SUPERELLIPTIC CURVES WITH MANY AUTOMORPHISMS AND CM JACOBIANS

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ABSTRACT. Let $\mathcal{C}$ be a smooth, projective, genus $g \geq 2$ curve, defined over $\mathbb{C}$. Then $\mathcal{C}$ has *many automorphisms* if its corresponding moduli point $\mathfrak{p} \in \mathcal{M}_g$ has a neighborhood U in the complex topology, such that all curves corresponding to points in $U \setminus \{\mathfrak{p}\}$ have strictly fewer automorphisms than $\mathcal{C}$. We compute completely the list of superelliptic curves $\mathcal{C}$ for which the superelliptic automorphism is normal in the automorphism group $\mathrm{Aut}(\mathcal{C})$ and $\mathcal{C}$ has many automorphisms. For each of these curves, we determine whether its Jacobian has complex multiplication. As a consequence, we prove the converse of Streit's complex multiplication criterion for these curves.

1. INTRODUCTION

An abelian variety $\mathcal{A}$ has complex multiplication (or is of CM-type) over a field k if $\mathrm{End}_k^0(\mathcal{A})$ contains a commutative, semisimple $\mathbb{Q}$-algebra of dimension $2 \dim \mathcal{A}$. They were first studied by M. Deuring [1, 2] for elliptic curves and generalized to Abelian varieties by Shimura and Tanyama in [3]. By abuse of terminology, a curve is said to have complex multiplication (or to be of CM-type) when its Jacobian is of CM-type. Since the CM property is a property of the Jacobian, it is an invariant of the curve. A natural question is whether there is anything special about the points in the moduli space $\mathcal{M}_g$ of genus $g \geq 2$ curves, for which the Jacobian is of CM-type; see [4]. Since CM-curves can be defined over a number field, by Belyi's theorem they admit a non-constant meromorphic function φ with exactly three branch points. Hence, it is natural to restrict to the case of curves with many automorphisms since for such curves this map φ is a Galois covering. F. Oort asked if curves with many automorphisms (cf. Section 2.3) are all of CM-type. The answer to this question is negative, as explained in [5], where a full history of the problem and recent developments are given. Other papers where the question of determining whether a curve has Jacobian of CM type is considered include [6] by Wolfart, [7] by Frediani–Penegini–Porru, and most influentially for us, [8] by Müller–Pink, which we will come back to later in the introduction.

Let $\mathcal{C}$ be a smooth, projective, genus $g \geq 2$ curve defined over k, $\mathfrak{p} \in \mathcal{M}_g$ its corresponding moduli point, and $G := \mathrm{Aut}(\mathcal{C})$ the automorphism group of $\mathcal{C}$ over the algebraic closure of k. For our purposes we will assume $k = \mathbb{C}$. We say that $\mathcal{C}$ has *many automorphisms* if its corresponding point $\mathfrak{p} \in \mathcal{M}_g$ has a neighborhood U (in the complex topology) such that all curves corresponding to points in $U \setminus \{\mathfrak{p}\}$ have automorphism group strictly smaller than G's corresponding point $\mathfrak{p} \in \mathcal{M}_g$

2010 *Mathematics Subject Classification.* Primary 14H37; Secondary 14H45, 14K22.

Key words and phrases. complex multiplication; superelliptic curves.

The first author was supported by the National Science Foundation under DMS Grant No. 1900396.

has a neighborhood U (in the complex topology) such that all curves corresponding to points in $U \setminus \{\mathfrak{p}\}$ have automorphism group strictly smaller than G. They shouldn't be confused with curves with *large automorphism group* which are curves with automorphism group $|G| > 4(g-1)$. Not all curves with large automorphism group are curves with many automorphisms.

As mentioned above, Oort asked if such curves are of CM-type, and this is not true in general. However it remains an interesting question to determine which curves with many automorphisms are of CM-type. In general, for a given $g \geq 2$ we can determine the full list of automorphism groups that occur; see [9] and [10] for a complete survey on automorphism groups of algebraic curves. It is difficult from the group $\mathrm{Aut}(\mathcal{C})$ alone to determine if $\mathcal{C}$ is of CM-type without knowing anything about an equation of the curve. However, there is only one class of curves for which we can determine the equation of the curves explicitly starting from the automorphism group, namely the superelliptic curves. Hence, it is a natural choice to try to determine which superelliptic curves with many automorphisms are of CM-type.

In [8], the authors solved this problem for hyperelliptic curves. Their main tool is a formula of Streit from [11], which gives conditions on the characters of the group of automorphisms of the curve. More precisely, let $\chi_{\mathcal{C}}$ be the character of $\mathrm{Aut}(\mathcal{C})$ on $H^0(\mathcal{C}, \omega_{\mathcal{C}})$, and let $\mathrm{Sym}^2 \chi_{\mathcal{C}}$ be the character of $\mathrm{Aut}(\mathcal{C})$ on $\mathrm{Sym}^2 H^0(\mathcal{C}, \omega_{\mathcal{C}})$. By χ_{triv} we denote the character of the trivial representation on $\mathbb{C}$. Streit showed that if $\langle \mathrm{Sym}^2 \chi_{\mathcal{C}}, \chi_{triv} \rangle = 0$ then $\mathrm{Jac}\, \mathcal{C}$ has complex multiplication; see [11]. We say that $\mathcal{C}$ *satisfies Streit's criterion* if this inner product is 0.

For $\mathcal{C}$ hyperelliptic, the Müller and Pink in [8] determine a formula which computes $\mathrm{Sym}^2 \chi_{\mathcal{C}}$, and through this formula are able to determine precisely if a hyperelliptic curve with many automorphisms is or is not of CM-type. They prove their formula using the fact that it is easy to write a monomial basis of holomorphic differentials for hyperelliptic function fields. As a consequence, the converse of Streit's criterion holds for hyperelliptic curves with many automorphisms and reduced automorphism group isomorphic to A_4, S_4, or A_5. In other words, no such curve that fails Streit's criterion can have complex multiplication.

Using a similar approach we are able to prove a similar formula for superelliptic curves (cf. Prop. 4.8). Let $\mathcal{C} : y^n = f(x)$ be a smooth superelliptic curve defined over $\mathbb{C}$ (in particular, $f(x)$ is assumed to be a *separable polynomial*, see Section 3), and let τ be the order n automorphism given by $y \mapsto \zeta_n y$, where ζ_n is a primitive nth root of unity. Let N be the normalizer of τ in $\mathrm{Aut}(\mathcal{C})$, and let $\overline{N} = N/\langle \tau \rangle$, which naturally lies in $\mathrm{Aut}(\mathcal{C}/\langle \tau \rangle) = \mathrm{Aut}(\mathbb{P}^1) \cong PGL_2(\mathbb{C})$. For each $\overline{\sigma} \in \overline{N}$, let m be its order, and let $\zeta_{\overline{\sigma}}$ be either ratio of the eigenvalues when $\overline{\sigma}$ is thought of as an element of $PGL_2(\mathbb{C})$. Observe that $\zeta_{\overline{\sigma}}$ is a primitive mth root of unity. Let $\zeta_{n,\overline{\sigma}}$ be a primitive mnth root of unity such that $\zeta_{n,\overline{\sigma}}^n = \zeta_{\overline{\sigma}}$. Define

$$k_{\overline{\sigma}} = \begin{cases} 1 \text{ if } \overline{\sigma} \text{ fixes a branch point of } \mathcal{C} \to \mathbb{P}^1 \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $A \subseteq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ be the set of ordered pairs defined below in (Eq. (4)). If

$$\sum_{\overline{\sigma} \in \overline{N}} \sum_{i=0}^{n-1} \left(\sum_{(a,b) \in A} \zeta_{\overline{\sigma}}^{2(a+1)} \zeta_n^{2(b+1)i} \zeta_{n,\overline{\sigma}}^{2(b-n+1)k_{\overline{\sigma}}} + \left(\sum_{(a,b) \in A} \zeta_{\overline{\sigma}}^{a+1} \zeta_n^{(b+1)i} \zeta_{n,\overline{\sigma}}^{(b-n+1)k_{\overline{\sigma}}} \right)^2 \right)$$

vanishes, then $\mathcal{C}$ satisfies Streit's criterion, and thus has CM. However, not all superelliptic curves with many automorphisms satisfy Streit's criterion; see Table 1. We in fact prove the converse of Streit's criterion for superelliptic curves with many automorphisms; any such curve not satisfying Streit's criterion does not have CM (Cor. 5.14). To do this we use stable reduction (cf. Section 5.2) and the theory of semistable models as in [12], as well as the so-called *criterion of Müller-Pink*; see Section 5.3. Our computations were done using Sage [13] and GAP [14] and are made available at arxiv.org/abs/2006.12685.

It is still an open question whether these results can be generalized to larger collections of curves, for example for generalized superelliptic curves as listed in [15].

In general, it is believed that for a curve, having lots of automorphisms places upward pressure on the endomorphism group of its Jacobian. Hence, perhaps an interesting family of curves to investigate would be all curves with large automorphism groups *and* many automorphisms (in the sense of the beginning of the introduction); these were determined in [9]. It would also be interesting to obtain a theoretical explanation for Cor. 5.14.

2. Preliminaries

Throughout, we work over the field $\mathbb{C}$. An Abelian variety is an absolutely irreducible projective variety which is a group scheme. A morphism of Abelian varieties $\mathcal{A}$ to $\mathcal{B}$ is a *homomorphism* if and only if it maps the identity element of $\mathcal{A}$ to the identity element of $\mathcal{B}$.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian varieties. We denote the $\mathbb{Z}$-module of homomorphisms $\mathcal{A} \mapsto \mathcal{B}$ by $\operatorname{Hom}(\mathcal{A}, \mathcal{B})$ and the ring of endomorphisms $\mathcal{A} \mapsto \mathcal{A}$ by $\operatorname{End} \mathcal{A}$. It is more convenient to work with the $\mathbb{Q}$-vector spaces $\operatorname{Hom}^0(\mathcal{A}, \mathcal{B}) := \operatorname{Hom}(\mathcal{A}, \mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\operatorname{End}^0 \mathcal{A} := \operatorname{End} \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Q}$. Determining $\operatorname{End} \mathcal{A}$ or $\operatorname{End}^0 \mathcal{A}$ is an interesting problem on its own; see [4].

The ring of endomorphisms of a generic Abelian variety $\mathcal{A}$ over $\mathbb{C}$ is "as small as possible", that is, $\operatorname{End}(\mathcal{A}) = \mathbb{Z}$. In general, $\operatorname{End}^0(\mathcal{A})$ is a $\mathbb{Q}$-algebra of dimension at most $4 \dim(\mathcal{A})^2$. Indeed, $\operatorname{End}^0(\mathcal{A})$ is a semi-simple algebra, and by duality one can apply a complete classification due to Albert of *possible* algebra structures on $\operatorname{End}^0(\mathcal{A})$, see [16, pg. 202]. We say that an abelian variety $\mathcal{A}$ has *complex multiplication* (CM) if $\operatorname{End}^0(\mathcal{A})$ contains a commutative, semisimple $\mathbb{Q}$-algebra of dimension $2 \dim \mathcal{A}$.

Since every abelian subvariety of an abelian variety is an isogeny factor, the following result follows from [17, I, Rem. 3.5]

Lemma 2.1. *If $\mathcal{A}$ is an Abelian variety with CM, then every abelian subvariety of $\mathcal{A}$ also has CM.*

2.1. **Curves and their Jacobians.** Throughout, $\mathcal{C}$ is a smooth, projective curve defined over $\mathbb{C}$. We will denote its group of automorphisms by $\operatorname{Aut}(\mathcal{C})$. It is a well known fact that $|\operatorname{Aut}(\mathcal{C})| \leq 84(g-1)$. Roughly speaking, one gets a stratification of $\mathcal{M}_g$ by strata of curves with the same automorphism group, and the generic curve of genus $g > 2$ has trivial automorphism group.

We state the following Corollary of Lem. 2.1, which is used repeatedly.

Corollary 2.2. *If $\psi : \mathcal{C} \to \mathcal{Y}$ is a finite morphism of curves and* $\operatorname{Jac} \mathcal{C}$ *has CM, then so does* $\operatorname{Jac} \mathcal{Y}$.

