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LOCAL AND GLOBAL HEIGHTS ON WEIGHTED PROJECTIVE VARIETIES

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Abstract. We investigate local and global weighted heights a-la Weil for weighted projective spaces via Cartier and Weil divisors and extend the definition of weighted heights on weighted projective spaces from [5] to weighted varieties and closed subvarieties. We prove that any line bundle on a weighted variety admits a locally bounded weighted M-metric. Using this fact, we define local and global weighted heights for weighted varieties in weighted projective spaces and their closed subschemes, and show their fundamental properties.

1. Introduction

Let $\mathfrak{q} = (q_0, \cdots, q_n)$ be a tuple of weights and $\mathbb{P}^n_{\mathfrak{q},k}$ the weighted projective space over a field k. In [5] was introduced a new height on $\mathbb{P}^n_{\mathfrak{q},k}$, called weighted height, and proved that such height satisfies basic properties of projective heights. This definition of weighted heights was motivated not only by its computational advantages, but also because such heights are more natural since they are defined on $\mathbb{P}^n_{\mathfrak{q},k}$ and not on some projective space $\mathbb{P}^n_k$ via the Veronese embedding. Such heights have been used in several computations in the moduli space of curves, rational functions; see [6,8,16] and are a very useful tool in using machine learning techniques in algebraic and arithmetic geometry. However, no complete theory of such heights exists. For example, weighted heights in [5] were not defined analytically via Cartier divisors, local weighted heights via line bundles, global weighted heights for closed subschemes. To the knowledge of authors this has not been done before.

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The goal of this paper is to introduce and develop the theory of weighted heights, inspired by Weil's approach. We achieve this by providing all the necessary tools for understanding and introducing weighted heights, which have not been extensively covered in the literature. To accomplish this, we focus on developing the theory of Cartier divisors on weighted projective varieties, exploring the analytic structure of weighted varieties, investigating weighted blow-ups, and introducing both local and global weighted heights an showing their fundamental properties. In our other work [19], we state some different versions of Vojta's conjecture for weighted varieties in terms of weighted local and global heights, and give an application to the greatest common divisor problem.

This paper is organized as follows. In Sec. 2 we recall some of the basic setup for Weil height machinery on projective spaces and varieties. In Thm. 2.2 we summarize all properties of local Weil heights and in Thm. 2.3 the properties of global Weil heights for such varieties. Such setup will be important later in the paper to draw an analogy between Weil heights and weighted heights.

In Sec. 3 we establish notation for weighted projective varieties and define Zariski topology, Veronese embedding, and singular locus of weighted projective varieties. Moreover, we introduce weighted blow-ups and exceptional divisors on weighted projective varieties.

In Sec. 4 we develop the theory of weighted heights a-la Weil. We introduce Cartier divisors on weighted projective varieties and show that results carry over easily to weighted projective varieties. Moreover, we show that any line bundle on a weighted variety $\mathcal{X}$ admits a locally bounded weighted M-metric. Given $\nu \in M_k$, the **local weighted height** $\zeta_{\widehat{D}}(-, \nu)$ **with respect to** $\widehat{D}$ on weighted variety $\mathcal{X}$ is defined as

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = -\log \|g_D(\mathbf{x})\|_v,$$

for $\mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D)$, where $v \in M$ such that $\nu = v|_k$. Properties of local weighted heights are proved in Thm. 4.4 as they are similar to properties of projective heights. The **global weighted height** $\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x})$ with respect to $\widehat{\mathcal{L}}$ is defined by

$$\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x}) := \sum_{u \in M_K} \zeta_{\widehat{\mathcal{L}_g}}(\mathbf{x}, u),$$

where $\zeta_{\widehat{\mathcal{L}_g}}(\mathbf{x}, u) = -\log \|g(\mathbf{x})\|_u$, and its properties are described in Thm. 4.5. In Sec. 4.5 we introduce weighted local and global heights associated to closed subschemes of weighted projective varieties.

Notation: Since our goal is to provide all the technical details of the theory of weighted heights, in analogy to that of projective heights there is a real possibility of mixing up notation between different heights. Below we give a list of notation of Weil heights and weighted heights. Throughout the paper, the projective space (resp. weighted projective space) over a field k is denoted by $\mathbb{P}^n_k$ (resp. $\mathbb{P}^n_{\mathfrak{q},k}$).

