartheta_7^2}{\vartheta_3^2 \vartheta_9^2}. \tag{5.5}$$

Conversely, given three distinct complex numbers $(\lambda_1, \lambda_2, \lambda_3)$ different from $0, 1, \infty$ there is complex abelian surface A with period matrix $[\mathbb{I}_2|\tau]$ such that $A = \operatorname{Jac}(\mathcal{C})$ where $\mathcal{C}$ is the genus-two curve with period matrix τ.

Proof. A proof can be found in [61, Lemma 8]. $\square$

We also have the following:

Lemma 5.5. *The following equations relate theta functions and branch points:*

$$\begin{aligned}
\vartheta_1^4 &= \kappa\, \lambda_1 \lambda_3 (\lambda_2 - 1)(\lambda_3 - \lambda_1) & \vartheta_2^4 &= \kappa\, \lambda_3 (\lambda_2 - \lambda_1)(\lambda_3 - 1) \\
\vartheta_3^4 &= \kappa\, \lambda_2 (\lambda_2 - 1)(\lambda_3 - \lambda_1) & \vartheta_4^4 &= \kappa\, \lambda_1 \lambda_2 (\lambda_2 - \lambda_1)(\lambda_3 - 1) \\
\vartheta_5^4 &= \kappa\, \lambda_2 (\lambda_1 - 1)(\lambda_3 - \lambda_1)(\lambda_3 - 1) & \vartheta_6^4 &= \kappa\, \lambda_3 (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_2 - \lambda_1) \\
\vartheta_7^4 &= \kappa\, \lambda_2 \lambda_3 (\lambda_1 - 1)(\lambda_3 - \lambda_2) & \vartheta_8^4 &= \kappa\, \lambda_1 (\lambda_2 - 1)(\lambda_3 - 1)(\lambda_3 - \lambda_2) \\
\vartheta_9^4 &= \kappa\, \lambda_1 (\lambda_1 - 1)(\lambda_3 - \lambda_2), & \vartheta_{10}^4 &= \kappa\, (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2),
\end{aligned} \tag{5.6}$$

where $\kappa \neq 0$ is a non-zero constant.

Proof. The proof follows immediately using Thomae's formula. $\square$

We can now express the invariants t_i in terms of theta functions:

Proposition 5.6. *For a configuration of six lines tangent to a conic associated with a genus-two curve $\mathcal{C}$ with level-two structure, the period matrix $\tau \in \mathfrak{A}_2(2)$ determines a point $\varpi \in \mathbf{H}_2/\Gamma_{\mathcal{T}}(1+i)$ such that*

$$\Big[t_1(\varpi) : \dots : t_{10}(\varpi)\Big] = \Big[\vartheta_1^4(\tau) : \ \vartheta_2^4(\tau) : \ \dots : \vartheta_{10}^4(\tau)\Big] \in \mathbb{P}^9, \quad R = 0. \tag{5.7}$$

Proof. For the lines in Equations (5.1) we compute the period matrix τ for $\mathrm{Jac}(\mathcal{C})$ using Lemma (5.4). Setting $\varpi = \tau$ yields a $\mathcal{T}$-invariant point in $\mathbf{H}_2/\Gamma_{\mathcal{T}}(1+i)$. By construction, the modular forms $\theta_i^2(\varpi)$ equal t_i for $1 \leq i \leq 10$ and can be computed directly from Equations (2.1) for the lines in Equations (5.1). On the other hand, we can also compute the fourth powers of theta functions directly using Equations (5.6) to confirm Equation (5.7). □

Remark 5.7. *Proposition 5.6 is a special case of a statement in* [48, Lemma 2.1.1(vi)] *where it was shown that under the restriction to $\mathbb{H}_2/\Gamma_2(2)$ we have $\theta_i(\varpi) = \vartheta_i^2(\tau)$.*

For the Siegel three-fold $\mathfrak{A}_2 = \mathbb{H}_2/\Gamma_2$, i.e., the set of isomorphism classes of principally polarized abelian surfaces, the even Siegel modular forms of $\mathfrak{A}_2$ form a polynomial ring in four free generators of degrees 4, 6, 10 and 12 usually denoted by $\psi_4, \psi_6, \chi_{10}$ and χ_{12}, respectively. Igusa showed in [30] that for the full ring of modular forms, one needs an additional generator χ_{35} which is algebraically dependent on the others. In fact, its square is a polynomial in the even generators given in [30, p. 849].

Let I_2, I_4, I_6, I_{10} denote Igusa invariants of the binary sextic $Y^2 = F(X)$ as defined in [47, Sec. 2.3]. Igusa [30, p.848] proved that the relation between the Igusa invariants of a binary sextic $Y^2 = F(X)$ defining a genus-two curve $\mathcal{C}$ with period matrix τ for $\mathrm{Jac}(\mathcal{C})$ and the even Siegel modular forms are as follows:

$$\begin{aligned} I_2(F) &= -2^3 \cdot 3 \, \frac{\chi_{12}(\tau)}{\chi_{10}(\tau)} \ , \\ I_4(F) &= \ \ 2^2 \, \psi_4(\tau) \ , \\ I_6(F) &= -\frac{2^3}{3} \, \psi_6(\tau) - 2^5 \, \frac{\psi_4(\tau) \, \chi_{12}(\tau)}{\chi_{10}(\tau)} \ , \\ I_{10}(F) &= -2^{14} \, \chi_{10}(\tau) \ . \end{aligned} \tag{5.8}$$

Conversely, the point $[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{P}(2,4,6,10)$ in weighted projective space equals

$$\big[2^3 3^2 \chi_{12} : 2^2 3^2 \psi_4 \chi_{10}^2 : 2^3 3^2 \big(12 \psi_4 \chi_{12} + \psi_6 \chi_{10}\big) \chi_{10}^2 : 2^2 \chi_{10}^6\big]. \tag{5.9}$$

We have the following:

Proposition 5.8. *For a configuration of six lines tangent to a conic associated with the binary sextic $Y^2 = F(X)$ defining a genus-two curve $\mathcal{C}$, the period matrix τ*

determines a point $\varpi \in \mathbf{H}_2/\Gamma_{\mathcal{T}}$ *such that*

$$\begin{aligned}\Big[J_2(\varpi) : J_3(\varpi) : J_4(\varpi) : J_5(\varpi) : J_6(\varpi)\Big] &= \Big[\psi_4(\tau) :\ \psi_6(\tau) :\ 0 :\ 2^{12}3^5\chi_{10}(\tau) :\ 2^{12}3^6\chi_{12}(\tau)\Big] \\ &= \Big[\tfrac{1}{4}I_4(F) : \tfrac{1}{8}(I_2I_4 - 3I_6)(F) : 0 : -\tfrac{243}{4}I_{10}(F) : \tfrac{243}{32}I_2I_{10}(F)\Big]\end{aligned} \tag{5.10}$$

as points in $\mathbb{P}(2, 3, 4, 5, 6)$. *The discriminant of the Satake sextic restricts to*

$$\mathrm{Disc}(\mathcal{S}) = 2^{64}3^{30}\frac{\chi_{35}^2(\tau)}{\chi_{10}(\tau)}. \tag{5.11}$$

Proof. For the lines in Equations (5.1) we compute the period matrix τ for $\mathrm{Jac}(\mathcal{C})$ using Lemma (5.4). Setting $\varpi = \tau$ and forgetting the level-two structure, yields a $\mathcal{T}$-invariant point in $\mathbf{H}_2/\Gamma_{\mathcal{T}}$. By construction, the modular forms $J_k(\varpi)$ equal J_k for $2 \le k \le 6$ and can be computed directly from Equations (2.32) for the lines in Equations (5.1). On the other hand, we can compute the Igusa invariants I_2, I_4, I_6, I_{10} of the binary sextic $Y^2 = F(X)$ as defined in [47, Sec. 2.3] for the genus-two curve (5.2) to confirm Equation (5.10). We then use Equation (5.9) to convert to expressions in terms of $\psi_4, \psi_6, \chi_{10}$ and χ_{12}. □

To summarize, when the six lines are tangent to a conic, the K3 surface $\mathcal{Y}$ becomes the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ of the Jacobian variety $\mathrm{Jac}(\mathcal{C})$ of a generic genus-two curve $\mathcal{C}$. In [11, 12] it was proved that the K3 surface $\mathcal{X}$ in turn is the Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$, i.e., a K3 surface which carries a Nikulin involution such that quotienting by this involution and blowing up the fixed points, recovers the Kummer surface $\mathcal{Y}$ *and* the rational quotient map of degree two induces a Hodge isometry[1] between the transcendental lattices $T_{\mathcal{X}}(2)$[2] and $T_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})}$. In particular, the Shioda-Inose surface $\mathcal{X}$ and the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ have Picard rank greater or equal to 17. Proposition 5.8 then has the following corollary:

Corollary 5.9. *Configurations of six lines tangent to a conic give rise to a three-parameter family of Kummer surfaces* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ *of the Jacobian varieties* $\mathrm{Jac}(\mathcal{C})$ *of generic genus-two curves* $\mathcal{C}$. *Moreover, the corresponding three-parameter family of Shioda-Inose surfaces* $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ *associated with* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ *is obtained by setting* $\epsilon = 0$ *and* $\zeta = 1$ *in Equation (4.1).*

We also have the following:

Corollary 5.10. *Along the locus* $J_4 = 0$, *the lattice polarization of the K3 surfaces* $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon = 0, \zeta = 1)$ *extends to a canonical* $H \oplus E_8(-1) \oplus E_7(-1)$ *lattice polarization.*

---

[1] A Hodge isometry between two transcendental lattices is an isometry preserving the Hodge structure.