2.2. **Hurwitz spaces.** We consider finite covers $\eta : \mathcal{C} \to \mathbb{P}^1$ of degree n, with branch locus $\{x_1, \ldots, x_r\}$. Pulling back rational functions via η identifies $\mathbb{C}(\mathbb{P}^1)$ with a subfield of $\mathbb{C}(\mathcal{C})$. First, we introduce the equivalence: $\eta \sim \eta'$ if there are isomorphisms $\alpha : \mathcal{C} \to \mathcal{C}'$ and $\beta \in \mathrm{Aut}(\mathbb{P}^1)$ with

$$\beta \circ \eta = \eta' \circ \alpha.$$

The *monodromy group* of η is the Galois group of the Galois closure L of $\mathbb{C}(\mathcal{C})/\mathbb{C}(\mathbb{P}^1)$. We assume that exactly $r \geq 3$ points in $\mathbb{P}^1$ are ramified (i.e. their preimages contain fewer than n points).

Fix a point $x_0 \in \mathbb{P}^1$ that is not a branch point of η. Label the points of $\eta^{-1}(x_0)$ from 1 to n, and fix a "bouquet" of loops γ_i based at x_0 around each of the r branch points of η such that $\prod_{i=1}^r \gamma_i = 1$ and the γ_i generate $\pi_1(\mathbb{P}^1 \backslash \{x_1, \ldots, x_r\}, x_0)$. By the theory of covers of Riemann surfaces, one associates to this data a tuple $(\sigma_1, \ldots, \sigma_r)$ of elements in S_n where σ_i gives the monodromy action of γ_i on $\eta^{-1}(x_0)$. Then $\sigma_1 \cdots \sigma_r = 1$, the number $e_i = \mathrm{ord}(\sigma_i)$ is the ramification order of the ith ramification point P_i in L, and the monodromy group $G = \langle \sigma_1, \ldots, \sigma_r \rangle$ is a transitive group in S_n. We call such a tuple $\sigma = (\sigma_1, \ldots, \sigma_r)$ the *ramification type* or the *signature* of the covering η and remark that once the γ_i and the x_i are fixed, such tuples are determined up to conjugation in S_n. The genus of $\mathcal{C}$ is determined by the signature because of the Hurwitz genus formula.

Let $\mathcal{H}_\sigma$ be the set of pairs $([\eta, (p_1, \ldots, p_r)])$, where $[\eta, (p_1, \ldots, p_r)]$ is an equivalence class modulo automorphisms of $\mathbb{P}^1$ of covers of signature σ with an ordering $p_1, \ldots, p_r$ of the branch points. The set $\mathcal{H}_\sigma$ carries the structure of a scheme; in fact it is a quasi-projective variety called the *Hurwitz space*. We have the forgetful morphism

$$\Phi_\sigma : \mathcal{H}_\sigma \to \mathcal{M}_g$$

mapping $([\eta], (p_1, \ldots, p_r))$ to the isomorphism class $[\mathcal{C}]$ in the moduli space $\mathcal{M}_g$. Each component of $\mathcal{H}_\sigma$ has the same image in $\mathcal{M}_g$.

Define the *moduli dimension of* σ (denoted by $\dim(\sigma)$) as the dimension of $\Phi_\sigma(\mathcal{H}_\sigma)$; i.e., the dimension of the locus of genus g curves admitting a cover to $\mathbb{P}^1$ of type σ.

2.3. **Curves with many automorphisms.** Let $\mathcal{C}$ be a genus $g \geq 2$ curve defined over $\mathbb{C}$, $\mathfrak{p} \in \mathcal{M}_g$ its corresponding moduli point, and $G := \mathrm{Aut}(\mathcal{C})$. Recall from the introduction that $\mathcal{C}$ has many automorphisms if it has more automorphisms than any nearby curve in the moduli space in the complex topology. The next lemma gives various well-known equivalent ways of characterizing this property.

Lemma 2.3 ([5, Lemma 4.4] or [8, Theorem 2.1])**.** *Let $\mathcal{C}$ have genus ≥ 2 as above, let $G := \mathrm{Aut}(\mathcal{C})$, and let $\eta : \mathcal{C} \to \mathbb{P}^1$ the corresponding map with signature σ. Then the following are equivalent:*

(1) $\mathcal{C}$ has many automorphisms.
(2) There exists a subgroup $H < G$ such that $g(\mathcal{C}/H) = 0$ and $\mathcal{C} \to \mathcal{C}/H$ has exactly three branch points.
(3) The quotient $\mathcal{C}/G$ has genus 0 and $\mathcal{C} \to \mathcal{C}/G$ has exactly three branch points.
(4) The signature σ has moduli dimension 0.

Question 2.4 (F. Oort)**.** If $\mathcal{C}$ has many automorphisms, does $\mathrm{Jac}\,\mathcal{C}$ have complex multiplication?

Wolfart answered this question for all curves of genus $g \leq 4$; see [6, §5].

3. Superelliptic curves with many automorphisms

The term *superelliptic curve* has been used differently by many authors. Most use it to mean a smooth projective curve $\mathcal{C}$ with affine equation of the form $y^n = f(x)$, where $f(x) \in \mathbb{C}[x]$ has discriminant $\Delta(f) \neq 0$. If $H := \langle \tau \rangle \subseteq \mathrm{Aut}(\mathcal{C})$ is the subgroup generated by $\tau(y) = \zeta_n y$, then it is sometimes further required that H be normal (or central) in $\mathrm{Aut}(\mathcal{C})$.

We will follow the definition in [15]. Specifically, a *superelliptic curve* is a smooth projective curve $\mathcal{C}$ of genus ≥ 2 with affine equation $y^n = \prod_{i=1}^{r}(x - a_i)$, with the a_i distinct complex numbers such that

(i) If H is as above, then H is *normal* in $\mathrm{Aut}(\mathcal{C})$.
(ii) Either $n \mid r$ or $\gcd(n, r) = 1$ (this guarantees that all branch points have index n).

Remark 3.1. In fact, if $\mathcal{C}$ is a superelliptic curve with many automorphisms, we have that $n \mid r$ or $r \equiv -1 \pmod{n}$, see Prop. 3.9.

If $\mathcal{C}$, H, and τ are as above, we call τ a *superelliptic automorphism (of level n)* and H a *superelliptic group (of level n)* of $\mathcal{C}$.

Suppose $\mathcal{C}$ is a superelliptic curve, with superelliptic group H and corresponding H-cover $\pi : \mathcal{C} \to \mathbb{P}^1$. If $G = \mathrm{Aut}(\mathcal{C})$, then there is a short exact sequence $1 \to H \to G \to \overline{G} \to 1$, where $\overline{G}$ is a group of Möbius transformations keeping the set of branch points of π invariant. We call $\overline{G}$ the *reduced automorphism group* of $\mathcal{C}$.

We also define a *pre-superelliptic curve* to be a curve satisfying all the requirements of a superelliptic curve except possibly for (i) above. In this case, if N is the normalizer of H in $\mathrm{Aut}(\mathcal{C})$, then we have a similar exact sequence $1 \to H \to N \to \overline{N} \to 1$, and we call $\overline{N}$ the reduced automorphism group of $\mathcal{C}$. In this case, H is called a *pre-superelliptic group* and τ is called a *pre-superelliptic automorphism*.

Remark 3.2. Because verifying that a curve is pre-superelliptic does not depend on computing its entire automorphism group, it can be significantly easier than verifying that a curve is superelliptic. This is why we work with pre-superelliptic curves throughout the paper.

As an immediate consequence of [15, Lem. 1], we obtain the following proposition.

Proposition 3.3. *If $\mathcal{C}$ is a pre-superelliptic curve, then any pre-superelliptic group is in fact* central *in its normalizer.*

Equations for superelliptic curves were described first in [18]. For more on arithmetic aspects of such curves we refer to [19].

3.1. **Superelliptic curves with many automorphisms.** In the rest of this subsection, we construct a list containing all superelliptic curves with many automorphisms. As above, let $\mathcal{C}$ be a pre-superelliptic curve defined over $\mathbb{C}$ with automorphism group $G := \mathrm{Aut}(\mathcal{C})$, and pre-superelliptic group H generated by τ of order ≥ 2. Let $N \subseteq G$ be the normalizer of H, and let $\overline{N} := N/H$ be the reduced automorphism group.

Definition 3.4. We say that a pre-superelliptic curve $\mathcal{C}$ *has property ($\star$) with respect to a pre-superelliptic group H* if N is as above and $\mathcal{C} \to \mathcal{C}/N$ is branched at exactly three points. We will usually omit mention of H when it is clear from context.

Remark 3.5. In Table 1, we will construct the list of all pre-superelliptic curves $\mathcal{C}$ with property ($\star$). Since $N = G$ for a superelliptic curve, Lem. 2.3 shows that any superelliptic curve with property ($\star$) in fact has many automorphisms. So our list contains all superelliptic curves with many automorphisms.

For the rest of this paper C_n and D_{2n} denote the respectively cyclic group of order n and the dihedral group of order $2n$.

Proposition 3.6. *Let $\mathcal{C}$ be a pre-superelliptic curve, let $N \subseteq \mathrm{Aut}(\mathcal{C})$ be the normalizer of a pre-superelliptic group H of level n, and let $\overline{N} = N/H$. Then $\overline{N}$ is isomorphic to either C_m, D_{2m}, A_4, S_4, or A_5. If $\mathcal{C}$ has property ($\star$), then furthermore:*

(i) *If $\overline{N} \cong C_m$ then $\mathcal{C}$ has equation $y^n = x^m + 1$ or $y^n = x(x^m + 1)$.*
(ii) *If $\overline{N} \cong D_{2m}$ then $\mathcal{C}$ has equation $y^n = x^{2m} - 1$ or $y^n = x(x^{2m} - 1)$.*
(iii) *If $\overline{N} \cong A_4$ then $\mathcal{C}$ has equation $y^n = f(x)$ where*

$$f(x) = x^4 + 2i\sqrt{3}x^2 + 1 \ \text{or}\ f(x) = x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1).$$

Furthermore, the A_4-orbit of ∞ consists of itself and the roots of $x(x^4 - 1)$.
(iv) *If $\overline{N} \cong S_4$ then $\mathcal{C}$ has equation $y^n = f(x)$ where $f(x)$ is one of the following: $r_4(x)$, $s_4(x)$, $t_4(x)$, $r_4(x)s_4(x)$, $r_4(x)t_4(x)$, $s_4(x)t_4(x)$, $r_4(x)s_4(x)t_4(x)$, where*

$$r_4(x) = x^{12} - 33x^8 - 33x^4 + 1,\ s_4(x) = x^8 + 14x^4 + 1,\ t_4(x) = x(x^4 - 1).$$

Furthermore, the S_4-orbit of ∞ consists of itself and the roots of $t_4(x)$.
(v) *If $\overline{N} \cong A_5$ then $\mathcal{C}$ has equation $y^n = f(x)$ where $f(x)$ is one of the following: $r_5(x)$, $s_5(x)$, $t_5(x)$, $r_5(x)s_5(x)$, $r_5(x)t_5(x)$, $s_5(x)t_5(x)$, $r_5(x)s_5(x)t_5(x)$, where*

$$\begin{aligned} r_5(x) &= x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1 \\ s_5(x) &= x(x^{10} + 11x^5 - 1) \\ t_5(x) &= x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1. \end{aligned}$$

Furthermore, the A_5-orbit of ∞ consists of itself and the roots of $s_5(x)$.

Remark 3.7. The notation r_4, s_4, t_4, r_5, s_5, t_5 in Prop. 3.6 above is consistent with that used in [20] from where [8] took the list of hyperelliptic curves with extra automorphisms and also with [18], where there is a complete list of superelliptic curves with extra automorphisms.

Proof. The first statement holds because $\overline{N} \subseteq PGL_2(\mathbb{C})$, using the well-known classification of finite subgroups of $PGL_2(\mathbb{C})$; see [21].

We have the following diagram:

$$\Phi : \mathcal{C} \xrightarrow{H} \mathbb{P}^1_x \xrightarrow{\overline{N}} \mathbb{P}^1_z = \mathcal{C}/N.$$

The group $\overline{N}$ is the monodromy group of the cover $\phi : \mathbb{P}^1_x \to \mathbb{P}^1_z$. Let $y^n = f(x)$ be the equation of $\mathcal{C}$, where $f(x)$ is a separable polynomial. Now, the map ϕ is given by the rational function z in x, which has degree $|\overline{N}|$.