Terminology in projective space	$\mathbb{P}^n_k$	$\mathbb{P}^n_{\mathfrak{q},k}$
multiplicative height over k	H_k	$\mathcal{S}_k$
logarithmic height over k	h_k	$\mathfrak{s}_k$
absolute multiplicative height	H	$\mathcal{S}$
absolute logarithmic height	h	$\mathfrak{s}$
local Weil height with respect to the divisor $\widehat{D}$	$\lambda_{\widehat{D}}(\mathbf{x}, \nu)$	$\zeta_{\widehat{D}}(\mathbf{x}, \nu)$
global Weil height with respect to the line bundle $\widehat{\mathcal{L}}$	$h_{\widehat{\mathcal{L}}}(\mathbf{x})$	$\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x})$
local height associated to exceptional divisor of $\mathcal{Y}$	$\lambda_{\mathcal{Y}}(\mathbf{x}, \nu)$	$\zeta_{\mathcal{Y}}(\mathbf{x}, \nu)$
global height associated to exceptional divisor of $\mathcal{Y}$	$h_{\mathcal{Y}}(\mathbf{x})$	$\mathfrak{s}_{\mathcal{Y}}(\mathbf{x})$
absolute logarithmic height on $\mathcal{X}$ wrt divisor D	$h_{\mathcal{X},D}$	$\mathfrak{s}_{\mathcal{X},D}$
absolute logarithmic local height on $\mathcal{X}$ wrt divisor D	$\lambda_{\mathcal{X},D}$	$\zeta_{\mathcal{X},D}$
Singular locus of $\mathbb{P}^n_{\mathfrak{q},k}$		$\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},k})$

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2. Preliminaries on Weil projective heights

In this section, we review Weil heights on varieties in usual projective spaces. One can find more details on the subjects in [7].

Let k be an algebraic number field of degree $m = [k : \mathbb{Q}]$ and $\bar{k}$ be an algebraically closed field containing k. We denote by $\mathcal{O}_k$ the ring of algebraic integers in k. Let $\mathcal{X}$ be a variety over k, i.e. an integral separated scheme of finite type over $\mathrm{Spec}(k)$ and $\mathcal{O}_{\mathcal{X}}$ the ring sheaf of regular functions on $\mathcal{X}$. We will use $\mathcal{X}$ to mean $\mathcal{X}(\bar{k})$ and $\mathcal{X}(k)$ for the set of k-rational points on $\mathcal{X}$.

Denote by M_k the set of all places of k, i.e. the equivalent classes of absolute values on k. It is a disjoint union of M_k^0, the set of all non-archimedian places, and M_k^∞, the set of all Archimedean places of k. More precisely, if $\nu \in M_k^0$, then $\nu = \nu_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset \mathcal{O}_k$ over a prime number p such that $\nu_{\mathfrak{p}}|_{\mathbb{Q}}$ is the p-adic absolute value. If $\nu \in M_k^\infty$, then $\nu = \nu_\infty$ and $\nu_\infty|_{\mathbb{Q}}$ is the usual absolute value $|\cdot|_\infty$ on $\mathbb{Q}$. The **local degree** n_ν at $\nu \in M_k$ is defined by $n_\nu = [k_\nu : \mathbb{Q}_\nu]$, where k_ν and $\mathbb{Q}_\nu$ are the completions with respect to ν. For each $\nu \in M_k$, we let

$|\cdot|_\nu$ be a representative of the equivalence class which is the n_ν-th power of the one that extends a normalized absolute value over $\mathbb{Q}$. Since k is a number field, then for every $x \in k^*$ we have the **product formula** $\prod_{\nu \in M_k} |x|_\nu = 1$. Given a finite field extension K/k, we denote by M_K the set of places v on K such that $v\,|_k = \nu$, for some $\nu \in M_k$. Then, we have the **degree formula** as

$$\sum_{v \in M_K,\ v|_k = \nu} [K_v : k_\nu] = [K : k].$$

2.1. **Heights.** For $x \in k^*$, the **multiplicative** and **logarithmic height** are defined by

$$H_k(x) = \prod_{\nu \in M_k} \max\{1, |x|_\nu\} \quad \text{and} \quad h_k(x) = \log H_k(x) = \sum_{\nu \in M_k} \log |x|_\nu. \tag{1}$$

For $\tilde{x} = (x_0, \cdots, x_n) \in k^{n+1}$ and $v \in M_k$, we let

$$|\tilde{x}|_\nu := \max\{|x_i|_\nu : 0 \leq i \leq n\}.$$

One extends such definitions to the projective space $\mathbb{P}^n(k)$ by defining the **multiplicative** and **logarithmic height** of $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$ by

$$H_k(\mathbf{x}) = \prod_{\nu \in M_k} \max_{0 \leq i \leq n} \{|x_i|_\nu\}, \quad \text{and} \quad h_k(\mathbf{x}) = \log H_k(\mathbf{x}) = \sum_{\nu \in M_k} \max_{0 \leq i \leq n} \{\log |x_i|_\nu\}. \tag{2}$$

They are independent of the choice of the coordinates and therefore well defined.

