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On Isogenies Among Certain Abelian Surfaces

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Abstract. We construct a three-parameter family of nonhyperelliptic and bielliptic plane genus-three curves whose associated Prym variety is two-isogenous to the Jacobian variety of a general hyperelliptic genus-two curve. Our construction is based on the existence of special elliptic fibrations with the section on the associated Kummer surfaces that provide a simple geometric interpretation for the rational double cover induced by the two-isogeny between the Abelian surfaces.

1. Introduction

A smooth projective curve is called hyperelliptic if it admits a map of degree two onto a curve of genus zero. Within the (coarse) moduli space of irreducible projective curves of genus-three $\mathcal{M}_3$, we denote the hyperelliptic locus by $\mathcal{M}_3^h$ and the isomorphism class of such hyperelliptic curve $\mathcal{H}$ by $[\mathcal{H}] \in \mathcal{M}_3^h$. It is known that $\mathcal{M}_3^h$ is an irreducible five-dimensional subvariety of $\mathcal{M}_3$ (the hyperelliptic involution on an irreducible smooth projective curve of genus g is unique if $g \geq 2$). Within the moduli space $\mathcal{M}_3$, we also define the bielliptic locus

$$\mathcal{M}_3^b = \{[\mathcal{D}] \in \mathcal{M}_3 | \mathcal{D} \text{ is bielliptic}\},$$

where bielliptic means that irreducible projective curve $\mathcal{D}$ of genus three admits a degree-two morphism $\pi_{\mathcal{E}}^{\mathcal{D}} : \mathcal{D} \to \mathcal{E}$ onto an elliptic curve $\mathcal{E}$. We denote by $[\mathcal{D}] \in \mathcal{M}_3$ the isomorphism class of $\mathcal{D}$ and by τ the involution, that is, the element of $\mathrm{Aut}(\mathcal{D})$ that interchanges the sheets of $\pi_{\mathcal{E}}^{\mathcal{D}}$ so that $\mathcal{E} \cong \mathcal{D}/\langle\tau\rangle$. For such a bielliptic genus-three curve $\mathcal{D}$ with a bielliptic involution τ, the Prym variety $\mathrm{Prym}(\mathcal{D}, \pi_{\mathcal{E}}^{\mathcal{D}})$ is defined as the connected component of the kernel of the induced norm map $\pi_{\mathcal{E},\star}^{\mathcal{D}}$.

We recall from [11] that $\mathcal{M}_3^b$ is an irreducible four-dimensional subvariety of $\mathcal{M}_3$, and it is the unique component of maximal dimension of the singular locus of $\mathcal{M}_3$. (By the Castelnuovo–Severi inequality it follows that bielliptic curves of genus $g \geq 6$ admit precisely one bielliptic structure and that bielliptic curves of genus $g \geq 4$ cannot be hyperelliptic.) The following proposition was proven in [3].

Proposition 1.1. (1) $[\mathcal{H}] \in \mathcal{M}_3^b \cap \mathcal{M}_3^h$ *iff* $\mathcal{H}$ *is a double cover of a genus-two curve* $\mathcal{C}$.
(2) $\mathcal{M}_3^b \cap \mathcal{M}_3^h$ *is an irreducible three-dimensional rational subvariety of* $\mathcal{M}_3^b$.

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On the other hand, among the smooth genus-three curves, there are the ones that are given as plane quartics in $\mathbb{P}^2$. However, smooth plane quartics are never hyperelliptic. We can ask whether the following Abelian surfaces are related by isogeny: (1) the Jacobian variety $\mathrm{Jac}(\mathcal{C})$ associated with a hyperelliptic and bielliptic curve $\mathcal{H}$ in $\mathcal{M}_3^b \cap \mathcal{M}_3^h$ covering a smooth genus-two curve $\mathcal{C}$ and (2) the Prym variety $\mathrm{Prym}(\mathcal{D}, \pi_{\mathcal{E}}^{\mathcal{D}})$ associated with a bielliptic plane genus-three curve $\mathcal{D}$ in $\mathcal{M}_3^b$. That is, we ask for what curves $\mathcal{H}$ and $\mathcal{D}$, there is an isogeny $\Psi : \mathrm{Prym}(\mathcal{D}, \pi_{\mathcal{E}}^{\mathcal{D}}) \to \mathrm{Jac}(\mathcal{C})$ where curves are embedded as divisors representing the respective polarization.

Barth [4] studied Abelian surfaces $\mathfrak{A}$ with $(1, 2)$-polarization line bundle $\mathcal{L}$ and proved their close connection with Prym varieties of smooth bielliptic genus-three curves. An excellent summary of Barth's construction was given by Garbagnati [16; 17; 18]. Abelian surfaces with $(1, 2)$-polarization were also discussed in [29; 30; 5]. Bielliptic genus-three curves and Abelian surfaces with $(1, 2)$-polarization have also appeared as spectral curves of Lax representations of certain algebraic integrable systems, most importantly (for us) the Kovalevskaya top [1; 2; 19; 14]. On the other hand, Kovalevskaya presented in her celebrated paper [24] a separation of variables of the corresponding integrable system using a certain (hyperelliptic) genus-two curve, nowadays commonly referred to as *Kowalewski curve,* whose Jacobian is associated with the integrals of motion of the Kovalevskaya top.

Barth's seminal work proved that the linear system $|\mathcal{L}|$ is a pencil on $\mathfrak{A}$ of bielliptic genus-three curves. Horozov and van Moerbeke [19] wrote down a specific Lefschetz pencil of bielliptic genus-three curves $\mathcal{D}_\lambda$ over $\mathbb{P}^1 \ni \lambda$, generically smooth and with twelve double points. However, the members of the pencil are generically *not* plane genus-three curves. The construction of the pencil is based on Barth's elegant geometric description for Prym varieties of bielliptic genus-three curves as intersection of quadrics in a projective space [4; 9]. However, less attention has been given in this context to the elliptic fibrations with section that the associated Kummer surfaces admit. The first two authors studied several elliptic fibrations on the Kummer surface associated with an Abelian surface with $(1, 2)$-polarization in [7], using the results of Mehran [30; 29; 31] and Garbagnati [16]. Among these fibrations is a particular elliptic fibration with twelve singular fibers, which is directly induced by the linear system $|\mathcal{L}|$ on $\mathfrak{A}$.

In this paper, we construct a three-parameter nonhyperelliptic and bielliptic genus-three curve whose associated Prym variety is two-isogenous to the Jacobian variety of the general three-parameter hyperelliptic genus-two curve. We consider the genus-two curve $\mathcal{C}$ to be in Rosenhain form

$$\mathcal{C}: \quad Y^2 = X(X-1)(X-\lambda_1)(X-\lambda_2)(X-\lambda_3) \tag{1.1}$$

with parameters λ_1, λ_2, λ_3. We define the subgroup $\Gamma_2(2n) = \{M \in \Gamma_2 | M \equiv \mathbb{I} \bmod 2n\}$ and Igusa's congruence subgroups $\Gamma_2(2n, 4n) = \{M \in \Gamma_2(2n) |$

$\mathrm{diag}(B) = \mathrm{diag}(C) \equiv \mathbb{I} \mod 4n\}$ of the Siegel modular group $\Gamma_2 = \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})\}$ such that

$$\begin{aligned} \Gamma_2/\,\Gamma_2(2) \cong S_6, \qquad \Gamma_2(2)/\,\Gamma_2(2,4) \cong (\mathbb{Z}/2\mathbb{Z})^4, \\ \Gamma_2(2,4)/\,\Gamma_2(4,8) \cong (\mathbb{Z}/2\mathbb{Z})^9, \end{aligned} \tag{1.2}$$

where S_6 is the permutation group of six elements. Then $\lambda_1, \lambda_2, \lambda_3$ are modular with respect to $\Gamma_2(2)$. We define a modular form l such that $l^2 = \lambda_1\lambda_2\lambda_3$ and three modular forms $m^{(i,j,k)}$ such that $(m^{(i,j,k)})^2 = (\lambda_i - \lambda_j)(\lambda_i - \lambda_k)/[(1-\lambda_j)(1-\lambda_k)]$ with $\{i, j, k\} = \{1, 2, 3\}$. Then l is modular with respect to $\Gamma_2(2,4)$, and $m^{(i,j,k)}$ are modular with respect to $\Gamma_2(4,8)$.

The main result of this paper is the following:

THEOREM 1.2. *Consider the plane bielliptic genus-three curves $\mathscr{D}_{[s_0^*:s_1^*]}$ given by*

$$\begin{aligned} \left(w^2 - u^2 - \frac{C(s_0^*, s_1^*)}{E(s_0^*, s_1^*)}uv - \frac{D(s_0^*, s_1^*)}{E(s_0^*, s_1^*)}v^2\right)^2 \\ = u^4 + B(s_0^*, s_1^*)u^2v^2 + A^2(s_0^*, s_1^*)v^4, \end{aligned} \tag{1.3}$$

where $[u : v : w] \in \mathbb{P}^2$, A, B, C, D, E are polynomials in $[s_0 : s_1] \in \mathbb{P}^1$ with coefficients in $\mathbb{Z}[l, \lambda_1, \lambda_2, \lambda_3]$ defined in the Appendix, and $[s_0^* : s_1^*] \in \mathbb{P}^1$ is one of the six special points given by

$$[s_0^* : s_1^*] = [(1 + \lambda_i - \lambda_j - \lambda_k)l : (\lambda_i - \lambda_j\lambda_k) \pm m^{(i,j,k)}(1-\lambda_j)(\lambda_0 - \lambda_k)] \tag{1.4}$$

with $\{i, j, k\} = \{1, 2, 3\}$ such that $E(s_0^*, s_1^*) \neq 0$.

Then the curves $\mathscr{D}_{[s_0^*:s_1^*]}$ are smooth and irreducible and admit a degree-two covering $\pi^{\mathscr{D}}_{\mathscr{E}} : \mathscr{D}_{[s_0^*:s_1^*]} \to \mathscr{E}_{[s_0^*:s_1^*]}$ onto a smooth elliptic curve $\mathscr{E}_{[s_0^*:s_1^*]} \cong \mathcal{D}^*/\langle\tau\rangle$, where τ is the bielliptic involution. Moreover, the Prym variety $\mathrm{Prym}(\mathscr{D}_{[s_0^*:s_1^*]}, \pi^{\mathscr{D}}_{\mathscr{E}})$ *is an Abelian surface that admits a* $(1, 2)$*-isogeny*

$$\Psi : \quad \mathrm{Prym}(\mathscr{D}_{[s_0^*:s_1^*]}, \pi^{\mathscr{D}}_{\mathscr{E}}) \to \mathrm{Jac}(\mathcal{C})$$

onto the principally polarized Abelian surface $\mathrm{Jac}(\mathcal{C})$ *of $\mathcal{C}$ in Equation* (1.1).

The geometry underlying Theorem 1.2 is the following: if we choose 6 points in $\mathbb{P}^1$, partitioned into 2 and 4, then we obtain three double covers of $\mathbb{P}^1$ branched respectively at the marked sets of 2, 4, and all 6 points. We label them $\mathcal{R}$, $\mathcal{E}$, $\mathcal{C}$ with genus 0, 1, and 2, respectively. These three curves have a common double cover $\mathcal{H}$, which can be obtained as the fiber product over $\mathbb{P}^1$ of any two of the three. This is a Galois cover of $\mathbb{P}^1$ with group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the three intermediate curves $\mathcal{R}$, $\mathcal{E}$, $\mathcal{H}$ are the quotients by the three $\mathbb{Z}/2\mathbb{Z}$ subgroups. The curve $\mathcal{H}$ is hyperelliptic via the map to $\mathcal{R}$ and bielliptic via the map to $\mathcal{E}$. Its Jacobian decomposes as $\mathrm{Jac}\,\mathcal{H} \cong \mathrm{Jac}\,\mathcal{C} \times \mathcal{E}$. On the other hand, Recillas' famous trigonal construction [38] relates to such a tower $\{\mathcal{R}, \mathcal{E}, \mathcal{H}\}$ a nonhyperelliptic and bielliptic genus-three curve $\mathcal{D}$ such that the Prym of the latter is two-isogenous to $\mathrm{Jac}\,\mathcal{C}$; see [13].

It is amazing that we can use two special elliptic fibrations with section on the Kummer surfaces associated with $\operatorname{Jac}\mathcal{C}$ and $\operatorname{Prym}(\mathcal{D}, \pi^{\mathcal{D}}_{\mathcal{E}})$ to provide a simple geometric interpretation for the rational double cover induced by the two-isogeny between the Abelian surfaces. We then work backward and obtain explicit expressions for the coefficients of suitable normal forms for $\mathcal{D}$ and $\mathcal{C}$ in terms of Siegel modular forms. This is the content of Theorem 1.2. Applications of isogenies of Pryms to hyperelliptic Jacobians are of central importance in cryptography; see [15, Section 9] for further details.

The paper is structured as follows: in Section 2, we consider an Abelian surface $\mathfrak{A}$ with polarization of type $(1, 2)$. On the Kummer surface $\operatorname{Kum}(\mathfrak{A})$, we identify a special elliptic fibration, alongside with a set of generators for the Mordell–Weil group and symplectic automorphisms in Theorem 2.8, which turn out to be crucial for the proof of Theorem 1.2. In Section 3, we determine a convenient normal form for a hyperelliptic and bielliptic genus-three curve that is the double cover of a general genus-two curve. We then generalize this construction to pencils and establish a connection to the aforementioned elliptic pencil on $\operatorname{Kum}(\mathfrak{A})$, providing explicit formulas for the coefficients of all normal forms in terms of suitable modular forms. In Section 4, we give a geometric description of plane bielliptic genus-three curves and determine a criterion for the quotient (elliptic) curves to have a rational level-two structure and branch locus. In Section 5, we carry out the proof of Theorem 1.2: using the results of Section 2, we identify six special members of the fibration induced by the pencil $|\mathcal{L}|$ on $\mathfrak{A}$, where the elliptic fiber satisfies the conditions of Proposition 4.8, and its double cover is a smooth bielliptic plane quartic curve. Using the results of Section 3 the plane bielliptic genus-three curve can then be related back to the Rosenhain normal form of a general genus-two curve to prove our theorem.

2. Abelian and Kummer Surfaces

Polarizations on an Abelian surface $\mathfrak{A} \cong \mathbb{C}^2/\Lambda$ are known to correspond to positive definite Hermitian forms H on $\mathbb{C}^2$ satisfying $E = \operatorname{Im} H(\Lambda, \Lambda) \subset \mathbb{Z}$. In turn, such a Hermitian form determines a line bundle $\mathcal{L}$ in the Néron–Severi group $\operatorname{NS}(\mathfrak{A})$. Then we may always choose a basis of Λ such that E is given by a matrix $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$ with $D = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ where $d_1, d_2 \in \mathbb{N}$, $d_1, d_2 \geq 0$, and d_1 divides d_2. The pair (d_1, d_2) gives the *type* of the polarization.

If $\mathfrak{A} = \operatorname{Jac}(\mathcal{C})$ is the Jacobian of a smooth curve $\mathcal{C}$ of genus two, then the Hermitian form associated with the divisor class $[\mathcal{C}]$ is a polarization of type $(1, 1)$, a *principal polarization.* Conversely, a principally polarized Abelian surface is either the Jacobian of a smooth curve of genus two or the product of two complex elliptic curves with product polarization.

Let $\mathfrak{A}$ an Abelian surface defined over $\mathbb{C}$ and $-\mathbb{I}$ be the minus identity involution on $\mathfrak{A}$. The quotient $\mathfrak{A}/\langle -\mathbb{I}\rangle$ has 16 ordinary double points, and its minimum resolution, denoted $\operatorname{Kum}(\mathfrak{A})$, is known as the *Kummer surface* of $\mathfrak{A}$. Thus there is an even set of 16 disjoint rational curves K_i for $0 \leq i \leq 15$ such that $K_i \circ K_j = -2\delta_{ij}$. The double points are the images of the order-two points

$\{\mathsf{P}_0, \ldots, \mathsf{P}_{15}\}$ on $\mathfrak{A}$, that is, elements of $\mathfrak{A}[2]$, and the disjoint rational curves $\{K_0, \ldots, K_{15}\}$ are the exceptional divisors introduced in the blowup process. The minimal primitive sublattice that contains all these curves is called the *Kummer lattice.* In particular, they form an even set in the Néron–Severi lattice. We recall that an *even set* of rational curves is a set of disjoint (-2)-rational smooth curves $\{K_0, \ldots, K_{15}\}$ such that there exists a divisor δ in the Néron–Severi lattice with $K_0 + \cdots + K_{15} \sim 2\delta$, where $\sim$ denotes linear equivalence. Since they form an even set, the class $\hat{K} = \frac{1}{2}(K_0 + \cdots + K_{15})$ is an element of this lattice with $\hat{K}^2 = -8$. However, the classes K_i and $\hat{K}$ do not generate, over $\mathbb{Z}$, the minimal primitive lattice containing these curves. The Néron–Severi lattice $\mathrm{NS}(\mathrm{Kum}\,\mathfrak{A})$ is generated over $\mathbb{Q}$ by the classes K_i and one additional class H with $H^2 = 8$ and $H \circ K_i = 0$ for $0 \leq i \leq 15$.

2.1. *Abelian Surfaces with* $(1, 2)$*-Polarization*

Let us now consider the generic Abelian surface $\mathfrak{A}$ with a $(1, 2)$-polarization. Let this polarization of type $(d_1, d_2) = (1, 2)$ be given by an ample symmetric line bundle $\mathcal{L}$ such that $\mathcal{L}^2 = 4$. We also assume that the Picard number $\rho(\mathfrak{A}) = 1$ such that the Néron–Severi group of $\mathfrak{A}$ is generated by $\mathcal{L}$ [6]. The line bundle $\mathcal{L}$ defines an associated rational map $\phi = \phi_{\mathcal{L}} : \mathfrak{A} \to \mathbb{P}^{d_1 d_2 - 1} = \mathbb{P}^1$. Since $h^0(\mathfrak{A}, \mathcal{L}) = 2$, the linear system $|\mathcal{L}|$ is a pencil on $\mathfrak{A}$, and the map $\phi_{\mathcal{L}}$ is a rational map $\phi_{\mathcal{L}} : \mathfrak{A} \to \mathbb{P}^1$. As $\mathcal{L}^2 = 4$, each curve in $|\mathcal{L}|$ has self-intersection equal to 4. Since we assumed that $\rho(\mathfrak{A}) = 1$, the Abelian surface $\mathfrak{A}$ cannot be a product of two elliptic curves or isogenous to a product of two elliptic curves.

It was proven in [6, Prop. 4.1.6, Lemma 10.1.2] that the linear system $|\mathcal{L}|$ has exactly four base points if $(d_1, d_2) = (1, 2)$. To characterize these four base points, Barth [4] proved that the base points form the translation group $T(\mathcal{L}) = \{P \in \mathfrak{A} \mid t_P^* \mathcal{L} = \mathcal{L}\}$ where elements of $\mathfrak{A}$ act by translation $t_p(x) = x + P$. Moreover, he proved that $T(\mathcal{L}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and that the base points all have order two on the Abelian surface $\mathfrak{A}$, and we denote them by $\{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3\}$. A curve in the pencil $|\mathcal{L}|$ is never singular at any of the base points $\{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3\}$; see [5, Lemma 3.2]. Barth's seminal duality theorem in [4] can then be stated as follows.

Theorem 2.1 (Barth). *In the situation above, let $\mathcal{D} \in |\mathcal{L}|$ be a smooth genus-three curve in the pencil $|\mathcal{L}|$. There exists a bielliptic involution τ on $\mathcal{D}$ with degree-two quotient map $\pi^{\mathcal{D}}_{\mathcal{E}} : \mathcal{D} \to \mathcal{E} = \mathcal{D}/\langle\tau\rangle$ such that $\mathfrak{A}$ is naturally isomorphic to the Prym variety* $\mathrm{Prym}(\mathcal{D}, \pi^{\mathcal{D}}_{\mathcal{E}})$ *and the involution $-\mathbb{I}$ restricts to τ.*

Conversely, if $\mathcal{D}$ is a smooth bielliptic genus-three curve with degree-two quotient map $\pi^{\mathcal{D}}_{\mathcal{E}} : \mathcal{D} \to \mathcal{E} = \mathcal{D}/\langle\tau\rangle$, then $\mathcal{D}$ is embedded in $\mathrm{Prym}(\mathcal{D}, \pi^{\mathcal{D}}_{\mathcal{E}})$ *as a curve of self-intersection four. The Prym variety* $\mathrm{Prym}(\mathcal{D}, \pi^{\mathcal{D}}_{\mathcal{E}})$ *is an Abelian surface with a polarization of type* $(1, 2)$.