Let $S = \{q_1, q_2, q_3\}$ be the set of branch points of $\Phi : \mathcal{C} \to \mathbb{P}^1_z$, let W be the set of branch points of $\pi : \mathcal{C} \to \mathbb{P}^1_x$ (that is, the roots of f and possibly ∞). Since $\mathcal{C} \to \mathbb{P}^1_z$ is Galois and $|H| \geq 2$, there exists a non-empty set $T \subseteq S$ such that $W = \phi^{-1}(T)$.

Write the rational function z in lowest terms as a ratio of polynomials $\frac{\Psi(x)}{\Upsilon(x)}$. We write $z - q_i = \frac{\Gamma_i(x)}{\Upsilon(x)}$ in lowest terms, for each branch point q_i, $i = 1, 2, 3$, where

$\Gamma_i(x) \in \mathbb{C}[x]$. Hence,

$$\Gamma_i(x) = \Psi(x) - q_i \cdot \Upsilon(x)$$

is a degree $|\overline{N}|$ polynomial and the multiplicity of each root of $\Gamma_i(x)$ corresponds to the ramification index for each q_i (if $q_i = \infty$, then $\Gamma_i(x) := \Upsilon(x)$). The roots of $\Gamma_i(x)$ are the preimages of q_i under ϕ. So letting $\gamma_i(x) = \operatorname{rad}(\Gamma_i(x))$ (the radical of $\Gamma_i(x)$), we conclude that the equation of $\mathcal{C}$ is given by $y^n = f(x)$, where

$$f(x) = \prod_{q_i \in T} \gamma_i(x). \tag{1}$$

The rest of the proof proceeds similarly to [18, §4]. In particular, for

$$\overline{N} \in \{C_m, D_{2m}, A_4, S_4, A_5\},$$

we can make a change of variables in x and z so that $z = \phi(x)$ is given by the appropriate entry in the first 5 rows of [18, Table 1]. We now go case by case.

i) $\overline{N} \cong C_m$: In this case,

$$\phi(x) = x^m,$$

which has branch points $q_1 = \infty$ and $q_2 = 0$. Hence $\gamma_1(x) = 1$ and $\gamma_2(x) = x$. After a change of variables in z, we may assume without loss of generality that $q_3 = -1$, which yields $\gamma_3(x) = x^m + 1$. Since the covering $\mathcal{C} \longrightarrow \mathbb{P}^1_z$ has three branch points, we must have $q_3 \in T$. From Eq. (1), we have $f(x) = x^m + 1$ or $f(x) = x(x^m + 1)$. This proves (i).

ii) $\overline{N} \cong D_{2m}$: In this case,

$$\phi(x) = x^m + \frac{1}{x^m} = \frac{x^{2m} + 1}{x^m},$$

which has branch points $q_1 = \infty$, $q_2 = 2$, and $q_3 = -2$. Hence $\gamma_1(x) = x$, $\gamma_2(x) = x^m - 1$, and $\gamma_3(x) = x^m + 1$. The involution in the dihedral group permutes the branch points q_2 and q_3, so $q_2 \in T$ if and only if $q_3 \in T$. But if neither is in T, then Eq. (1) shows that $\mathcal{C}$ has equation $y^n = x$, contradicting the assumption that $g(\mathcal{C}) \geq 2$. So $T = \{q_1, q_2, q_3\}$ or $T = \{q_2, q_3\}$. From Eq. (1), we have the two possible equations

$$y^n = x^{2m} - 1, \quad y^n = x(x^{2m} - 1).$$

This proves (ii).

iii) $\overline{N} \cong A_4$: In this case,

$$\phi(x) = \frac{x^{12} - 33x^8 - 33x^4 + 1}{x^2(x^4 - 1)^2},$$

which has branch points $q_1 = \infty$ of index 2, $q_2 = 6i\sqrt{3}$ of index 3, and $q_3 = -6i\sqrt{3}$ of index 3, where $i^2 = -1$. Hence

$$t_4 := \gamma_1 = x(x^4 - 1), \quad \gamma_2 = x^4 + 2i\sqrt{3}x^2 + 1, \quad \gamma_3 = x^4 - 2i\sqrt{3}x^2 + 1,$$

with ∞ in the fiber of q_1 as well. The branch points $q_1 = \infty$, $q_2 = 6i\sqrt{3}$, and $q_3 = -6i\sqrt{3}$ are the branch points of the covering $\pi : \mathcal{C} \to \mathbb{P}^1$. Let s_4 and t_4 be as in Eq. (2) below. If neither q_2 nor q_3 is in T, then Eq. (1) shows that the equation of $\mathcal{C}$ is $y^n = t_4$. Observe that $\gamma_2\gamma_3 = s_4$. So if both q_2 and q_3 are in T, then Eq. (1) shows that the equation of $\mathcal{C}$ is either $y^n = s_4 t_4$ or $y^n = s_4$. In all cases, the reduced automorphism group is actually S_4, so we may assume that exactly one of q_2 or q_3 is in T. Since the two choices are conjugate, we may assume $q_2 \in T$ but $q_3 \notin T$.

So $T = \{q_1, q_2\}$ or $\{q_2\}$. By Eq. (1), we have the two possible equations

$$y^n = (x^4 + 2i\sqrt{3}x^2 + 1), \quad y^n = x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1).$$

The last assertion of (iii) is true because $\infty \in \phi^{-1}(q_1)$. In the Table 1 we denote by $p_4 := x^4 + 2i\sqrt{3}x^2 + 1$.

iv) $\overline{N} \cong S_4$**:** In this case,

$$\phi(x) = \frac{(x^8 + 14x^4 + 1)^3}{108\, x^4 (x^4 - 1)^4},$$

which has branch points $q_1 = 1$, $q_2 = 0$ and $q_3 = \infty$. Also, $\infty \in \phi^{-1}(q_3)$. Then

$$\begin{aligned} & r_4(x) : \gamma_1(x) = x^{12} - 33x^8 - 33x^4 + 1 \\ & s_4(x) := \gamma_2(x) = x^8 + 14x^4 + 1 \\ & t_4(x) := \gamma_3(x) = x(x^4 - 1), \end{aligned} \tag{2}$$

Every possible case occurs here. So by Eq. (1), the equation of the curve $\mathcal{C}$ is $y^n = f(x)$ for $f(x)$ one of

$$r_4(x), s_4(x), t_4(x), r_4(x)s_4(x), r_4(x)t_4(x), s_4(x)t_4(x), r_4(x)s_4(x)t_4(x).$$

The last assertion of (iv) is true because $\infty \in \phi^{-1}(q_3)$.

v) $\overline{N} \cong A_5$**:** In this case,

$$\phi(x) = \frac{(-x^{20} + 228x^{15} - 494^{10} - 228x^5 - 1)^3}{(x(x^{10} + 11x^5 - 1))^5},$$

which has branch points of $q_1 = 0$, $q_2 = 1728$, and $q_3 = \infty$. One computes

$$\begin{aligned} & r_5(x) := \gamma_1(x) = x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1 \\ & s_5(x) := \gamma_2(x) = x(x^{10} + 11x^5 - 1) \\ & t_5(x) := \gamma_3(x) = x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1, \end{aligned} \tag{3}$$

and one notes that $\infty \in \phi^{-1}(q_2)$ as well. Every possible case occurs here. So by Eq. (1), the equation of the curve $\mathcal{C}$ is $y^n = f(x)$ for $f(x)$ one of

$$r_5(x), s_5(x), t_5(x), r_5(x)s_5(x), r_5(x)t_5(x), s_5(x)t_5(x), r_5(x)s_5(x)t_5(x).$$

The last assertion of (v) is true because $\infty \in \phi^{-1}(q_2)$. □

Remark 3.8. It is clear from the proof of Prop. 3.6 that in all cases, the group $\overline{N}$ permutes the branch locus of $\pi : \mathcal{C} \to \mathbb{P}^1_x$.

Proposition 3.9. (i) *Let $\mathcal{C}$ be a pre-superelliptic curve with property ($\star$) given by an affine equation $y^n = f(x)$ as in Prop. 3.6. Suppose $\overline{N} \in \{A_4, S_4, A_5\}$ as in Prop. 3.6(iii), (iv), or (v). Then $n \mid (\deg f(x) + \delta)$, where*

$$\delta = \begin{cases} 1 & t_4(x) \mid f(x) \text{ or } s_5(x) \mid f(x) \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $\deg(f) + \delta$ is the number of branch points of the map $\mathcal{C} \to \mathbb{P}^1$ given by projection to the x-coordinate.

(ii) *Conversely, if $\mathcal{C}$ is a smooth projective curve given by an affine equation $y^n = f(x)$ as in Prop. 3.6(iii), (iv), or (v) with $n \mid (\deg(f(x) + \delta)$, then $\mathcal{C}$ is a pre-superelliptic curve with property ($\star$).*

Proof. We first note that by Prop. 3.6(iii), (iv), and (v), the orbit of ∞ under $\overline{N}$ consists of the roots of $t_4(x)$ (in cases (iii) and (iv)) or of $s_5(x)$ (in case (v)). So ∞ is a branch point of $\mathcal{C} \to \mathbb{P}^1$ if and only if $t_4(x) \mid f(x)$ or $s_5(x) \mid f(x)$; that is, if and only if $\delta = 1$.

Since $f(x)$ is separable, the monodromy action induced by a small counterclockwise loop around any root of $f(x)$ takes a point (a, b) to $(a, e^{2\pi i/n}b)$. In particular, if $\tau \in H := \mathrm{Aut}(\mathcal{C}/\mathbb{P}^1)$ is the isomorphism $(a, b) \mapsto (a, e^{2\pi i/n}b)$, then τ has order n and the signature of $\mathcal{C} \to \mathbb{P}^1$ is

$$\begin{cases} (\tau, \dots, \tau) & \delta = 0 \\ (\tau, \dots, \tau, \sigma) & \delta = 1, \end{cases}$$

for some $\sigma \in H$ corresponding to the branch point ∞. Now, the product of all entries in the signature must be the identity. In the first case, this implies that the number of branch points is divisible by n, so $\deg f(x)$ is divisible by n. In the second case, since ∞ is permuted with other branch points of the cover and H is central in N (Prop. 3.3), we have that $\sigma = \tau$. Thus the number of branch points, which is now $\deg f(x) + 1$, is also divisible by n. This completes the proof of part (i).

Let $\pi : \mathcal{C} \to \mathbb{P}^1$ be the projection to the x-coordinate. To prove (ii), it suffices to show that if $\overline{\alpha} \in \mathrm{Aut}(\mathbb{P}^1)$ preserves the branch locus of π, then $\overline{\alpha}$ lifts to an element α of $\mathrm{Aut}(\mathcal{C})$. By the proof of part (i), the signature of π is $(\tau, \dots, \tau)$ for some $\tau \in H$. After a change of variables, we may assume that ∞ is not a branch point, so the affine equation is $y^n = f(x)$ where $f(x)$ is separable. Now, $\overline{\alpha}$ permutes the roots of $f(x)$, since ∞ is not a branch point. Thus it is clear that $\overline{\alpha}$ lifts to an automorphism of $\mathcal{C}$ given by acting trivially on y. This proves (ii). □

The following corollary is immediate.

Corollary 3.10. *If $\mathcal{C}$ is a superelliptic curve with many automorphisms and H is a superelliptic group, then the cover $\mathcal{C} \to \mathcal{C}/H$ has signature $(\tau, \dots, \tau)$ for some superelliptic automorphism τ.*

Let $\mathcal{C}$ be a pre-superelliptic curve with property $(\star)$. As a consequence of Prop. 3.6 and Prop. 3.9, there are finitely many such curves with reduced automorphism group A_4, S_4, or A_5, and the list of such curves is exactly Table 1 below. In particular, all superelliptic curves with many automorphisms and reduced automorphism group not isomorphic to C_m or D_{2m} appear in Table 1.

For the rest of the paper, our goal is to determine which pre-superelliptic curves with property $(\star)$ (and thus, which superelliptic curves with many automorphisms) are of CM-type.