[2] The notation $T_{\mathcal{X}}(2)$ indicates that the bilinear pairing on the transcendental lattice $T_{\mathcal{X}}$ is multiplied by 2.

Proof. It was proved in [11] that the family in Equation (4.1) with $\varepsilon = 0, \zeta = 1$ is endowed with a canonical $H \oplus E_8(-1) \oplus E_7(-1)$ lattice polarization. They also found the parameters $(\alpha, \beta, \gamma, \delta)$ in terms of the standard even Siegel modular forms $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ (cf. [29]) given by

$$(\alpha, \beta, \gamma, \delta) = \big(\psi_4, \psi_6, 2^{12}3^5\, \chi_{10}, 2^{12}3^6\, \chi_{12}\big) \ , \tag{5.12}$$

which agrees with Equation (5.10) and Equation (4.32). □

The different Jacobian elliptic fibrations, the Satake sextic, and further confluences of singular fibers were investigated in [43, 47].

6. Relation to string theory

Using the results from Section 2 through Section 4, the goal for the remainder of this article is to determine the duality map (and thus the quantum-exact effective interactions) between a particular dual F-theory/heterotic string pair in eight space-time dimensions after restriction to a natural four-dimensional sub-space in the full eighteen dimensional moduli space. In fact, we will be restricting to the sub-space, which describes the partial higgsing of the gauge algebra $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_8$ to $\mathfrak{e}_7 \oplus \mathfrak{e}_7$[3] of the associated low energy effective eight-dimensional supergravity theory, and then establish the F-theory/heterotic string correspondence explicitly.

We let $L^{2,4}$ be the lattice of signature $(2, 4)$ which is the orthogonal complement of $E_7(-1) \oplus E_7(-1)$ in the unique integral even unimodular lattice $\Lambda^{2,18}$ of signature $(2, 18)$, which is

$$\Lambda^{2,18} = H \oplus H \oplus E_8(-1) \oplus E_8(-1) \ . \tag{6.1}$$

By insisting that the Wilson lines values associated to the $E_7(-1) \oplus E_7(-1)$ sub-lattice are trivial, we restrict to heterotic vacua parameterized by the quotient of the symmetric space[4] for $O(2, 4)$ by the automorphism group $O(L^{2,4})$, i.e., the space

$$\mathcal{D}_{2,4}/O(L^{2,4}). \tag{6.3}$$

The space $\mathcal{D}_{2,4}$ is also known as *bounded symmetric domain of type IV*. An eight-dimensional effective theory for the heterotic string compactified on $\mathbf{T}^2$ has a complex scalar field which then takes values in the Narain space (6.3); see [56]. In the large volume limit, the space (6.3) decomposes as a product of spaces parameterizing the Kähler and complex structures on $\mathbf{T}^2$ as well as two complex Wilson line expectation values around the two generators of $\pi_1(\mathbf{T}^2)$; see [57] for details. However, the decomposition is not preserved when the moduli vary arbitrarily. Families of

[3] As we shall see our methods also works for the heterotic string with gauge algebra $\mathfrak{g} = \mathfrak{so}(32)$ restricted to $\mathfrak{so}(24) \oplus \mathfrak{su}(2)^{\oplus 2}$

[4] By $\mathcal{D}_{p,q}$ we denote the symmetric space for $O(p, q)$, i.e.,

$$\mathcal{D}_{p,q} = (O(p) \times O(q))\backslash O(p, q). \tag{6.2}$$

heterotic models employing the full $O(L^{2,4})$ symmetry are therefore considered *non-geometric* compactifications, because the Kähler and complex structures on $\mathbf{T}^2$, and the Wilson line values, are not distinguished under the $O(L^{2,4})$-equivalences but instead are mingled together. We further restrict to a certain index-two sub-group $O^+(L^{2,4}) \subset O(L^{2,4})$ in the construction above with the corresponding degree-two cover given by

$$\mathcal{D}_{2,4}/O^+(L^{2,4}). \tag{6.4}$$

The group $O^+(L^{2,4})$ is the maximal sub-group whose action preserves the complex structure on the symmetric space, and thus is the maximal sub-group for which modular forms are holomorphic. Therefore, the space (6.4) is the natural sub-space of the moduli space of non-geometric heterotic models whose quantum-exact effective heterotic description is geometrically captured by the ring of holomorphic modular forms on the bounded symmetric domain of type IV. These heterotic theories on a torus $\mathbf{T}^2$ have two complex moduli and two non-vanishing complex Wilson lines.

On the side of F-theory, the above restriction to the moduli space in Equation (6.4) should correspond to the K3 surfaces admitting a $H \oplus E_7(-1) \oplus E_7(-1)$ lattice polarization. The H summand in the lattice polarization is generated by the classes of the smooth fiber and the section of the elliptic fibration and contains a pseudo-ample class. The universal family of such K3 surfaces is the Clingher-Doran family of K3 surfaces in Equation (4.1). We determined equations for three important elliptic fibrations with sections on these K3 surfaces in Theorem 4.1 and Lemmas 4.4, 4.6, 4.7. We also identified the coarse moduli space $\mathfrak{M}$ of the Clingher-Doran family to be the quotient space of $\mathbf{H}_2$ by the modular group $\Gamma_{\mathcal{T}}$; see Theorem 2.24 and Corollary 4.11. As we will show, the ring of modular forms of even characteristic relative to $\Gamma_{\mathcal{T}}$ coincides with the ring of modular forms on the bounded symmetric domain of type IV. Therefore, we will have established a particular dual F-theory/heterotic string pair.

We will carry out the following steps: (1) we will prove in Section 6.2 that the function field of the Narain moduli space of quantum-exact heterotic string compactifications with two non-vanishing Wilson lines coincides with the ring of modular forms relative to $\Gamma_{\mathcal{T}}$ of even characteristic constructed in Theorem 2.24; (2) we will then provide Weierstrass models, given in Equation (6.8) and Equation (6.17), that define two elliptic fibrations with section on K3 surfaces with $H \oplus E_7(-1) \oplus E_7(-1)$ lattice polarization, with the coefficients in the equation being modular forms relative to $\Gamma_{\mathcal{T}}$ of even characteristic (in Section 6.2 and Section 6.5). These two Weierstrass equations are the F-theory models dual to non-geometric vacua of the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ and $\mathfrak{so}(32)$ heterotic string, respectively. The unbroken gauge algebra is either $\mathfrak{e}_7 \oplus \mathfrak{e}_7$ or $\mathfrak{so}(24) \oplus \mathfrak{su}(2)^{\oplus 2}$ ensuring that *two* Wilson line expectation values are non-zero. (3) In Section 6.3 we will prove that our F-theory models can be used to construct *supersymmetric families* of non-geometric heterotic compactifications. This is of great importance for building new non-geometric compactifications of the heterotic string by further compatifying to lower space-time dimensions. (4) In Section 6.4 we derive a condition for the F-theory models dual to the $\mathfrak{e}_7 \oplus \mathfrak{e}_7$ heterotic string to consistently admit the simpler (but naively expected) elliptic fibration with two fibers of Kodaira type III^* and a trivial Mordell-Weil group. We find that conditions similar to equations governing global and local anomaly cancellation ensure that pointlike instantons are avoided and the elliptic fibration extends across (a double cover of) the parameter space. (5) Finally, in Section 6.6 we elaborate on the impact of these applications.