2.2. *An Elliptic Fibration on* $(1,2)$*-Polarized Kummer Surfaces*

We denote the exceptional curves associated with the base points on $\mathrm{Kum}(\mathfrak{A})$ by $\{K_0, K_1, K_2, K_3\}$. The map $\phi_{\mathcal{L}} : \mathfrak{A} \to \mathbb{P}^1$ induces an elliptic fibration π : $\mathrm{Kum}(\mathfrak{A}) \to \mathbb{P}^1$ with section O as follows: first, a fibration is obtained by blowing up the base points of the pencil $|\mathcal{L}|$. The fibers of this fibration are the strict transform of the curves $\mathcal{D} \in |\mathcal{L}|$, and so the general fiber is a smooth genus-three curve. The involution τ lifts to an involution on this fibration whose fixed points are the exceptional curves over $\{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3\}$. We then take as the general fiber of π the quotient of the general fiber of $\phi_{\mathcal{L}}$ by the bielliptic involution. Since a curve in the pencil $|\mathcal{L}|$ is never singular at any of the base points $\{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3\}$, we can take as zero-section O the exceptional curve over P_0 such that the divisor class of the section is $[\mathsf{O}] = K_0$. Garbagnati [16; 17; 18] proved the following:

Proposition 2.2 (Garbagnati). *The fibration π has twelve singular fibers of Kodaira type I_2 and no other singular fibers. The Mordell Weil group satisfies* $\mathrm{MW}(\pi, \mathsf{O})_{\mathrm{tor}} = (\mathbb{Z}/2\mathbb{Z})^2$ *and* $\mathrm{rank}\,\mathrm{MW}(\pi, \mathsf{O}) = 3$. *The smooth fiber class F with $F^2 = 0$ and $F \circ K_0 = 1$ is given by*

$$F = \frac{H - K_0 - K_1 - K_2 - K_3}{2}.$$

The twelve nonneutral components of the reducible fibers of Kodaira type A_1 represent the classes $K_4, \dots, K_{15}$ of the Kummer lattice and are not intersected by the class of the zero section given by K_0. The remaining four classes K_i with $0 \le i \le 3$ satisfy $F \circ K_i = 1$ and $K_j \circ K_i = 0$ with $4 \le j \le 15$. Thus they represent sections of the elliptic fibration with section (π, O), which we still denote by K_i, and intersect only neutral components of the reducible fibers given by the divisor classes $F - K_j$ with $1 \le i \le 3$ and $4 \le j \le 15$.

In [8; 7] the authors introduced explicit normal forms for the elliptic fibration with section (π, O), given as the affine Weierstrass model

$$Y^2 = X(X^2 - 2B(s)X + (B(s)^2 - 4A(s)^2)), \tag{2.1}$$

where $A(s)$ and $B(s)$ are certain *even* polynomials of degree four in s – we will determine them in Corollary 3.12 and Equation (3.28) – such that there are no singular fibers over $s = 0, \infty$, and

$$A(s) = s^4 A(1/s), \qquad B(s) = s^4 B(1/s),$$

and the discriminant of the elliptic fiber given by $\Delta = 16A(s)^2(B(s)^2 - 4A(s)^2)^2$ has twelve roots of order two. Moreover, the elliptic fibration is invariant under the action of the hyperelliptic involution $(s, X, Y) \mapsto (s, X, -Y)$, which we denote by $p \mapsto -p$ for a point $p \in F$ in a fiber F given by Equation (2.1), and three additional involutions given by

$$\begin{aligned} \jmath_1: &\quad (s, X, Y) \mapsto (s' = -s, X, Y), \\ \jmath_2: &\quad (s, X, Y) \mapsto \left(s'' = \frac{1}{s}, \frac{X}{s^4}, \frac{Y}{s^6}\right), \\ \jmath_3: &\quad (s, X, Y) \mapsto \left(s''' = -\frac{1}{s}, \frac{X}{s^4}, -\frac{Y}{s^6}\right). \end{aligned} \tag{2.2}$$

The involutions $s \mapsto -s$ and $s \mapsto 1/s$ and their composition map singular fibers of Equation (2.1) to singular fibers and smooth fibers to smooth fibers. The zero section O, given as the point at infinity in each fiber, and the two-torsion sections $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3$, given by

$$\begin{aligned} &\mathsf{T}_1 : (X, Y) = (0, 0), \qquad \mathsf{T}_2 : (X, Y) = (B - 2A, 0), \\ &\mathsf{T}_3 : (X, Y) = (B + 2A, 0), \end{aligned} \tag{2.3}$$

are invariant under the involutions $\jmath_1, \jmath_2, \jmath_3$ and the hyperelliptic involution. The two-torsion sections intersect the nonneutral components of eight reducible fibers of type A_1 each (which we represent as sets $W_k = \{K_i \mid i \in I_k\}$ for index sets I_k such that $|W_k| = 8$ for $k = 1, 2, 3$) partitioning the twelve rational curves K_j with $4 \leq j \leq 15$ into three sets of eight curves with pairwise intersections consisting of four curves, that is, $|W_j \cap W_k| = 4$ and $W_1 \cap W_2 \cap W_3 = \emptyset$. None of the twelve reducible fibers is invariant under the action of the involutions $\jmath_1, \jmath_2$. However, the sets W_k and $W_j \cap W_k$ for $1 \leq j, k \leq 3$ are invariant under $\jmath_1, \jmath_2$. We may define the divisors $\bar{K}_{W_k} = \frac{1}{2}\sum_{n \in I_k} K_n$ with $1 \leq k \leq 3$, which are known to be elements of the Kummer lattice [16; 17; 18] with $\bar{K}_{W_j} \circ \bar{K}_{W_k} = -2 - 2\delta_{jk}$ for $1 \leq j, k \leq 3$. We also define the divisors $\bar{K}_{W_j \cap W_k} = \frac{1}{2}\sum_{n \in I_j \cap I_k} K_n$ with $\bar{K}^2_{W_j \cap W_k} = -2$. By construction the elements $\bar{K}_{W_k}$ and $\bar{K}_{W_j \cap W_k}$ for $1 \leq j, k \leq 3$ are invariant under the action of the involutions $\jmath_1, \jmath_2$. The twelve singular fibers of fibration (2.1) arise when two-torsion sections collide. This happens as follows in Table 1.

We have the following:

Corollary 2.3. *The divisor classes of the two-torsion sections* T_k *are given by*

$$[\mathsf{T}_k] = 2F + K_0 - \bar{K}_{W_k} \quad \textit{for } 1 \leq k \leq 3. \tag{2.4}$$

Proof. The proof follows from $[\mathsf{T}_k] \circ F = 1$, $[\mathsf{T}_k] \circ K_0 = 0$, $[\mathsf{T}_k] \circ K_j = 1$ for $j \in I_k$ and $[\mathsf{T}_k] \circ K_j = 0$ for $j \notin I_k$, and $[\mathsf{T}_k] \circ K_l = 2$ for $1 \leq l \leq 3$. □

Table 1 Collision of torsion sections.

colliding sections	equation	# of points	fiber components
$\mathsf{T}_1 = \mathsf{T}_2$	$B - 2A = 0$	4	$W_1 \cap W_2$
$\mathsf{T}_1 = \mathsf{T}_3$	$B + 2A = 0$	4	$W_1 \cap W_3$
$\mathsf{T}_2 = \mathsf{T}_3$	$A = 0$	4	$W_2 \cap W_3$

In [8; 7], three nontorsion sections S_1, S_2, S_3 of the elliptic fibration (π, O) of minimal height were constructed explicitly. We will review an explicit construction of these sections in Section 3.4. For two arbitrary sections S' and S'' of the elliptic fibration, we define the *height pairing* using the formula

$$\langle S', S''\rangle = \chi^{\text{hol}} + \mathsf{O} \circ S' + \mathsf{O} \circ S'' - S' \circ S'' - \sum_{\{s \mid \Delta=0\}} C_s^{-1}(S', S''), \tag{2.5}$$

where the holomorphic Euler characteristic is $\chi^{\text{hol}} = 2$, and the inverse Cartan matrix C_s^{-1} of a fiber of type A_1 located over point s of the discriminant locus $\Delta = 0$ contributes $(\frac{1}{2})$ if and only if both S' and S'' intersect the nonneutral component. It turns out that the sections S_1 and S_2 do not intersect the zero section O and intersect the nonneutral components of six reducible fibers of type A_1 each (which we represent as complementary sets $V_k = \{K_i \mid i \in J_k\}$ for index sets J_k such that $|V_k| = 6$ for $k = 1, 2$) partitioning the twelve rational curves K_j with $4 \le j \le 15$ into two disjoint sets of six curves. We also set $W_1' = V_1 \cup \{K_0, K_1\}$, $I_1' = J_1 \cup \{0, 1\}$ and $W_2' = V_2 \cup \{K_2, K_3\}$, $I_2' = J_2 \cup \{2, 3\}$ and define the divisors $\bar{K}_{W_k'} = \frac{1}{2} \sum_{n \in I_k'} K_n$ with $1 \le k \le 2$. The sets V_1 and V_2 are invariant under the action of the involution $\jmath_1$ and interchanged under the action of $\jmath_2$. The section S_3 intersects the nonneutral components of all reducible fibers and the zero section such that $\mathsf{S}_3 \circ \mathsf{O} = 2$.

We have the following:

PROPOSITION 2.4. *The sections* $\{\mathsf{O}, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ *form a basis of the Mordell–Weil group of sections. In particular, we have*

$$\mathrm{MW}(\pi, \mathsf{O}) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus \langle 1\rangle^{\oplus 2} \oplus \langle 2\rangle. \tag{2.6}$$

Proof. Given an explicit form of the sections $\{\mathsf{O}, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$, we computed the intersection pairings for their divisor classes. The results are part of Table 2. The height pairings of the corresponding sections of the elliptic fibration (π, O) are given in Table 2. We observe from Table 2 that the pairwise orthogonal sections S_1, S_2, S_3 of height less than or equal to two generate a rank-three sublattice of the Mordell–Weil group of sections. It was proved in [16, Prop. 2.2.4] that the transcendental lattice of the Kummer surface $\mathrm{Kum}(\mathfrak{A})$ is isometric to $H(2) \oplus H(2) \oplus \langle -8\rangle$ such that the determinant of the discriminant form equals 2^7. This is in numerical agreement with the determinant of the discriminant form for the Néron–Severi lattice obtained from an elliptic fibration with section, twelve singular fibers of Kodaira type I_2, and a Mordell–Weil group of sections given by Equation (2.6). $\square$

We recall that an automorphism of finite order on a complex K3 surface is called symplectic if it acts trivially on the holomorphic two-form of the K3 surface, and it is called antisymplectic if it acts as multiplication by (-1). These notions were introduced by Nikulin in [36]. We have the following:

Table 2 Intersection and height pairings.

$\circ$	F	O	T_1	T_2	T_3	S'_1	S'_2	S'_3	S_1	S_2	S_3
F	0	1	1	1	1	1	1	1	1	1	1
O	1	−2	0	0	0	0	0	0	0	0	2
T_1	1	0	−2	0	0	2	2	2	0	0	0
T_2	1	0	0	−2	0	2	2	2	0	0	0
T_3	1	0	0	0	−2	2	2	2	0	0	0
S'_1	1	0	2	2	2	−2	0	0	0	2	4
S'_2	1	0	2	2	2	0	−2	0	1	1	2
S'_3	1	0	2	2	2	0	0	−2	1	3	2
S_1	1	0	0	0	0	0	1	1	−2	2	1
S_2	1	0	0	0	0	2	1	3	2	−2	1
S_3	1	2	0	0	0	4	2	2	1	1	−2

$\langle\bullet,\bullet\rangle$	O	T_1	T_2	T_3	S'_1	S'_2	S'_3	S_1	S_2	S_3
O	0	0	0	0	0	0	0	0	0	0
T_1	0	0	0	0	0	0	0	0	0	0
T_2	0	0	0	0	0	0	0	0	0	0
T_3	0	0	0	0	0	0	0	0	0	0
S'_1	0	0	0	0	4	2	2	2	0	0
S'_2	0	0	0	0	2	4	2	1	1	2
S'_3	0	0	0	0	2	2	4	1	−1	2
S_1	0	0	0	0	2	1	1	1	0	0
S_2	0	0	0	0	0	1	−1	0	1	0
S_3	0	0	0	0	0	2	2	0	0	2

Table 3 Action of involutions torsion and non-torsion sections.

	O	T_1	T_2	T_3	S_1	S_2	S_3
$\jmath_1$	O	T_1	T_2	T_3	S_1	S_2	$-\mathsf{S}_3$
$\jmath_2$	O	T_1	T_2	T_3	S_2	S_1	S_3
$\jmath_3$	O	T_1	T_2	T_3	$-\mathsf{S}_2$	$-\mathsf{S}_1$	S_3

LEMMA 2.5. *The involutions $\jmath_1$, $\jmath_2$, $\jmath_3$ are three commuting antisymplectic involutions of the elliptic fibration with section (π, O) with $\jmath_3 = -\jmath_1 \jmath_2$. The involutions $\jmath_l$ for $1 \le l \le 3$ act on the sections $\{\mathsf{O}, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ as follows in Table 3.*

Proof. We check that the involutions are antisymplectic by using an explicit representative of the holomorphic two-form for the affine Weierstrass model in Equation (2.1) given by $ds \wedge dX/Y$. The rest of the statement follows by explicit computation. □

We should emphasize that the operations $+$ and $-$, when used with sections of a Jacobian elliptic fibration, are operations with respect to the group law in the Mordell–Weil group $\mathrm{MW}(\pi, \mathsf{O})$, that is, the fiberwise application of the elliptic curve group law. In contrast, before the symbols were used in the context of divisors in the Néron–Severi group. We have the following:

PROPOSITION 2.6. *There are four possible choices for sections* $\{\mathsf{S}_1', \mathsf{S}_2', \mathsf{S}_3'\}$ *of the elliptic fibration with section* (π, O) (*up to permutation and the action of the hyperelliptic involution*) *such that the divisor classes* K_0, K_1, K_2, K_3 *are represented as*

$$K_0 = [\mathsf{O}], \qquad K_1 = [\mathsf{S}_1'], \qquad K_2 = [\mathsf{S}_2'], \qquad K_3 = [\mathsf{S}_3']. \tag{2.7}$$

The sections are obtained as linear combinations of the nontorsion sections S_1, S_2, S_3 *generating* $\mathrm{MW}(\pi, \mathsf{O})$ *using the elliptic-curve group law in each fiber* F *given by Equation* (2.1) *as follows in Table* 4.

Proof. We explicitly compute $2\mathsf{S}_1$, $\mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3$, $\mathsf{S}_1 - \mathsf{S}_2 + \mathsf{S}_3$ using the elliptic-curve group law. Since these are sections of the elliptic fibration, we find that the intersection pairing with the smooth fiber F always equals one. We then check that the three sections intersect only neutral components of the reducible fiber, that is, the components $F - K_j$ for $4 \le j \le 15$. We finally check that the three sections do not mutually intersect nor intersect the zero section O. For $\mathsf{S}_1' = 2\mathsf{S}_1$, $\mathsf{S}_2' = \mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3$, $\mathsf{S}_3' = \mathsf{S}_1 - \mathsf{S}_2 + \mathsf{S}_3$, the intersection pairings of all aforementioned divisor classes and height pairings of the corresponding sections are given in Table 2. The sections of the table are then obtained by acting with involutions $\jmath_1, \dots, \jmath_3$ and the hyperelliptic involution. Using the height pairing, we check that these are the only possibilities. □

REMARK 2.7. The different choices in Proposition 2.6 are permuted by automorphisms that fix the ample class; see Theorem 2.8.

Table 4 Action of involutions on the sections of $(\pi : \mathscr{E} \to \mathbb{P}^1, \mathsf{O})$.

#	*action*	S_1'	S_2'	S_3'	$\sum_{i=1}^3 \mathsf{S}_i'$
1	$\pm\,\mathrm{id}$	$\pm 2\mathsf{S}_1$	$\pm(\mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3)$	$\pm(\mathsf{S}_1 - \mathsf{S}_2 + \mathsf{S}_3)$	$\pm 2(2\mathsf{S}_1 + \mathsf{S}_3)$
2	$\pm\jmath_1$	$\pm 2\mathsf{S}_1$	$\pm(\mathsf{S}_1 + \mathsf{S}_2 - \mathsf{S}_3)$	$\pm(\mathsf{S}_1 - \mathsf{S}_2 - \mathsf{S}_3)$	$\pm 2(2\mathsf{S}_1 - \mathsf{S}_3)$
3	$\pm\jmath_2$	$\pm 2\mathsf{S}_2$	$\pm(\mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3)$	$\pm(-\mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3)$	$\pm 2(2\mathsf{S}_2 + \mathsf{S}_3)$
4	$\mp\jmath_3$	$\pm 2\mathsf{S}_2$	$\pm(\mathsf{S}_1 + \mathsf{S}_2 - \mathsf{S}_3)$	$\pm(-\mathsf{S}_1 + \mathsf{S}_2 - \mathsf{S}_3)$	$\pm 2(2\mathsf{S}_2 - \mathsf{S}_3)$

Using the elliptic-curve group law on each fiber $F_s \ni (s, X, Y)$ given by Equation (2.1), the three involutions in Equation (2.2), and a choice of sections $\{\mathsf{S}'_1, \mathsf{S}'_2, \mathsf{S}'_3\}$ in Proposition 2.4, we define involutions of the elliptic fibration with section (π, O) mapping smooth or singular fibers to smooth or singular fibers, respectively:

$$\begin{aligned} \imath_1: \quad & (s, X, Y) \mapsto (s', X', Y') = -\jmath_1(s, X, Y) + \mathsf{S}'_1|_{F_{s'}}, \\ \imath_2: \quad & (s, X, Y) \mapsto (s'', X'', Y'') = -\jmath_2(s, X, Y) + \mathsf{S}'_2|_{F_{s''}}, \\ \imath_3: \quad & (s, X, Y) \mapsto (s''', X''', Y''') = -\jmath_3(s, X, Y) + \mathsf{S}'_3|_{F_{s'''}}. \end{aligned} \tag{2.8}$$

By a slight abuse of notation, we also denote the involutions more intuitively by $p \mapsto \imath_l(p) := -\jmath_l(p) + K_l$ for $p \in F$ and $1 \le l \le 3$. We have the following:

THEOREM 2.8. *The involutions $\imath_1, \imath_2, \imath_3$ are three commuting symplectic involutions of the elliptic fibration with section (π, O) on $\mathrm{Kum}(\mathfrak{A})$ such that $\imath_3 = \imath_1 \circ \imath_2$. The involutions act on the divisor classes $\{F, K_0, K_1, K_2, K_3\}$ as follows in Table 5.*

Proof. Each involution $\imath_l$ is a composition of the involution $\jmath_l$, an inversion given by the hyperelliptic involution, and a shift on the fiber. Since $\jmath_l$ is antisymplectic by Lemma 2.5, the involution $\imath_l$ is symplectic. We check by explicit computation that the involutions $\imath_l$ commute and satisfy $\imath_3 = \imath_1 \circ \imath_2$. The rest of the statement follows using the explicit representation of each class K_l for $1 \le l \le 3$ in Equation (2.7). $\square$

We have the following consequence.

COROLLARY 2.9. *For the Abelian surface $\mathfrak{A}$ with polarization of type $(1, 2)$ given by a line bundle $\mathcal{L}$, the translation group $T(\mathcal{L}) = \{P \in \mathfrak{A} \mid t_P^*\mathcal{L} = \mathcal{L}\} \cong (\mathbb{Z}/2\mathbb{Z})^2$ induces the group of symplectic involutions $\{\mathrm{id}, \imath_1, \imath_2, \imath_3\}$ given by Equation (2.8) on the elliptic fibration with section (π, O) on the Kummer surface $\mathrm{Kum}(\mathfrak{A})$.*

Proof. Denote the four base points of the linear system $|\mathcal{L}|$ by $\{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3\}$ and identify $\mathsf{P}_0 = 0$ and the action by translation as follows in Table 6.

The action of t_{P_i} on the Abelian surface descends to a symplectic automorphism of $\mathrm{Kum}(\mathfrak{A})$. Since $\mathsf{P}_i \in T(\mathcal{L})$, the action of t_{P_i} on the Abelian surface descends to an automorphism that preserves the elliptic fibration with section (π, O) and maps the zero section O to the section representing the image of the base

Table 5 Action of translations on divisor classes.

	F	K_0	K_1	K_2	K_3
$\imath_1$	F	K_1	K_0	K_3	K_2
$\imath_2$	F	K_2	K_3	K_0	K_1
$\imath_3$	F	K_3	K_2	K_1	K_0

Table 6 Action of translations on the base points of the linear system $|\mathcal{L}|$.

	0	P_1	P_2	P_3
t_{P_1}	P_1	0	P_3	P_2
t_{P_2}	P_2	P_3	0	P_1
t_{P_3}	P_3	P_2	P_1	0

point P_i on Kum($\mathfrak{A}$). By Theorem 2.8 this is the group of symplectic involutions $\{\text{id}, \imath_1, \imath_2, \imath_3\}$ given by Equation (2.8). □

3. Bielliptic and Hyperelliptic Genus-Three Curves

In this section, we construct a bidouble cover of $\mathbb{P}^1$ introducing the curves $\mathcal{H}$, $\mathcal{C}$, and $\mathcal{E} \cong \mathcal{Q}$, which are used in the construction of the bielliptic curve $\mathcal{D}$ and in Section 5 to show that the Prym variety of $\mathcal{D}$ is two-isogenous to the Jacobian variety of $\mathcal{C}$.