Table 1: Pre-superelliptic curves with property $(\star)$, exceptional reduced automorphism groups $\overline{N}$, and equation $y^n = f(x)$. This table contains all superelliptic curves with many automorphisms.

Nr.	$\overline{N}$	sig. $\overline{N}$	n	g	$f(x)$	CM?	Justification
$\mathcal{C}_0$	A_4	(2,3,3)	4	3	p_4	YES	Prop. 4.9
$\mathcal{C}_1$			2	4		YES	Prop. 4.9
$\mathcal{C}_2$	A_4	(2,3,3)	5	16	$p_4 t_4$	NO	Prop. 5.11
$\mathcal{C}_3$			10	36		NO	Cor. 5.12

$\mathcal{C}_4$			2	5		YES	Prop. 4.9
$\mathcal{C}_5$			3	10		NO	Prop. 5.11
$\mathcal{C}_6$	S_4	(2,4,4)	4	15	r_4	NO	Prop. 5.11
$\mathcal{C}_7$			6	25		NO	Cor. 5.12
$\mathcal{C}_8$			12	55		NO	Cor. 5.12
$\mathcal{C}_9$			2	3		NO	Prop. 5.1
$\mathcal{C}_{10}$	S_4	(2,4,4)	4	9	s_4	NO	Prop. 5.1
$\mathcal{C}_{11}$			8	21		NO	Prop. 5.1
$\mathcal{C}_{12}$			2	2		YES	Prop. 4.9
$\mathcal{C}_{13}$	S_4	(2,4,4)	3	4	t_4	YES	Prop. 4.9
$\mathcal{C}_{14}$			6	10		YES	Prop. 4.9
$\mathcal{C}_{15}$			2	9		NO	Prop. 5.1
$\mathcal{C}_{16}$			4	27		NO	Prop. 5.1
$\mathcal{C}_{17}$	S_4	(2,4,4)	5	36	r_4s_4	NO	Prop. 5.5
$\mathcal{C}_{18}$			10	81		NO	Prop. 5.1
$\mathcal{C}_{19}$			20	171		NO	Prop. 5.1
$\mathcal{C}_{20}$			2	8		YES	Prop. 4.9
$\mathcal{C}_{21}$			3	16		NO	Prop. 5.5
$\mathcal{C}_{22}$	S_4	(2,4,4)	6	40	r_4t_4	NO	Cor. 5.6
$\mathcal{C}_{23}$			9	64		NO	Cor. 5.6
$\mathcal{C}_{24}$			18	136		NO	Cor. 5.6
$\mathcal{C}_{25}$			2	6		NO	Prop. 5.1
$\mathcal{C}_{26}$	S_4	(2,4,4)	7	36	s_4t_4	NO	Prop. 5.11
$\mathcal{C}_{27}$			14	78		NO	Prop. 5.1
$\mathcal{C}_{28}$			2	12		NO	Prop. 5.1
$\mathcal{C}_{29}$	S_4	(2,4,4)	13	144	$r_4s_4t_4$	NO	Prop. 5.11
$\mathcal{C}_{30}$			26	300		NO	Prop. 5.1
$\mathcal{C}_{31}$			2	9		NO	Prop. 5.1
$\mathcal{C}_{32}$			4	27		NO	Prop. 5.1
$\mathcal{C}_{33}$	A_5	(2,3,5)	5	36	r_5	NO	Prop. 5.11
$\mathcal{C}_{34}$			10	81		NO	Prop. 5.1
$\mathcal{C}_{35}$			20	171		NO	Prop. 5.1
$\mathcal{C}_{36}$			2	5		NO	Prop. 5.1
$\mathcal{C}_{37}$			3	10		YES	Prop. 4.9
$\mathcal{C}_{38}$	A_5	(2,3,5)	4	15	s_5	NO	Prop. 5.1
$\mathcal{C}_{39}$			6	25		NO	Prop. 5.1
$\mathcal{C}_{40}$			12	55		NO	Prop. 5.1
$\mathcal{C}_{41}$			2	14		YES	Prop. 4.9
$\mathcal{C}_{42}$			3	28		NO	Prop. 5.5
$\mathcal{C}_{43}$			5	56		NO	Prop. 5.11
$\mathcal{C}_{44}$	A_5	(2,3,5)	6	70	t_5	NO	Cor. 5.6
$\mathcal{C}_{45}$			10	126		NO	Cor. 5.12
$\mathcal{C}_{46}$			15	196		NO	Cor. 5.6
$\mathcal{C}_{47}$			30	406		NO	Cor. 5.6
$\mathcal{C}_{48}$			2	15		NO	Prop. 5.1
$\mathcal{C}_{49}$			4	45		NO	Prop. 5.1
$\mathcal{C}_{50}$	A_5	(2,3,5)	8	105	r_5s_5	NO	Prop. 5.1

$\mathcal{C}_{51}$			16	225		NO	Prop. 5.1
$\mathcal{C}_{52}$			32	465		NO	Prop. 5.1
$\mathcal{C}_{53}$			2	24		NO	Prop. 5.1
$\mathcal{C}_{54}$			5	96		NO	Prop. 5.5
$\mathcal{C}_{55}$	A_5	(2,3,5)	10	216	r_5t_5	NO	Prop. 5.1
$\mathcal{C}_{56}$			25	576		NO	Cor. 5.6
$\mathcal{C}_{57}$			50	1176		NO	Prop. 5.1
$\mathcal{C}_{58}$			2	20		NO	Prop. 5.1
$\mathcal{C}_{59}$			3	40		NO	Prop. 5.5
$\mathcal{C}_{60}$			6	100		NO	Prop. 5.1
$\mathcal{C}_{61}$	A_5	(2,3,5)	7	120	s_5t_5	NO	Prop. 5.5
$\mathcal{C}_{62}$			14	260		NO	Prop. 5.1
$\mathcal{C}_{63}$			21	400		NO	Cor. 5.6
$\mathcal{C}_{64}$			42	820		NO	Prop. 5.1
$\mathcal{C}_{65}$			2	30		NO	Prop. 5.1
$\mathcal{C}_{66}$	A_5	(2,3,5)	31	900	$r_5s_5t_5$	NO	Prop. 5.11
$\mathcal{C}_{67}$			62	1830		NO	Prop. 5.1

Remark 3.11. Each row of the Table 1 can be considered a family of genus $g \geq 2$ of complex dimension $\deg f + \delta - 3$ (where δ is as in Prop. 3.9) admitting an action of C_n with total ramification, and among them there is a special point which is pre-superelliptic with property ($\star$). Notice that some values for g appear twice. For example, for $g = 5$ we have hyperelliptic curves $\mathcal{C}_4$ and $\mathcal{C}_{36}$ from which $\mathcal{C}_4$ has CM and $\mathcal{C}_{36}$ does not.

4. Positive CM Results

In this section, we confirm that all pre-superelliptic curves with property ($\star$) and reduced automorphism group C_m or D_{2m} have CM, and we show that the curves in Table 1 marked "YES" have CM as well. Throughout, if $\mathcal{C}$ is a pre-superelliptic curve, we write H for a pre-superelliptic group, N for the normalizer of H in $\mathrm{Aut}(\mathcal{C})$, and $\overline{N} := N/H$ for the reduced automorphism group of $\mathcal{C}$.

4.1. **Quotients of Fermat curves.** A *Fermat curve* is a projective curve with affine equation $x^a + y^a + z^a = 0$ for some $a \in \mathbb{N}$. We show in the proof of Thm. 4.2 below that any pre-superelliptic curve with property ($\star$) and reduced automorphism group $\overline{G} = C_m$ or $\overline{G} = D_{2m}$ is isomorphic to a quotient of a Fermat curve. Such curves indeed have CM:

Lemma 4.1. *If $\mathcal{C}$ is the quotient of a Fermat curve then* $\mathrm{Jac}(\mathcal{C})$ *has CM.*

Proof. It is well-known that Fermat curves have CM Jacobians, see, e.g., [22, Ch. VI.1]. The lemma follows since any quotient of a curve with CM Jacobian has CM Jacobian Cor. 2.2. □

Theorem 4.2. *Suppose $\mathcal{C}$ is a pre-superelliptic curve with property ($\star$).*
i) If $\overline{N}$ is cyclic then $\mathrm{Jac}(\mathcal{C})$ *has CM.*
ii) If $\overline{N}$ is dihedral then $\mathrm{Jac}(\mathcal{C})$ *has CM.*

Proof. First note that for any $a \in \mathbb{N}$, the smooth proper curve with affine equation $x^a = y^a \pm 1$ is isomorphic to a Fermat curve over $\mathbb{C}$.

If $\overline{N} \cong C_m$ or $\overline{N} \cong D_{2m}$, then by Prop. 3.6(i) and (ii), $\mathcal{C}$ has (affine) equation $y^n = x^r \pm 1$ or $y^n = x(x^r \pm 1)$ for some n and r. In the first case, $\mathcal{C}$ is clearly a quotient of the Fermat curve $\mathcal{Y}$ with affine equation $u^{rn} = v^{rn} \pm 1$ under the automorphism group generated by $u \mapsto \zeta_r u$ and $v \mapsto \zeta_n v$. In the second case, the quotient of the Fermat curve with affine equation $u^{rn} = v^{rn} \pm 1$ by the automorphism group generated by $(u, v) \mapsto (\zeta_{rn}^{-1} u, \zeta_n v)$ and $(u, v) \mapsto (\zeta_r u, v)$ is $\mathcal{C}$, as we see by setting $x = v^n$ and $y = u^r v$. By Lem. 4.1, $\mathcal{C}$ has CM, proving part (i). □

Remark 4.3. In Prop. 4.11 below, we give another proof of Thm. 4.2.

4.2. **Streit's criterion.** Let $\mathcal{C}$ be a smooth projective curve defined over $\mathbb{C}$, with $\sigma \in \mathrm{Aut}(\mathcal{C})$. Let $\chi_{\mathcal{C}}(\sigma)$ be the character of σ on $H^0(\mathcal{C}, \omega_{\mathcal{C}})$, and let $\mathrm{Sym}^2 \chi_{\mathcal{C}}(\sigma)$ be the character of σ on the $\mathrm{Aut}(\mathcal{C})$-representation $\mathrm{Sym}^2 H^0(\mathcal{C}, \omega_{\mathcal{C}})$.

Lemma 4.4 (Streit [11])**.** *If* $\langle \mathrm{Sym}^2 \chi_{\mathcal{C}}, \chi_{triv} \rangle = 0$*, then* $\mathrm{Jac}(\mathcal{C})$ *has CM.*

Remark 4.5. If Γ is any subgroup of $\mathrm{Aut}(\mathcal{C})$, it suffices to verify Streit's criterion considering $\mathrm{Sym}^2 H^0(\mathcal{C}, \omega_{\mathcal{C}})$ as a Γ-representation. This is because $\langle \mathrm{Sym}^2 \chi_{\mathcal{C}}, \chi_{triv} \rangle_\Gamma = 0$ implies that $\langle \mathrm{Sym}^2 \chi_{\mathcal{C}}, \chi_{triv} \rangle_G = 0$, since the former means that $\mathrm{Sym}^2 H^0(\mathcal{C}, \omega_{\mathcal{C}})$ has no Γ-invariant vectors whereas the latter means it has no G-invariant vectors.

Remark 4.6. In [23], the authors give a geometric interpretation for Streit's criterion. Specifically, if $\mathcal{C}$ is a curve with many automorphisms, then $\mathrm{Jac}\,\mathcal{C}$ lies in a special subvariety of $\mathcal{A}_g$ of dimension $\langle \mathrm{Sym}^2 \chi_{\mathcal{C}}, \chi_{triv} \rangle$, see [23, Thm. 3.9 and its proof]. Since special subvarieties of dimension zero are CM points, Streit's criterion holds. Special subvarieties of higher dimension must contain CM points, but they may also contain non-CM points, so one cannot conclude anything when Streit's criterion fails.

Now, suppose $\mathcal{C}$ is pre-superelliptic, and for $\sigma \in N$, write $\bar{\sigma}$ for the image of σ in $\overline{N}$. By Remark 4.5, to verify Streit's criterion, we may consider $\mathrm{Sym}^2 \chi_{\mathcal{C}}$ as an N-representation. To help our calculation, we record the following lemma.