6.1. **F-theory and F-theory/heterotic string duality.** F-theory [54, 55, 65] provides a general and non-perturbative approach to constructing vacua of string theory. In a standard compactification of the type IIB string, the axio-dilaton field τ is constant and no D7-branes are present. Vafa's idea in proposing F-theory [65] was to allow a variable axio-dilaton field τ and D7-brane sources, defining at a new class of models in which the string coupling is never weak. These compactifications of the type IIB string in which the axio-dilaton field varies over a base are referred to as *F-theory models*. They depend on the following key ingredients [54, 55]: an $\mathrm{SL}_2(\mathbb{Z})$ symmetry of the physical theory, a complex scalar field τ with positive imaginary part on which $\mathrm{SL}_2(\mathbb{Z})$ acts by fractional linear transformations, and D7-branes serving as the source for the multi-valuedness of τ. In this way, F-theory models correspond to torus fibrations over some compact base manifold. A well-known duality in string theory asserts that compactifying M-theory on a torus $\mathbf{T}^2$ with complex structure parameter τ and area A is dual to the type IIB string compactified on a circle of radius $A^{-3/4}$ with axio-dilaton field τ [2, 64]. In turn, this gives a connection between F-theory models and geometric compactifications of M-theory: after compactifying an F-theory model further on S^1 without breaking supersymmetry, one obtains a model which is dual to M-theory compactified on the total space of the torus fibration. The geometric M-theory model preserves supersymmetry exactly when the total space of the family is a Calabi–Yau manifold. In this way, we recover the familiar condition for supersymmetric F-theory models in eight dimensions: the total space of the fibration is a K3 surface.

Most of the literature has been focused on a subclass of such torus fibrations that are simpler to analyze, i.e., the class of genus-one fibration with a section or Jacobian elliptic fibrations. As pointed out by Witten [69], this subclass of models is physically simpler to treat, because the existence of a section implies the absence of NS-NS and R-R fluxes in F-theory. Geometrically, the restriction to fibrations with a section facilitated model building with various non-Abelian gauge symmetries using the Tate algorithm [33, 40]. In particular, insertions of seven-branes in an F-theory model correspond to singular fibers in the M-theory model. Through work of Kodaira [37] and Néron [58], all possible singular fibers in one-parameter families of elliptic curves have been classified. When interpreted from a physics point of view, this classification gives a catalog of the different types of seven-branes which must be inserted; see [46].

More recently, the geometry and physics of F-theory compactifications on genus-one fibrations without section has also been investigated using methods of toric hypersurfaces [9] and birational geometry [34, 35], and by describing their M-theory dual descriptions [1]. The set of genus-one fibrations with the same τ function and $\mathrm{SL}_2(\mathbb{Z})$ representation, known as the Tate-Shafarevich group, supplies an important additional degree of freedom in the construction of general F-theory models. The construction of a normal form for a certain class of K3 surfaces that admit a genus-one fibration without section, but no elliptic fibration and are closely connected to the so called CHL string was carried out in [14]. The CHL string is obtained from the heterotic string with gauge algebra $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ on a torus T^2 as a certain $\mathbb{Z}/2\mathbb{Z}$ quotient. A detailed discussion of the CHL string and the dual eight-dimensional F-theory models can be found in [7, 41, 71].

However, there is another part within the analysis of F-theory models that is greatly aided by the existence of a fibration with a section, namely the investigation of F-theory/heterotic string duality: this duality particularly predicts that F-theory compactified on an elliptically fibered K3 surface agrees on the quantum level with heterotic string theory compactified on an elliptic curve [16, 22]. Since there is no microscopic description of such an F-theory, the F-theory/heterotic string duality provides new insights into the physics of F-theory compactifications. Most importantly, the spectral cover construction [22] has shed light on the relationship between non-Abelian gauge groups in F-theory and the heterotic string using geometrically engineering of supersymmetric gauge theories [20]. As a result the correspondence between F- theory compactifications and the low energy effective gauge theory particle spectra is well understood. On the heterotic side, the F-theory/heterotic string duality has shed light on important non-perturbative aspects of the heterotic string, for example NS5-branes states and small instantons [60, 70], that admit a geometric description in the dual F-theory.

While F-theory/heterotic string duality is well established for particle spectra [70], it has been less explored on the level of moduli spaces for (quantum-exact) effective interactions. This is due to the fact that the F-theory/heterotic string duality is often formulated in a certain limit, the so called stable degeneration limit of the F-theory geometry and a large fiber limit of the heterotic compactification space [3, 5], which describes the duality only at the boundary of the moduli spaces of the dual theories. Beyond the matching of dimensions of the moduli spaces, an exact correspondence between the two moduli spaces is only known in the large volume limit on the heterotic side, which corresponds to the stable degeneration limit on the F-theory side [5, 19, 22]. In fact it was proven in [16] that corresponding regions of the two moduli spaces can be identified by means of classical geometric data when quantum corrections are small. Clingher and Doran [10] and McOrist et al. [53] gave an explicit description of the duality when all the Wilson lines are turned off. Clingher and Doran [11, 12] and Malmendier and Morrison [46] extended this result to one non-vanishing Wilson line in the heterotic theory. Upon further compatifying to lower space-time dimensions, these geometric F-theory scenarios give rise to non-geometric compactifications of the heterotic string as argued in [26, 46, 53]. Further F-theory model building was carried out in [21, 23].

6.2. **The $\mathfrak{e}_7 \oplus \mathfrak{e}_7$-string.** In this section, we prove that the ring of modular forms relative to $\Gamma_{\mathcal{T}}$ of even characteristic constructed in Theorem 2.24 is exactly the holomorphic function field of the Narain moduli space of heterotic string compactifications with two non-vanishing Wilson lines. We also determine the F-theory model dual to the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic theory on $\mathbf{T}^2$ with two non-vanishing Wilson lines that has the unbroken gauge group $\mathfrak{e}_7 \oplus \mathfrak{e}_7$.

In Section 2.1, we introduced the space $\mathbf{H}_2$ of complex two-by-two matrices ϖ over $\mathbb{C}$ such that the hermitian matrix $(\varpi - \varpi^\dagger)/(2i)$ is positive definite; see Equation (2.11). On these elements, the modular group $\Gamma_{\mathcal{T}}$ introduced in Equation (2.15) acts by matrix multiplication for elements in $U(2,2)$ and as matrix transposition, generated by the additional element $\mathcal{T} \cdot \varpi = \varpi^t$. In [48, Prop. 1.5.1] it was proved that there is an isomorphism $\Gamma_{\mathcal{T}} \cong O^+(L^{2,4})$ that induces an isomorphism

$$\mathbf{H}_2 \cong \mathcal{D}_{2,4}. \tag{6.5}$$

Generally, $O^+(L^{2,n})$ is the index-two sub-group given by the condition that the upper left minor of order two is positive; see [67] for details. The group $O^+(L^{2,4})$ contains the special orthogonal sub-group $SO^+(L^{2,4})$ of all elements of determinant one. In our situation, this group $SO^+(L^{2,4})$ is precisely the index-two sub-group $\Gamma^+_{\mathcal{T}}$ introduced in Equation (2.23): an isomorphism $SO^+(L^{2,4}) \cong \Gamma^+_{\mathcal{T}}$ is given by mapping the generators of $\Gamma^+_{\mathcal{T}}$ to generators of $SO^+(L^{2,4})$. In fact, we map the generators $G_1\mathcal{T}$ and $G_2, \dots, G_5$ in Lemma 2.13 to the generators explicitly given in [48, p. 393] and denote the latter by $\mathcal{G}_k \in SO^+(L^{2,4})$ for $k = 1, \dots, 5$.[5] Moreover, the elements G_1 and $\mathcal{T}$ are mapped to reflections $\mathcal{R}_{G_1}$ and $\mathcal{R}_{\mathcal{T}}$ in $O^+(L^{2,4})$ associated with roots of square -2 and -4, respectively, such that $\mathcal{G}_1 = \mathcal{R}_{G_1} \cdot \mathcal{R}_{\mathcal{T}}$. We also find $\mathcal{G}_3 = \mathcal{R}_{G_3} \cdot \mathcal{R}_{\mathcal{T}}$ for another reflection $\mathcal{R}_{G_3}$.[6] Note that reflections belong to $O^+(L^{2,4})$, but not to $SO^+(L^{2,4})$. In fact, the generators $\mathcal{R}_{G_1}, \mathcal{R}_{\mathcal{T}} \in O^+(L^{2,4})$ together with $\mathcal{G}_k \in SO^+(L^{2,4})$ for $k = 1, \dots, 5$ determine the full isomorphism $\Gamma_{\mathcal{T}} \cong O^+(L^{2,4})$.