The intersection $\mathcal{M}_3^h \cap \mathcal{M}_3^b$ is exactly the locus of curves with automorphism group V_4 (the Klein 4-group) inside the hyperelliptic locus. Such curves are usually called *hyperelliptic curves with extra involutions*. In [39] the locus $\mathcal{M}_3^b \cap \mathcal{M}_3^h$ was explicitly described in terms of invariants. We will construct a curve $\mathcal{H} \in \mathcal{M}_3^b \cap \mathcal{M}_3^h$ by choosing four out of the six Weierstrass points of $\mathcal{C}$ to be the images of four pairs of points on the curve $\mathcal{H}$ such that all eight Weierstrass points of $\mathcal{H}$ in the preimage are fixed under the hyperelliptic involution, and each pair is kept fixed by the bielliptic involution. For a genus-two curve $\mathcal{C}$ given as sextic $Y^2 = f_6(X, Z)$, a class in $\mathcal{M}_2(2)$, that is, the moduli space of genus-two curves with level-two structure, is given by the ordered tuple $(\lambda_1, \lambda_2, \lambda_3)$ after we sent the three remaining roots to $0, \infty, 1$. We then choose the points $(1, \lambda_1, \lambda_2, \lambda_3)$ to be the images of the eight ramification points of $\mathcal{H}$.

3.1. A Normal Form

We assume that the smooth genus-two curve $\mathcal{C}$ in Proposition 1.1 is in Rosenhain normal form, that is, for $[X : Z : Y] \in \mathbb{P}(1, 1, 3)$, the curve is given by

$$Y^2 = XZ\prod_{i=0}^{3}(X - \lambda_i Z) \tag{3.1}$$

with the hyperelliptic map $\pi^{\mathcal{C}} : \mathcal{C} \to \mathbb{P}^1$ given by $[X : Z : Y] \mapsto [X : Z]$. The hyperelliptic involution on $\mathcal{C}$ has six fixed points $\mathfrak{p}_i = [\lambda_i : 1 : 0]$ for $i = 0, \ldots, 3$, $\mathfrak{p}_4 = [0 : 1 : 0]$, and $\mathfrak{p}_5 = [1 : 0 : 0]$. To simplify our discussion in the situation of pencils, we will use λ_0 rather than choosing $\lambda_0 = 1$. Since $\mathcal{C}$ is smooth, we will assume that $\lambda_i \neq 0$ and $\lambda_i \neq \lambda_j$ for $0 \leq i < j \leq 3$. The lambdas are ratios of squares of even theta functions $\theta_i^2 = \theta_i^2(0, \tau)$ with zero elliptic argument, modular argument $\tau \in \mathbb{H}_2/\Gamma_2(2)$, and $1 \leq i \leq 10$, where we are using the same standard notation for even theta functions as in [21; 20; 9]. We have a choice of $6! = 720$

such expressions. In each case, there is a ratio of *squares* of theta functions such that $l^2 = \lambda_0\lambda_1\lambda_3\lambda_3$. In the following, we use the convention from [37].

LEMMA 3.1. *If $\mathcal{C}$ is a genus-two curve with period matrix τ and nonvanishing discriminant, then $\mathcal{C}$ is equivalent to the curve in Equation (3.1) with Rosenhain parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ given by*

$$\lambda_0 = 1, \qquad \lambda_1 = \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}, \qquad \lambda_2 = \frac{\theta_3^2\theta_8^2}{\theta_4^2\theta_{10}^2}, \qquad \lambda_3 = \frac{\theta_1^2\theta_8^2}{\theta_2^2\theta_{10}^2}. \tag{3.2}$$

Conversely, given three distinct complex numbers $(\lambda_1, \lambda_2, \lambda_3)$ different from 0, 1, ∞, *the complex Abelian surface* $\mathrm{Jac}(\mathcal{C})$ *has the period matrix* $[\mathbb{I}_2|\tau]$ *where $\mathcal{C}$ is the genus-two curve with period matrix τ.*

REMARK 3.2. We define

$$\begin{aligned} l &= \frac{\theta_1^2\theta_3^2\theta_8^2}{\theta_2^2\theta_4^2\theta_{10}^2}, \qquad & m^{(1,2,3)} &= \frac{\theta_1\theta_3\theta_6^2}{\theta_2\theta_4\theta_5^2}, \\ m^{(2,1,3)} &= i\,\frac{\theta_3\theta_8\theta_6^2}{\theta_4\theta_{10}\theta_7^2}, \qquad & m^{(3,1,2)} &= \frac{\theta_1\theta_8\theta_6^2}{\theta_2\theta_{10}\theta_9^2}, \end{aligned} \tag{3.3}$$

so that $l^2 = \lambda_0\lambda_1\lambda_3\lambda_3$ and $(m^{(i,j,k)})^2 = (\lambda_i - \lambda_j)(\lambda_i - \lambda_k)/[(\lambda_0 - \lambda_i)(\lambda_0 - \lambda_j)]$ with $\{i, j, k\} = \{1, 2, 3\}$. The latter identities follow from the well-known Frobenius identities for theta functions; see [27; 28].

We define the subgroup $\Gamma_2(2n) = \{M \in \Gamma_2 | M \equiv \mathbb{I} \mod 2n\}$ and Igusa's congruence subgroups $\Gamma_2(2n, 4n) = \{M \in \Gamma_2(2n) | \operatorname{diag}(B) = \operatorname{diag}(C) \equiv \mathbb{I} \mod 4n\}$ of the Siegel modular group $\Gamma_2 = \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})\}$ such that

$$\begin{aligned} \Gamma_2/\Gamma_2(2) &\cong S_6, \qquad \Gamma_2(2)/\Gamma_2(2,4) \cong (\mathbb{Z}/2\mathbb{Z})^4, \\ \Gamma_2(2,4)/\Gamma_2(4,8) &\cong (\mathbb{Z}/2\mathbb{Z})^9, \end{aligned} \tag{3.4}$$

where S_6 is the permutation group of six elements. The following lemma was proven in [9].

LEMMA 3.3. $\lambda_1, \lambda_2, \lambda_3$ *are modular with respect to* $\Gamma_2(2)$, l *is a modular with respect to* $\Gamma_2(2, 4)$, *and* $m^{(i,j,k)}$ *is modular with respect to* $\Gamma_2(4, 8)$ *for* $\{i, j, k\} = \{1, 2, 3\}$.

By Proposition 1.1 a hyperelliptic and bielliptic genus-three curve $\mathcal{H}$ in the preimage of the curve $\mathcal{C}$ defined in Equation (3.1), that is, the parameters λ_i in Equations (3.2) under the map $\mathcal{M}_3^b \cap \mathcal{M}_3^h \to \mathcal{M}_2$, is given by the equation

$$y^2 = \prod_{i=0}^{3}(x^2 - \lambda_i z^2) \tag{3.5}$$

with $[x : z : y] \in \mathbb{P}(1, 1, 4)$. On $\mathcal{H}$, there are *two* involutions, the hyperelliptic involution $\imath^{\mathcal{H}} : [x : z : y] \mapsto [x : z : -y]$ and the bielliptic involution $\tau^{\mathcal{H}} : [x : z : y] \mapsto [-x : z : y]$.

$$\begin{tikzcd}
& \mathcal{H} \arrow[dl, "\pi^{\mathcal{H}}_{\mathcal{C}}"'] \arrow[d, "\pi^{\mathcal{H}}_{\mathcal{Q}}"] \arrow[dr] & \\
\mathcal{C} \arrow[dr, "\pi^{\mathcal{C}}"'] & \mathcal{Q} \arrow[d, "\pi^{\mathcal{Q}}"] & \mathbb{P}^1 \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

Figure 1 Quotients of bielliptic, hyperelliptic curve $\mathcal{H}$.

It is easy to check that the composition $\tau^{\mathcal{H}} \circ \imath^{\mathcal{H}}$ is fixed-point-free. An unramified double cover $\pi^{\mathcal{H}}_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is given by

$$\pi^{\mathcal{H}}_{\mathcal{C}} : \quad [x : z : y] \mapsto [X : Z : Y] = [x^2 : z^2 : xyz]. \tag{3.6}$$

The images of the four pairs of hyperelliptic fixed points and the two pairs of bielliptic fixed points under $\pi^{\mathcal{H}}_{\mathcal{C}}$ are exactly the Weierstrass points of the genus-two curve $\mathcal{C}$. It is easily proved that every unramified double cover of a hyperelliptic genus two curve is obtained in this way [19, p. 387]; in particular, the cover is always hyperelliptic.

The quotient genus-one curve $\mathcal{Q} = \mathcal{H}/\langle \tau^{\mathcal{H}} \rangle$ obtained from the bielliptic involution is the quartic curve

$$y^2 = \prod_{i=0}^{3} (X - \lambda_i Z) \tag{3.7}$$

with $[X : Z : y] \in \mathbb{P}(1, 1, 2)$, and the double cover $\pi^{\mathcal{H}}_{\mathcal{Q}} : \mathcal{H} \to \mathcal{Q}$ is given by

$$\pi^{\mathcal{H}}_{\mathcal{Q}} : \quad [x : \pm z : y] = [-x : \mp z : y] \mapsto [X : Z : y] = [x^2 : z^2 : y].$$

The four branch points of $\pi^{\mathcal{H}}_{\mathcal{Q}}$ are precisely the images of the bielliptic fixed points. The situation is summarized in Figure 1. Here the map $\mathbb{P}^1 \to \mathbb{P}^1$ is given by $[x : z] \mapsto [X : Z] = [x^2 : z^2]$. Moreover, in the introduction the genus-one curve in the bidouble cover is denoted $\mathcal{E}$. Here it is called $\mathcal{Q}$, and we prove in Section 5 that it is isomorphic to a curve $\mathcal{E}$ with a certain given equation.

We have the following:

PROPOSITION 3.4. *The quotient $\mathcal{Q} = \mathcal{H}/\langle \tau^{\mathcal{H}} \rangle$ in Equation* (3.7) *of the hyperelliptic and bielliptic genus-three curve in Equation* (3.5) *is isomorphic to the elliptic curve*

$$\mathcal{E} : \quad \rho^2 \eta = \xi(\xi^2 - 2b\xi\eta + (b^2 - 4a^2)\eta^2) \tag{3.8}$$

with $[\xi : \eta : \rho] \in \mathbb{P}^2$ and coefficients

$$\begin{aligned} a &= (\lambda_0 - \lambda_1)(\lambda_2 - \lambda_3), \\ b &= 4\lambda_0\lambda_1 + 4\lambda_2\lambda_3 - 2\lambda_0\lambda_2 - 2\lambda_0\lambda_3 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3. \end{aligned} \tag{3.9}$$

The elliptic curve (3.8) *has two-torsion points* $[\xi : \eta : \rho] = [0 : 1 : 0]$, $[b \pm 2a : 1 : 0]$ *and the neutral element* $[0 : 0 : 1]$.

Proof. The proof follows by an explicit computation. □

Later, we will also use the existence of certain rational points on $\mathcal{E}$ in Equation (3.8), that stems from the fact that $\mathcal{E}$ is isomorphic to the genus-one curve $\mathcal{Q} = \mathcal{H}/\langle\tau^{\mathcal{H}}\rangle$ with four bielliptic branch points. We have the following:

LEMMA 3.5. *On the elliptic curve $\mathcal{E}$ in Proposition 3.4, there are the rational points p_1 with coordinates given by*

$$[\xi : \eta : \rho] = [4(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3) : 1 : 8(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)] \qquad (3.10)$$

and p_2 with

$$\begin{aligned}&[\xi : \eta : \rho]\\ &\quad = [4\lambda_0\lambda_1(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3) : \lambda_0^2 : 8l(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)].\end{aligned} \qquad (3.11)$$

Using the group law on $\mathcal{E}$, we obtain the rational points $2\mathsf{p}_1$ with coordinates

$$\begin{aligned}\xi &= (\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3)^2, \qquad \eta = 1,\\ \rho &= (\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3)(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3)(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3),\end{aligned} \qquad (3.12)$$

and rational points $\mathsf{p}_1 \pm \mathsf{p}_2$ with coordinates

$$\begin{aligned}\xi &= 4(\lambda_0\lambda_1 + \lambda_2\lambda_3 \mp 2l), \qquad \eta = 1,\\ \rho &= 8(\pm l(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \lambda_0\lambda_1\lambda_2 - \lambda_0\lambda_1\lambda_3 - \lambda_0\lambda_2\lambda_3 - \lambda_1\lambda_2\lambda_3).\end{aligned} \qquad (3.13)$$

Proof. The points $\pm\mathsf{p}_1$ and $\pm\mathsf{p}_2$ are the images of the four branch points of $\pi^{\mathcal{H}}_{\mathcal{Q}}$, namely $[X : Z : y] = [1 : 0 : \pm 1]$ and $[0 : 1 : \pm l]$ on the genus-one curve in Equation (3.7), respectively. The rest of the proof follows by explicit computation. □

Moreover, we have the following:

PROPOSITION 3.6. *Given a smooth genus-two curve $\mathcal{C}$, the hyperelliptic and bielliptic genus-three curve $\mathcal{H}$ in Equation (3.5) and the elliptic curve $\mathcal{E}$ in Equation (3.8) satisfy*

$$\operatorname{Jac}(\mathcal{H}) \cong \operatorname{Prym}(\mathcal{H}, \pi^{\mathcal{H}}_{\mathcal{E}}) \times \mathcal{E}, \qquad \operatorname{Prym}(\mathcal{H}, \pi^{\mathcal{H}}_{\mathcal{E}}) \cong \operatorname{Jac}(\mathcal{C}),$$

where $\operatorname{Prym}(\mathcal{H}, \pi^{\mathcal{H}}_{\mathcal{E}})$ is the Prym variety associated with $\pi^{\mathcal{H}}_{\mathcal{E}}$.

Proof. The involution $\tau^{\mathcal{H}}$ extends to the Jacobian variety $\operatorname{Jac}(\mathcal{H})$. Therefore it contains two Abelian subvarieties, the elliptic curve $\mathcal{E}$, and the two-dimensional Prym-variety $\operatorname{Prym}(\mathcal{H}, \pi^{\mathcal{H}}_{\mathcal{E}})$, which is antisymmetric with respect to the extended involution. On the other hand, the étale double cover $\pi^{\mathcal{H}}_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ satisfies $\pi^{\mathcal{H}}_{\mathcal{C}} \circ \tau^{\mathcal{H}} = \imath^{\mathcal{C}}$, that is, it is equivariant with respect to the bielliptic involution on $\mathcal{H}$ and the hyperelliptic involution on $\mathcal{C}$. The claim follows. □

3.2. Göpel Groups and Double Covers

We denote the space of two-torsion points on an Abelian variety $\mathfrak{A}$ by $\mathfrak{A}[2]$. In the case of the Jacobian of a genus-two curve, every nontrivial two-torsion point can be expressed using differences of Weierstrass points of $\mathcal{C}$. Concretely, the sixteen order-two points of $\mathrm{Jac}(\mathcal{C})[2]$ are obtained using the embedding of the curve into the connected component of the identity in the Picard group, that is, $\mathcal{C} \hookrightarrow \mathrm{Jac}(\mathcal{C}) \cong \mathrm{Pic}^0(\mathcal{C})$ with $\mathfrak{p} \mapsto [\mathfrak{p} - \mathfrak{p}_5]$. We obtain 15 elements $\mathsf{P}_{ij} \in \mathrm{Jac}(\mathcal{C})[2]$ with $0 \le i < j \le 5$ as

$$\begin{aligned} \mathsf{P}_{i5} &= [\mathfrak{p}_i - \mathfrak{p}_5] \quad \text{for } 0 \le i < 5, \\ \mathsf{P}_{ij} &= [\mathfrak{p}_i + \mathfrak{p}_j - 2\mathfrak{p}_5] \quad \text{for } 0 \le i < j \le 4, \end{aligned} \tag{3.14}$$

and set $\mathsf{P}_0 = \mathsf{P}_{55} = [0]$. For $\{i, j, k, l, m, n\} = \{0, \dots, 5\}$, the group law on $\mathrm{Jac}(\mathcal{C})[2]$ is given by the relations

$$\begin{aligned} \mathsf{P}_0 + \mathsf{P}_{ij} &= \mathsf{P}_{ij}, & \mathsf{P}_{ij} + \mathsf{P}_{ij} &= \mathsf{P}_0, \\ \mathsf{P}_{ij} + \mathsf{P}_{kl} &= \mathsf{P}_{mn}, & \mathsf{P}_{ij} + \mathsf{P}_{jk} &= \mathsf{P}_{ik}. \end{aligned} \tag{3.15}$$

The space $A[2]$ of two-torsion points on an Abelian variety $\mathfrak{A}$ admits a symplectic bilinear form, called the *Weil pairing*. The Weil pairing is induced by the pairing

$$\langle [\mathfrak{p}_i - \mathfrak{p}_j], [\mathfrak{p}_k - \mathfrak{p}_l] \rangle = \#\{\mathfrak{p}_i, \mathfrak{p}_j\} \cap \{\mathfrak{p}_k, \mathfrak{p}_l\} \mod 2.$$

We call a two-dimensional maximal isotropic subspace of $A[2]$ with respect to the Weil pairing, that is, a subspace such that the symplectic form vanishes on it, a *Göpel group* in $A[2]$. Such a maximal subgroup is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

We give the following characterization of the choices involved in our construction of the curves $\mathcal{H}$ and $\mathcal{E}$ in Figure 1.

PROPOSITION 3.7. *For a smooth genus-two curve $\mathcal{C}$, there are 15 inequivalent hyperelliptic and bielliptic genus-three curves $\mathcal{H}_{ij}$ for $0 \le i < j \le 5$ that are unramified double covers of $\mathcal{C}$. The double covers $\mathcal{H}_{ij} \to \mathcal{C}$ are in one-to-one correspondence with nontrivial elements of* $\mathsf{P}_{ij} \in \mathrm{Jac}(\mathcal{C})[2]$. *Moreover, isomorphisms $\mathcal{Q}_{ij} \cong \mathcal{E}$, understood as isomorphisms between genus-one curves with marked Weierstrass points, are in one-to-one correspondence with Göpel groups* $G \subset \mathrm{Jac}(\mathcal{C})[2]$ *such that* $\mathsf{P}_{ij} \in G$.

Proof. We constructed the curve $\mathcal{H} \in \mathcal{M}_3^b \cap \mathcal{M}_3^h$ by choosing four out of the six Weierstrass points of $\mathcal{C}$ to be the images of four pairs of points on the curve $\mathcal{H}$ such that all eight Weierstrass points of $\mathcal{H}$ in the preimage are fixed under the hyperelliptic involution, and each pair is kept fixed by the bielliptic involution. That is, the construction of $\mathcal{H}$ was determined by $\{\mathfrak{p}_4, \mathfrak{p}_5\}$. The unordered pair represents a divisor class $[\mathfrak{p}_4 - \mathfrak{p}_5]$ with $2[\mathfrak{p}_4 - \mathfrak{p}_5] \equiv 0$. Therefore $[\mathfrak{p}_4 - \mathfrak{p}_5] \in \mathrm{Jac}(\mathcal{C})[2]$. We check that the resulting curve for any two different Weierstrass points also has a different j-invariant. It is easy to see that the elliptic curve $\mathcal{E}$, together with the set of two-torsion points $\{[0 : 1 : 0], [B \pm 2A : 1 : 0]\}$, depends on the partition of Weierstrass points of $\mathcal{E}$ or, equivalently, on a partition of the Weierstrass points of $\mathcal{C}$. From every partition of Weierstrass points, we obtain three elements $\mathsf{P}_{ij}, \mathsf{P}_{kl}, \mathsf{P}_{mn} \in \mathrm{Jac}(\mathcal{C})[2]$ with $\{i, j, k, l, m, n\} = \{0, \dots, 5\}$, each generating a $(\mathbb{Z}/2\mathbb{Z})$ subgroup. Because the only relation between these classes is given by $\mathsf{P}_{ij} + \mathsf{P}_{kl} + \mathsf{P}_{mn} = 0$, the classes generate a subgroup in $\mathrm{Jac}(\mathcal{C})[2]$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Because the pairs of Weierstrass points are all disjoint, the associated subgroup is in fact isotropic with respect to the Weil pairing. □

REMARK 3.8. For the smooth genus-two curve $\mathcal{C}$ in Equation (3.1), the hyperelliptic and bielliptic genus-three curve $\mathcal{H}$ in Figure 1 corresponds to the divisor $\mathsf{P}_{45} \in \mathrm{Jac}(\mathcal{C})[2]$.

3.3. *Pencils of Hyperelliptic Curves*

We start with the hyperelliptic and bielliptic genus-three curve $\mathcal{H}$ in the preimage of $\mathcal{M}_3^b \cap \mathcal{M}_3^h \to \mathcal{M}_2$ given by Equation (3.5). The automorphism $\imath^{\mathcal{H}} \times \imath^{\mathcal{H}}$ of $\mathcal{H} \times \mathcal{H}$ induces an automorphism on the symmetric square $\mathrm{Sym}^2(\mathcal{H})$, which by a slight abuse of notation we will denote by $\imath^{\mathcal{H}} \times \imath^{\mathcal{H}}$ as well. We have the following:

LEMMA 3.9. *On the variety* $\mathscr{H} = \mathrm{Sym}^2(\mathcal{H})/\langle \imath^{\mathcal{H}} \times \imath^{\mathcal{H}} \rangle$, *there is a pencil over* $\mathbb{P}^1 \ni [s_0 : s_1]$ *of hyperelliptic and bielliptic genus-three curves* $\mathscr{H}_{[s_0:s_1]}$ *given by*

$$\mathscr{H}_{[s_0:s_1]}: \quad y^2 = \prod_{i=0}^{3} \left(x^2 - \frac{(s_0 + \lambda_i s_1)^2}{\lambda_i} z^2 \right) \tag{3.16}$$

with $[x : z : y] \in \mathbb{P}(1, 1, 4)$. *In particular, the central fiber over* $[s_0 : s_1] = [0 : 1]$ *is isomorphic to* $\mathcal{H}$.