For any finite extension K of k and $v \in M_K$, we normalize the absolute value $|\cdot|_v$ such that its restriction $|\cdot|_\nu$ on k satisfies $|\cdot|_\nu = |\cdot|_v^{[K_\nu : k_\nu]}$. Using the degree formula, for $x \in k^*$ we have

$$H_k(x) = H_K(x)^{1/[K:k]}, \quad \text{and} \quad h_k(x) = \frac{1}{[K:k]} h_K(x), \tag{3}$$

and hence for all $\mathbf{x} \in \mathbb{P}^n(k)$,

$$H_k(\mathbf{x}) = H_K(\mathbf{x})^{1/[K:k]}, \quad \text{and} \quad h_k(\mathbf{x}) = \frac{1}{[K:k]} h_K(\mathbf{x}). \tag{4}$$

The **field of definition** of $\mathbf{x} \in \mathbb{P}^n(\bar{k})$ is $k(\mathbf{x}) := k\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$, for any i such that $x_i \neq 0$. The **absolute multiplicative** and **logarithmic global Weil heights** of $x \in \bar{k}^*$ are defined by

$$H(x) = H_K(x)^{1/[K:k]} \quad \text{and} \quad h(x) = \frac{1}{[K:k]} h_K(\mathbf{x}),$$

and for $\mathbf{x} \in \mathbb{P}^n(\bar{k})$ by

$$H(\mathbf{x}) = H_K(\mathbf{x})^{1/[K:k]}, \quad \text{and} \quad h(\mathbf{x}) = \frac{1}{[K:k]} h_K(\mathbf{x}), \tag{5}$$

where K is a number field containing $k(\mathbf{x})$. The absolute height is independent of the choice of K. We call $h(\mathbf{x})$ the **global Weil height** on $\mathbb{P}^n(\bar{k})$.

2.2. **M-bounded sets, functions, and M-metrized line bundles.** Let $M = M_{\bar{k}}$ be the set of places on $\bar{k}$ extending those of M_k, i.e., if $v \in M$ then $\nu = v|_k$ the restriction of v over k belongs to M_k.

A function $\gamma : M_k \to \mathbb{R}$ is called **M_k-constant** if $\gamma(\nu) = 0$ for all but finitely many $\nu \in M_k$. We extend each M_k-constant γ to a function $\gamma : M \to \mathbb{R}$ by setting $\gamma(v) = \gamma(v|_k)$. Given any variety $\mathcal{X}$, by an **M_k-function** on $\mathcal{X}$ we mean a map $\lambda : \mathcal{X} \times M \to \mathbb{R}$ such that $\lambda(\mathbf{x}, v)$ is M_k-constant or $\lambda(\mathbf{x}, v) = \infty$ for all $\mathbf{x} \in \mathcal{X}$ and $v \in M$. Two M_k-functions λ_1 and λ_2 on $\mathcal{X}$ are called equivalent, and denoted by $\lambda_1 \sim \lambda_2$, if there is an M_k-constant function γ such that

$$|\lambda_1(\mathbf{x}, v) - \lambda_2(\mathbf{x}, v)| \leq \gamma(v) \text{ for all } (\mathbf{x}, v) \in \mathcal{X} \times M.$$

We say that an M_k-function λ is **M_k-bounded** if $\lambda \sim 0$.

For an affine variety $\mathcal{X}$, a set $E \subset \mathcal{X} \times M$ is called an **affine M_k-bounded set** if there are coordinate function $x_1, \cdots, x_n$ on $\mathcal{X}$ and an M_k-bounded constant function γ such that

$$|x_i(\mathbf{x})|_v \leq e^{\gamma(v)} \text{ for all } 0 \leq i \leq n, \text{ and } (\mathbf{x}, v) \in E.$$

The set E is bounded by a finite set of absolute values and it is integral with respect to the rest of absolute values. This definition is independent of choice of the coordinates x_i on $\mathcal{X}$. By definition, any finite union of affine M-bounded sets is again an affine M-bounded.

For an arbitrary variety $\mathcal{X}$, we say that $E \subset \mathcal{X} \times M$ is a **M_k-bounded set** if there exists a finite cover $\{U_i\}$ of affine open subsets of $\mathcal{X}$ and M_k-bounded sets $E_i \subset U_i \times M$ such that $E = \bigcup E_i$.

A function $\lambda : \mathcal{X} \times M \to \mathbb{R}$ is called **locally M_k-bounded above** if for every M_k bounded subset $E \subset \mathcal{X} \times M$, there exists an M_k-constant γ such that $\lambda(\mathbf{x}, v) \leq \gamma(v)$ holds for $(\mathbf{x}, v) \in E$. The **locally M_k-bounded below** and **locally M_k-bounded** functions are defined similarly.