The element $\mathcal{T}$ acts trivially on the five modular forms J_k of weights $2k$ for $k = 2, \dots, 6$. Thus, they all have *even characteristic* with respect to the action of $\mathcal{T}$. We proved in Theorem 2.24 that they freely generate the ring of modular forms relative to $\Gamma_{\mathcal{T}}$ with character $\chi_{2k}(g) = \det(G)^k$ for all $g = G\mathcal{T}^n \in \Gamma_{\mathcal{T}}$. By a result of Vinberg [67], the ring of modular forms relative to $O^+(L^{2,4})$ turns out to be exactly this ring of modular forms relative to $\Gamma_{\mathcal{T}}$ of even characteristic.

The space $\mathbf{H}_2$ is a generalization of the Siegel upper-half space $\mathbb{H}_2$. In fact, elements invariant under transposition $\mathcal{T}$ are precisely the two-by-two symmetric matrices over $\mathbb{C}$ whose imaginary part is positive definite, i.e., elements of the Siegel upper-half plane

[5] In [48] G_1 was mapped to $\mathcal{G}_1$ which is *not* compatible with the identification $SO^+(L^{2,4}) \cong \Gamma^+_{\mathcal{T}}$.

[6] In [48] the roots associated with $\mathcal{R}_{G_1}$, $\mathcal{R}_{\mathcal{T}}$, and $\mathcal{R}_{G_3}$ were denoted by $\alpha(1,2,3)$, β_1, and β_6.

$\mathbb{H}_2 \cong \mathcal{D}_{2,3}$, on which the modular group $\mathrm{Sp}_4(\mathbb{Z}) \cong SO^+(L^{2,3})$ acts. For the sub-space

$$\mathcal{D}_{2,3}/O^+(L^{2,3}) \hookrightarrow \mathcal{D}_{2,4}/O^+(L^{2,4}), \tag{6.6}$$

another result of Vinberg [68] proves that the ring of $O^+(L^{2,3})$-modular forms corresponds to the ring of Siegel modular forms of *even weight*. Igusa [29] showed that this ring of even modular forms is generated by the Siegel modular forms ψ_4, ψ_6, χ_{10}, χ_{12} of respective weights $4, 6, 10, 12$.

Matrix transposition $\mathcal{T}$ acts as -1 on the $\Gamma_{\mathcal{T}}$-modular forms of *odd characteristic*, and the fixed locus of $\mathcal{T}$ must be contained in the vanishing locus of any $\Gamma_{\mathcal{T}}$-modular form of odd characteristic. Modular forms of odd characteristic are generated by the unique (up to scaling) modular form $\Theta(\varpi)$ of odd characteristic introduced in Theorem 2.8. In Theorem 2.24 we found the relation $J_4(\varpi) = (\Theta(\varpi)/25)^2$. Therefore, the fixed locus of $\mathcal{T}$ coincides with the vanishing locus of $J_4(\varpi)$. In fact, we showed in Proposition 5.8 that in the case $\varpi = \tau \in \mathbb{H}_2$ we obtain

$$\Big[J_2(\varpi) : J_3(\varpi) : J_4(\varpi) : J_5(\varpi) : J_6(\varpi)\Big] = \Big[\psi_4(\tau) : \psi_6(\tau) : 0 : 2^{12}3^5\chi_{10}(\tau) : 2^{12}3^6\chi_{12}(\tau)\Big], \tag{6.7}$$

that is, $J_4 = (\Theta(\varpi)/25)^2$ vanishes and the other $\Gamma_{\mathcal{T}}$-modular forms restrict to the generators of the ring of Siegel modular forms.

Going back to the moduli space in Equation (6.4), the key geometric fact for the construction of F-theory models is that Equation (4.22) defines an elliptically fibered K3 surface $\mathcal{X}$ with section whose periods determine a point $\varpi \in \mathbf{H}_2$, with the coefficients in the equation being modular forms relative to $\Gamma_{\mathcal{T}}$ of even characteristic. In Remark 3.3 we outlined the construction and some analytic aspects of the period map. Re-writing Equation (4.22) in terms of the generators J_k with $k = 2, \dots, 6$ of the ring of modular forms yields

$$\begin{aligned} Y^2 = X^3 - T^2 \left(3\,J_2(\varpi)\,T^2 + J_5(\varpi)\,T + \frac{1}{3}\,J_4(\varpi)^2\right) X \\ + T^3 \left(T^4 - 2\,J_3(\varpi)\,T^3 + \big(J_2J_4 + J_6\big)(\varpi)\,T^2 + \frac{1}{3}\,J_4J_5(\varpi)\,T + \frac{2}{27}J_4(\varpi)^3\right). \end{aligned} \tag{6.8}$$

Under the restriction given by Equation (6.7) and a simple rescaling of variables, one obtains

$$Y^2 = X^3 - T^3\left(\frac{1}{48}\psi_4(\tau)\,T + 4\,\chi_{10}(\tau)\right) X + T^5\left(T^2 - \frac{1}{864}\psi_6(\tau)\,T + \chi_{12}(\tau)\right). \tag{6.9}$$

The latter equation was exactly the equation upon which the analysis of non-geometric heterotic string vacua with *one* Wilson line parameter was based in [46].

The explicit form of the F-theory/heterotic string duality on the moduli space in Equation (6.4) has two parts: starting from $\varpi \in \mathbf{H}_2$, we always obtain the equation of a Jacobian elliptic fibration on K3 surface given by Equation (6.8). Conversely, we can start with any Jacobian elliptic fibration on a K3 surface $\mathcal{X}$ given by the equation

$$\begin{aligned} Y^2 = X^3 + a\,T^2 X + b\,T^3 + c\,T^3 X + c\,d\,T^4 \\ + e\,T^4 X + (d\,e + f)\,T^5 + g\,T^6 + T^7. \end{aligned} \tag{6.10}$$

We can then determine a point in $\mathcal{D}_{2,4}$ by calculating the periods of the holomorphic two-form $\omega_{\mathcal{X}} = dT \wedge dX/Y$ over a basis of the lattice $H \oplus E_7(-1) \oplus E_7(-1)$ in $H^2(\mathcal{X}, \mathbb{Z})$ which in turn determines a point $\varpi \in \mathbf{H}_2$ using the isomorphism in Equation (6.5). For some non-vanishing scale factor λ, we obtain

$$\begin{aligned} c = -\lambda^{10} J_5(\varpi), \quad d = -\frac{\lambda^8}{3} J_4(\varpi), \quad e = -3\,\lambda^4 J_2(\varpi), \\ f = \lambda^{12} J_6(\varpi), \quad g = -2\,\lambda^6 J_3(\varpi), \end{aligned} \tag{6.11}$$

and $a = -3\,d^2, \quad b = -2\,d^3$. The Weierstrass equation (6.10) is therefore the F-theory model dual to the heterotic theory with two non-vanishing Wilson lines that has the unbroken gauge group $\mathfrak{e}_7 \oplus \mathfrak{e}_7$.

6.3. **Condition for five-branes and supersymmetry.** In this section, we describe how our results from the previous section are used to construct *families* of non-geometric heterotic compactifications that are supersymmetric.

We start with a compact manifold $\mathfrak{Z}$ as parameter space and a line bundle $\Lambda \to \mathfrak{Z}$. Choose sections $c(z)$, $d(z)$, $e(z)$, $f(z)$, and $g(z)$ of the bundles $\Lambda^{\otimes 10}$, $\Lambda^{\otimes 8}$, $\Lambda^{\otimes 4}$, $\Lambda^{\otimes 12}$, and $\Lambda^{\otimes 6}$, respectively; then, for each point $z \in \mathfrak{Z}$, there is a non-geometric heterotic compactification given by Equation (6.10) with $c = c(z)$, $d = d(z)$, etc., and $a = -3\,d(z)^2$, $b = -2\,d(z)^3$ and moduli $\varpi \in \mathbf{H}_2$ and $O^+(L^{2,4})$ symmetry such that Equations (6.11) hold.