Proof. If we set $y = y^{(1)} y^{(2)}/l$, $x = x^{(1)} z^{(2)} + x^{(2)} z^{(1)}$, $s_0 z = x^{(1)} x^{(2)}$, and $s_1 z = z^{(1)} z^{(2)}$, Equation (3.16) becomes the product of two copies of Equation (3.5). Since the variables are invariant under the product of the hyperelliptic involutions on each copy of $\mathcal{H}$, the statements follows. □

We make the following:

REMARK 3.10. The bielliptic and hyperelliptic involution on the curve in Equation (3.5) both lift to involutions on the fibers of the pencil (3.16).

We define two pencils $\mathscr{Q}_{[s_0:s_1]}$ and $\mathscr{C}_{[s_0:s_1]}$ of genus-one and genus-two curves over $\mathbb{P}^1 \ni [s_0 : s_1]$, respectively as follows:

$$\begin{aligned} \mathscr{Q}_{[s_0:s_1]}: &\quad y^2 = \prod_{i=0}^{3} \left(X - \frac{(s_0 + \lambda_i s_1)^2}{\lambda_i} Z \right), \\ \mathscr{C}_{[s_0:s_1]}: &\quad Y^2 = XZ \prod_{i=0}^{3} \left(X - \frac{(s_0 + \lambda_i s_1)^2}{\lambda_i} Z \right), \end{aligned} \tag{3.17}$$

with $[X : Z : y] \in \mathbb{P}(1, 1, 2)$ and $[X : Z : Y] \in \mathbb{P}(1, 1, 3)$. The pencils are constructed such that the diagram of Figure 1 holds for every fiber over $[s_0 : s_1]$, and central fibers over $[s_0 : s_1] = [0 : 1]$ are exactly the curves $\mathcal{H}$, $\mathcal{C}$, and $\mathcal{Q}$, respectively. That is, we have

$$\mathscr{Q}_{[0:1]} = \mathcal{Q}, \qquad \mathscr{C}_{[0:1]} = \mathcal{C}, \qquad \mathscr{H}_{[0:1]} = \mathcal{H}.$$

We define another pencil $\mathscr{Q}'_{[t_0:t_1]}$ of genus-one curves over $\mathbb{P}^1 \ni [t_0 : t_1]$ by

$$\mathscr{Q}'_{[t_0:t_1]}: \quad Y^2 = t_0 t_1 \prod_{i=0}^{3} \left(x - \frac{t_0 + \lambda_i^2 t_1}{\lambda_i} z \right) \tag{3.18}$$

with $[x : z : Y] \in \mathbb{P}(1, 1, 2)$, and a two-to-one map $\mathscr{Q} \to \mathscr{Q}'$ by setting

$$\begin{aligned}([s_0 : s_1], [X : Z : y]) &\mapsto ([t_0 : t_1], [x : z : Y]) \\ &= ([s_0^2 : s_1^2], [X - 2s_0 s_1 Z : Z : s_0 s_1 y]).\end{aligned} \tag{3.19}$$

From these pencils we obtain the total spaces of fibrations (without multiple fibers)

$$\mathscr{C} = \coprod_{[s_0:s_1] \in \mathbb{P}^1} \mathscr{C}_{[s_0:s_1]}, \qquad \mathscr{Q} = \coprod_{[s_0:s_1] \in \mathbb{P}^1} \mathscr{Q}_{[s_0:s_1]}, \qquad \mathscr{Q}' = \coprod_{[t_0:t_1] \in \mathbb{P}^1} \mathscr{Q}'_{[t_0:t_1]}.$$

In the next section, we show that the total spaces $\mathscr{Q}$ and $\mathscr{Q}'$ are in fact singular models for certain Kummer surfaces. Singular fibers for pencils of genus-two curves were classified by Namikawa and Ueno [33; 34; 35; 42]. We have the following immediate:

PROPOSITION 3.11. *The pencil $\mathscr{C} \to \mathbb{P}^1$ has twelve singular fiber of Namikawa–Ueno type I_{2-0-0} and four singular fibers of type I_{4-0-0} with modulus point $\left(\begin{smallmatrix} \tau_1 & * \\ * & \infty \end{smallmatrix}\right)$.*

Comparing Equation (3.17) with Equation (3.7), we introduce the functions

$$\Lambda_i(s_0, s_1) = \frac{(s_0 + \lambda_i s_1)^2}{\lambda_i}, \qquad L(s_0, s_1) = \frac{\prod_{i=0}^{3}(s_0 + \lambda_i s_1)}{l} \tag{3.20}$$

for $0 \le i \le 3$ such that $L^2 = \Lambda_0 \Lambda_1 \Lambda_2 \Lambda_3$. Using Proposition 3.4, we immediately have the following:

COROLLARY 3.12. *The pencil $\mathscr{Q}$ is isomorphic to the elliptic fibration $\pi : \mathscr{E} \to \mathbb{P}^1$,*

$$\mathscr{E}_{[s_0:s_1]}: \quad \rho^2 \eta = \xi(\xi^2 - 2B(s_0, s_1)\xi\eta + (B^2(s_0, s_1) - 4A^2(s_0, s_1))\eta^2), \tag{3.21}$$

with section $\mathsf{O} : [\xi : \eta : \rho] = [0 : 0 : 1]$ and

$$\begin{aligned} A(s_0, s_1) &= (\Lambda_0(s_0, s_1) - \Lambda_1(s_0, s_1))(\Lambda_2(s_0, s_1) - \Lambda_3(s_0, s_1)), \\ B(s_0, s_1) &= 4\Lambda_0(s_0, s_1)\Lambda_1(s_0, s_1) + 4\Lambda_2(s_0, s_1)\Lambda_3(s_0, s_1) \\ &\quad - 2\Lambda_0(s_0, s_1)\Lambda_2(s_0, s_1) - 2\Lambda_0(s_0, s_1)\Lambda_3(s_0, s_1) \\ &\quad - 2\Lambda_1(s_0, s_1)\Lambda_2(s_0, s_1) - 2\Lambda_1(s_0, s_1)\Lambda_3(s_0, s_1). \end{aligned} \tag{3.22}$$

In particular, A and B are even polynomials of degree four such that there are no singular fibers over $[s_0 : s_1] = [0 : 1], [1 : 0]$, *and*

$$l^2 A(s_0, s_1) = A(ls_1, s_0), \qquad l^2 B(s_0, s_1) = B(ls_1, s_0).$$

Similarly, we obtain the following:

COROLLARY 3.13. *The pencil* $\mathscr{Q}'$ *is isomorphic to the elliptic fibration* $\pi' : \mathscr{E}' \to \mathbb{P}^1$,

$$\begin{aligned} \mathscr{E}'_{[t_0:t_1]}: \quad \rho'^2\eta' = \xi'(\xi'^2 - 2B'(t_0,t_1)\xi'\eta' \\ + (B'^2(t_0,t_1) - 4A'^2(t_0,t_1))\eta'^2), \end{aligned} \tag{3.23}$$

with section $\mathsf{O}' : [\xi' : \eta' : \rho'] = [0 : 0 : 1]$ *and polynomials*

$$A'(t_0,t_1) = t_0 t_1 A\big(\sqrt{t_0}, \sqrt{t_1}\big), \qquad B'(t_0,t_1) = t_0 t_1 B\big(\sqrt{t_0}, \sqrt{t_1}\big),$$

which are well-defined polynomials because of Corollary 3.12. Moreover, the two-to-one map in Equation (3.19) extends to a double cover $\psi : \mathscr{E} \to \mathscr{E}'$ *given by*

$$\begin{aligned} \psi : \quad ([s_0 : s_1], [\xi : \eta : \rho]) \mapsto ([t_0 : t_1], [\xi' : \eta' : \rho']) \\ = ([s_0^2 : s_1^2], [s_0^2 s_1^2 \xi : \eta : s_0^3 s_1^3 \rho]). \end{aligned} \tag{3.24}$$

Proof. Making the point $[x : z : Y] = [t_0 + \lambda_0^2 t_1 : \lambda_0 : 0]$ the neutral element of an elliptic curve and the point $[x : z : Y] = [t_0 + \lambda_1^2 t_1 : \lambda_1 : 0]$ a two-torsion point, we can bring Equation (3.18) into the normal form in Equation (3.23). □

3.4. Relation Between Elliptic Pencils and Kummer Surfaces

Corollary 3.12 proves that the elliptic fibration with section $(\pi : \mathscr{E} \to \mathbb{P}^1, \mathsf{O})$ is equivalent to the pencil $\mathscr{Q}$ given by

$$\mathscr{Q}_{[s_0:s_1]}: \quad y^2 = \prod_{i=0}^{3} (X - \Lambda_i(s_0,s_1)Z). \tag{3.25}$$

Therefore Proposition 3.4 and Lemma 3.5 can be applied in *each* fiber by replacing $\lambda_i \mapsto \Lambda_i(s_0,s_1)$ for $0 \le i \le 3$ and $l \mapsto L(s_0,s_1)$. Three two-torsion sections for the elliptic fibration (π, O) are given by $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3 : [\xi : \eta : \rho] = [0 : 1 : 0], [B \pm 2A : 1 : 0]$. Two nontorsion sections, which by slight abuse of notation we still denote by $\mathsf{p}_1, \mathsf{p}_2$, are obtained by assigning the points p_1 and p_2 in Lemma 3.5 in each fiber. The existence of a third nontorsion section p_3 in the pencil is easily verified by assigning the point $[X : Z : y] = [4s_0s_1 : 1 : L(-s_0, s_1)]$ in each fiber $\mathscr{Q}_{[s_0:s_1]}$ and then converting to coordinates $[\xi : \eta : \rho]$.

We also define the sections $\{\mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ as follows in Table 7.

PROPOSITION 3.14. *For the elliptic fibration with section* $(\pi : \mathscr{E} \to \mathbb{P}^1, \mathsf{O})$ *in Equation (3.21) and in Proposition 2.2, the sections* $\{\mathsf{O}, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ *are generators of the Mordell–Weil group* $\mathrm{MW}(\pi, \mathsf{O}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus \langle 1 \rangle^{\oplus 2} \oplus \langle 2 \rangle$. *In particular, we have*

$$\mathsf{p}_1 = \mathsf{S}_1, \qquad \mathsf{p}_2 = -\mathsf{S}_2 + \mathsf{S}_3, \qquad \mathsf{p}_3 = \mathsf{S}_2 + \mathsf{S}_3. \tag{3.26}$$

Table 7 Sections of the fibration $(\pi : \mathscr{E} \to \mathbb{P}^1, \mathsf{O})$.

sec.	$[\xi : \eta : \rho]$
S_1	$[4\lambda_0\lambda_1 \prod_{i=2}^3(\lambda_i - \lambda_0)(s_0^2 - \lambda_0\lambda_i s_1^2) : 8\prod_{i=1}^3(\lambda_i - \lambda_0)(s_0^2 - \lambda_0\lambda_i s_1^2) : \lambda_0^2 l^2]$
S_2	$[4l \prod_{i=2}^3(\lambda_i - \lambda_1)(s_0^2 - \lambda_i\lambda_1 s_1^2)$: $8\prod_{i=1}^3(\lambda_0 - \lambda_i)\prod_{1\le j<k\le 3}(s_0^2 - \lambda_j\lambda_k s_1^2) : l^3]$
S_3	$[4l s_0 s_1 \prod_{i=0}^1\prod_{j=2}^3(s_0^2 - \lambda_i\lambda_j s_1^2)$: $-8\prod_{i=1}^3(s_0^2 - \lambda_0\lambda_i s_1^2)\prod_{1\le j<k\le 3}(s_0^2 - \lambda_j\lambda_k s_1^2) : l^3 s_0^3 s_1^3]$

Proof. Using our previous definitions and results in Lemma 3.1, we set

$$\mu = \frac{\theta_1\theta_3\theta_8}{\theta_2\theta_4\theta_{10}}, \tag{3.27}$$

so that $\mu^4 = \lambda_0\lambda_1\lambda_2\lambda_3$. If we use the affine chart given by $s_0 = 1$, $s_1 = s/\mu$, $\xi = X$, $\eta = 1$, $\rho = Y$ in Corollary 3.12, then we obtain the Weierstrass model (2.1) and two involutions $s \mapsto -s$ and $s \mapsto 1/s$; see Equations (2.2). In fact, the coefficients in Equation (2.1) are obtained from Corollary 3.12 by setting

$$A(s) := A\left(s_0 = 1, s_1 = \frac{s}{\mu}\right), \qquad B(s) := B\left(s_0 = 1, s_1 = \frac{s}{\mu}\right). \tag{3.28}$$

A straightforward computation shows that the sections $\{\mathsf{O}, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ have exactly the intersection and height pairings given by Table 2 and form a basis of the Mordell–Weil group $\mathrm{MW}(\pi, \mathsf{O})$. Using the elliptic-curve group law in each fiber, we find that the sections $\{\mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ satisfy relations (3.26). □

We turn to the symmetric square $\mathrm{Sym}^2(\mathcal{C})$ associated with a smooth genus-two curve $\mathcal{C}$. The automorphism $\imath^{\mathcal{C}} \times \imath^{\mathcal{C}}$ of $\mathcal{C} \times \mathcal{C}$ again induces an automorphism on the symmetric square $\mathrm{Sym}^2(\mathcal{C})$, which by a slight abuse of notation we denote by $\imath^{\mathcal{C}} \times \imath^{\mathcal{C}}$ as well. The variety $\mathrm{Sym}^2(\mathcal{C})/\langle \imath^{\mathcal{C}} \times \imath^{\mathcal{C}}\rangle$ admits a birational model, which can be easily derived: in terms of the variables $z_1 = Z^{(1)}Z^{(2)}$, $z_2 = X^{(1)}Z^{(2)} + X^{(2)}Z^{(1)}$, $z_3 = X^{(1)}X^{(2)}$, and $z_4 = Y^{(1)}Y^{(2)}$ with $[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}(1, 1, 1, 3)$, it is given by the equation

$$z_4^2 = z_1 z_3 \prod_{i=0}^{3}(\lambda_i^2 z_1 - \lambda_i z_2 + z_3). \tag{3.29}$$

DEFINITION 3.15. The hypersurface in $\mathbb{P}(1, 1, 1, 3)$ given by Equation (3.29) is called the Shioda sextic and was described in [41].

We easily check the following:

LEMMA 3.16. *The Shioda sextic in Equation* (3.29) *is birational to the Kummer surface* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ *associated with the Jacobian* $\mathrm{Jac}(\mathcal{C})$ *of a genus-two curve* $\mathcal{C}$ *in Rosenhain normal form* (3.1).

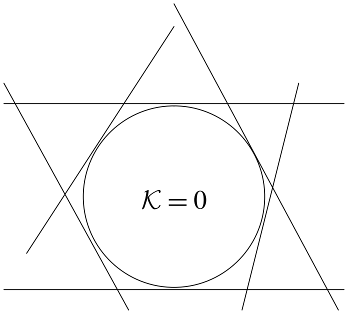

Figure 2 Double cover branched along reducible sextic.

REMARK 3.17. Equation (3.29) defines a double cover of $\mathbb{P}^2 \ni [z_1 : z_2 : z_3]$ branched along six lines given by

$$\lambda_i^2 z_1 - \lambda_i z_2 + z_3 = 0 \quad \text{with } 0 \le i \le 3, \qquad z_1 = 0, \qquad z_3 = 0, \qquad (3.30)$$

The six lines are tangent to the common conic $\mathcal{K} : z_2^2 - 4z_1z_3 = 0$. Conversely, any six lines tangent to a common conic can always be brought into the form of Equations (3.30). A picture is provided in Figure 2.

Equation (3.29) is birationally equivalent to Equation (3.18), as can be seen by setting

$$[z_1 : z_2 : z_3 : z_4] = [t_1 z : x : t_0 z : lzY],$$

and is in turn is equivalent to the elliptic fibration with section $(\pi' : \mathscr{E}' \to \mathbb{P}^1, \mathsf{O}')$ in Corollary 3.13. We make the following:

REMARK 3.18. The elliptic fibration with section $(\pi' : \mathscr{E}' \to \mathbb{P}^1, \mathsf{O}')$ has six singular fibers of Kodaira type I_2, two singular fibers of type I_0^* (over $[t_0 : t_1] = [0 : 1], [1 : 0]$), and a Mordell–Weil group $\mathrm{MW}(\pi', \mathsf{O}') \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus \langle 1 \rangle$. The elliptic fibration is induced by a pencil of lines in $\mathbb{P}^2$ passing through one of intersection point between two lines in Figure 2. This is precisely the fibration on $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ described in [41].

We have established that $\mathscr{E}'$ defines a pencil on $\mathrm{Sym}^2(\mathcal{C})/\langle \imath^{\mathcal{C}} \times \imath^{\mathcal{C}} \rangle$. The minimal resolution of $\mathscr{E}'$ is the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ associated with the Jacobian $\mathrm{Jac}(\mathcal{C})$ of a smooth genus-two curve $\mathcal{C}$. The involution $-\mathbb{I}$ on $\mathrm{Jac}(\mathcal{C})$ restricts to the hyperelliptic involution on each factor of $\mathcal{C}$ in $\mathrm{Sym}^2(\mathcal{C})$. The Weierstrass model in Remark 3.18 with two singular fibers of Kodaira type I_0^* over $[t_0 : t_1] = [0 : 1]$ and $[t_0 : t_1] = [1 : 0]$ extends to an elliptic fibration with section on the Kummer surface and two reducible fibers of type D_4. We then have the following:

LEMMA 3.19. *The map $\psi : \mathscr{E} \to \mathscr{E}'$ in Equation (3.24) extends to rational double cover between the minimal resolutions $\widehat{\psi} : \widehat{\mathscr{E}} \dashrightarrow \widehat{\mathscr{E}'} \cong \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ that is* *branched along the eight noncentral components of the two reducible fibers of type D_4.*

Proof. The proof is straightforward since the rational map is explicitly given. □

Mehran [30] proved that there are 15 distinct isomorphism classes of rational double covers of $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ and computed the 15 even eights (up to taking complements) on the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$, which give rise to all distinct 15 isomorphism classes of rational double covers [30, Prop. 4.2]. An *even eight* is an even set (as defined in Section 2) of eight exceptional curves. Each even eight is enumerated by points $\mathsf{P}_{ij} \in \mathrm{Jac}(\mathcal{C})[2]$ with $0 \le i < j \le 5$ and given as a sum

$$\Delta_{ij} = K_{0i} + \cdots + \widehat{K_{ij}} + \cdots + K_{i5} + K_{0j} + \cdots + \widehat{K_{ij}} + \cdots + K_{j5},$$

where $K_{00} = 0$, and K_{ij} are the exceptional divisors obtained by resolving the nodes p_{ij}, that is, the images of the points P_{ij}, and the hat indicates divisors that are not part of the even eight. Moreover, Mehran proved that every rational map $\psi_\Delta : \mathrm{Kum}(\mathfrak{A}) \dashrightarrow \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ from a $(1,2)$-polarized to a principally polarized Kummer surface is induced by an isogeny $\Psi_\Delta : \mathfrak{A} \to \mathrm{Jac}(\mathcal{C})$ of Abelian surfaces of degree two [30] and that all inequivalent $(1,2)$-polarized Abelian surfaces $\mathfrak{A}$ are obtained in this way. We have the following:

PROPOSITION 3.20. *There exists an Abelian surface $\mathfrak{A}$ with a polarization of type $(1,2)$ such that the variety $\mathscr{E}$ is birational to the Kummer surface* $\mathrm{Kum}(\mathfrak{A})$. *In particular, the map ψ is induced by an isogeny $\Psi : \mathfrak{A} \to \mathrm{Jac}(\mathcal{C})$ of degree two.*

Proof. It was shown in [8; 7] that the eight noncentral components of the two reducible fibers of type D_4 form an even eight on $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$. In fact, the sum of the components in the even eight labeled by p_{45} that forms the ramification locus of $\widehat{\psi}$ is given by

$$\Delta_{45} = K_{04} + K_{14} + K_{24} + K_{34} + K_{05} + K_{15} + K_{25} + K_{35}.$$

The result then follows from Lemma 3.19 and [30, Prop. 5.1]. □

REMARK 3.21. The construction in Proposition 3.20 was based on a double cover branched along the even eight Δ_{45} labeled by the point $\mathsf{P}_{45} \in \mathrm{Jac}(\mathcal{C})[2]$. This is in agreement with the construction of the hyperelliptic and bielliptic genus-three curves $\mathcal{H}$ in Figure 1, which was based on the divisor $\mathsf{P}_{45} \in \mathrm{Jac}(\mathcal{C})[2]$; see Remark 3.8.