Lemma 4.7. *We have*

$$\mathrm{Sym}^2 \chi_{\mathcal{C}}(\sigma) = \frac{1}{2} \left(\chi_{\mathcal{C}}(\sigma^2) + \chi_{\mathcal{C}}(\sigma)^2 \right).$$

Proof. This is a basic result of representation theory. □

We now have the following:

Proposition 4.8. *Let* $\mathcal{C} : y^n = f(x)$ *be a smooth pre-superelliptic curve defined over* $\mathbb{C}$*. For each* $\overline{\sigma}$ *in the reduced automorphism group* $\overline{N}$*, let* m *be its order, and let* $\zeta_{\overline{\sigma}}$ *be either ratio of the eigenvalues when* $\overline{\sigma}$ *is thought of as an element of* $PGL_2(\mathbb{C})$ *(*$\zeta_{\overline{\sigma}}$ *is a primitive* m*th root of unity). Define*

$$k_{\overline{\sigma}} = \begin{cases} 1 & \text{if } \overline{\sigma} \text{ fixes a branch point of } \mathcal{C} \to \mathbb{P}^1 \\ 0 & \text{otherwise.} \end{cases}$$

Let ζ_n*, be a primitive* n*th root of unity, and let* $\zeta_{n,\overline{\sigma}}$ *be a primitive* mn*th root of unity such that* $\zeta_{n,\overline{\sigma}}^n = \zeta_{\overline{\sigma}}$*. Let* A *be the set of ordered pairs defined below in Eq. (4). If*

$$\sum_{\overline{\sigma} \in \overline{N}} \sum_{i=0}^{n-1} \left(\sum_{(a,b) \in A} \zeta_{\overline{\sigma}}^{2(a+1)} \zeta_n^{2(b+1)i} \zeta_{n,\overline{\sigma}}^{2(b-n+1)k_{\overline{\sigma}}} + \left(\sum_{(a,b) \in A} \zeta_{\overline{\sigma}}^{a+1} \zeta_n^{(b+1)i} \zeta_{n,\overline{\sigma}}^{(b-n+1)k_{\overline{\sigma}}} \right)^2 \right)$$

vanishes, then $\mathcal{C}$ has CM.

Proof. Suppose $\mathcal{C}$ has genus g and $\deg(f) = d$. From [24], a basis for the space of holomorphic differentials on $\mathcal{C}$ is given by

$$x^a y^b \left(\frac{dx}{y^{n-1}}\right),$$

as (a, b) ranges through the set

$$A := \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0, 0 \leq b < n, 0 \leq an + bd \leq 2g - 2\}. \tag{4}$$

By the Hurwitz formula, $2g - 2 = -n - \gcd(n, d) + d(n - 1)$, so we can also write A as

$$\{(a, b) \in \mathbb{Z}^2 \mid 0 \leq b < n, 0 \leq a \leq d - 1 - \frac{d(1 + b) + \gcd(n, d)}{n}\}. \tag{5}$$

Let $\sigma \in N$, $\overline{\sigma}$ its image in $\overline{N}$, and m be the order of $\overline{\sigma}$. If τ is a pre-superelliptic automorphism of $\mathcal{C}$ and $H = \langle\tau\rangle$, then σH is a coset consisting of n different automorphisms of $\mathcal{C}$, say $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$, all projecting to $\overline{\sigma} \in \overline{\mathrm{Aut}}(\mathcal{C}) \subseteq \mathrm{Aut}(\mathbb{P}^1)$. Now, $\overline{\sigma}$ acts on $\mathbb{P}^1$ with two fixed points, and after a change of coordinate we may assume that they are 0 and ∞. After possibly replacing x by $1/x$, we may assume that $\overline{\sigma}$ acts on the coordinate x via $x \mapsto \zeta_{\overline{\sigma}} x$. After this change of variables, there is a polynomial $h \in k[x]$ such that the equation for $\mathcal{C}$ is given by $y^n = x^{k_{\overline{\sigma}}} h(x^m)$, where $k_{\overline{\sigma}} = 1$ if the fixed point 0 of $\overline{\sigma}$ is a ramification point of $\mathcal{C} \to \mathbb{P}^1$, and $k_{\overline{\sigma}} = 0$ otherwise, as in the statement of the proposition.

Fix a primitive nth root of unity ζ_n, as well as a primitive mnth root $\zeta_{n,\overline{\sigma}}$ as in the statement of the proposition, so $\zeta_{n,\overline{\sigma}}^n = \zeta_m$. Since $\overline{\sigma}(x) = \zeta_{\overline{\sigma}} x$ and $y^n = x^{k_{\overline{\sigma}}} h(x^m)$, we have that

$$\sigma(y) = \zeta_n^i \zeta_{n,\overline{\sigma}}^{k_{\overline{\sigma}}} y,$$

for some $i \in \{1, \ldots, n\}$. After reordering $\sigma_1, \ldots, \sigma_n$, we may assume that $\sigma_i(y) = \zeta_n^i \zeta_{n,\overline{\sigma}}^{k_{\overline{\sigma}}} y$ for each i. In particular, $x^a y^b \frac{dx}{y^{n-1}}$ is an eigenvector for every σ_i. Its eigenvalue is

$$\zeta_{\overline{\sigma}}^{a+1} \zeta_n^{i(b-n+1)} \zeta_{n,\overline{\sigma}}^{(b-n+1)k_{\overline{\sigma}}} = \zeta_{\overline{\sigma}}^{a+1} \zeta_n^{i(b+1)} \zeta_{n,\overline{\sigma}}^{(b-n+1)k_{\overline{\sigma}}}$$

We thus have

$$\sum_{\sigma \mapsto \overline{\sigma}} \chi_X(\sigma) = \sum_{i=0}^{n-1} \sum_{(a,b) \in A} \zeta_{\overline{\sigma}}^{a+1} \zeta_n^{(b+1)i} \zeta_{n,\overline{\sigma}}^{(b-n+1)k_{\overline{\sigma}}}. \tag{6}$$

Likewise,

$$\sum_{\sigma \mapsto \overline{\sigma}} \chi_X(\sigma^2) = \sum_{i=0}^{n-1} \sum_{(a,b) \in A} \zeta_{\overline{\sigma}}^{2(a+1)} \zeta_n^{2(b+1)i} \zeta_{n,\overline{\sigma}}^{2(b-n+1)k_{\overline{\sigma}}}. \tag{7}$$

Also,

$$\sum_{\sigma \mapsto \overline{\sigma}} \chi_X(\sigma)^2 = \sum_{i=0}^{n-1} \left(\sum_{(a,b) \in A} \zeta_{\overline{\sigma}}^{a+1} \zeta_n^{(b+1)i} \zeta_{n,\overline{\sigma}}^{(b-n+1)k_{\overline{\sigma}}} \right)^2. \tag{8}$$

By Lem. 4.7, $\operatorname{Sym}^2 \chi_{\mathcal{C}}(\sigma) = \frac{1}{2}(\chi_{\mathcal{C}}(\sigma^2) + \chi_{\mathcal{C}}(\sigma)^2)$. Combining this with Eq. (7) and Eq. (8), we have

$$\sum_{\sigma \mapsto \overline{\sigma}} \operatorname{Sym}^2 \chi_{\mathcal{C}}(\sigma) = \frac{1}{2}\left(\sum_{i=0}^{n-1}\left(\sum_{(a,b)\in A} \zeta_{\overline{\sigma}}^{2(a+1)} \zeta_n^{2(b+1)i} \zeta_{n,\overline{\sigma}}^{2(b-n+1)k_{\overline{\sigma}}} + \right.\right.$$
$$\left.\left. + \left(\sum_{(a,b)\in A} \zeta_{\overline{\sigma}}^{a+1} \zeta_n^{(b+1)i} \zeta_{n,\overline{\sigma}}^{(b-n+1)k_{\overline{\sigma}}}\right)^2\right)\right).$$

Combining the above with Lem. 4.4 we claim the result. ☐

Proposition 4.9. *The curves $\mathcal{C}_1$, $\mathcal{C}_4$, $\mathcal{C}_{12}$, $\mathcal{C}_{13}$, $\mathcal{C}_{14}$, $\mathcal{C}_{20}$, $\mathcal{C}_{37}$, $\mathcal{C}_{41}$ all have CM.*

Proof. The GAP program `streit_program.gap`[1] computes the sum in Prop. 4.8 for any superelliptic curve , presented as in Prop. 3.6, with n, $\overline{N}$, and $f(x)$ as inputs. The program is modelled on that of Pink and Müller used in [8]. To calculate $k_{\overline{\sigma}}$, we use the embedding of $\overline{N}$ into $PGL_2(\mathbb{C})$ from Prop. 3.6 and its proof. The rest of the calculation is straightforward. For all of the curves in the proposition, the sum in Prop. 4.8 comes to 0. ☐

Remark 4.10. The curves $\mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_{12}, \mathcal{C}_{20}$, and $\mathcal{C}_{41}$ are all hyperelliptic, and Streit's criterion was already verified for them in [8]. These correspond to X_4, X_7, X_5, X_9, and X_{14} respectively in that paper.

Proposition 4.11. *Let $\mathcal{C}$ be a pre-superelliptic curve with property ($\star$), and assume that $\overline{N} = C_m$ or D_{2m}. Then $\mathcal{C}$ satisfies Streit's criterion. That is, $\langle \operatorname{Sym}^2 \chi_{\mathcal{C}}, \chi_{triv}\rangle = 0$.*

Proof. Let $V = \operatorname{Sym}^2(H^0(\mathcal{C}, \omega_{\mathcal{C}}))$. By Remark 4.5, it suffices to verify Streit's criterion for V as an N-representation. So it is enough to show that no non-trivial points of V are fixed by N.

Let us first assume that $\overline{N} = C_m$. By Prop. 3.6, the affine equation of $\mathcal{C}$ is $y^n = x^k(x^m+1)$, where $k \in \{0, 1\}$. The set $\{v_{a,b} := x^a y^b dx/y^{n-1} \mid (a,b) \in A\}$ is a basis of simultaneous eigenvectors for the action of N on $H^0(\mathcal{C}, \omega_{\mathcal{C}})$, where A is as in Eq. (4). For each $v_{a,b}$ in this basis, let $\lambda_{a,b}(g)$ be its eigenvalue under the action of $g \in N$. It suffices to prove that there is no set $\{(a_1, b_1), (a_2, b_2)\}$ of indices such that $\lambda_{a_1,b_1}(g)\lambda_{a_2,b_2}(g)$ takes the constant value 1 on N.

Suppose $\sigma(x, y) = (\zeta_m x, \zeta_{mn}^k y)$, and $\tau(x, y) = (x, \zeta_n y)$, where ζ_{mn} is a primitive mnth root of unity with $\zeta_{mn}^n = \zeta_m$. The elements σ and τ generate N. Then

$$\lambda_{a_1,b_1}(\sigma^i\tau^j)\lambda_{a_2,b_2}(\sigma^i\tau^j) = \zeta_m^{i(a_1+a_2+2)}\zeta_n^{j(b_1+b_2+2-2n)}\zeta_{mn}^{ik(b_1+b_2+2-2n)}.$$

This is independent of j only if $b_1 + b_2 + 2 = n$, in which case it equals

$$\zeta_m^{i(a_1+a_2+2-k)}.$$

By Eq. (5), we have

$$a_1 + a_2 + 2 \le 2d - \frac{d(b_1 + b_2 + 2) + \gcd(n, d)}{n} = d - \frac{\gcd(n, d)}{n}, \tag{9}$$

[1]available at arxiv.org/abs/2006.12685

where $d = m + k$. So $0 < a_1 + a_2 + 2 - k < m$, which means that $\zeta_m^{i(a_1+a_2+2-k)}$ is not independent of i. Thus the eigenvalue cannot take the constant value 1 on N.