Accordingly, for the corresponding heterotic models the gauge algebra is enhanced to $\mathfrak{e}_7 \oplus \mathfrak{e}_7$. Appropriate five-branes must still be inserted on $\mathfrak{Z}$ as dictated by the geometry of the corresponding family of K3 surfaces. The change in the singularities and the lattice polarization for the fibration (6.8) was determined in Lemma 4.14 and occur along three loci of co-dimension one. In Corollary 4.16 we proved that each locus is the vanishing locus of a polynomial in the modular forms J_k's, i.e., $\mathrm{Disc}(\mathcal{S}) = 0$, $J_4 = 0$, and $\mathrm{Disc}(\mathcal{A}) = 0$, respectively. We proved in Corollary 2.22 that each locus is the fixed locus of elements in $\Gamma_{\mathcal{T}} \setminus \Gamma_{\mathcal{T}}^+$. Using the isomorphism $\Gamma_{\mathcal{T}} \cong O^+(L^{2,4})$ it is trivial to write down the reflections in $O^+(L^{2,4}) \backslash SO^+(L^{2,4})$ corresponding to $\mathrm{Disc}(\mathcal{S}) = 0$, $J_4 = 0$, and $\mathrm{Disc}(\mathcal{A}) = 0$ explicitly: they are $\mathcal{R}_{\mathcal{T}}$ and orthogonal conjugates of $\mathcal{R}_{\mathcal{G}_1}$ and $\mathcal{R}_{\mathcal{G}_3}$ defined in Section 6.2.

From the point of view of K3 geometry, given as a reflection in a lattice element δ of square -2 we have the following: if the periods are preserved by the reflection in δ, then δ must belong to the Néron-Severi lattice of the K3 surface. That is, the Néron-Severi lattice is enlarged by adjoining δ. In Corollary 4.16 we proved that there are three ways an enlargement can happen: the lattice $H \oplus E_7(-1) \oplus E_7(-1)$ of rank sixteen can be extended to $H \oplus E_7(-1) \oplus E_7(-1) \oplus \langle -2 \rangle$, $H \oplus E_8(-1) \oplus E_7(-1)$, or $H \oplus E_8(-1) \oplus D_7(-1)$, each of rank seventeen.

From the heterotic side, these five-brane solitons are easy to see: when $\mathrm{Disc}(\mathcal{S}) = 0$, we have a gauge symmetry enhancement from $\mathfrak{e}_7 \oplus \mathfrak{e}_7$ to include an additional $\mathfrak{su}(2)$, and the parameters of the theory include a Coulomb branch for that gauge theory on which the Weyl group $W_{\mathfrak{su}(2)} = \mathbb{Z}_2$ acts. Thus, there is a five-brane solution in which the field has a $\mathbb{Z}_2$ ambiguity encircling the location in the moduli space of enhanced gauge symmetry. When $J_4 = 0$, we have an enhancement to $\mathfrak{e}_8 \oplus \mathfrak{e}_7$ gauge symmetry, and, when $\mathrm{Disc}(\mathcal{A}) = 0$ an enhancement to $\mathfrak{e}_8 \oplus \mathfrak{so}(14)$. Further enhancements to $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ gauge symmetry or $\mathfrak{e}_8 \oplus \mathfrak{so}(16)$ occur along $J_4 = J_5 = 0$ or $\mathrm{Disc}(\mathcal{A}) = \mathrm{Res}(\mathcal{A}, \mathcal{B}) = 0$, respectively.

To understand when such families of compactifications are supersymmetric, we mirror the discussion in [46]. A heterotic compactification on $\mathbf{T}^2$ with parameters given by $\varpi \in \mathbf{H}_2$ is dual to the F-theory compactification on the elliptically fibered K3 surface $\mathcal{X}(\varpi)$ defined by Equation (6.8). For sections $c(z)$, $d(z)$, $e(z)$, $f(z)$, and $g(z)$ of line bundles over $\mathfrak{Z}$, we have a criterion for when F-theory compactified on the elliptically fibered manifold (6.10) is supersymmetric: this is the case if and only if the total space defined by Equation (6.10) – now considered as an elliptic fibration over a base space locally given by variables T and z – is itself a Calabi–Yau manifold. The base space of the elliptic fibration is a $\mathbb{P}^1$-bundle $\pi : \mathfrak{W} \to \mathfrak{Z}$ which takes the form $\mathfrak{W} = \mathbb{P}(\mathcal{O} \oplus \mathcal{M})$ where $\mathcal{M} \to \mathfrak{Z}$ is the normal bundle of $\Sigma_0 := \{T = 0\}$ in $\mathfrak{W}$. Monomials of the form T^n are then considered sections of the line bundles $\mathcal{M}^{\otimes n}$. We also set $\Sigma_\infty := \{T = \infty\}$ such that $-K_{\mathfrak{W}} = \Sigma_0 + \Sigma_\infty + \pi^{-1}(-K_{\mathfrak{Z}})$.

When the elliptic fibration (6.10) is written in Weierstrass form, the coefficients of X^1 and X^0 must again be sections of $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 6}$, respectively, for a line bundle $\mathcal{L} \to \mathfrak{W}$. The condition for supersymmetry of the total space is that the line bundle $\mathcal{L}$ is the anti-canonical bundle of the base, $\mathcal{L} = \mathcal{O}_{\mathfrak{W}}(-K_{\mathfrak{W}})$. Restricting the various terms in Equation (6.10) to Σ_0, we find relations

$$
\begin{aligned}
(\mathcal{L}|_{\Sigma_0})^{\otimes 4} &= \Lambda^{\otimes 4} \otimes \mathcal{M}^{\otimes 4} = \Lambda^{\otimes 10} \otimes \mathcal{M}^{\otimes 3} = \Lambda^{\otimes 16} \otimes \mathcal{M}^{\otimes 2}, \\
(\mathcal{L}|_{\Sigma_0})^{\otimes 6} &= \mathcal{M}^{\otimes 7} = \Lambda^{\otimes 6} \otimes \mathcal{M}^{\otimes 6} \\
&= \Lambda^{\otimes 12} \otimes \mathcal{M}^{\otimes 5} = \Lambda^{\otimes 18} \otimes \mathcal{M}^{\otimes 4} = \Lambda^{\otimes 24} \otimes \mathcal{M}^{\otimes 3}
\end{aligned}
\tag{6.12}
$$

Thus, it follows that $\mathcal{M} = \Lambda^{\otimes 6}$ and $\mathcal{L}|_{\Sigma_0} = \Lambda^{\otimes 7}$ (up to torsion) and the $\mathbb{P}^1$-bundle takes the form $\mathfrak{W} = \mathbb{P}(\mathcal{O} \oplus \Lambda^{\otimes 6})$. Since Σ_0 and Σ_∞ are disjoint, the condition for supersymmetry is equivalent to $\Lambda = \mathcal{O}_{\mathfrak{Z}}(-K_{\mathfrak{Z}})$.

Therefore, we derived that supersymmetric families of non-geometric heterotic compactifications are obtained from Equation (6.10) when promoting $c(z)$, $d(z)$, $e(z)$, $f(z)$, and $g(z)$ to sections of the bundles $\Lambda^{\otimes 10}$, $\Lambda^{\otimes 8}$, $\Lambda^{\otimes 4}$, $\Lambda^{\otimes 12}$, and $\Lambda^{\otimes 6}$, respectively, over the parameter space $\mathfrak{Z}$, and the bundle Λ satisfies $\Lambda = \mathcal{O}_{\mathfrak{Z}}(-K_{\mathfrak{Z}})$. The location of the five-brane solitons are then controlled by the classical geometric invariants $\mathrm{Disc}(\mathcal{S}) = 0$, $J_4 = 0$, and $\mathrm{Disc}(\mathcal{A}) = 0$ introduced before.

6.4. **Double covers and pointlike instantons.** To a reader familiar with elliptic fibrations, it might come as a surprise that the Weierstrass model we considered in

Equation (6.10) did not simply have two fibers of Kodaira type III^* and a trivial Mordell-Weil group. On each K3 surface endowed with a $H \oplus E_7(-1) \oplus E_7(-1)$ lattice polarization in Corollary 4.11, such a fibration exists: we constructed it in Lemma 4.4. However, it is not guaranteed that the fibration extends across any parameter space.

To see this, assume $J_6 \neq 0$ and that there is an $\mathfrak{a}$ such that $\operatorname{Disc}(\mathcal{A}) = J_5^2 - 4\, J_4 J_6 = \mathfrak{a}^2$. Then, Equation (4.13) can be brought into the form

$$y^2 = x^3 - s^3 \left(\frac{J_5 - \mathfrak{a}}{2\, J_6}\, s^2 + 3\, J_2\, s + \frac{J_5 + \mathfrak{a}}{2} \right) x + s^5 \Big(s^2 - 2\, J_3\, s + J_6 \Big). \tag{6.13}$$

For $J_4 = 0$ it follows $J_5^2 = \mathfrak{a}^2$, and the choice of square root $\mathfrak{a} = \pm J_5$ determines whether either the III^* fiber over $s = 0$ or the one over $s = \infty$ is extended to a fiber of Kodaira type II^*; see Lemma 4.5. The situation is very different for the elliptic fibration with section used for the family construction in Section 6.3: it follows from our results in Lemma 4.7 and Corollary 4.11 that the previously used elliptic fibration with section which has one fiber of Kodaira type I_2^* or worse and another fiber of type precisely II^*. Because of the presence of a II^* fiber, the Mordell-Weil group is always trivial, including all cases with gauge symmetry enhancement. From a physics point of view as argued in [46], assuming that one fiber is fixed and of Kodaira type II^* avoids "pointlike instantons" on the heterotic dual after compactification to dimension six or below, at least for general moduli.