We make the following crucial remark.

REMARK 3.22. The involution $-\mathbb{I}$ on the Abelian surface $\mathfrak{A}$ with $(1,2)$-polarization does <u>not</u> restrict to the bielliptic involution on each factor of $\mathcal{H}$ in $\mathscr{H}$. Therefore the generic Prym variety associated with the bielliptic quotient map $\mathscr{H}_{[s_0:s_1]} \to \mathscr{Q}_{[s_0:s_1]}$ of a general fiber is not isomorphic to $\mathfrak{A}$. Instead, it is isomorphic to $\mathrm{Jac}(\mathcal{C})$ by Proposition 3.6 and is only two-isogenous to $\mathfrak{A}$ by Proposition 3.20.

4. Plane Bielliptic Curves

In this section, we provide a geometric characterization of plane bielliptic genus-three curves $\mathcal{D}$ and their bielliptic quotients. The precise characterization of the associated branch loci turns out to be critical to relate the bielliptic genus-three curve $\mathcal{D}$ to a genus-two curve $\mathcal{C}$ such that the Prym variety of the former is isogenous to the Jacobian variety of the latter.

Let $\mathcal{D}$ be a bielliptic curve. Then there is an involution $\tau \in \operatorname{Aut}(\mathcal{D})$ such that $\mathcal{D}/\langle\tau\rangle$ is a genus one curve. For $g = 3$, there are two loci in the moduli space $\mathcal{M}_3$ such that the automorphism group has precisely order two, namely the hyperelliptic locus $\mathcal{M}_3^h$ and the $\mathcal{M}_3^b$ of dimensions five and four, respectively. In [3] a normal form for bielliptic genus-three curves was determined. From [26, Table 1] we see that a generic curve $[\mathcal{D}] \in \mathcal{M}_3^b$ has a degree-two cover $\pi^{\mathcal{D}} : \mathcal{D} \to \mathbb{P}^1$ ramified at four points. The curve has the equation

$$w^4 + w^2(u^2 + av^2) + bu^4 + cu^3v + du^2v^2 + euv^3 + gv^4 = 0, \tag{4.1}$$

where $e = 1$ or $g = 1$.

Precise equations, in terms of invariants of binary sextics, describing the locus $\mathcal{M}_3^h \cap \mathcal{M}_3^b$ can be easily obtained; see [39]. The same cannot be said for the locus $\mathcal{M}_3^b$; see [40]. However, there is a geometric description of the locus $\mathcal{M}_3^b$, which seems to have been known from the XIX century, and it was pointed out to us by I. Dolgachev.

4.1. Characterization of Plane Bielliptic Curves

Let $\mathcal{D}$ be a canonical curve of genus three over $\mathbb{C}$ with a bielliptic involution $\tau : \mathcal{D} \to \mathcal{D}$. In its canonical plane model given in [3], τ is induced by a projective involution $\tilde{\tau}$ whose set of fixed points consists of a point $u_0 \in \mathbb{P}^2$ and a line ℓ_0. The intersection $\ell_0 \cap \mathcal{D}$ is the fixed points of τ on $\mathcal{D}$, namely the branch points of the degree-two cover $\pi^{\mathcal{D}} : \mathcal{D} \to \mathbb{P}^1$. The following characterization is originally due to Kowalevskaya; see Dolgachev [12].

THEOREM 4.1 (Kowalevskaya). *The point u_0 is the intersection point of four distinct bitangents of $\mathcal{D}$. Conversely, if a plane quartic has four bitangents intersecting at a point u_0, then there exists a bielliptic involution τ of $\mathcal{D}$ such that the projective involution $\tilde{\tau}$ has u_0 as its isolated fixed point.*

We give another characterization of bielliptic quartic curves.

THEOREM 4.2. *$\mathcal{D}$ is bielliptic if and only if the following conditions are satisfied*:

(i) *There exists a line ℓ_0 intersecting $\mathcal{D}$ at four distinct points $p_1, \dots, p_4$ with tangent lines ℓ_i at the points p_i that intersect at one point u_0.*

(ii) *Let $P_{u_0}(\mathcal{D})$ be the cubic polar of $\mathcal{D}$ with respect to the point u_0, and let Q be the conic component of $P_{u_0}(\mathcal{D})$ (note that the line ℓ_0 from above is a line component of $P_{u_0}(\mathcal{D})$). Then ℓ_0 is the polar line of $\mathcal{D}$ with respect to u_0.*

Proof. Suppose $\mathcal{D}$ is bielliptic so that its equation $f(u,v,w)=0$ can be written in the form

$$\begin{aligned} f(u,v,w) &= w^4 - 2a_2(u,v)w^2 + a_4(u,v) \\ &= (w^2 - a_2(u,v))^2 + (a_4(u,v) - a_2(u,v)^2) = 0. \end{aligned} \tag{4.2}$$

We check that for the curve in Equation (4.2), the involution $\tau : [u:v:w] \mapsto [u:v:-w]$ is induced by a projective involution $\tilde{\tau}$ whose set of fixed points consists of the point $u_0 = [0:0:1] \in \mathbb{P}^2$ and the line $\ell_0 = V(w)$. The polar cubic $P_{u_0}(\mathcal{D})$ has the equation $q = w(w^2 - a_2(u,v)) = 0$. It is the union of the line ℓ_0 and the conic $Q = V(w^2 - a_2(u,v))$. The line ℓ_0 intersects $\mathcal{D}$ at the points $p_i : [\beta_i : \alpha_i : 0]$, where $a_4(\beta_i, \alpha_i) - a_2(u,v)^2 = 0$. The tangent lines at the points p_i are $\ell_i = V(\alpha_i u - \beta_i v)$. By the main property of polars, $P_{u_0}(\mathcal{D})$ intersects $\mathcal{D}$ at the points p such that the tangent line of $\mathcal{D}$ at p contains the point u_0. Thus the tangent lines ℓ_i at $p_i \in \mathcal{D}$ all pass through the point u_0, which, given the normalization of the curve $\mathcal{D}$ in Equation (4.2), is $u_0 = [0:0:1]$. Thus part (i) is verified.

Using Equation (4.2), we compute the line polar $P_{u_0^3}(\mathcal{D}) = V(\partial^3(F)/\partial w^3)$ of $\mathcal{D}$. It coincides with the line ℓ_0. On other hand,

$$\begin{aligned} P_{u_0^3}(\mathcal{D}) &= P_{u_0^2}(P_{u_0}(\mathcal{D})) = P_{u_0^2}(qw) = P_{u_0}(q + P_{u_0}(q)w) \\ &= 2P_{u_0}(q) + P_{u_0^2}(q)w = w, \end{aligned}$$

where we identify the polar curves with the corresponding partial derivatives. This implies that $V(P_{u_0}(q)) = V(w) = \ell_0$. This checks property (ii).

Let us prove the converse. Choose coordinates to assume that $\ell_0 = V(w)$ and the intersection point of the four tangent lines to is $u_0 = [0:0:1]$. The cubic polar $P_{u_0}(\mathcal{D})$ must contain the line component equal to ℓ_0. Write the equation of $\mathcal{D}$ in the form

$$a_0 w^4 + a_1(u,v)w^3 + a_2(u,v)w^2 + a_3(u,v)w + a_4(u,v) = 0.$$

Then we get

$$\begin{aligned} P_{u_0}(\mathcal{D}) &= V(4a_0 w^3 + 3a_1(u,v)w^2 + a_2(u,v)w + a_3(u,v)), \\ P_{u_0^2}(\mathcal{D}) &= V(12a_0 w^2 + 6a_1(u,v)w + a_2(u,v)), \\ P_{u_0^3}(\mathcal{D}) &= 24a_0 w + 6a_1(u,v). \end{aligned}$$

Since w divides the equation of the polar cubic, we obtain $a_3(u,v) = 0$. If $a_0 = 0$, then $u_0 \in \mathcal{D}$, and the line polar $P_{u_0^3}(\mathcal{D})$ vanishes at u_0. But this polar is the tangent line of $\mathcal{D}$ at u_0. This implies that $\mathcal{D}$ is singular at u_0. So, we may assume that $a_0 \neq 0$. Thus the first condition implies that $\mathcal{D}$ can be written in the form

$$w^4 + a_1(u,v)w^3 + a_2(u,v)w^2 + a_4(u,v) = 0.$$

As in the first part of the proof, we obtain that $a_1(u,v) = 0$ if and only if condition (ii) is satisfied. Thus $\mathcal{D}$ can be written in the form (4.2) and hence is bielliptic. □

For any general line ℓ, let $\ell_1, \dots, \ell_4$ be the tangents of $\mathcal{D}$ at the points $a_1 + \cdots + a_4 = \mathcal{D} \cap \ell$ with $\ell_i \cap \mathcal{D} = 2a_i + c_i + d_i$. Adding up, we see that

$$\sum (c_i + d_i) \sim 4K_{\mathcal{D}} - 2\sum a_i \sim 4K_{\mathcal{D}} - 2K_{\mathcal{D}} = 2K_{\mathcal{D}},$$

where $K_{\mathcal{D}}$ is a canonical divisor. This shows that there exists a conic $S(\ell)$ that cuts out on $\mathcal{D}$ the divisor $\sum (c_i + d_i)$ of degree eight. This conic is called the *satellite conic* of ℓ (see [10]). The map

$$\begin{aligned} S : \mathbb{P}^2 &\longrightarrow \mathbb{P}^5, \\ \ell &\mapsto S(\ell), \end{aligned} \tag{4.3}$$

is given by polynomials of degree 10 whose coefficients are polynomials in coefficients of $\mathcal{D}$ of degree seven. Since $2\ell + S(\ell)$ and $T = \ell_1 + \cdots + \ell_4$ cut out on $\mathcal{D}$ the same divisor, we obtain that the equation of $\mathcal{D}$ can be written in the form

$$F = l_1 \cdots l_4 + l^2 q = 0,$$

where $\ell_i = V(l_i)$, $\ell = V(l)$, and $S(\ell) = V(q)$.

Using an automorphism of $\mathbb{P}^2$, we can assume that the line ℓ is given by $\ell = V(w)$. For a general quartic, given by

$$\begin{aligned} &ax^4 + by^4 + cz^4 + 6fy^2z^2 + 6gx^2z^2 + 6hx^2y^2 + 12lx^2yz + 12mxy^2z \\ &\quad + 12nxyz^2 + 4x^3ya_1 + 4x^3za_2 + 4xy^3b_0 + 4xz^3c_0 + 4y^3zb_2 + 4yz^3c_1 = 0, \end{aligned}$$

an expression of the satellite conic is

$$\begin{aligned} S = {} & x^2(9af^2 - 16ab_2c_1) + 2xy(18a_1f^2 - 32a_1b_2c_1) \\ & + 2xz(18a_2f^2 - 32a_2b_2c_1) + y^2(54f^2h - 96b_2c_1h) \\ & + 2xy(54f^2l - 108fmn - 96b_2c_1l + 72c_1m^2 + 72b_2n^2) \\ & + z^2(54fg^2 - 96b_2c_1g). \end{aligned}$$

For the bielliptic curve in (4.1), this satellite conic is

$$S = -u^2ace + \frac{1}{4}u^2ad^2 - v^2ce + \frac{1}{4}v^2d^2 - w^2cea + \frac{1}{4}w^2da^2.$$

A line ℓ is called a *bielliptic line* if the four tangents ℓ_i intersect at a common point u_0. Choose the coordinates such that $u_0 = [0 : 0 : 1]$ and $l = w$. Then the equation of $\mathcal{D}$ is of the form

$$F = w^2(a_0w^2 + a_1(u, v)w + a_2(u, v)) + a_4(u, v) = 0.$$

It is a bielliptic curve if and only if $a_1(u, v) = 0$. This is equivalent to $P_{u_0}(S(\ell)) = \ell$. Thus we have obtained the following:

THEOREM 4.3. *Suppose ℓ is a bielliptic line. Then $\mathcal{D}$ is bielliptic if and only if the polar line of the satellite conic $S(\ell)$ with respect to the point u_0 coincides with ℓ.*

Let ℓ be a bielliptic line. The polar cubic of $P_{u_0}(\mathcal{D})$ passes through $\mathcal{D} \cap \ell$, and hence it contains ℓ as an irreducible component. In particular, $P_{u_0}(\mathcal{D})$ is singular. Recall that the locus of points $u \in \mathbb{P}^2$ such that $P_u(\mathcal{D})$ is a singular cubic is the

Steinerian curve $\mathrm{St}(\mathcal{D})$. If $\mathcal{D}$ is general enough, then the degree of $\mathrm{St}(\mathcal{D})$ is equal to 12, and it has 24 cusps and 21 nodes. The cusps correspond to points such that the polar cubic is cuspidal, and the nodes correspond to points such that the polar cubic is reducible. The line components define the set of 21 bielliptic lines. In [10] the 21 lines are described as singular points of multiplicity 4 of the curve of degree 24 in the dual plane parameterizing lines so that the tangents to $\mathcal{D}$ at three intersection points of $\mathcal{D}$ and ℓ are concurrent.

According to [10, p. 327], the equation of the satellite conic $S(\ell)$ is equal to

$$S = \mathcal{D}_{7,2,10} + \ell \cdot \mathcal{D}_{7,1,9} + \ell^2 \cdot \mathcal{D}_{7,0,8} = 0,$$

where $\mathcal{D}_{a,b,c} \in S^a(S^4(V^*)^*) \otimes S^b(V) \otimes S^c(V^*)$ is a comitant of degree a in coefficients of $\mathcal{D}$, of degree b in coordinates of the plane $\mathbb{P}(V^*)$ (we use Grothendieck's notation), and of degree c in coordinates of the dual plane. Thus the vanishing of $a_1(u, v)$ from above is equivalent to the vanishing of the comitant $\mathcal{D}_{7,1,9}$. Cohen [10] gives an explicit equation of $\mathcal{D}_{7,1,9}$.

THEOREM 4.4. *$\mathcal{D}$ is bielliptic if and only if $\mathcal{D}_{7,1,9}$, considered as a map $\mathbb{P}(V) \longrightarrow \mathbb{P}(V^*)$, has one of the 21 lines corresponding to the nodes of* $\mathrm{St}(\mathcal{D})$ *as its indeterminacy point. The rational map is given by polynomials of degree nine with polynomial coefficients in coefficients of $\mathcal{D}$ of degree seven.*

This gives equations of the locus of bielliptic curves in $\mathcal{M}_3$. It is unknown to us if this locus has ever been explicitly determined in terms of the invariants of the ternary quartics.

4.2. Ramification Locus

In this section, we determine explicit equations for plane bielliptic genus-three curves based on their characterization in Theorem 4.2. All computations in this section are carried out over an arbitrary field K. We start with the plane bielliptic genus-three curve $\mathcal{D}$ given in Equation (4.2), that is,

$$\mathcal{D}: \quad w^4 - 2a_2(u, v)w^2 + a_4(u, v) = 0 \tag{4.4}$$

with $[u : v : w] \in \mathbb{P}^2$ and general homogeneous polynomials a_2 and a_4 of degree two and four, respectively, and the bielliptic involution $\tau : [u : v : w] \mapsto [u : v : -w]$. It follows from [3, Corollary 2.2] that any such smooth curve $\mathcal{D}$ is the canonical model of a bielliptic nonhyperelliptic curve of genus three. The bielliptic quotient $\mathcal{D}/\langle\tau\rangle$ is the genus-one curve $\mathcal{Q}$ given by

$$\mathcal{Q}: \quad W^2 = a_2(u, v)^2 - a_4(u, v) = c_4u^4 + c_3u^3v + c_2u^2v^2 + c_1uv^3 + c_0v^4 \tag{4.5}$$

with $W = w^2 - a_2(u, v)$ and $[u, v, W] \in \mathbb{P}(1, 1, 2)$. Using a standard technique, as explained, for example, in [25, App. A], we convert this genus-one curve to the Weierstrass form given any K-rational point on the curve. We have the following:

LEMMA 4.5. *Given a K-rational point, the bielliptic quotient $\mathcal{D}/\langle\tau\rangle$ is isomorphic to the elliptic curve $\mathcal{E}$ given by*

$$\mathcal{E}: \quad \rho^2\eta = \xi^3 + f\xi^2\eta + g\xi\eta^2 + h\eta^3 \tag{4.6}$$

with $[\xi : \eta : \rho] \in \mathbb{P}^2$ *and*

$$\begin{aligned} f &= 3c_1^2 - 8c_0c_2, \qquad h = (c_1^3 - 4c_0c_1c_2 + 8c_0^2c_3)^2,\\ g &= 3c_1^4 - 16c_0c_1^2c_2 + 16c_0^2(c_2^2 + c_1c_3) - 64c_0^3c_4. \end{aligned}$$

Proof. By a change of coordinates we can assume that the K-rational point is given by $[u, v, W] = [0 : 1 : \alpha]$, that is, $c_0 = \alpha^2$. Using the transformation

$$u = -\frac{4c_0\xi\eta v}{\tilde{\rho}}, \qquad w = \alpha v^2 \frac{\tilde{\rho}^2 - 2c_1\xi\eta\tilde{\rho} - 2\xi^3\eta - 2(c_1^2 - 4c_0c_2)\xi^2\eta^2}{\tilde{\rho}^2},$$

followed by the transformation

$$\tilde{\rho} = \rho\eta + c_1\xi\eta + (c_1^3 - 4c_0c_1c_2 + 8c_0^2c_3)\eta^2,$$

proves the lemma. □

REMARK 4.6. The elliptic curve (4.6) remains well defined, independently of the existence of a K-rational point. However, in general, there is only an isomorphism

$$\mathrm{Jac}(\mathcal{D}/\langle\tau\rangle) \cong \mathcal{E}.$$

The existence of a K-rational point is required for an isomorphism $\mathcal{D}/\langle\tau\rangle \cong \mathcal{E}$. The Jacobian was first found by Hermite as the determinant of a symmetric matrix that defines a conic bundle which degenerates over $\mathcal{E}$; see [32].

The following lemma is easily verified.

LEMMA 4.7. *The elliptic curve $\mathcal{E}$ with full K-rational two-torsion given by*

$$\mathcal{E}: \quad \rho^2\eta = \xi(\xi^2 - 2b\xi\eta + (b^2 - 4a^2)\eta^2) \tag{4.7}$$

with $[\xi : \eta : \rho] \in \mathbb{P}^2$ *is isomorphic to the genus-one curve*

$$\mathcal{Q}: \quad W^2 = u^4 + bu^2v^2 + a^2v^4, \tag{4.8}$$

where $[u : v : W] \in \mathbb{P}(1, 1, 2)$. *An isomorphism* $\varphi : \mathcal{E} \xrightarrow{\cong} \mathcal{Q}$ *is given by*

$$[\xi : \eta : \rho] \mapsto [u : v : W] = [\rho\eta : -2\xi\eta : (\rho^2\eta + 2b\xi^2\eta - 2\xi^3)\eta] \tag{4.9}$$

and by mapping points $\mathsf{T}_1 : [\xi : \eta : \rho] = [0 : 1 : 0]$ *and* $\mathsf{O} : [0 : 0 : 1]$ *to* $[u : v : W] = [1 : 0 : 1]$ *and* $[1 : 0 : -1]$, *respectively.*

For the elliptic curve $\mathcal{E}$ in Equation (4.7), the flex-point is the point at infinity $[\xi : \eta : \rho] = [0 : 0 : 1]$, which is also the base point O for the elliptic-curve group law. The point $\mathsf{T}_1 : [\xi : \eta : \rho] = [0 : 1 : 0]$ is a nontrivial two-torsion point. We have the following:

PROPOSITION 4.8. *The plane bielliptic genus-three curve*

$$\mathcal{D}: \quad (w^2 - a_2(u,v))^2 = u^4 + bu^2v^2 + a^2v^4, \tag{4.10}$$

where $[u:v:w] \in \mathbb{P}^2$, a_2 *is a homogeneous polynomial of degree two, and* a, b *are* K*-rational numbers such that* $a(b^2-4a^2) \neq 0$, *admits the bielliptic involution* $\tau : [u:v:w] \mapsto [u:v:-w]$ *and a degree-two cover given by*

$$\pi^{\mathcal{D}}_{\mathcal{Q}}: \quad \mathcal{D} \to \mathcal{Q}, \qquad [u:v:w] \mapsto [u:v:W = w^2 - a_2(u,v)] \tag{4.11}$$

onto the genus-one curve $\mathcal{Q}$ *in Equation* (4.8)*. The branch locus of the bielliptic involution* τ *is isomorphic via* φ *to a collection of points* $\{\mathrm{pt}_1, \mathrm{pt}_2, \mathrm{pt}_3, \mathrm{pt}_4\} \subset \mathcal{E}$ *in Equation* (4.9) *satisfying*

$$\xi^3 - a_2(\rho, -2\xi)\eta - (b^2-4a^2)\xi\eta^2 = 0. \tag{4.12}$$

In particular, we have $\sum_{i=1}^4 \mathrm{pt}_i = \mathsf{O}$*. Conversely, the elliptic curve* $\mathcal{E}$ *in Equation* (4.9) *and* $\{\mathrm{pt}_1, \mathrm{pt}_2, \mathrm{pt}_3, \mathrm{pt}_4\} \subset \mathcal{E}$ *with* $\sum_{i=1}^4 \mathrm{pt}_i = \mathsf{O}$ *determine Equation* (4.10) *uniquely.*

Proof. The first part follows by explicit computation using Lemma 4.7 and the group law on $\mathcal{E}$. Conversely, the elliptic curve in Equation (4.6) is isomorphic to the general genus-one quotient curve given by Lemma 4.5 iff we impose $h = 0$. The condition $h = 0$ allows us to express the coefficients $c_2\alpha^2$, $c_3\alpha^4$, and $c_4\alpha^6$ with $c_0 = \alpha^2$ as simple rational functions of A, B, c_1. The general isomorphism $\varphi : \mathcal{E} \to \mathcal{Q}$ is given by

$$[\xi:\eta:\rho] \mapsto [u:v:W] = [(2c_1\xi+\rho)\eta : -2\xi\eta : (\rho^2\eta + 2b\xi^2\eta - 2\xi^3)\eta] \tag{4.13}$$

such that

$$\begin{aligned}\mathcal{Q}: \quad W^2 = {} & u^4 + bu^2v^2 + a^2v^4 \\ & + c_1(2u + c_1v)(2u^2 + 2c_1uv + (c_1^2 + b)v^2)v,\end{aligned} \tag{4.14}$$

where $[u:v:W] \in \mathbb{P}(1,1,2)$, and $c_1 \in K$ is an arbitrary coefficient.