Now, assume $\overline{N} = D_{2m}$. By Prop. 3.6, the affine equation of $\mathcal{C}$ is $y^n = x^k(x^{2m} - 1)$, where $k \in \{0, 1\}$. The group N is generated by

$$\sigma(x, y) = (\zeta_m x, \zeta_{mn}^k y), \quad \tau(x, y) = (x, \zeta_n y), \quad \rho(x, y) = (1/x, \zeta_{2n} y/x^{2(m+k)/n}),$$

where ζ_{2n} is any $2n$th root of unity and ζ_m, ζ_n, and ζ_{mn} are as before. Let T be the index two subgroup of N generated by σ and τ. The same $v_{a,b}$ as in the C_m case form a basis of simultaneous eigenvectors for the action of T on $H^0(\mathcal{C}, \omega_{\mathcal{C}})$. If we again set $\lambda_{a,b}$ to be the respective eigenvalues, then exactly as in the C_m case, $\lambda_{a_1,b_1}(t)\lambda_{a_2,b_2}(t)$ takes the constant value 1 as t ranges over T only if $b_1 + b_2 + 2 = n$ and $a_1 + a_2 + 2 - k$ is divisible by m. Furthermore, since $d = 2m + k$, we know from Eq. (9) that $a_1 + a_2 + 2 - k$ is divisible by m only if $a_1 + a_2 + 2 - k = m$.

The only eigenvector with eigenvalue 1 for the action of T on $\mathrm{Sym}^2(H^0(\mathcal{C}, \omega_{\mathcal{C}}))$ is

$$\begin{aligned} \omega :=& x^{a_1+a_2} y^{b_1+b_2-2n+2} (dx)^2 = x^{m+k-2} y^{-n} (dx)^2 \\ =& \frac{x^{m+k-2}}{x^k(x^{2m}-1)} (dx)^2 = \frac{x^m}{x^{2m}-1} \left(\frac{dx}{x}\right)^2. \end{aligned}$$

One sees immediately that $\rho(\omega) = -\omega$, so ω is not fixed under N, which completes the proof. □

Note that Prop. 4.11 gives another proof of Thm. 4.2.

5. Negative CM results

In this section, we show that the remaining curves in Table 1 do not have CM.

5.1. **Bootstrapping the hyperelliptic case.** The following proposition is a direct consequence of the main result of [8].

Proposition 5.1. *None of the curves $\mathcal{C}_i$ in Table 1 for*

$$\begin{aligned} i \in \{&9, 10, 11, 15, 16, 18, 19, 25, 27, 28, 30, 31, 32, 34, 35, 36, 38, \\ &39, 40, 48, 49, 50, 51, 52, 53, 55, 57, 58, 60, 62, 64, 65, 67\} \end{aligned}$$

has CM.

Proof. The curves $\mathcal{C}_i$ for $i \in S := \{9, 15, 25, 28, 31, 36, 48, 53, 58, 65\}$ are all hyperelliptic, and were shown not to have CM in [8, Table 1]. For each of the other curves $\mathcal{C}_i$ in the proposition, there exists $j \in S$ such that $\mathcal{C}_i$ has $\mathcal{C}_j$ as a quotient by an automorphism fixing x and multiplying y by an appropriate root of unity. Since $\mathcal{C}_j$ does not have CM, neither does $\mathcal{C}_i$. □

5.2. **Using stable reduction.** It is well-known that if $\mathcal{C}$ is a CM curve defined over a number field K, then its Jacobian $\operatorname{Jac}\mathcal{C}$ has potentially good reduction modulo all primes of K ([25, Theorem 6]). For any such prime $\mathfrak{p}$, let $K_{\mathfrak{p}}$ be the corresponding completion of K. Assume the genus of $\mathcal{C}$ is at least 2. The *stable reduction theorem* states that there exists a finite extension $L/K_{\mathfrak{p}}$ for which $\mathcal{C} \times_K L$ has a *stable model* $\mathcal{C}^{st}$ over $\operatorname{Spec}\mathcal{O}_L$, where $\mathcal{O}_L$ is the ring of integers of L (see, e.g., [26, Corollary 2.7]). Specifically, $\mathcal{C}^{st} \to \operatorname{Spec}\mathcal{O}_L$ is a flat relative curve whose generic fiber is isomorphic to $\mathcal{C}$ and whose special fiber $\overline{\mathcal{C}}$ (called the *stable reduction* of $\mathcal{C}$ modulo $\mathfrak{p}$) is reduced, has smooth irreducible components, has only ordinary double points for singularities, and has the property that each irreducible component of genus zero contains at least three singular points of $\overline{\mathcal{C}}$. One forms the *dual graph* $\Gamma_{\overline{\mathcal{C}}}$ of $\overline{\mathcal{C}}$ by taking the vertices of $\Gamma_{\overline{\mathcal{C}}}$ to correspond to the irreducible components of $\overline{\mathcal{C}}$, with an edge between two vertices for each point where the two corresponding components intersect. A model $\mathcal{C}^{ss}$ of of $\mathcal{C} \times_K L$ is called a *semistable model* if it satisfies all the properties of a stable model, except possibly the requirement on genus zero irreducible components. The dual graph of the special fiber of any $\mathcal{C}^{ss}$ is homeomorphic to $\Gamma_{\overline{X}}$.

One has a similar construction for smooth curves $\mathcal{Y}/K$ with marked points. To wit, if $\lambda_1, \ldots, \lambda_m$ are points of $\mathcal{Y}(K)$ and $2g + m \geq 3$, there exists a unique stable model $\mathcal{Y}^{st}$ of the marked curve $(\mathcal{Y}, \{\lambda_1, \ldots, \lambda_m\})$, which is defined as above, except that we require only that each genus zero component of the special fiber $\overline{\mathcal{Y}}$ contain at least three points that are *either* singular points of $\overline{\mathcal{Y}}$ *or* specializations of marked points. We also require that the marked points specialize to distinct smooth points of $\overline{\mathcal{Y}}$. Note that if $\mathcal{Y} = \mathbb{P}^1$, then $\Gamma_{\overline{\mathcal{Y}}}$ is always a tree.

By [27, Chapter 9, §2], $\operatorname{Jac} \mathcal{C}$ has potentially good reduction if and only if $\Gamma_{\overline{\mathcal{C}}}$ is a tree (i.e., has trivial first homology). Thus, if we can find a prime $\mathfrak{p}$ for which $\Gamma_{\overline{\mathcal{C}}}$ is not a tree, then $\operatorname{Jac} \mathcal{C}$ does not have CM. We will use this criterion for several of the curves in Table 1, generalizing [28, §10] to the superelliptic case.

For the rest of Section 5.2, let K be a complete discrete valuation field with residue field k, and let $n \in \mathbb{N}$ with $\operatorname{char}(k) \nmid n$. Let $\mathcal{C} \to \mathbb{P}^1_K$ be the (potentially) $\mathbb{Z}/n$-cover of $\mathbb{P}^1_K$ given by the affine equation

$$y^n = \prod_i (x - \alpha_i),$$

where the α_i are pairwise distinct elements of K. Let B be the set of branch points of $\mathcal{C} \to \mathbb{P}^1_K$ (so B consists of the α_i, as well as ∞ if $n \nmid \deg(\prod_i (x - \alpha_i))$. Let $\mathcal{Y}^{st}$ be the stable model of the marked curve $(\mathbb{P}^1_K, B)$, and let $\Gamma_{\overline{\mathcal{Y}}}$ be the dual graph of its special fiber. Assume that $g(\mathcal{C}) \geq 2$, which in turn implies $|B| \geq 3$.

Lemma 5.2. *There exists a finite extension L/K with valuation ring $\mathcal{O}_L$, such that the normalization of $\mathcal{Y}^{st} \times_{\mathcal{O}_K} \mathcal{O}_L$ in $L(\mathcal{C})$ is a semistable model of $\mathcal{C}$ over L.*

Proof. This follows from [12, Corollary 3.6] and its proof, with $(\mathbb{P}^1_K, B)$ playing the role of (X_{L_0}, D_{L_0}). □

The normalization from Lem. 5.2 induces a map

$$\pi : \Gamma_{\overline{\mathcal{C}}} \to \Gamma_{\overline{\mathcal{Y}}} \tag{10}$$

of graphs.

Lemma 5.3. *Let π be as in Eq. (10).*

(i) *If v is a leaf of $\Gamma_{\overline{\mathcal{Y}}}$ (i.e., a vertex incident to only one edge), then $|\pi^{-1}(v)| \leq n/2$.*
(ii) *Suppose there exists an edge e of $\Gamma_{\overline{\mathcal{Y}}}$ such that removing e splits $\Gamma_{\overline{\mathcal{Y}}}$ into two trees T_1 and T_2 where n divides the number of elements of B specializing to each of T_1 and T_2. Then $|\pi^{-1}(e)| = n$.*

Proof. By the definition of marked stable model, the irreducible component J of $\overline{\mathcal{Y}}$ corresponding to v contains the specialization of at least one element b of B. Since the cover $\mathcal{C} \to \mathbb{P}^1_K$ is potentially Galois and ramification indices are at least 2, the preimage of b in $\mathcal{C}$ contains at most $n/2$ points. Since every irreducible component of $\overline{\mathcal{C}}$ lying above J contains the specialization of one of these points, there are at most $n/2$ such irreducible components. By the construction of π, this proves part (i).

Now, assume e, T_1, and T_2 are as in part (ii). Let v be the unique vertex of T_2 incident to e, and let J be the corresponding irreducible component of $\overline{\mathcal{Y}}$. For $i \in \{1, 2\}$, write B_i for the subset of B consisting of branch points specializing to T_i. Choose three distinct elements α, β, and γ of B, such that $\alpha \in B_1$ and such that (α, β, γ) corresponds to v via [12, Proposition 4.2(3)]. Note that in the construction of [12], α can be chosen freely, and then it is automatic that $\beta, \gamma \in B_2$. This is because if, say, $\beta \in B_1$, then in the language of [12], we would have $\overline{\lambda}_t(\beta) = \overline{\lambda}_t(\alpha)$, which contradicts the definition of λ_t above [12, Proposition 4.2]). The same holds for γ.

As in [12, Notation 4.4], the triple (α, β, γ) gives rise to a coordinate x_v on $\mathbb{P}^1_K$ whose reduction $\overline{x}_v$ is a coordinate on J such that $\overline{x}_v = 0$ at the point of J corresponding to the edge e.[2] Since $\overline{x}_v$ is a coordinate on J, we in fact have that an element $b \in B$ satisfies $\overline{x}_v(b) = 0$ if and only if $b \in B_2$.

As in [12, §4.3], we can write f in terms of the variable x_v, multiply by an appropriate element of K, and then reduce modulo a uniformizer of K to obtain an element $\overline{f}_v \in k(\overline{x}_v)$. Since $\overline{x}_v(b) = 0$ if and only if $b \in B_2$, the order of $\overline{f}_v$ at $\overline{x}_v = 0$ is $|B_2|$, which is assumed to be divisible by n.

By [12, Proposition 4.5], the restriction of the cover $\overline{\mathcal{C}} \to \overline{\mathcal{Y}}$ above J is given birationally by the equation $y_v^n = \overline{f}_v$. Since $\mathrm{ord}_{\overline{x}_v=0}(\overline{f}_v)$ is divisible by n, the preimage of $\overline{x}_v = 0$ in $\overline{\mathcal{C}}$ has cardinality n. This means that $|\pi^{-1}(e)| = n$, proving part (ii). □

Proposition 5.4. *In the situation of Lem. 5.3(ii), the graph $\Gamma_{\overline{\mathcal{C}}}$ is not a tree.*

Proof. Let e be the edge from Lem. 5.3(ii). Consider the graph Γ constructed by removing $\pi^{-1}(e)$ from $\Gamma_{\overline{\mathcal{C}}}$. Then Γ is the disjoint union of $\pi^{-1}(T_1)$ and $\pi^{-1}(T_2)$. Since π is ultimately constructed from a normalization Lem. 5.2, every connected component of $\pi^{-1}(T_i)$ maps surjectively onto T_i for $i \in \{1, 2\}$. If v_i is a leaf of $\Gamma_{\overline{\mathcal{Y}}}$ in T_i, then Lem. 5.3(i) shows that $|\pi^{-1}(v_i)| \leq n/2$, so there are at most $n/2$ connected components in $\pi^{-1}(T_i)$. So if V (resp. E) is the number of vertices (resp. edges) in Γ, then $V \leq E + n/2 + n/2 = E + n$. Since V (resp. $E + n$) is the number of vertices (resp. edges) in $\Gamma_{\overline{\mathcal{C}}}$, this shows that the first homology of $\Gamma_{\overline{\mathcal{C}}}$ has dimension at least 1, so $\Gamma_{\overline{\mathcal{C}}}$ is not a tree. □

Proposition 5.5. *None of the curves $\mathcal{C}_{17}$, $\mathcal{C}_{21}$, $\mathcal{C}_{42}$, $\mathcal{C}_{54}$, $\mathcal{C}_{59}$, or $\mathcal{C}_{61}$ in Table 1 has CM Jacobian.*

Proof. For $\mathcal{C}_{17}$, the tree $\Gamma_{\overline{\mathcal{Y}}}$ as in Prop. 5.4 for the reduction modulo 3 is shown in [28, Figure 3, p. 43]. In this case, $n = 5$, and there are four edges which split the tree up into subtrees with 5 and 15 marked points. By Prop. 5.4, $\Gamma_{\overline{\mathcal{C}}_{17}}$ is not a tree. So $\mathrm{Jac}\,\mathcal{C}_{17}$ has bad reduction, and thus does not have CM.