For the fibration in Equation (6.13) to have the same property, we have to be able to choose a square root $\mathfrak{a} = \pm J_5$ consistently throughout the parameter space. For $J_4 = 0$ and $\mathfrak{a} = J_5$ (and similarly for $\mathfrak{a} = -J_5$ after mapping $s \mapsto 1/s$), the restriction given by Equation (6.7) and a simple rescaling of variables then yield the same equation that describes non-geometric heterotic string vacua with one Wilson line parameter already encountered before, i.e.,

$$y^2 = x^3 - s^3 \left(\frac{1}{48} \psi_4(\tau)\, s + 4\, \chi_{10}(\tau) \right) x + \, s^5 \left(s^2 - \frac{1}{864} \psi_6(\tau)\, s + \chi_{12}(\tau) \right). \tag{6.14}$$

If we vary non-geometric heterotic vacua given by Equation (6.13) over a parameter space $\mathfrak{Z}$ as in Section 6.3, the functions J_k are again sections of line bundles $\Lambda^{\otimes 2k} \to \mathfrak{Z}$. For the coefficient of $s^5 x$ in Equation (6.13) to be well defined, a necessary condition is $J_6 \neq 0$ over $\mathfrak{Z}$ which implies that J_6 is a trivializing section for the bundle $\Lambda^{\otimes 12}$; in particular, we have $\Lambda^{\otimes 12} \cong \mathcal{O}_{\mathfrak{Z}}$.

In Equation (2.37) we obtained

$$\mathfrak{a}^2 = \operatorname{Disc}(\mathcal{A}) = J_5(\varpi)^2 - 4\, J_4 J_6(\varpi) = 2^{-4} 3^{10} \prod_{i=1}^{10} \theta_i^2(\varpi), \tag{6.15}$$

where $\theta_i^2(\varpi)$ are the ten theta functions of weight two relative to $\Gamma_{\mathcal{T}}(1+i)$. In our situation, they combine to form a section $\mathfrak{a}^2$ of the line bundle $\Lambda^{\otimes 20}$. We want to take the square root of the line bundle $\Lambda^{\otimes 20}$, that is, construct a line bundle $\Lambda' \to \mathfrak{Z}$ with $(\Lambda')^{\otimes 2} = \Lambda^{\otimes 20}$ such that $\mathfrak{a}$ becomes a section of the new line bundle. The square root of a line bundle (if it exists) is not unique in general, and any two of them will differ by a two-torsion line bundle. If the Picard group of $\mathfrak{Z}$ is torsion free, then there is at most one square root. We already know that one square root exists, namely the line bundle $\Lambda^{\otimes 10} \to \mathfrak{Z}$. Therefore, setting $H^2(\mathfrak{Z}, \mathbb{Z}_2) = 0$ guarantees that the square root is isomorphic to $\Lambda^{\otimes 10}$.

If we further assume that the line bundle is effective, i.e., $\Lambda^{\otimes 10} \cong \mathcal{O}_{\mathfrak{Z}}(D)$ for some effective divisor D – which is equivalent to $\dim H^0(\mathfrak{Z}, \Lambda^{\otimes 10}) > 0$ – then the existence of the square root of $\Lambda^{\otimes 20}$ is equivalent to *the existence of a double cover* $\mathfrak{Y} \to \mathfrak{Z}$ branched along the zero locus of the holomorphic section given by $J_5 - \mathfrak{a} = 0$. The vanishing locus corresponds exactly to heterotic models where the gauge algebra is enhanced to $\mathfrak{e}_8 \oplus \mathfrak{e}_7$.

Using the condition for supersymmetry already established in Section 6.3, we will assume that

$$\begin{array}{llll} (1) & H^2(\mathfrak{Z}, \mathbb{Z}_2) = 0, & (2) & \Lambda = \mathcal{O}_{\mathfrak{Z}}(-K_{\mathfrak{Z}}), \\ (3) & \dim H^0(\mathfrak{Z}, \Lambda^{\otimes 10}) > 0, & (4) & \Lambda^{\otimes 12} \cong \mathcal{O}_{\mathfrak{Z}}. \end{array} \tag{6.16}$$

Therefore, we obtain a consistent and supersymmetric family of non-geometric heterotic vacua given by Equation (6.13) over the parameter space $\mathfrak{Y}$ which is a double cover of $\mathfrak{Z}$ branched along $J_5 - \mathfrak{a} = 0$. The conditions derived in Equation (6.16) are similar to equations governing global and local anomaly cancellation [44, 45].

6.5. **The $\mathfrak{so}(32)$-string.** Here, we describe the $\mathfrak{so}(32)$ heterotic theory on $\mathbf{T}^2$ with two non-vanishing Wilson lines that has the unbroken gauge group $\mathfrak{so}(24) \oplus \mathfrak{su}(2)^{\oplus 2}$ and its dual F-theory model.

We proved in Lemma 4.6 that a K3 surface $\mathcal{X}$ with lattice polarization $H \oplus E_7(-1) \oplus E_7(-1)$ also admits an *alternate* elliptic fibration related to the $\mathfrak{so}(32)$ heterotic string. We now establish the explicit form of the F-theory/heterotic string duality on the moduli space in Equation (6.4) for the $\mathfrak{so}(32)$ string: Equation (4.18) defines an elliptically fibered K3 surface $\mathcal{X}$ whose periods determine a point $\varpi \in \mathbf{H}_2$, with the coefficients in the equation being modular forms of even characteristic. Re-writing Equation (4.18) in terms of the generators of the ring of these modular forms yields

$$y^2 = x^3 + \Big(t^3 - 3\,J_2(\varpi)\,t - 2\,J_3(\varpi)\Big)\,x^2 + \Big(J_4(\varpi)\,t^2 - J_5(\varpi)\,t - J_6(\varpi)\Big)\,x. \tag{6.17}$$

Under the restriction given by Equation (6.7) and a simple rescaling of variables, one obtains again

$$y^2 = x^3 + \Big(t^3 - \frac{1}{48}\,\psi_4(\tau)\,t - \frac{1}{864}\,\psi_6(\tau)\Big)\,x^2 - \Big(4\,\chi_{10}(\tau)\,t - \chi_{12}(\tau)\Big)\,x. \tag{6.18}$$

Conversely, we can start with any Jacobian elliptic fibration given by the equation

$$y^2 = x^3 + \Big(t^3 + e\,t + g\Big)\,x^2 + \Big(-3\,d\,t^2 + c\,t + f\Big)\,x. \tag{6.19}$$

We then determine a point in $\mathcal{D}_{2,4}$ by calculating the periods of the holomorphic two-form $\omega_{\mathcal{X}} = dt \wedge dx/y$ over a basis of the period lattice $H \oplus E_7(-1) \oplus E_7(-1)$ in

$H^2(\mathcal{X},\mathbb{Z})$. The discriminant is $\Delta_{\mathcal{X}}(t)=\mathcal{A}(t)^2\left(\mathcal{B}(t)^2-4\,\mathcal{A}(t)\right)$ with

$$\begin{gathered}\mathcal{S}(t)=\mathcal{B}(t)^2-4\,\mathcal{A}(t),\\ \mathcal{B}(t)=t^3-3\,J_2(\varpi)\,t-2\,J_3(\varpi),\quad \mathcal{A}(t)=J_4(\varpi)\,t^2-J_5(\varpi)\,t+J_6(\varpi).\end{gathered}\tag{6.20}$$

The fiber over $t=\infty$ is of Kodaira type I_8^*. In addition, there are two fibers of Kodaira type I_2 over $\mathcal{A}(t)=0$. The gauge algebra is enhanced to $\mathfrak{so}(24)\oplus\mathfrak{su}(2)^{\oplus 2}$. For generic coefficients, the other factor in the discriminant contributes six fibers of type I_1. The loci of the I_1 fibers form the ramification locus of the Satake sextic in Equation (6.20). Moreover, the section $x=y=0$ defines an element of order two in the Mordell-Weil group. It follows as in [4, 5] that the gauge group of this model is $(\mathrm{Spin}(24)\times SU(2)\times SU(2))/\mathbb{Z}_2$.