The branch locus on $\mathcal{E}$ in Equation (4.6) uniquely defines a conic. This conic is given by

$$\mathcal{K}: \quad (1-\gamma)\xi^2 + 4\beta\gamma\rho\eta + 4\alpha\xi\eta - (1+\gamma)(b^2-4a^2)\eta^2 = 0 \tag{4.15}$$

with $\alpha, \beta, \gamma \in K$. If a plane curve of degree n intersects an elliptic curve in $3n$ points, then these points always sum up using the group law of the elliptic curve $\mathcal{E}$ in Equation (4.9). In our case, we expect six points $\mathrm{pt}_1, \dots, \mathrm{pt}_6 \in \mathcal{E}$ such that $[\mathrm{pt}_1 + \cdots + \mathrm{pt}_6 - 6\mathsf{O}] = 0 \in \mathrm{Pic}^0(\mathcal{E})$ as it is the divisor class of $\mathrm{Div}(\mathcal{K}/\mathcal{L}^6)$ where $\mathcal{L} : \eta = 0$ is the flex-line. However, the conic and the elliptic curve intersect at $\eta = 0$; we check this computing the resultant of $\mathcal{K}$ and the defining equation of $\mathcal{E}$. From Equation (4.15) we check that the intersection at $\eta = 0$ has order two, whence $\mathrm{pt}_5 = \mathrm{pt}_6 = \mathsf{O}$. Therefore the remaining four points $\mathrm{pt}_1, \dots, \mathrm{pt}_4$ satisfy $[\mathrm{pt}_1 + \cdots + \mathrm{pt}_4 - 4\mathsf{O}] = 0$. We set $a_2(u,v) = \gamma(u + (\beta + c_1)v)^2 - (\alpha + \beta^2\gamma - \gamma b/2)v^2$, and then the branching locus satisfies

$$\xi^3 - (b^2-4a^2)\xi\eta^2 = a_2(2c_1\xi + \rho, -2\xi)\eta. \tag{4.16}$$

In turn, the plane genus-three curve is given by setting

$$\begin{aligned} w^2 &= W + a_2(u,v) \\ &= -(\xi^3 - (b^2-4a^2)\xi\eta^2)\eta + a_2(2c_1\xi+\rho, -2\xi)\eta^2. \end{aligned} \tag{4.17}$$

Since $\Delta_{\mathcal{E}} = \Delta_{\mathcal{Q}} = 16a^2(b^2-4a^2)^2$, we can set $c_1 = 0$ without loss of generality. □

We have the following:

REMARK 4.9. The point O is a branch point if and only if $a_2(u,0) = u^2$. We then write $a_2(u,v) = (u+\beta v)^2 - (\alpha+\beta^2-b/2)v^2$ with $\alpha, \beta \in K$. The remaining points of the branch locus lie on the intersection of $\mathcal{E}$ with the line $2\alpha\xi + 2\beta\rho - (b^2-4a^2)\eta = 0$. If the point O is in the branch locus of $\pi^{\mathcal{D}}_{\mathcal{Q}}$, then the remaining points $\{\mathrm{pt}_1, \mathrm{pt}_2, \mathrm{pt}_3\}$ satisfy $\sum_{i=1}^3 \mathrm{pt}_i = \mathsf{O}$ on $\mathcal{E}$.

On $\mathcal{E}$, we have different involutions acting on points $p \in \mathcal{E}$: (1) the hyperelliptic involution $\imath^{\mathcal{E}} : p \mapsto -p$ given by $[\xi : \eta : \rho] \mapsto [\xi : \eta : -\rho]$; (2) the involution $\imath^{\mathcal{E}}_{\mathsf{T}_1} : p \mapsto p + \mathsf{T}_1$ obtained by translation by two-torsion T_1 and given by

$$\imath^{\mathcal{E}}_{\mathsf{T}_1} : \quad [\xi : \eta : \rho] \mapsto [(b^2-4a^2)\xi\eta : \xi^2 : -(b^2-4a^2)\rho\eta]; \tag{4.18}$$

and (3) the composition $\imath^{\mathcal{E}} \circ \imath^{\mathcal{E}}_{\mathsf{T}_1} = \imath^{\mathcal{E}}_{\mathsf{T}_1} \circ \imath^{\mathcal{E}} : p \mapsto -p + \mathsf{T}_1$. We have the following:

LEMMA 4.10. *The involutions act on $\mathcal{Q}$ as follows*:

$$\begin{aligned} \varphi \circ \imath^{\mathcal{E}} : &\quad [u : v : W] \mapsto [-u : v : W] = [u : -v : W], \\ \varphi \circ \imath^{\mathcal{E}}_{\mathsf{T}_1} : &\quad [u : v : W] \mapsto [-u : v : -W] = [u : -v : -W], \\ \varphi \circ (\imath^{\mathcal{E}} \circ \imath^{\mathcal{E}}_{\mathsf{T}_1}) : &\quad [u : v : W] \mapsto [u : v : -W]. \end{aligned} \tag{4.19}$$

Proof. The proof follows by computation. □

4.3. Biquadratic Quotients

We discuss the case in Proposition 4.8 for a branch locus on $\mathcal{E}$ in Equation (4.7), which consists of the points $\mathsf{O} : [\xi : \eta : \rho] = [0 : 0 : 1]$ and pt_1, pt_2, pt_3 with $\sum_{i=1}^3 \mathrm{pt}_i = \mathsf{O}$. We have the following:

LEMMA 4.11. *Given two K-rational points $q_1, q_2 \in \mathcal{E}$ such that $[u : v : W] = [R_i : 1 : S_i] = \varphi(q_i)$ for $1 \le i \le 2$ with $R_1^2 \neq R_2^2$, the point $q_3 = -q_1 - q_2 \in \mathcal{E}$ is a K-rational point with $[R_3 : 1 : S_3] = \varphi(q_3)$, and we have for a, b in Equation (4.7) the relations*

$$a^2 = R_1^2 R_2^2 + \frac{R_1^2 S_2^2 - R_2^2 S_1^2}{R_1^2 - R_2^2}, \qquad b = -\frac{R_1^4 - R_2^4 - S_1^2 + S_2^2}{R_1^2 - R_2^2}, \tag{4.20}$$

and

$$R_3 = \frac{R_2 S_1 - R_1 S_2}{R_1^2 - R_2^2},$$
$$S_3 = -R_1 R_2 + \frac{(R_1 S_1 - R_2 S_2)(R_2 S_1 - R_1 S_2)}{R_1^2 - R_2^2}. \tag{4.21}$$

Proof. Given two different K-rational points pt_1, pt_2 on the elliptic curve $\mathcal{E}$ in Equation (4.7) such that $[R_1 : 1 : S_1] = \varphi(q_1)$ and $[R_2 : 1 : S_2] = \varphi(q_2)$ with $R_1^2 \neq R_2^2$, the point $q_3 = -q_1 - q_2$ is K-rational. Since the coordinates $[R_i : 1 : S_i]$ for $1 \le i \le 3$ label points on $\mathcal{Q}$ satisfying Equation (4.8), we can easily derive Equation (4.20). We compute the coordinates for all points on $\mathcal{E}$ and check using the elliptic-curve group law that $q_1 + q_2 + q_3 = \mathsf{O}$. $\square$

For $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ and $\varepsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ and two distinct K-rational points $q_1, q_2 \in \mathcal{E}$ such that $[u : v : W] = [R_i : 1 : S_i] = \varphi(q_i)$ for $1 \le i \le 2$ with $R_1^2 \neq R_2^2$, we define the plane bielliptic genus-three curves $\mathcal{D}^\varepsilon$ given by

$$\begin{aligned} & w^4 - 2w^2\Bigg(u^2 - \left(\epsilon_1 R_1 + \epsilon_2 R_2 + \epsilon_3 \frac{\epsilon_2 S_1 - \epsilon_1 S_2}{\epsilon_1 R_1 - \epsilon_2 R_2}\right) uv \\ & \quad + \left(\epsilon_1 \epsilon_2 R_1 R_2 - \epsilon_3 \frac{R_1 S_2 - R_2 S_1}{\epsilon_1 R_1 - \epsilon_2 R_2}\right) v^2\Bigg) \\ & \quad - 2\left(\epsilon_1 R_1 + \epsilon_2 R_2 + \epsilon_3 \frac{\epsilon_2 S_1 - \epsilon_1 S_2}{\epsilon_1 R_1 - \epsilon_2 R_2}\right)(u - \epsilon_1 R_1 v)(u - \epsilon_2 R_2 v) \\ & \quad \times \left(u - \epsilon_3 \frac{R_2 S_1 - R_1 S_2}{R_1^2 - R_2^2} v\right) v = 0 \end{aligned} \tag{4.22}$$

with $[u : v : W] \in \mathbb{P}(1, 1, 2)$. We have the following:

LEMMA 4.12. *Given two K-rational points $q_1, q_2 \in \mathcal{E}$ such that $[R_i : 1 : S_i] = \varphi(q_i)$ for $1 \le i \le 2$ with $R_1^2 \neq R_2^2$, we set $q_3 = -q_1 - q_2 \in \mathcal{E}$ with $[R_3 : 1 : S_3] = \varphi(q_3)$. Then we have*:

(1) *For all $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$, the plane genus-three curves $\mathcal{D}^\varepsilon$ in Equation* (4.22) *admit the bielliptic involution $\imath_b^{\mathcal{D}} : [u : v : w] \mapsto [u : v : -w]$ and the degree-two quotient map $\pi_{\mathcal{Q}}^{\mathcal{D}^\varepsilon}$ given by*

$$\begin{aligned} \pi_{\mathcal{Q}}^{\mathcal{D}^\varepsilon} : \quad w \mapsto W = w^2 - u^2 + \left(\epsilon_1 R_1 + \epsilon_2 R_2 + \epsilon_3 \frac{\epsilon_2 S_1 - \epsilon_1 S_2}{\epsilon_1 R_1 - \epsilon_2 R_2}\right) uv \\ - \left(\epsilon_1 \epsilon_2 R_1 R_2 - \epsilon_3 \frac{R_1 S_2 - R_2 S_1}{\epsilon_1 R_1 - \epsilon_2 R_2}\right) v^2 \end{aligned} \tag{4.23}$$

onto the curve $\mathcal{Q}$ in Equation (4.8), *isomorphic to $\mathcal{E}$ in Equation* (3.8).

(2) *The branch points $\mathrm{pt}_i \in \mathcal{E}$ with $1 \le i \le 4$ of $\pi_{\mathcal{Q}}^{\mathcal{D}^\varepsilon}$ are given by Table* 8.

(3) *With respect to the elliptic-curve group law, we have $\sum_{i=1}^3 \mathrm{pt}_i = \mathsf{O}$.*

Table 8 Possible branch loci on $\mathcal{Q}$.

#	ϵ_1	ϵ_2	ϵ_3	pt_1	pt_2	pt_3	pt_4
1	1	1	1	q_1	q_2	q_3	O
2	−1	−1	−1	$-q_1$	$-q_2$	$-q_3$	O
3	1	−1	−1	q_1	T_1+q_2	T_1+q_3	O
4	−1	1	1	$-q_1$	T_1-q_2	T_1-q_3	O
5	−1	−1	1	T_1+q_1	T_1+q_2	q_3	O
6	1	1	−1	T_1-q_1	T_1-q_2	$-q_3$	O
7	−1	1	−1	T_1+q_1	q_2	T_1+q_3	O
8	1	−1	1	T_1-q_1	$-q_2$	T_1-q_3	O

Proof. Using Equations (4.20) and (4.21), $\mathcal{Q}$ in Equation (4.8) can be written as

$$\begin{aligned}\mathcal{Q}:\quad W^2 = {} & 2\left(R_1+R_2+\frac{S_1-S_2}{R_1-R_2}\right)(u-R_1v)(u-R_2v)(u-R_3v)v\\ & +\left(u^2-\left(R_1+R_2+\frac{S_1-S_2}{R_1-R_2}\right)uv\right.\\ & \left.+\left(R_1R_2-\frac{R_1S_2-R_2S_1}{R_1-R_2}\right)v^2\right)^2.\end{aligned} \tag{4.24}$$

Regardless of what signs ϵ_1, ϵ_2, ϵ_3 are chosen in Equation (4.22), the bielliptic quotient is always the same; namely, it coincides with the curve $\mathcal{Q}$ in Equation (4.24). Then (1) is immediate; for (2), we check that the branch points of the map $\pi^{\mathcal{D}^\varepsilon}_{\mathcal{Q}}$ in Equation (4.23) are the points with coordinates $[\epsilon_1R_1 : 1 : \epsilon_2\epsilon_3S_1]$, $[\epsilon_2R_2 : 1 : \epsilon_1\epsilon_3S_2]$, $[\epsilon_3R_3 : 1 : \epsilon_1\epsilon_2S_3]$, and O. Lemma 4.10 provides a geometric interpretation for these branch points; (3) follows from Lemma 4.11. □

We consider the plane bielliptic curve $\mathcal{D}$ given by

$$\mathcal{D}:\quad (e(w^2-u^2)-cuv-dv^2)^2 = e^2(u^4+bu^2v^2+a^2v^4) \tag{4.25}$$

with $[u:v:w]\in\mathbb{P}^2$ and $a,b,c,d,e\in K$. We discuss the singular locus of Equation (4.25). We have the following:

Lemma 4.13. *The plane genus-three curve in Equation* (4.25) *is irreducible and nonsingular if and only if* $\Delta_{\mathcal{E}}\Delta_{\mathcal{D}}\neq 0$, *where*

$$\begin{aligned}\Delta_{\mathcal{E}} &= 16a^2(b^2-4a^2)^2,\\ \Delta_{\mathcal{D}} &= -((c^2-be^2-4de)^2-12de^2(be+d))^3\\ &\quad +(54ac^2e^4-c^6+3(be+4d)c^4e\\ &\quad -3(b^2e^2+2bde+10d^2)c^2e^2+(be-2d)^3e^3)^2.\end{aligned} \tag{4.26}$$

Proof. For $e=0$, Equation (4.25) is singular, and $\Delta_{\mathcal{D}}=0$. We assume that $e\neq 0$. Then we can set $e=1$ since rescaling $c\mapsto ce$ and $d\mapsto de$ eliminates e from the equation. We check that for $b^2-4a^2=0$, Equation (4.25) factors and the curve is reducible. We assume that $b^2-4a^2=0\neq 0$. We check that the curve in Equation (4.25) has a singular point with $w=0$ iff $\Delta_{\mathcal{D}}=0$. $\Delta_{\mathcal{D}}$ is the iterated (reduced) discriminant of Equation (4.25) with respect to w and u (or v). For $w\neq 0$ and $u=0$, the curve in Equation (4.25) has a singular point if $a=0$. We assume that $\Delta_{\mathcal{E}}\neq 0$. Then there are no singular points with $w\neq 0$, $v\neq 0$, and $\Delta_{\mathcal{E}}\Delta_{\mathcal{D}}\neq 0$. □

REMARK 4.14. If $c=0$ or $e=0$, then $\Delta_{\mathcal{D}}=0$ in Equation (4.26).

The genus-one curve $\mathcal{Q}$ in Equation (4.8) is isomorphic via φ in Equation (4.9) to the elliptic curve $\mathcal{E}$ in Equation (4.7). We now make the latter coincide with the elliptic curve in Proposition 3.4 to obtain the following:

PROPOSITION 4.15. *Let $\mathcal{D}$ be the plane bielliptic curve given by*

$$\mathcal{D}:\quad \left(w^2-u^2-\frac{c}{e}uv-\frac{d}{e}v^2\right)^2=u^4+bu^2v^2+a^2v^4 \tag{4.27}$$

with $[u:v:w]\in\mathbb{P}^2$, a, b given in Equation (3.9), and the coefficients

$$\begin{aligned} c&=c(\lambda_0,\lambda_1,\lambda_2,\lambda_3,l), & d&=d(\lambda_0,\lambda_1,\lambda_2,\lambda_3,l),\\ e&=e(\lambda_0,\lambda_1,\lambda_2,\lambda_3,l), \end{aligned} \tag{4.28}$$

with polynomials c, d, e given in the Appendix such that $\Delta_{\mathcal{E}}\Delta_{\mathcal{D}}\neq 0$. Then the curve $\mathcal{D}$ is smooth and irreducible and admits the involution $\tau:[u:v:w]\mapsto[u:v:-w]$ and the degree-two cover

$$\pi^{\mathcal{D}}_{\mathcal{Q}}:\quad \mathcal{D}\to\mathcal{Q},\qquad [u:v:w]\mapsto\left[u:v:W=w^2-u^2-\frac{c}{e}uv-\frac{d}{e}v^2\right], \tag{4.29}$$

onto $\mathcal{Q}\cong\mathcal{E}$ with branch points $\{\mathsf{O},2\mathsf{p}_1,\mathsf{p}_1+\mathsf{p}_2,-3\mathsf{p}_1-\mathsf{p}_2\}\subset\mathcal{E}$, where $\mathcal{E}$ is the smooth elliptic curve given in Equation (4.7) with $\Delta_{\mathcal{E}}\neq 0$ and the K-rational points p_1, p_2 in Lemma 3.5.

Proof. It follows from Lemma 4.13 that $\mathcal{D}$ is smooth and irreducible, and from Remark 4.14 that Equation (4.29) is well defined. We apply Lemma 4.12 to the situation encountered in Proposition 3.4 with $q_1=2\mathsf{p}_1$, $q_2=\mathsf{p}_1+\mathsf{p}_2$, where the K-rational points p_1, p_2 were given in Lemma 3.5. Using the isomorphism $\varphi:\mathcal{E}\to\mathcal{Q}$, we compute the coordinates $[u:v:W]=[R_i:1:S_i]=\varphi(q_i)$ for $1\leq i\leq 2$. We then use Equations (4.20) and (4.21) to obtain formulas for the coefficients c, d, e. □

REMARK 4.16. Replacing $l\mapsto -l$ is equivalent to $\mathsf{p}_2\mapsto-\mathsf{p}_2$ due to Lemma 3.5. Thus Proposition 4.15 generalizes to branch points $\{\mathsf{O},2\mathsf{p}_1,\mathsf{p}_1+\epsilon_2\mathsf{p}_2,-3\mathsf{p}_1-\epsilon_2\mathsf{p}_2\}\subset\mathcal{E}$ with $\epsilon_2\in\{\pm 1\}$ when replacing $l\mapsto\epsilon_2 l$ in Equations (4.28).

REMARK 4.17. It follows from Lemma 4.10 that the inversion $q\mapsto -q$ on the elliptic curve is equivalent to $u\mapsto -u$. Moreover, for $[R_i:1:S_i]\mapsto[-R_i:1:S_i]=\varphi(-q_i)$ with $1\leq i\leq 2$, we have $[R_3:1:S_3]\mapsto[-R_3:1:S_3]$ in Equations

(4.21). Therefore a bielliptic plane quartic with branch points $\{\mathsf{O}, -2\mathsf{p}_1, -\mathsf{p}_1 - \epsilon_2\mathsf{p}_2, 3\mathsf{p}_1 + \epsilon_2\mathsf{p}_2\}$ is obtained by setting $c \mapsto -c$ in Equation (4.25).

We also briefly discuss the existence of an additional involution for the plane genus-three curve $\mathcal{D}$ in Equation (4.25). We have the following:

LEMMA 4.18. *The bielliptic plane genus-three curve $\mathcal{D}$ in Proposition 4.15 admits an additional involution of the form*

$$[u : v : w] \mapsto [\alpha^2 v : u : \alpha w]$$

iff $a = \pm d/e$ and $\alpha^2 = d/e$. In particular, such an involution exists for $\mathcal{D}$ if $\lambda_0\lambda_2 = \lambda_1\lambda_3$ or $\lambda_0\lambda_3 = \lambda_1\lambda_2$.