For $\mathcal{C}_{61}$, the tree $\Gamma_{\overline{\mathcal{Y}}}$ for reduction modulo 5 is shown in [28, Figure 11, p. 46]. In this case, $n = 7$, and there are four edges which split the tree up into subtrees with 7 and 35 marked points. Using Prop. 5.4 as before, $\mathrm{Jac}\,\mathcal{C}_{61}$ has bad reduction, and thus does not have CM.

[2]Specifically, we have $x_v = \frac{\beta-\gamma}{\beta-\alpha}\frac{x-\alpha}{x-\gamma}$.

For curves $\mathcal{C}_{21}$, $\mathcal{C}_{42}$, $\mathcal{C}_{54}$, and $\mathcal{C}_{59}$, the program `stable_reduction.sage`[3] gives the tree $\Gamma_{\overline{\mathcal{Y}}}$ for reduction modulo 2.

- For curve $\mathcal{C}_{21}$, there is an edge splitting $\Gamma_{\overline{\mathcal{Y}}}$ into two trees with 6 and 12 markings, and $n = 3$.
- For curve $\mathcal{C}_{42}$, there is an edge splitting $\Gamma_{\overline{\mathcal{Y}}}$ into two trees with 6 and 24 markings, and $n = 3$.
- For curve $\mathcal{C}_{54}$, there is an edge splitting $\Gamma_{\overline{\mathcal{Y}}}$ into two trees with 10 and 40 markings, and $n = 5$.
- For curve $\mathcal{C}_{59}$, there is an edge splitting $\Gamma_{\overline{\mathcal{Y}}}$ into two trees with 6 and 36 markings, and $n = 3$.

In all cases $i \in \{21, 42, 54, 59\}$, using Prop. 5.4 as before shows that $\Gamma_{\overline{\mathcal{C}}_i}$ is not a tree, which means that $\operatorname{Jac} \mathcal{C}$ has bad reduction, and thus does not have CM. □

Corollary 5.6. *None of the curves $\mathcal{C}_{22}$, $\mathcal{C}_{23}$, $\mathcal{C}_{24}$, $\mathcal{C}_{44}$, $\mathcal{C}_{46}$, $\mathcal{C}_{47}$, $\mathcal{C}_{56}$, or $\mathcal{C}_{63}$ in Table 1 has CM Jacobian.*

Proof. The curves $\mathcal{C}_{22}$, $\mathcal{C}_{23}$, and $\mathcal{C}_{24}$ have $\mathcal{C}_{21}$ as a quotient (via an automorphism multiplying y by an appropriate root of unity). Likewise, the curves $\mathcal{C}_{44}$, $\mathcal{C}_{46}$, and $\mathcal{C}_{47}$ have $\mathcal{C}_{42}$ as a quotient. The curve $\mathcal{C}_{56}$ has $\mathcal{C}_{54}$ as a quotient. The curve $\mathcal{C}_{63}$ has $\mathcal{C}_{61}$ as a quotient.

Since all quotients of a CM curve must be CM curves, Prop. 5.5 shows that none of the curves in the corollary is a CM curve. □

5.3. **Frobenius criterion.** To show that the remaining curves in Table 1 do not have CM, we use a criterion of Müller–Pink. Let A be an abelian variety defined over a number field K. For any prime $\mathfrak{p}$ of K where A has good reduction, let $f_{\mathfrak{p}} \in \mathbb{Q}[T]$ be the minimal polynomial of the Frobenius at $\mathfrak{p}$ acting on the Tate module of the reduction of A modulo $\mathfrak{p}$. Let $E_{f_{\mathfrak{p}}}$ equal $\mathbb{Q}[T]/f_{\mathfrak{p}}$. Since the Frobenius action is semisimple, the polynomial $f_{\mathfrak{p}}$ has no multiple factors.

If t is the image of T in $E_{f_{\mathfrak{p}}}$, then let $E'_{f_{\mathfrak{p}}} \subseteq E_{f_{\mathfrak{p}}}$ be the subring given by intersecting the rings of $\mathbb{Q}[t^n]_{n\in\mathbb{N}}$ inside $E_{f_{\mathfrak{p}}}$. Observe that, if $f_{\mathfrak{p}} = g(T^m)$ for some polynomial g and $m \in \mathbb{N}$, we can replace $f_{\mathfrak{p}}$ by $g(T)$ when computing $E'_{f_{\mathfrak{p}}}$.

The following criterion can be used to show that A does not have CM.

Proposition 5.7 ([8, Theorem 6.2 (a) ⇒ (b)])**.** *Maintain notation as above. If $\dim(A) = g$ and A has CM, then there exists a product of number fields E with $\dim_{\mathbb{Q}} E \le 2g$ such that for any good prime $\mathfrak{p}$, we have an embedding $E'_{f_{\mathfrak{p}}} \hookrightarrow E$.*

Remark 5.8. The paper of Müller–Pink uses the simpler (but weaker) criterion of [8, Corollary 6.7]. However, it appears to be difficult to use this criterion to prove Prop. 5.11 below.

Before our main application of Prop. 5.7, we prove two lemmas.

Lemma 5.9. *If $K_1, \dots, K_n$ and $L_1, \dots, L_m$, are characteristic 0 fields, then there exists a $\mathbb{Q}$-algebra embedding of $K_1 \times \cdots \times K_n$ into $L_1 \times \cdots \times L_m$ if and only if there exists a surjective map $\phi : \{1, \dots, m\} \to \{1, \dots, n\}$ such that for all $i \in \{1, \dots, m\}$, there exists an embedding $\gamma_i : K_{\phi(i)} \hookrightarrow L_i$.*

[3]available at arxiv.org/abs/2006.12685

Proof. An embedding $\pi : K_1 \times \cdots \times K_n \hookrightarrow L_1 \times \cdots \times L_m$ gives rise to a $\mathbb{Q}$-algebra morphism $\pi_i : K_1 \times \cdots \times K_n \to L_i$ for each i. Since the kernel is a prime ideal, this morphism is projection onto some K_j followed by an embedding $K_j \hookrightarrow L_i$. Set $j = \phi(i)$. The kernel of π is the intersection of the kernels of the π_i, which is trivial only if each j is equal to ϕ_i for some i, i.e., if ϕ is surjective. Thus the condition in the lemma is necessary for the existence of an embedding.

Conversely, if the condition in the lemma is satisfied, then we define the following $\mathbb{Q}$-algebra morphism, which is easily seen to be an embedding.

$$\begin{aligned}\pi : K_1 \times \cdots \times K_n &\hookrightarrow L_1 \times \cdots \times L_m\\ (r_1, \ldots, r_n) &\mapsto (\gamma_i(r_{\phi(1)}), \ldots, \gamma_i(r_{\phi(m)})).\end{aligned}$$

□

Lemma 5.10. *The curves $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_{26}$, $\mathcal{C}_{29}$, $\mathcal{C}_{33}$, $\mathcal{C}_{43}$, and $\mathcal{C}_{66}$ from Table 1 have quotients with the following affine equations, respectively, where i is a square root of -1:*

Curve	*Affine Birational Equation of Quotient Curve*
$\mathcal{C}_5$	$y^3 = x^6 - 33x^4 - 33x^2 + 1$
$\mathcal{C}_6$	$y^4 = x^6 - 33x^4 - 33x^2 + 1$
$\mathcal{C}_{26}$	$y^7 = (x^4 - 4ix^2 + 12)(x^2 - 2i)$
$\mathcal{C}_{29}$	$y^{13} = (x^2 - 4)^7(x + 14)(x - 34)$
$\mathcal{C}_{33}$	$y^5 = x^{10} + 10x^8 + 35x^6 - 228x^5 + 50x^4 - 1140x^3 + 25x^2 - 1140x + 496$
$\mathcal{C}_{43}$	$y^5 = x^6 + 522x^5 - 10005x^4 - 10005x^2 - 522x + 1$
$\mathcal{C}_{66}$	$y^{31} = (x^2 - 228x + 496)^2(x^2 + 522x - 10004)^2(x + 11)^2(x^2 + 4)$

Proof. For $\mathcal{C}_5$ and $\mathcal{C}_6$, the affine equation given is the obvious quotient by the automorphism fixing y and sending x to $-x$. Likewise, for $\mathcal{C}_{43}$, the affine equation given is the obvious quotient by the automorphism fixing y and sending x to $\zeta_5 x$.

For $\mathcal{C}_{26}$, consider the order 2 automorphism σ of $\mathbb{C}(\mathcal{C}_{26})$ given by $\sigma(x) = i/x$ and $\sigma(y) = iy/x^2$. The fixed subfield is generated by z and w, where $z = x + i/x$ and $w = y/x$. The affine equation of $\mathcal{C}_{26}$ is $y^7 = s_4 t_4$, which in w and z becomes

$$w^7 = \left(x^2 - \frac{1}{x^2}\right)\left(x^4 + 14 + \frac{1}{x^4}\right) = (z^2 - 2i)(z^4 - 4iz^2 + 12).$$

Changing back to x and y gives the equation in the table.

For $\mathcal{C}_{29}$, consider the automorphism group isomorphic to D_8 generated by σ and τ where $\sigma(x, y) = (ix, iy)$ and $\tau(x, y) = (1/x, y/x^2)$. The fixed subfield is generated by z and w, where $z = x^4 + 1/x^4$ and $w = y(x^8 - 1)/x^5$. The affine equation of $\mathcal{C}_{29}$ can be written as

$$y^{13} = x(x^8 - 1)(x^8 + 14x^4 + 1)(x^8 - 34x^4 + 1),$$

which in terms of w and z becomes

$$w^{13} = (z^2 - 4)^7(z + 14)(z - 34),$$

as can be easily checked. Changing back to x and y gives the equation in the table.

For $\mathcal{C}_{33}$, consider the order 2 automorphism σ of $\mathbb{C}(\mathcal{C}_{33})$ given by $\sigma(x) = -1/x$ and $\sigma(y) = y/x^4$. The fixed subfield is generated by z and w, where $z = x - 1/x$ and $w = y/x^2$. The affine equation of $\mathcal{C}_{33}$ is $y^5 = r_5$, which in w and z becomes

$$\begin{aligned} w^5 &= x^{10} - 228x^5 + 494 + \frac{228}{x^5} + \frac{1}{x^{10}} \\ &= z^{10} + 10z^8 + 35z^6 - 228z^5 + 50z^4 - 1140z^3 + 25z^2 - 1140z + 496, \end{aligned}$$

Substituting w, z in terms of x, y gives the equation in the table.

For $\mathcal{C}_{66}$, consider the automorphism group isomorphic to D_{10} generated by σ and τ where $\sigma(x, y) = (\zeta_5 x, \zeta_5 y)$ and $\tau(x, y) = (-1/x, y/x^2))$. Here ζ_5 is some primitive 5th root of unity. The fixed subfield is generated by z and w, where $z = x^5 - 1/x^5$ and $w = y^2/x^2$. The affine equation of $\mathcal{C}_{66}$ is $y^{31} = r_5 s_5 t_5$, which in terms of w and z becomes

$$w^{31} = (z^2 - 228z + 496)^2 (z^2 + 4)(z^2 + 522z - 10004)^2 (z + 11)^2$$

as can be checked tediously by hand or more easily with some basic computational assistance. Substituting w and z gives the equation in the table.