Further lattice enhancements were discussed in Lemma 4.13 and Corollary 4.16. For $\mathrm{Disc}(\mathcal{S})=0$, we have a gauge symmetry enhancement to include an additional $\mathfrak{su}(2)$. When $\mathrm{Disc}(\mathcal{A})=0$ the gauge algebra is enhanced to $\mathfrak{so}(24)\oplus\mathfrak{su}(4)$. For $J_4=0$, the gauge algebra is enhanced to $\mathfrak{so}(28)\oplus\mathfrak{su}(2)$. For $J_4=J_5=0$ the gauge group is enhanced to $\mathrm{Spin}(32)/\mathbb{Z}_2$. For $\mathrm{Disc}(\mathcal{A})=\mathrm{Res}(\mathcal{A},\mathcal{B})=0$ the gauge algebra is enhanced to $\mathfrak{so}(24)\oplus\mathfrak{so}(8)$. The intrinsic property of elliptically fibered K3 surfaces which leads to Equation (6.19) is the requirement that there be a two-torsion element in the Mordell–Weil group, and that one fiber in the fibration be of type I_n^* for some $n\geq 8$. Under these assumptions, following the same argument as in [53] we can always choose coordinates so that the Weierstrass equations is of the form (6.19). The Weierstrass equation is therefore the F-theory model dual to the heterotic theory with two non-vanishing Wilson lines that has the unbroken gauge group $\mathfrak{so}(24)\oplus\mathfrak{su}(2)^{\oplus 2}$.

6.6. **Conclusions.** The results of Sections 6.2 through Section 6.5 establish the F-theory/heterotic string correspondence on the natural sub-space on the full moduli space where the non-geometric heterotic description has two non-vanishing Wilson lines. Using the same methods as were used in [26] for the case of one Wilson line, our results thus allow to establish the F-theory/heterotic string duality in the entire non-geometric phase adiabatically fibered over a $\mathbb{P}^1$ with a precise limit to the semi-classical heterotic string in both eight and lower space-time dimensions. Crucial for such an extension is also the perfect match that we established in Section 5 between our results and the results in [46] when there is only one Wilson line parameter. In particular, our results imply that F-theory descriptions remain geometric over the entire moduli space dual to the heterotic string moduli space that continuously interpolates between the non-geometric quantum phase and the semi-classical phase with two non-vanishing Wilson lines.

To make further progress in analyzing the discussed class of non-geometric heterotic string compactifications in four and six dimensions, it would be interesting to study also global features of the six-line configurations underlying our description of

F-theory models. The geometry of double planes branched in six lines is interesting, and there already is a well established canon of mathematical literature on the subject; see [17, 36, 42]. With the lines in general position, such double planes are precisely the K3 surfaces identified in this article as F-theory models whose dual heterotic models we identified by the image under the inverse period map. However, the divisors parametrizing special line configurations also have a moduli interpretation, namely as a principally polarized abelian four-fold as established by Hermann in [27]. Interestingly, this abelian four-fold also coincides with the so called Kuga-Satake variety associated with the K3 surfaces up to isogeny. Therefore, it seems only prudent to ask for a detailed physical interpretation of this abelian four-fold itself in terms of non-geometric heterotic string vacua.

While the studied F-theory/heterotic quantum duality is based on the special class of F-theory/heterotic string models with two non-vanishing Wilson lines, it provides a rich and explicit testing ground for non-geometric string compactifications in general. Our technique of looking at the moduli space of elliptically fibered K3 surfaces obtained as double cover of the projective plane ramified along a sextic may also opens up a new method to arrive at more general non-geometric heterotic string theories beyond two Wilson line moduli. For example, it is natural to also analyze the K3 surfaces in the context of F-theory/heterotic string duality that are obtained as double cover of the plane ramified along a sextic which is made up of two lines and two conics, but not necessary tangent to some other curves. Generically, such K3 surfaces will have Picard number 14, and we expect them to describe F-theory models dual to heterotic models with four Wilson lines.

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Appendix A. Invariants of the quintic pencils

Using a 2-neighbor-step procedure twice starting with the natural fibration in Equation (3.7), we constructed on the K3 surface $\mathcal{Y}$ associated with the double cover branched along the six lines given by Equations (3.2) the following Weierstrass model:

$$Y^2 = X\Big(X^2 - 2\,\mathcal{B}(t)X + \mathcal{B}(t)^2 - 4\,\mathcal{A}(t)\Big), \tag{A.1}$$

where $\mathcal{B}(t) = t^3 - J_2'\,t - J_3'$ and $\mathcal{A}(t) = J_4't^2 - J_5't + J_6'$, and