Proof. The first statement is immediate. The second follows by computing $e^2a - d^2$ in terms of $\lambda_0, \ldots, \lambda_3, l$. □

REMARK 4.19. For $\lambda_0\lambda_1 = \lambda_2\lambda_3$, we find $\Delta_{\mathcal{D}} = 0$ in Equation (4.26), and the curve $\mathcal{D}$ is singular. This is easily understood when observing that the construction of $\mathcal{D}$ in Proposition 4.15 depends on two Weierstrass points corresponding to λ_0 and λ_1.

5. Proof of Theorem 1.2

Until now, we constructed a bidouble cover of $\mathbb{P}^1$ introducing the curves $\mathcal{H}$, $\mathcal{C}$, and $\mathcal{E} \cong \mathcal{Q}$ of genus three, two, and one in Section 3 and provided a precise geometric characterization of plane bielliptic genus-three curves $\mathcal{D}$, their bielliptic quotients, and the associated branch loci in Section 4. We now combine the results of the previous sections to prove our main theorem. In this section, we prove Corollary 5.3, which implies that, under certain conditions, the bielliptic genus-three curve $\mathcal{D}$ is smooth and irreducible. Moreover, we prove in Theorem 5.4 the existence of the $(1, 2)$-isogeny between the Prym variety of $\mathcal{D}$ and the Jacobian variety of a smooth genus-two curve $\mathcal{C}$ by using Theorem 2.1 and Propositions 2.6, 3.14, 3.20, 4.8, and 4.15.

We first determine on which fibers of the elliptic fibration with section (π, O) given by Equation (2.1) on the Abelian surfaces $\mathfrak{A}$ with $(1, 2)$-polarization line bundle $\mathcal{L}$, the branch locus with respect to the action induced by $-\mathbb{I}$ on $\mathfrak{A}$ consists of four points $\{\mathrm{pt}_1, \mathrm{pt}_2, \mathrm{pt}_3, \mathrm{pt}_4\} \subset \mathcal{E}$ such that $\sum_{i=1}^4 \mathrm{pt}_i = \mathsf{O}$. A normal form for the elliptic fibration and the generators $\{\mathsf{O}, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ of the Mordell–Weil group was provided in Corollary 3.12 and Section 3.4. As explained in Section 2.1, Barth's Theorem 2.1 asserts that $\mathfrak{A}$ is naturally isomorphic to the Prym variety $\mathrm{Prym}(\mathcal{D}, \pi^{\mathcal{D}}_{\mathcal{E}})$ of a smooth genus-three curve $\mathcal{D} \in |\mathcal{L}|$ with bielliptic involution τ such that $-\mathbb{I}$ restrict to τ, and the linear pencil $|\mathcal{L}|$ has precisely $T(\mathcal{L}) = \{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3\}$ as base points. The blowup in the base points is equivalent to the elliptic fibration with section (π, O) with sections $\{\mathsf{O}, \mathsf{S}'_1, \mathsf{S}'_2, \mathsf{S}'_3\}$ such that the divisor classes $\{K_0, K_1, K_2, K_3\}$ given by

$$K_0 = [\mathsf{O}], \qquad K_1 = [\mathsf{S}'_1], \qquad K_2 = [\mathsf{S}'_2], \qquad K_3 = [\mathsf{S}'_3] \tag{5.1}$$

Table 9 Special fibers in the elliptic fibration (π, O) on $\mathrm{Kum}(A)$.

#		$\mathsf{O} = \mathsf{T}_0$	T_1	T_2	T_3	$\{\mathsf{O}, \mathsf{S}_1', \mathsf{S}_2', \mathsf{S}_3'\}$
1	relation: constraint:	$2\mathsf{S}_1 + \mathsf{S}_3 = \mathsf{O}$, $p_4(s_0, s_1) = 0$	$2\mathsf{S}_1 + \mathsf{S}_3 = \mathsf{T}_1$, $p_2^{(1,2,3)}(s_0, s_1) = 0$	$2\mathsf{S}_1 + \mathsf{S}_3 = \mathsf{T}_2$, $p_2^{(2,1,3)}(s_0, s_1) = 0$	$2\mathsf{S}_1 + \mathsf{S}_3 = \mathsf{T}_3$, $p_2^{(3,1,2)}(s_0, s_1) = 0$	$\pm\{\mathsf{O}, -\mathsf{S}_1 + \mathsf{S}_2 + \mathsf{T}_i$, $-\mathsf{S}_1 - \mathsf{S}_2 + \mathsf{T}_i, 2\mathsf{S}_1\}$
2	relation: constraint:	$2\mathsf{S}_1 - \mathsf{S}_3 = \mathsf{O}$, $p_4(s_0, -s_1) = 0$	$2\mathsf{S}_1 - \mathsf{S}_3 = \mathsf{T}_1$, $p_2^{(1,2,3)}(s_0, -s_1) = 0$	$2\mathsf{S}_1 - \mathsf{S}_3 = \mathsf{T}_2$, $p_2^{(2,1,3)}(s_0, -s_1) = 0$	$2\mathsf{S}_1 - \mathsf{S}_3 = \mathsf{T}_3$, $p_2^{(3,1,2)}(s_0, -s_1) = 0$	$\pm\{\mathsf{O}, -\mathsf{S}_1 + \mathsf{S}_2 + \mathsf{T}_i$, $-\mathsf{S}_1 - \mathsf{S}_2 + \mathsf{T}_i, 2\mathsf{S}_1\}$
3	relation: constraint:	$2\mathsf{S}_2 + \mathsf{S}_3 = \mathsf{O}$, $p_4(ls_1, s_0) = 0$	$2\mathsf{S}_2 + \mathsf{S}_3 = \mathsf{T}_1$, $p_2^{(1,2,3)}(ls_1, s_0) = 0$	$2\mathsf{S}_2 + \mathsf{S}_3 = \mathsf{T}_2$, $p_2^{(2,1,3)}(ls_1, s_0) = 0$	$2\mathsf{S}_2 + \mathsf{S}_3 = \mathsf{T}_3$, $p_2^{(3,1,2)}(ls_1, s_0) = 0$	$\pm\{\mathsf{O}, -\mathsf{S}_2 + \mathsf{S}_1 + \mathsf{T}_i$, $-\mathsf{S}_2 - \mathsf{S}_1 + \mathsf{T}_i, 2\mathsf{S}_2\}$
4	relation: constraint:	$2\mathsf{S}_2 - \mathsf{S}_3 = \mathsf{O}$, $p_4(ls_1, -s_0) = 0$	$2\mathsf{S}_2 - \mathsf{S}_3 = \mathsf{T}_1$, $p_2^{(1,2,3)}(ls_1, -s_0) = 0$	$2\mathsf{S}_2 - \mathsf{S}_3 = \mathsf{T}_2$, $p_2^{(2,1,3)}(ls_1, -s_0) = 0$	$2\mathsf{S}_2 - \mathsf{S}_3 = \mathsf{T}_3$, $p_2^{(3,1,2)}(ls_1, -s_0) = 0$	$\pm\{\mathsf{O}, -\mathsf{S}_2 + \mathsf{S}_1 + \mathsf{T}_i$, $-\mathsf{S}_2 - \mathsf{S}_1 + \mathsf{T}_i, 2\mathsf{S}_2\}$

are the four exceptional curves of the blowup; see Proposition 2.6. To that end, the sum of the sections $\{\mathsf{S}_1', \mathsf{S}_2', \mathsf{S}_3'\}$ representing the divisor classes K_1, K_2, K_3 in Proposition 2.4 has to vanish. This will happen in certain smooth and certain singular fibers of the elliptic fibration, and we are interested in the former. We have the following:

PROPOSITION 5.1. *Table 9 lists all points in the base curve of the elliptic fibration with section* (π, O) *on* $\mathrm{Kum}(\mathfrak{A})$ *where the sum of sections* $\{\mathsf{S}_1', \mathsf{S}_2', \mathsf{S}_3'\}$ *representing divisor classes* K_1, K_2, K_3 *vanishes. Table 9 is based on the four possible choices for* $\{\mathsf{S}_1', \mathsf{S}_2', \mathsf{S}_3'\}$ *determined by Proposition 2.4. The polynomials* $p_4(s_0, s_1)$ *and* $p_2^{(1,2,3)}(s_0, s_1)$ *are the polynomials of degree* 4 *and* 2, *respectively, given by*

$$\begin{aligned} p_4(s_0, s_1) &= 2\lambda_0\lambda_1\lambda_2\lambda_3 l s_1^4 - \lambda_0\lambda_1\lambda_2\lambda_3(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)s_1^3 s_0 \\ &\qquad + (\lambda_0\lambda_1\lambda_2\lambda_3 + \lambda_0\lambda_1\lambda_2 + \lambda_0\lambda_1\lambda_3 + \lambda_0\lambda_2\lambda_3)s_1 s_0^3 - 2l s_0^4, \\ p_2^{(1,2,3)}(s_0, s_1) &= \lambda_0\lambda_1\lambda_2\lambda_3(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3)s_1^2 - 2l(\lambda_0\lambda_1 - \lambda_2\lambda_3)s_0 s_1 \\ &\qquad + \lambda_0\lambda_1(\lambda_2 + \lambda_3) - \lambda_2\lambda_3(\lambda_0 + \lambda_1), \end{aligned} \tag{5.2}$$

and $p_2^{(2,1,3)}$ *and* $p_2^{(3,1,2)}$ *are obtained by interchanging* $\lambda_1 \leftrightarrow \lambda_2$ *and* $\lambda_1 \leftrightarrow \lambda_3$, *respectively, in* $p_2^{(1,2,3)}$. *The parameters* $\lambda_0, \lambda_1, \lambda_2, \lambda_3, l$ *are the moduli of a general genus-two curve given in Lemma 3.1.* (*We remind the reader that we write* λ_0 *and* $l^2 = \lambda_0\lambda_1\lambda_3\lambda_3$ *rather than substituting* $\lambda_0 = 1$.)

Proof. The conditions $2\mathsf{S}_1 \pm \mathsf{S}_3 = \mathsf{T}_i$ and $2\mathsf{S}_2 \pm \mathsf{S}_3 = \mathsf{T}_i$ for $0 \le i \le 3$ result from making the last column in the table in Proposition 2.6 vanish. Staying away from singular fibers of the elliptic fibration described in Section 2.2, the rest of the statement follows from explicit computation using the group law on the smooth elliptic fibers. □

The marked cells in Table 9 determine points in the base curve of the elliptic fibration with section $(\pi : \mathscr{E} \to \mathbb{P}^1, \mathsf{O})$ where the sum of sections $\{\mathsf{O}, \mathsf{S}_1', \mathsf{S}_2', \mathsf{S}_3'\}$

vanishes. In particular, these points can be explicitly expressed in terms of modular forms *and* the sections in terms of the rational points p_1, p_2 in Lemma 3.5. We have the following:

COROLLARY 5.2. *For the six points in $\mathbb{P}^1$ given by*

$$\begin{aligned}&[s_0^* : s_1^*]\\ &\quad = [(\lambda_0 + \lambda_i - \lambda_j - \lambda_k)l : \lambda_0\lambda_i - \lambda_j\lambda_k \pm m^{(i,j,k)}(\lambda_0 - \lambda_j)(\lambda_0 - \lambda_k)], \quad (5.3)\end{aligned}$$

the sections $\{\mathsf{O}, \mathsf{S}'_1, \mathsf{S}'_2, \mathsf{S}'_3\}$ *coincide with the points* $\{\mathsf{O}, 2\mathsf{p}_1, \mathsf{p}_1 + \mathsf{p}_2, -3\mathsf{p}_1 - \mathsf{p}_2\}$ *in fibers* $\mathscr{E}_{[s_0^*:s_1^*]}$ *given by Equation* (3.21)*. Here* $m^{(i,j,k)}$ *satisfy*

$$(m^{(i,j,k)})^2 = \frac{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}{(\lambda_0 - \lambda_i)(\lambda_0 - \lambda_j)}$$

for $\{i, j, k\} = \{1, 2, 3\}$, *the point* $2\mathsf{p}_1$ *has the coordinates*

$$\begin{aligned}\xi &= (\Lambda_0 + \Lambda_1 - \Lambda_2 - \Lambda_3)^2, \qquad \eta = 1,\\ \rho &= (\Lambda_0 + \Lambda_1 - \Lambda_2 - \Lambda_3)(\Lambda_0 - \Lambda_1 - \Lambda_2 + \Lambda_3)\\ &\quad \times (\Lambda_0 - \Lambda_1 + \Lambda_2 + \Lambda_3),\end{aligned} \tag{5.4}$$

and $\mathsf{p}_1 + \mathsf{p}_2$ *has the coordinates*

$$\begin{aligned}\xi &= 4(\Lambda_0\Lambda_1 + \Lambda_2\Lambda_3 - 2L), \qquad \eta = 1,\\ \rho &= 8(L(\Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3) - \Lambda_0\Lambda_1\Lambda_2\\ &\quad - \Lambda_0\Lambda_1\Lambda_3 - \Lambda_0\Lambda_2\Lambda_3 - \Lambda_1\Lambda_2\Lambda_3),\end{aligned} \tag{5.5}$$

where $\Lambda_i = \Lambda_i(s_0^*, s_1^*)$ *for* $0 \le i \le 3$ *and* $L = L(s_0^*, s_1^*)$ *are given in Equation* (3.20).

Proof. We check that the points in Equation (5.3) satisfy $p_2^{(1,2,3)}(s_0^*, s_1^*) = 0$ in Proposition 5.1. The relation between the generators of the Mordell–Weil group $\mathrm{MW}(\pi, \mathsf{O})$ and the points p_1 and p_2 is given in Equation (3.26). Using Proposition 2.6, it follows

$$\begin{aligned}&\mathsf{S}'_1 = 2\mathsf{S}_1 = 2\mathsf{p}_1, \qquad \mathsf{S}'_2 = \mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3 = \mathsf{p}_1 + \mathsf{p}_3,\\ &\mathsf{S}'_3 = \mathsf{S}_1 - \mathsf{S}_2 + \mathsf{S}_3 = \mathsf{p}_1 + \mathsf{p}_2.\end{aligned}$$

Then the condition $2\mathsf{S}_1 + \mathsf{S}_3 = \mathsf{T}_i$ with $1 \le i \le 3$ in Proposition 5.1 implies $\sum \mathsf{S}'_i = 2\mathsf{T}_i = \mathsf{O}$. Because $p_2^{(2,1,3)}$ and $p_2^{(3,1,2)}$ are obtained by interchanging $\lambda_1 \leftrightarrow \lambda_2$ and $\lambda_1 \leftrightarrow \lambda_3$, the statements follow for the other points in the base curve as well. □

By replacing $\lambda_i \mapsto \Lambda_i(s_0, s_1)$ for $0 \le i \le 3$ and $l \mapsto L(s_0, s_1)$ in Proposition 4.15 we obtain the coefficients A, B, C, D, E from a, b, c, d, e; A, B are given by Equation (3.22) and

$$\begin{aligned}C(s_0, s_1) &= c(\Lambda_0(s_0, s_1), \Lambda_1(s_0, s_1), \Lambda_2(s_0, s_1), \Lambda_3(s_0, s_1), L(s_0, s_1)),\\ D(s_0, s_1) &= d(\Lambda_0(s_0, s_1), \Lambda_1(s_0, s_1), \Lambda_2(s_0, s_1), \Lambda_3(s_0, s_1), L(s_0, s_1)),\\ E(s_0, s_1) &= e(\Lambda_0(s_0, s_1), \Lambda_1(s_0, s_1), \Lambda_2(s_0, s_1), \Lambda_3(s_0, s_1), L(s_0, s_1)).\end{aligned} \tag{5.6}$$

Then a pencil $\mathscr{D}$ of plane bielliptic genus-three curves $\mathscr{D}_{[s_0:s_1]}$ is given by

$$\begin{aligned}(E(s_0,s_1)(w^2-u^2)-C(s_0,s_1)uv-D(s_0,s_1)v^2)^2\\ = E(s_0,s_1)^2(u^4+B(s_0,s_1)u^2v^2+A^2(s_0,s_1)v^4)\end{aligned} \tag{5.7}$$

with $[u:v:w]\in\mathbb{P}^2$ and $[s_0:s_1]\in\mathbb{P}^1$. We immediately have the following:

COROLLARY 5.3. *The plane genus-three curves $\mathscr{D}_{[s_0:s_1]}$ in Equation (5.7) are irreducible and nonsingular for all $[s_0:s_1]\in\mathbb{P}^1$ with $\Delta_{\mathscr{E}}(s_0,s_1)\Delta_{\mathscr{D}}(s_0,s_1)\neq 0$, where*

$$\begin{aligned}\Delta_{\mathscr{E}}(s_0,s_1) &= 16A^2(B^2-4A^2)^2,\\ \Delta_{\mathscr{D}}(s_0,s_1) &= -((C^2-BE^2-4DE)^2-12DE^2(BE+D))^3\\ &\quad + (54AC^2E^4-C^6+3(BE+4D)C^4E\\ &\quad - 3(B^2E^2+2BDE+10D^2)C^2E^2+(BE-2D)^3E^3)^2,\end{aligned} \tag{5.8}$$

and $A=A(s_0,s_1)$, $B=B(s_0,s_1)$, and so on. In particular, then we have $E(s_0,s_1)\neq 0$.

Proof. The proof follows from Lemma 4.13 by replacing $a\mapsto A(s_0,s_1)$, $b\mapsto B(s_0,s_1)$, and so on. $\square$

Then each smooth and irreducible curve $\mathscr{D}_{[s_0:s_1]}$ in the pencil $\mathscr{D}$ admits the bielliptic involution $\tau:[u:v:w]\in\mathbb{P}^2\mapsto[u:v:-w]$ interchanging the sheets of the degree-two cover

$$\begin{aligned}\pi^{\mathscr{D}}_{\mathscr{E}}:\quad & \mathscr{D}_{[s_0:s_1]}\to\mathscr{E}_{[s_0:s_1]}\\ & [u:v:w]\mapsto\left[u:v:W=w^2-u^2-\frac{C(s_0,s_1)}{E(s_0,s_1)}uv-\frac{D(s_0,s_1)}{E(s_0,s_1)}v^2\right]\end{aligned} \tag{5.9}$$

onto the elliptic curve $\mathscr{E}_{[s_0:s_1]}\cong\mathscr{D}_{[s_0:s_1]}/\langle\tau\rangle$ in Equation (3.21). It follows from Barth's Theorem 2.1 that the Prym varieties for the bielliptic curves $\mathscr{D}_{[s_0:s_1]}$ are Abelian surfaces with polarization of type $(1,2)$. We have the following:

THEOREM 5.4. *The Prym varieties of the smooth plane bielliptic genus-three curves $\mathscr{D}_{[s_0^*:s_1^*]}$ obtained as fibers of the pencil in Equation (5.7) over $[s_0^*:s_1^*]\in\mathbb{P}^1$ in Equation (5.3) with $\Delta_{\mathscr{E}}(s_0^*,s_1^*)\Delta_{\mathscr{D}}(s_0^*,s_1^*)\neq 0$ admit a $(1,2)$-isogeny*

$$\Psi:\quad \mathrm{Prym}(\mathscr{D}_{[s_0^*:s_1^*]},\pi^{\mathscr{D}}_{\mathscr{E}})\to\mathrm{Jac}(\mathcal{C})$$

onto the principally polarized Abelian surface $\mathrm{Jac}(\mathcal{C})$ for $\mathcal{C}$ in Equation (3.1).