□

Proposition 5.11. *None of the curves $\mathcal{C}_2$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_{26}$, $\mathcal{C}_{29}$, $\mathcal{C}_{33}$, $\mathcal{C}_{43}$, or $\mathcal{C}_{66}$ in Table 1 has CM Jacobian.*

Proof. For $i \in \{5, 6, 26, 29, 33, 43, 66\}$, let $\mathcal{C}'_i$ be the quotient curve of $\mathcal{C}_i$ from Lem. 5.10. It suffices to prove that neither $\mathcal{C}_2$ nor any of these $\mathcal{C}'_i$ has CM. For each curve, we use the the program `frobenius_polynomials.sage`[4] to compute the algebras $E'_{f,\mathfrak{p}}$ for various p, and then we show that it is impossible for all the $E'_{f,\mathfrak{p}}$ to embed into a $\mathbb{Q}$-algebra of the correct dimension.

For all cases other than $\mathcal{C}'_{29}$ and $\mathcal{C}'_{66}$, the curve is $\mathcal{C}'_i$ is itself a pre-superelliptic curve, and we can compute the Frobenius minimal polynomial using the superelliptic curve package in Sage (which only works on these types of curves); see [13]. For $\mathcal{C}'_{29}$ and $\mathcal{C}'_{66}$, we instead use the function `ZetaFunction` from Magma, and take the reciprocal polynomial of radical of the numerator. The results are in the following charts.

Table 2: Curve $\mathcal{C}_2$, genus $g_2 := 16$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
7	$\mathbb{Q}(\sqrt{-3}) \times \mathbb{Q}(\sqrt{-5 \cdot 23})$
13	$\mathbb{Q}(\sqrt{-3})$
31	$K_2 \times L_2$
43	$\mathbb{Q}(\sqrt{-3}) \times \mathbb{Q}(\sqrt{-5 \cdot 127})$
67	$\mathbb{Q}(\sqrt{-3}) \times \mathbb{Q}(\sqrt{-5 \cdot 11 \cdot 13})$

Here, $[K_2 : \mathbb{Q}] = [L_2 : \mathbb{Q}] = 8$. Additionally, one verifies that the only quadratic field contained in K_2 is $\mathbb{Q}(\sqrt{5})$ and the only quadratic fields contained in

[4] available at arxiv.org/abs/2006.12685

L_2 are those contained in $\mathbb{Q}(\sqrt{-3}, \sqrt{5})$. Using Lem. 5.9, one sees that a minimum-dimensional product of number fields containing all these $E'_{f,\mathfrak{p}}$ is

$$K_2(\sqrt{-3}) \times L_2 \times \mathbb{Q}(\sqrt{-3}, \sqrt{-5 \cdot 23}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-5 \cdot 127}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-5 \cdot 11 \cdot 13}),$$

which has dimension $16+8+4+4+4 > 32 = 2g_2$. By Prop. 5.7, $\mathcal{C}_2$ does not have CM.

Table 3: Curve $\mathcal{C}'_5$, genus $g_5 := 4$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
7	$\mathbb{Q}(\sqrt{-3})$
17	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-2})$
23	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-33})$
29	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-39})$

Using Lem. 5.9, one sees that a minimum-dimensional product of number fields containing all of these $E'_{f,\mathfrak{p}}$ is

$$\mathbb{Q}(\sqrt{-2}, \sqrt{-3}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-33}) \times \mathbb{Q}(\sqrt{-3}, \sqrt{-39}),$$

which has dimension $12 > 8 = 2g_5$. By Prop. 5.7, $\mathcal{C}'_5$ does not have CM.

Table 4: Curve $\mathcal{C}'_6$, genus $g_6 := 7$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
5	$\mathbb{Q}(\sqrt{-1})$
7	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-6})$
17	$\mathbb{Q}(\sqrt{-1}) \times \mathbb{Q}(\sqrt{-33})$
19	$\mathbb{Q}(\sqrt{-1}) \times \mathbb{Q}(\sqrt{-21})$
29	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-13})$

Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing all of these $E'_{f,\mathfrak{p}}$ is

$$\mathbb{Q}(\sqrt{-1}, \sqrt{-6}) \times \mathbb{Q}(\sqrt{-1}, \sqrt{-13}) \times \mathbb{Q}(\sqrt{-1}, \sqrt{-21}) \times \mathbb{Q}(\sqrt{-1}, \sqrt{-33}),$$

which has dimension $16 > 14 = 2g_6$. By Prop. 5.7, $\mathcal{C}'_6$ does not have CM.

Table 5: Curve $\mathcal{C}'_{26}$, genus $g_{26} := 15$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
5	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-2 \cdot 7 \cdot 11})$
17	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-2 \cdot 5 \cdot 7 \cdot 13})$
29	$\mathbb{Q}(\sqrt{-7}) \times \mathbb{Q}(\zeta_7) \times L_{26}$
37	$\mathbb{Q}(\sqrt{-7}) \times \mathbb{Q}(\sqrt{-7})$
61	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-2 \cdot 193})$
73	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-2 \cdot 7 \cdot 23 \cdot 113 \cdot 211 \cdot 1571})$

Here $[L_{26} : \mathbb{Q}] = 12$ and the only quadratic field contained in L_{26} is $\mathbb{Q}(\sqrt{-7})$. Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing all of these $E'_{f,\mathfrak{p}}$ is

$$\mathbb{Q}(\sqrt{-7}, \sqrt{-2 \cdot 7 \cdot 11}) \times \mathbb{Q}(\sqrt{-7}, \sqrt{-2 \cdot 5 \cdot 7 \cdot 13}) \times \mathbb{Q}(\sqrt{-7}, \sqrt{-2 \cdot 193}) \times$$
$$\times \mathbb{Q}(\sqrt{-7}, \sqrt{-a}) \times \mathbb{Q}(\zeta_7) \times L_{26},$$

where $a = 2 \cdot 7 \cdot 23 \cdot 113 \cdot 211 \cdot 1571$. This has dimension $34 > 30 = 2g_{26}$. By Prop. 5.7, $\mathcal{C}'_{26}$ does not have CM.

Table 6: Curve $\mathcal{C}'_{29}$, genus $g_{29} := 18$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
19	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-3 \cdot 7 \cdot 3847})$
53	$K_{29} \times L_{29}$

Here $[K_{29} : \mathbb{Q}] = 12$ and $[L_{29} : \mathbb{Q}] = 24$, and the only quadratic field contained in K_{29} or L_{29} is $\mathbb{Q}(\sqrt{13})$. Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing $E'_{f,\mathfrak{p}}$ for $p \in \{19, 53\}$ is

$$K_{29}(\sqrt{-3 \cdot 7 \cdot 3847}) \times L_{29},$$

which has dimension $48 > 36 = 2g_{29}$. By Prop. 5.7, $\mathcal{C}'_{29}$ does not have CM.

Table 7: Curve $\mathcal{C}'_{33}$, genus $g_{33} := 16$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
7	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-6})$
13	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-1})$
17	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-5 \times 41})$
37	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-3 \times 47})$
43	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-3 \times 83})$
61	$\mathbb{Q}(\zeta_5)$

Using Lem. 5.9, one shows that a minimum-dimensional product of number fields containing all of these $E'_{f,\mathfrak{p}}$ is

$$\mathbb{Q}(\zeta_5, \sqrt{-6}) \times \mathbb{Q}(\zeta_5, \sqrt{-1}) \times \mathbb{Q}(\zeta_5, \sqrt{-5 \cdot 41}) \times \mathbb{Q}(\zeta_5, \sqrt{-3 \cdot 47}) \times \mathbb{Q}(\zeta_5, \sqrt{-3 \cdot 83}).$$

This has dimension $40 > 32 = 2g_{33}$. By Prop. 5.7, $\mathcal{C}'_{33}$ does not have CM.

Table 8: Curve $\mathcal{C}'_{43}$, genus $g_{43} := 10$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
7	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-10})$
11	$K_{43} \times L_{43}$
13	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-3})$
17	$\mathbb{Q} \times \mathbb{Q}(\sqrt{-2 \cdot 3 \cdot 7 \cdot 19})$

Here, $[K_{43} : \mathbb{Q}] = 12$ and $[L_{43} : \mathbb{Q}] = 4$. Additionally, one verifies that the only quadratic field contained in K_{43} is $\mathbb{Q}(\sqrt{-5})$ and the only quadratic field contained in L_{43} is $\mathbb{Q}(\sqrt{-43 \cdot 1361})$. Using Lem. 5.9, a minimum-dimensional product of number fields containing all these $E'_{f,\mathfrak{p}}$ is

$$K_{43} \times L_{43} \times \mathbb{Q}(\sqrt{-3}) \times \mathbb{Q}(\sqrt{-10}) \times \mathbb{Q}(\sqrt{-2 \cdot 3 \cdot 7 \cdot 19}),$$

which has dimension $22 > 20 = 2g_{43}$. By Prop. 5.7, $\mathcal{C}'_{43}$ does not have CM. This completes the proof.

Table 9: Curve $\mathcal{C}'_{66}$, genus $g_{66} := 90$

$\mathfrak{p}$	$E'_{f,\mathfrak{p}}$
13	$\mathbb{Q} \times L_{13}$
17	$\mathbb{Q} \times L_{17}$
37	$\mathbb{Q} \times L_{37}$
47	$K_{47} \times L_{47}$

Here, $[K_{13} : \mathbb{Q}] = [L_{13} : \mathbb{Q}] = 4$, and $[L_{37} : \mathbb{Q}] = 20$, $[K_{47} : \mathbb{Q}] = 6$, and $[L_{47} : \mathbb{Q}] = 30$. Furthermore, there is an embedding $K_{47} \hookrightarrow L_{47}$, the fields L_{13} and L_{17} are linearly disjoint, and K_{47} is linearly disjoint from L_i for $i \in \{13, 17, 37\}$. All of this is verified in `frobenius_polynomials.sage`. Since $K_{47} \hookrightarrow L_{47}$, Lem. 5.9 shows that any product of number fields containing all these $E'_{f,\mathfrak{p}}$ must have an embedding of K_{47} into each factor. Another application of Lem. 5.9 shows that a minimum-dimensional product of number fields containing all these $E_{f,\mathfrak{p}}$ is

$$K_{47}L_{13} \times K_{47}L_{17} \times K_{47}L_{37} \times L_{47},$$

which has dimension $198 > 180 = 2g_{66}$. By Prop. 5.7, $\mathcal{C}'_{43}$ does not have CM. This completes the proof.

□

Corollary 5.12. *None of the curves $\mathcal{C}_3$, $\mathcal{C}_7$, $\mathcal{C}_8$, or $\mathcal{C}_{45}$ in Table 1 has CM Jacobian.*

Proof. The curve $\mathcal{C}_3$ has $\mathcal{C}_2$ as a quotient. The curves $\mathcal{C}_7$ and $\mathcal{C}_8$ have $\mathcal{C}_5$ as a quotient, and the curve $\mathcal{C}_{45}$ has $\mathcal{C}_{43}$ as a quotient. Since all quotients of a CM curve must be CM curves, Prop. 5.7 shows that none of the curves in the corollary is a CM curve. □

Now we are ready to state the main theorem. Recall that if $\mathcal{C}$ is a smooth superelliptic curve with reduced automorphism group isomorphic to A_4, S_4, and A_5, then $\mathcal{C}$ is isomorphic to one of the curves in Table 1.

Theorem 5.13. *For each case in Table 1, whether or not* $\operatorname{Jac}\mathcal{C}$ *has CM is determined in the 6th column of the Table.*

Proof. This follows from Prop. 4.9, Prop. 5.1, Prop. 5.5, Cor. 5.6, Cor. 5.12, and Prop. 5.11. □

Corollary 5.14. *For superelliptic curves with many automorphisms, having complex multiplication is equivalent to satisfying Streit's Criterion (Lem. 4.4).*

Proof. One direction is immediate from Lem. 4.4. For the other, observe first that for each entry in Table 1 that is a CM curve, the fact that the Jacobian has CM is proven using Streit's criterion in Prop. 4.9. Combining this with Prop. 4.11 finishes the proof. □

Acknowledgments: We would like to thank Andrew Sutherland and Richard Pink for their comments and suggestions when this paper was written. We also thank the referees for their careful readings and many useful comments.

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