$$\begin{aligned}
2J_2' \;=\;& 2\,a^2d^2 - 2\,abcd + 2\,b^2c^2 - 2\,a^2d + bca + adb + adc - 2\,ad^2 - 2\,b^2c - 2\,bc^2 + bcd + 2\,a^2\\
& -2\,ba - 2\,ca + ad + 2\,b^2 + bc - 2\,db + 2\,c^2 - 2\,cd + 2\,d^2,\\
-4J_3' \;=\;& -4\,a^3d^3 + 6\,a^2bcd^2 + 6\,ab^2c^2d - 4\,b^3c^3 + 6\,a^3d^2 - 6\,a^2bcd - 3\,a^2bd^2\\
& -3\,a^2cd^2 + 6\,a^2d^3 - 3\,ab^2c^2 - 6\,ab^2cd - 6\,abc^2d - 6\,abcd^2 + 6\,b^3c^2 + 6\,b^2c^3\\
& -3\,b^2c^2d + 6\,a^3d - 3\,a^2bc - 6\,a^2bd - 6\,a^2cd - 6\,a^2d^2 - 6\,ab^2c - 3\,ab^2d - 6\,abc^2 + 60\,abcd\\
& -6\,abd^2 - 3\,ac^2d - 6\,acd^2 + 6\,ad^3 + 6\,b^3c - 6\,b^2c^2 - 6\,b^2cd + 6\,bc^3 - 6\,bc^2d - 3\,bcd^2 - 4\,a^3\\
& +6\,a^2b + 6\,a^2c - 3\,a^2d + 6\,ab^2 - 6\,bca - 6\,adb + 6\,ac^2 - 6\,adc - 3\,ad^2 - 4\,b^3 - 3\,b^2c\\
& +6\,b^2d - 3\,bc^2 - 6\,bcd + 6\,bd^2 - 4\,c^3 + 6\,c^2d + 6\,cd^2 - 4\,d^3,\\
16\,J_4' \;=\;& 81\,\left(bca - adb - adc + bcd + ad - bc\right)^2,\\
-\tfrac{8}{81}J_5' \;=\;& -2\,b^2c^2da^3 + b^2cd^2a^3 + b^2d^3a^3 + bc^2d^2a^3 - 4\,bcd^3a^3 + c^2d^3a^3 + b^3c^3a^2 + b^3c^2da^2\\
& -2\,b^3cd^2a^2 + b^2c^3da^2 + 4\,b^2c^2d^2a^2 + b^2cd^3a^2 - 2\,bc^3d^2a^2 + bc^2d^3a^2 - 4\,b^3c^3da\\
& +b^3c^2d^2a + b^2c^3d^2a - 2\,b^2c^2d^3a + b^3c^3d^2 + b^2c^2a^3 + b^2cda^3 - 2\,b^2d^2a^3 + bc^2da^3\\
& +4\,bcd^2a^3 + bd^3a^3 - 2\,c^2d^2a^3 + cd^3a^3 - 2\,b^3c^2a^2 + b^3cda^2 + b^3d^2a^2 - 2\,b^2c^3a^2\\
& -4\,b^2c^2da^2 - 4\,b^2cd^2a^2 - 2\,b^2d^3a^2 + bc^3da^2 - 4\,bc^2d^2a^2 + 4\,bcd^3a^2 + c^3d^2a^2\\
& -2\,c^2d^3a^2 + b^3c^3a + 4\,b^3c^2da + b^3cd^2a + 4\,b^2c^3da - 4\,b^2c^2d^2a + b^2cd^3a + bc^3d^2a\\
& +bc^2d^3a + b^3c^3d - 2\,b^3c^2d^2 - 2\,b^2c^3d^2 + b^2c^2d^3 - 4\,bcda^3 + bd^2a^3 + cd^2a^3 - 2\,a^3d^3\\
& +b^2c^2a^2 + 4\,b^2cda^2 + b^2d^2a^2 + 4\,bc^2da^2 - 4\,a^2bcd^2 + bd^3a^2 + c^2d^2a^2 + cd^3a^2 + b^3c^2a\\
& -4\,b^3cda + b^2c^3a - 4\,ab^2c^2d + 4\,b^2cd^2a - 4\,bc^3da + 4\,bc^2d^2a - 4\,bcd^3a - 2\,b^3c^3 + b^3c^2d\\
& +b^2c^3d + b^2c^2d^2 + a^3d^2 + a^2bcd - 2\,a^2bd^2 - 2\,a^2cd^2 + a^2d^3 - 2\,ab^2c^2 + ab^2cd\\
& +abc^2d + abcd^2 + b^3c^2 + b^2c^3 - 2\,b^2c^2d,\\
\tfrac{16}{81}J_6' \;=\;& -4\,b^2c^2da^4 + 4\,bc^2da^4 + 4\,b^2cda^4 - 10\,bcda^4 + 4\,bc^2d^4a^3 - 22\,bc^2d^3a^3 - 4\,b^3cd^3a^3\\
& -22\,b^2cd^3a^3 - 10\,b^2c^3d^2a^3 + 16\,bc^3d^2a^3 - 10\,b^3c^2d^2a^3 + 4\,b^2c^2d^2a^3 + 16\,b^3cd^2a^3 - 4\,b^3c^3da^3\\
& +16\,b^2c^3da^3 - 10\,bc^3da^3 + 16\,b^3c^2da^3 - 10\,b^3cda^3 + 4\,b^2c^2d^4a^2 - 10\,bc^2d^4a^2 - 10\,b^2cd^4a^2\\
& +4\,b^2cd^4a^3 + 12\,bcd^4a^3 - 4\,bc^3d^3a^3 + 12\,b^2c^2d^3a^3 + 12\,bcd^4a^2 - 10\,b^2c^3d^3a^2 + 16\,bc^3d^3a^2\\
& -10\,b^3c^2d^3a^2 + 4\,b^4cd^2a - 10\,b^4c^4da + 12\,b^3c^4da + 12\,b^2c^4da - 10\,bc^4da + 12\,b^4c^3da + 12\,b^4c^2da\\
& -10\,b^4cda + 4\,b^2c^2d^3a^2 + 16\,b^3cd^3a^2 + 4\,b^2c^4d^2a^2 - 4\,bc^4d^2a^2 + 12\,b^3c^3d^2a^2 + 4\,b^2c^3d^2a^2\\
& +4\,b^4c^2d^2a^2 + 4\,b^3c^2d^2a^2 - 4\,b^4cd^2a^2 + 4\,b^3c^4da^2 - 10\,b^2c^4da^2 + 4\,bc^4da^2 + 4\,b^4c^3da^2\\
& -22\,b^3c^3da^2 - 10\,b^4c^2da^2 + 4\,b^4cda^2 - 4\,b^2c^2d^4a + 4\,bc^2d^4a + 4\,b^2cd^4a - 10\,bcd^4a - 4\,b^3c^3d^3a\\
& +16\,b^2c^3d^3a - 10\,bc^3d^3a + 16\,b^3c^2d^3a - 10\,b^3cd^3a + 4\,b^3c^4d^2a - 10\,b^2c^4d^2a + 4\,bc^4d^2a + 4\,b^4c^3d^2a\\
& -22\,b^3c^3d^2a - 10\,b^4c^2d^2a - 10\,bcd^4a^4 + 4\,bc^2d^3a^4 + 4\,b^2cd^3a^4 + 12\,bcd^3a^4 + 4\,b^2c^2d^2a^4\\
& -10\,bc^2d^2a^4 - 10\,b^2cd^2a^4 + 12\,bcd^2a^4 + 4\,b^2cd^2a^3 + 12\,bc^2da^3 + 12\,b^2cda^3 + 4\,bc^2d^3a^2 + 4\,b^2cd^3a^2\\
& +4\,b^2c^3da^2 + 12\,bc^3da^2 + 4\,b^3c^2da^2 + 12\,b^3cda^2 + 12\,bc^2d^3a + 12\,b^2cd^3a + 4\,b^2c^3d^2a + 12\,bc^3d^2a\\
& +4\,b^3c^2d^2a + 12\,b^3cd^2a + 4\,bcd^3a^3 - 22\,bcd^2a^3 - 22\,b^2c^2da^3 + 4\,bcda^3 - 22\,bcd^3a^2\\
& -22\,bc^3d^2a^2 + 12\,b^2c^2d^2a^2 + 4\,bc^2d^2a^2 - 22\,b^3cd^2a^2 + 4\,b^2cd^2a^2 + 4\,b^2c^2da^2 - 10\,bc^2da^2\\
& -10\,b^2cda^2 - 22\,b^2c^2d^3a + 4\,bcd^3a + 4\,b^2c^2d^2a - 10\,bc^2d^2a - 10\,b^2cd^2a + 4\,b^3c^3da - 22\,b^2c^3da\\
& +16\,a^2bcd^2 + 16\,ab^2c^2d + 4\,bc^3da - 22\,b^3c^2da + 4\,b^3cda + 4\,bc^2d^2a^3 - 4\,b^3c^2a - 4\,b^2c^3d\\
& +4\,c^3d^2a^2 - 10\,b^2d^3a^2 + 4\,b^2c^2a^3 + 4\,b^2d^2a^2 - 10\,b^3c^2d^2 + 4\,b^2c^2a^2 + 16\,b^3c^3a + 4\,c^2d^2a^2\\
& -10\,b^2c^3d^2 - 4\,bd^3a^2 + 4\,b^3d^2a^2 + 16\,cd^3a^3 - 10\,b^3c^2a^2 + 16\,bd^3a^3 - 4\,cd^2a^3 + 16\,b^2d^3a^3\\
& -4\,cd^3a^2 - 10\,b^2c^3a^2 + 4\,b^2c^2d^3 - 10\,c^2d^3a^2 + 16\,b^3c^3d - 10\,c^2d^2a^3 + 16\,b^3c^3d^2 - 4\,b^3c^2d\\
& +4\,b^2c^2d2 - 4\,bd^2a^3 - 4\,b^2c^3a + 16\,b^3c^3a^2 - 10\,b^2d^2a^3 + 16\,c^2d^3a^3 + b^4c^2 + 4\,b^4c^4 - 4\,b^3c^4\\
& -4\,b^4c^3 + d^2a^4 + d^4a^2 + 4\,d^4a^4 - 4\,d^3a^4 - 4\,d^4a^3 + b^2c^4 + 2a^3d^3 + 2b^3c^3 + 4\,b^4c^2d\\
& +2b^3c^3a^3 + b^4c^4d^2 - 10\,bd^3a^4 - 4\,b^3d^2a^3 + 4b^3d^3a^3 + b^2c^2a^4 + 4\,b^2c^4a + 4\,bd^4a^4\\
& -10\,cd^3a^4 - 4\,c^3d^3a^2 + 4\,b^2d^4a^2 + b^4d^2a^2 - 10\,b^3c^4a - 4\,b^3c^4a^2 + 4\,bd^2a^4 + 4\,b^2d^2a^4\\
& +2b^3c^3d^3 - 10\,cd^4a^3 - 4\,b^3c^2a^3 + 4\,c^2d^4a^2 + 4\,cd^4a^2 - 4\,b^4c^3a^2 - 10\,b^3c^4d + 4\,cd^4a^4 - 10\,bd^4a^3\\
& +4\,bd^4a^2 + 2\,c^3d^3a^3 + 4\,b^2c^4d + 4\,cd^2a^4 - 4\,b^3d^3a^2 + 4\,b^4c^2a - 10\,b^4c^3d - 4\,c^2d^4a^3 + b^4c^4a^2\\
& -4\,b^3c^4d^2 - 4\,b^2c^3d^3 + 4\,b^4c^2a^2 - 4\,b^2c^3a^3 + 4\,b^4c^2d^2 - 4\,b^3c^2d^3 + 4\,b^4c^4a - 10\,b^4c^3a + 4\,b^2c^4d^2\\
& +2\,c^4d^2a^2 - 4\,b^4c^3d^2 - 4\,b^2d^3a^4 - 4\,b^2d^4a^3 - 4\,c^2d^3a^4 + 4\,c^2d^2a^4\\
& +b^2d^4a^4 + b^2c^2d^4 + 4\,b^4c^4d - 4\,c^3d^2a^3 + c^2d^4a^4 + 4\,b^2c^4a^2.
\end{aligned} \tag{A.2}$$

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