Proof. It follows from Proposition 3.14 that the elliptic fibration $\pi:\mathscr{E}\to\mathbb{P}^1$ in Equation (3.21) is the special elliptic fibration $\pi:\mathrm{Kum}(\mathfrak{A})\to\mathbb{P}^1$ with section O in Proposition 2.2 on the Kummer surface of the Abelian surface $\mathfrak{A}$ with a polarization of type $(1,2)$. By Barth's Theorem 2.1 the elliptic fibration is induced by a pencil of bielliptic genus-three curves. The bielliptic involution τ has fixed points $\{\mathsf{P}_0,\mathsf{P}_1,\mathsf{P}_2,\mathsf{P}_3\}$. We proved in Proposition 2.6 that the branch points of the bielliptic involution are given by the sections $\{\mathsf{O}, \mathsf{S}_1', \mathsf{S}_2', \mathsf{S}_3'\}$ that represent divisor classes

$$K_0 = [\mathsf{O}], \qquad K_1 = [\mathsf{S}_1'], \qquad K_2 = [\mathsf{S}_2'], \qquad K_3 = [\mathsf{S}_3']. \tag{5.10}$$

The double points are the images of the order-two points $\{\mathsf{P}_0, \dots, \mathsf{P}_{15}\}$ on $\mathfrak{A}$, that is, elements of $\mathfrak{A}[2]$, and the disjoint rational curves $\{K_0, \dots, K_{15}\}$ are the exceptional divisors introduced in the blowup process. We proved in Corollary 5.2 that over the points $[s_0^* : s_1^*] \in \mathbb{P}^1$ in Equation (5.3) the sections add up to zero with respect to the elliptic-curve group law and coincide with $\{\mathsf{O}, 2\mathsf{p}_1, \mathsf{p}_1 + \mathsf{p}_2, -3\mathsf{p}_1 - \mathsf{p}_2\}$ in fibers $\mathscr{E}_{[s_0^*:s_1^*]}$. It follows from Proposition 4.15 that a plane bielliptic genus-three curve covering $\mathscr{E}_{[s_0^*:s_1^*]}$ with the same branch locus is obtained from Equation (4.25) by replacing $\lambda_i \mapsto \Lambda_i(s_0^*, s_1^*)$ for $0 \le i \le 3$ and $l \mapsto L(s_0^*, s_1^*)$ in Proposition 4.15. By Proposition 4.8 this model is unique. Moreover, it follows from [3, Corollary 2.2] that any such smooth curve is the canonical model of a bielliptic nonhyperelliptic curve of genus three. It follows from Proposition 3.20 that the polarization of type $(1,2)$ on the Abelian surface $\mathfrak{A}$ is induced by an isogeny $\Psi : \mathfrak{A} \to \operatorname{Jac}(\mathcal{C})$ onto the principally polarized Abelian surface $\operatorname{Jac}(\mathcal{C})$ for the genus-two curve in Equation (3.1). □

A tedious computation gives the following:

Corollary 5.5. *The coefficients A, B, C, D, E of the bielliptic genus-three curves $\mathscr{D}_{[s_0^*:s_1^*]}$ are polynomials in $\mathbb{Z}[\lambda_0, \lambda_1, \lambda_2, \lambda_3, l, l^{-1}, m^{(i,j,k)}]$.*

We determine some symmetries of these functions. We have the following:

Lemma 5.6. *For $\{i, j, k\} = \{1, 2, 3\}$,*

$$\begin{aligned}[s_0^* : s_1^*] = [&(\lambda_0 + \lambda_i - \lambda_j - \lambda_k)l : \\ &\lambda_0\lambda_i - \lambda_j\lambda_k \pm m^{(i,j,k)}(\lambda_0 - \lambda_j)(\lambda_0 - \lambda_k)],\end{aligned} \tag{5.11}$$

and $\lambda_0\lambda_i = \lambda_j\lambda_k$, we have $C = D = E = 0$ in Equation (5.7).

Corollary 5.7. *For $\{i, j, k\} = \{1, 2, 3\}$,*

$$\begin{aligned}[s_0^* : s_1^*] = [&(\lambda_0 + \lambda_i - \lambda_j - \lambda_k)l : \\ &\lambda_0\lambda_i - \lambda_j\lambda_k \pm m^{(i,j,k)}(\lambda_0 - \lambda_j)(\lambda_0 - \lambda_k)],\end{aligned} \tag{5.12}$$

and one of the additional relations given in Table 10, the smooth and irreducible bielliptic plane genus-three curve $\mathscr{D}_{[s_0^*:s_1^*]}$ in Theorem 5.4 admits an additional involution of the form

$$[u : v : w] \mapsto [\alpha^2 v : u : \alpha w],$$

with $\alpha^2 = D(s_0^*, s_1^*)/E(s_0^*, s_1^*)$. In particular, the Jacobian variety of the smooth genus-two curve $\mathcal{C}$ is two-isogenous to a product of two elliptic curves, that is, $\operatorname{Jac}(\mathcal{C}) \sim_2 \mathcal{E}_1 \times \mathcal{E}_2$.

Table 10 Additional relations between Rosenhain roots.

(i,j,k)	*relation*
$(1,2,3)$	$\lambda_0\lambda_2=\lambda_1\lambda_3$ *or* $\lambda_0\lambda_3=\lambda_1\lambda_2$
$(2,1,3)$	$\lambda_0\lambda_1=\lambda_2\lambda_3$
$(3,1,2)$	$\lambda_0\lambda_1=\lambda_2\lambda_3$

Proof. The first part follows from Lemma 4.18 and a tedious computation after replacing $a\mapsto A(s_0^*,s_1^*)$, $b\mapsto B(s_0^*,s_1^*)$, and so on. For the second part, we compute the Igusa–Clebsch invariants of $\mathcal{C}$, denoted by $[I_2:I_4:I_6:I_{10}]\in\mathbb{P}(2,4,6,10)$, using the same normalization as in [27; 28]. Then we can ask what the Igusa invariants of a genus-two curve $\mathcal{C}$ defined by a sextic curve are in terms of $\underline{\tau}$ such that $(\underline{\tau},\mathbb{I}_2)\in\mathrm{Mat}(2,4;\mathbb{C})$ is the period matrix of the principally polarized Abelian surface $\mathrm{Jac}(\mathcal{C})$. This allows us to compute the Siegel modular forms $\psi_4,\psi_6,\chi_{10},\chi_{12}$ for $\mathrm{Jac}(\mathcal{C})$, as introduced by Igusa in [22]. Igusa [23] also proved that the ring of Siegel modular forms is generated by $\psi_4,\psi_6,\chi_{10},\chi_{12}$ and by one more cusp form χ_{35} of odd weight 35 whose square is the following polynomial [22, p. 849]. We check that the additional relation implies $\chi_{35}(\mathcal{C})^2=0$. On the other hand, it is well known that for any smooth genus-two curve $\mathcal{C}$ with $\chi_{35}(\mathcal{C})=0$, its Jacobian is two-isogenous to a product of two elliptic curves, that is, $\mathrm{Jac}(\mathcal{C})\sim_2\mathcal{E}_1\times\mathcal{E}_2$; see [22]. □

Proof of Main Theorem 1.2. Corollary 5.3 proves that for $\Delta_{\mathscr{E}}(s_0^*,s_1^*)\Delta_{\mathscr{D}}(s_0^*,s_1^*)\neq 0$, the curves $\mathscr{D}_{[s_0^*:s_1^*]}$ are smooth and irreducible. Theorem 5.4 proves the existence of a $(1,2)$-isogeny. Lemma 3.1 and Remark 3.2 provide explicit formulas for $\lambda_0,\lambda_1,\lambda_2,\lambda_3,l,m^{(i,j,k)}$ in terms of theta functions. □

Appendix: Coefficients of Plane Bielliptic Genus-Three Curves

The plane bielliptic genus-three curve $\mathcal{D}$ in Proposition 4.15 is given by

$$\left(w^2-u^2-\frac{c}{e}uv-\frac{d}{e}v^2\right)^2=u^4+bu^2v^2+a^2v^4 \tag{A.1}$$

with $[u:v:w]\in\mathbb{P}^2$ and the coefficients

$$\begin{aligned} a&=(\lambda_0-\lambda_1)(\lambda_2-\lambda_3),\\ b&=4\lambda_0\lambda_1+4\lambda_2\lambda_3-2\lambda_0\lambda_2-2\lambda_0\lambda_3-2\lambda_1\lambda_2-2\lambda_1\lambda_3,\end{aligned} \tag{A.2}$$

$$\begin{aligned} c&=c(\lambda_0,\lambda_1,\lambda_2,\lambda_3,l)\\ &=-4\prod_{i=0}^{1}\prod_{j=1}^{2}(\lambda_i-\lambda_j)\left(\sum_{i=0}^{3}\lambda_i^2-2\sum_{0\le i<j\le 3}\lambda_i\lambda_j+8l\right),\end{aligned} \tag{A.3}$$

$$\begin{aligned} d&=d(\lambda_0,\lambda_1,\lambda_2,\lambda_3,l)\\ &=-\lambda_0^5\lambda_1\lambda_2-\lambda_0^5\lambda_1\lambda_3+2\lambda_0^5\lambda_2\lambda_3-4\lambda_0^4\lambda_1^2\lambda_2-4\lambda_0^4\lambda_1^2\lambda_3+5\lambda_0^4\lambda_1\lambda_2^2\end{aligned}$$

$$
\begin{aligned}
&+ 5\lambda_0^4\lambda_1\lambda_3^2 - \lambda_0^4\lambda_2^2\lambda_3 - \lambda_0^4\lambda_2\lambda_3^2 + 10\lambda_0^3\lambda_1^3\lambda_2 + 10\lambda_0^3\lambda_1^3\lambda_3 - 5\lambda_0^3\lambda_1^2\lambda_2^2 \\
&- 2\lambda_0^3\lambda_1^2\lambda_2\lambda_3 - 5\lambda_0^3\lambda_1^2\lambda_3^2 - 5\lambda_0^3\lambda_1\lambda_2^3 + \lambda_0^3\lambda_1\lambda_2^2\lambda_3 + \lambda_0^3\lambda_1\lambda_2\lambda_3^2 \\
&- 5\lambda_0^3\lambda_1\lambda_3^3 - 5\lambda_0^3\lambda_2^3\lambda_3 + 10\lambda_0^3\lambda_2^2\lambda_3^2 - 5\lambda_0^3\lambda_2\lambda_3^3 - 4\lambda_0^2\lambda_1^4\lambda_2 - 4\lambda_0^2\lambda_1^4\lambda_3 \\
&- 5\lambda_0^2\lambda_1^3\lambda_2^2 - 2\lambda_0^2\lambda_1^3\lambda_2\lambda_3 - 5\lambda_0^2\lambda_1^3\lambda_3^2 + 10\lambda_0^2\lambda_1^2\lambda_2^3 + 10\lambda_0^2\lambda_1^2\lambda_3^3 \\
&- \lambda_0^2\lambda_1\lambda_2^4 + \lambda_0^2\lambda_1\lambda_2^3\lambda_3 + \lambda_0^2\lambda_1\lambda_2\lambda_3^3 - \lambda_0^2\lambda_1\lambda_3^4 + 5\lambda_0^2\lambda_2^4\lambda_3 - 5\lambda_0^2\lambda_2^3\lambda_3^2 \\
&- 5\lambda_0^2\lambda_2^2\lambda_3^3 + 5\lambda_0^2\lambda_2\lambda_3^4 - \lambda_0\lambda_1^5\lambda_2 - \lambda_0\lambda_1^5\lambda_3 + 5\lambda_0\lambda_1^4\lambda_2^2 + 5\lambda_0\lambda_1^4\lambda_3^2 \\
&- 5\lambda_0\lambda_1^3\lambda_2^3 + \lambda_0\lambda_1^3\lambda_2^2\lambda_3 + \lambda_0\lambda_1^3\lambda_2\lambda_3^2 - 5\lambda_0\lambda_1^3\lambda_3^3 - \lambda_0\lambda_1^2\lambda_2^4 + \lambda_0\lambda_1^2\lambda_2^3\lambda_3 \\
&+ \lambda_0\lambda_1^2\lambda_2\lambda_3^3 - \lambda_0\lambda_1^2\lambda_3^4 + 2\lambda_0\lambda_1\lambda_2^5 - 2\lambda_0\lambda_1\lambda_2^3\lambda_3^2 - 2\lambda_0\lambda_1\lambda_2^2\lambda_3^3 \\
&+ 2\lambda_0\lambda_1\lambda_3^5 - \lambda_0\lambda_2^5\lambda_3 - 4\lambda_0\lambda_2^4\lambda_3^2 + 10\lambda_0\lambda_2^3\lambda_3^3 - 4\lambda_0\lambda_2^2\lambda_3^4 - \lambda_0\lambda_2\lambda_3^5 \\
&+ 2\lambda_1^5\lambda_2\lambda_3 - \lambda_1^4\lambda_2^2\lambda_3 - \lambda_1^4\lambda_2\lambda_3^2 - 5\lambda_1^3\lambda_2^3\lambda_3 + 10\lambda_1^3\lambda_2^2\lambda_3^2 - 5\lambda_1^3\lambda_2\lambda_3^3 \\
&+ 5\lambda_1^2\lambda_2^4\lambda_3 - 5\lambda_1^2\lambda_2^3\lambda_3^2 - 5\lambda_1^2\lambda_2^2\lambda_3^3 + 5\lambda_1^2\lambda_2\lambda_3^4 - \lambda_1\lambda_2^5\lambda_3 - 4\lambda_1\lambda_2^4\lambda_3^2 \\
&+ 10\lambda_1\lambda_2^3\lambda_3^3 - 4\lambda_1\lambda_2^2\lambda_3^4 - \lambda_1\lambda_2\lambda_3^5 \\
&+ (8\lambda_0^4\lambda_1 - 4\lambda_0^4\lambda_2 - 4\lambda_0^4\lambda_3 - 8\lambda_0^3\lambda_1^2 + 4\lambda_0^3\lambda_2^2 + 4\lambda_0^3\lambda_3^2 - 8\lambda_0^2\lambda_1^3 \\
&+ 8\lambda_0^2\lambda_1^2\lambda_2 + 8\lambda_0^2\lambda_1^2\lambda_3 - 4\lambda_0^2\lambda_1\lambda_2^2 - 4\lambda_0^2\lambda_1\lambda_3^2 + 4\lambda_0^2\lambda_2^3 - 4\lambda_0^2\lambda_2^2\lambda_3 \\
&- 4\lambda_0^2\lambda_2\lambda_3^2 + 4\lambda_0^2\lambda_3^3 + 8\lambda_0\lambda_1^4 - 4\lambda_0\lambda_1^2\lambda_2^2 - 4\lambda_0\lambda_1^2\lambda_3^2 - 4\lambda_0\lambda_2^4 \\
&+ 8\lambda_0\lambda_2^2\lambda_3^2 - 4\lambda_0\lambda_3^4 - 4\lambda_1^4\lambda_2 - 4\lambda_1^4\lambda_3 + 4\lambda_1^3\lambda_2^2 + 4\lambda_1^3\lambda_3^2 + 4\lambda_1^2\lambda_2^3 \\
&- 4\lambda_1^2\lambda_2^2\lambda_3 - 4\lambda_1^2\lambda_2\lambda_3^2 + 4\lambda_1^2\lambda_3^3 - 4\lambda_1\lambda_2^4 + 8\lambda_1\lambda_2^2\lambda_3^2 - 4\lambda_1\lambda_3^4 \\
&+ 8\lambda_2^4\lambda_3 - 8\lambda_2^3\lambda_3^2 - 8\lambda_2^2\lambda_3^3 + 8\lambda_2\lambda_3^4)l, \qquad \text{(A.4)}
\end{aligned}
$$

and

$$
\begin{aligned}
e &= e(\lambda_0, \lambda_1, \lambda_2, \lambda_3, l) \\
&= (\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3)(2\lambda_0^2\lambda_1^2 - \lambda_0^3\lambda_1 - 2\lambda_0^2\lambda_1\lambda_2 \\
&\quad - 3\lambda_0^2\lambda_2\lambda_3 - \lambda_0\lambda_1^3 - 2\lambda_0\lambda_1^2\lambda_2 - 2\lambda_0\lambda_1^2\lambda_3 + 3\lambda_0\lambda_1\lambda_2^2 + 3\lambda_0\lambda_1\lambda_3^2 \\
&\quad + 2\lambda_0\lambda_2^2\lambda_3 + 2\lambda_0\lambda_2\lambda_3^2 - 3\lambda_1^2\lambda_2\lambda_3 + 2\lambda_1\lambda_2^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3^2 + \lambda_2^3\lambda_3 \\
&\quad - 2\lambda_0^2\lambda_1\lambda_3 - 2\lambda_2^2\lambda_3^2 + \lambda_2\lambda_3^3 + 4(\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2)l). \qquad \text{(A.5)}
\end{aligned}
$$

Here we assumed that the curve $\mathcal{D}$ is irreducible and smooth such that $\Delta_{\mathcal{E}}\Delta_{\mathcal{D}} \neq 0$ (and $e \neq 0$ in particular), where

$$
\begin{aligned}
\Delta_{\mathcal{E}} &= 16a^2(b^2 - 4a^2)^2, \\
\Delta_{\mathcal{D}} &= -((c^2 - be^2 - 4de)^2 - 12de^2(be + d))^3 \\
&\quad + (54ac^2e^4 - c^6 + 3(be + 4d)c^4e \\
&\quad - 3(b^2e^2 + 2bde + 10d^2)c^2e^2 + (be - 2d)^3e^3)^2.
\end{aligned} \qquad \text{(A.6)}
$$

The plane bielliptic genus-three curves $\mathscr{D}_{[s_0^*:s_1^*]}$ in Theorem 1.2 are given by

$$\left(w^2 - u^2 - \frac{C(s_0^*, s_1^*)}{E(s_0^*, s_1^*)}uv - \frac{D(s_0^*, s_1^*)}{E(s_0^*, s_1^*)}v^2\right)^2 = u^4 + B(s_0^*, s_1^*)u^2v^2 + A^2(s_0^*, s_1^*)v^4 \tag{A.7}$$

with $[u : v : w] \in \mathbb{P}^2$, $[s_0 : s_1] \in \mathbb{P}^1$, and the coefficients

$$\begin{aligned} A(s_0, s_1) &= (\Lambda_0(s_0, s_1) - \Lambda_1(s_0, s_1))(\Lambda_2(s_0, s_1) - \Lambda_3(s_0, s_1)), \\ B(s_0, s_1) &= 4\Lambda_0(s_0, s_1)\Lambda_1(s_0, s_1) + 4\Lambda_2(s_0, s_1)\Lambda_3(s_0, s_1) \\ &\quad - 2\Lambda_0(s_0, s_1)\Lambda_2(s_0, s_1) - 2\Lambda_0(s_0, s_1)\Lambda_3(s_0, s_1) \\ &\quad - 2\Lambda_1(s_0, s_1)\Lambda_2(s_0, s_1) - 2\Lambda_1(s_0, s_1)\Lambda_3(s_0, s_1), \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} C(s_0, s_1) &= c(\Lambda_0(s_0, s_1), \Lambda_1(s_0, s_1), \Lambda_2(s_0, s_1), \Lambda_3(s_0, s_1), L(s_0, s_1)), \\ D(s_0, s_1) &= d(\Lambda_0(s_0, s_1), \Lambda_1(s_0, s_1), \Lambda_2(s_0, s_1), \Lambda_3(s_0, s_1), L(s_0, s_1)), \\ E(s_0, s_1) &= e(\Lambda_0(s_0, s_1), \Lambda_1(s_0, s_1), \Lambda_2(s_0, s_1), \Lambda_3(s_0, s_1), L(s_0, s_1)), \end{aligned} \tag{A.9}$$

where the functions c, d, e are given in Equations (A.3)–(A.5) with λ_0, λ_1, λ_2, λ_3, l replaced by Λ_0, Λ_1, Λ_2, Λ_3, L with

$$\begin{aligned} \Lambda_i(s_0, s_1) &= \frac{(s_0 + \lambda_i s_1)^2}{\lambda_i}, \quad 0 \le i \le 3, \\ L(s_0, s_1) &= \frac{\prod_{i=0}^{3}(s_0 + \lambda_i s_1)}{l}. \end{aligned} \tag{A.10}$$

Moreover, the special point $[s_0 : s_1] = [s_0^* : s_1^*]$ to be used in Equation (A.7) is given by

$$\begin{aligned} [s_0^* : s_1^*] = [&(\lambda_0 + \lambda_i - \lambda_j - \lambda_k)l : \\ &(\lambda_0\lambda_i - \lambda_j\lambda_k) \pm m^{(i,j,k)}(\lambda_0 - \lambda_j)(\lambda_0 - \lambda_k)] \end{aligned} \tag{A.11}$$

for all $\{i, j, k\} = \{1, 2, 3\}$, where

$$\begin{aligned} &\lambda_0 = 1, \qquad \lambda_1 = \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}, \qquad \lambda_2 = \frac{\theta_3^2\theta_8^2}{\theta_4^2\theta_{10}^2}, \\ &\lambda_3 = \frac{\theta_1^2\theta_8^2}{\theta_2^2\theta_{10}^2}, \qquad l = \frac{\theta_1^2\theta_3^2\theta_8^2}{\theta_2^2\theta_4^2\theta_{10}^2}, \qquad m^{(1,2,3)} = \frac{\theta_1\theta_3\theta_6^2}{\theta_2\theta_4\theta_5^2}, \\ &m^{(2,1,3)} = i\,\frac{\theta_3\theta_8\theta_6^2}{\theta_4\theta_{10}\theta_7^2}, \qquad m^{(3,1,2)} = \frac{\theta_1\theta_8\theta_6^2}{\theta_2\theta_{10}\theta_9^2}, \end{aligned} \tag{A.12}$$

and the ten even theta functions $\theta_i^2 = \theta_i^2(0, \tau)$ with zero elliptic argument, modular argument $\tau \in \mathbb{H}_2/\Gamma_2(2)$, and $1 \le i \le 10$ follow the same standard notation for even theta functions as that used in [21; 20; 27; 28; 9]. Since the curves $\mathscr{D}_{[s_0^*:s_1^*]}$

are assumed to be irreducible and smooth, we have $\Delta_{\mathscr{E}}(s_0^*, s_1^*)\Delta_{\mathscr{D}}(s_0^*, s_1^*) \neq 0$, where

$$\begin{aligned}\Delta_{\mathscr{E}}(s_0, s_1) &= 16A^2(B^2 - 4A^2)^2,\\ \Delta_{\mathscr{D}}(s_0, s_1) &= -((C^2 - BE^2 - 4DE)^2 - 12DE^2(BE + D))^3\\ &\quad + (54AC^2E^4 - C^6 + 3(BE + 4D)C^4E\\ &\quad - 3(B^2E^2 + 2BDE + 10D^2)C^2E^2 + (BE - 2D)^3E^3)^2,\end{aligned} \tag{A.13}$$

and $A = A(s_0, s_1)$, $B = B(s_0, s_1)$, and so on.

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