Recall that a line bundle $\mathcal{L}$ on a variety $\mathcal{X}$ defined over k, is a covering map $\pi : \mathcal{L} \to \mathcal{X}$ such that for each $\mathbf{x} \in \mathcal{X}$, the fiber $\mathcal{L}_{\mathbf{x}} := \pi^{-1}(\mathbf{x})$ is a 1-dimensional vector space over k. An **M-metric** on a line bundle $\mathcal{L}$ is a norm $\| \cdot \| = (\| \cdot \|_v)$

such that for each $v \in M$, and each fiber $\mathcal{L}_{\mathbf{x}}$ assigns a function

$$\|\cdot\|_v : \mathcal{L}_{\mathbf{x}} \to \mathbb{R}_{\geq 0},$$

which is not identically zero and satisfies:

(i) $\|\lambda \cdot \xi\|_v = |\lambda|_v \cdot \|\xi\|_v$ for $\lambda \in \bar{k}$ and $\xi \in \mathcal{L}_{\mathbf{x}}$.
(ii) If $v_1, v_2 \in M$ agree on $k(\mathbf{x})$, then $\|\cdot\|_{v_1} = \|\cdot\|_{v_2}$ on $\mathcal{L}_{\mathbf{x}}(k(\mathbf{x}))$.

An M-metric $\|\cdot\| = (\|\cdot\|_v)$ on $\mathcal{L}$ is called **locally M-bounded** if for any regular function $g \in \mathcal{O}_{\mathcal{X}}(U)$ on an open set $U \subseteq \mathcal{X}$, the function $(\mathbf{x}, v) \mapsto \log \|g(\mathbf{x})\|_v$ on $U \times M$ is locally M_k-bounded.

We say that $\mathcal{L}$ is an **M-metrized line bundle** on $\mathcal{X}$ if $\mathcal{L}$ is equipped with an M-metric. The following result shows that there exist an M-metric on any line bundle on a variety in projective spaces; see [7, Prop. 2.7.5].

Lemma 2.1. *Any line bundle $\mathcal{L}$ on a variety $\mathcal{X} \subseteq \mathbb{P}^n_{\bar{k}}$ defined over k admits a locally bounded M-metric $\|\cdot\|$.*

Denote by $\widehat{\mathcal{L}}$ the pair $(\mathcal{L}, \|\cdot\|)$. Given two pairs $\widehat{\mathcal{L}_1} = (\mathcal{L}_1, \|\cdot\|_1)$ and $\widehat{\mathcal{L}_2} = (\mathcal{L}_2, \|\cdot\|_2)$, we define $\widehat{\mathcal{L}}_1 \otimes \widehat{\mathcal{L}}_2 := (\mathcal{L}_1 \otimes \mathcal{L}_2, \|\cdot\|)$, where

$$\|f \otimes g\| = \|f\|_1 \cdot \|g\|_2, \text{ for } f \in \mathcal{L}_{1,\mathbf{x}},\ g \in \mathcal{L}_{2,\mathbf{x}}, \quad \text{and } \mathbf{x} \in \mathcal{X}.$$

We say that $\widehat{\mathcal{L}_1}$ and $\widehat{\mathcal{L}_2}$ are **isometric** if there is an isomorphism between $\mathcal{L}_1$ and $\mathcal{L}_2$ which is fiber-wise an isometry.

Let $\widehat{\operatorname{Pic}(\mathcal{X})}$ denote the group of the isometric classes of pairs $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ where $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$. Then, the identity element of $\widehat{\operatorname{Pic}(\mathcal{X})}$ is $\mathcal{O}_{\mathcal{X}}$ with trivial metric $\|1\|_v = |1|_v$ and $\widehat{\mathcal{L}}^{-1} = (\mathcal{L}^{-1}, 1/\|\cdot\|)$ is the inverse of $\widehat{\mathcal{L}} \in \widehat{\operatorname{Pic}(\mathcal{X})}$. Given any morphisms $\phi : \mathcal{X}' \to \mathcal{X}$ of varieties over k, and $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\operatorname{Pic}(\mathcal{X})}$, the **pull-back** of $\widehat{\mathcal{L}}$ by ϕ is defined as $\widehat{\phi^*(\mathcal{L})} := (\phi^*(\mathcal{L}), (\|\cdot\|'_v))$, such that for $\mathbf{x} \in \mathcal{X}'$, any open subset U of $\mathcal{X}$ containing $\phi(\mathbf{x})$, and $g \in \mathcal{O}_{\mathcal{X}}(U)$ we have

$$\|\phi^*(g)(\mathbf{x})\|'_v = \|g(\phi(\mathbf{x}))\|_v.$$

The pull-back induces a group homomorphism between $\widehat{\operatorname{Pic}(\mathcal{X})}$ and $\widehat{\operatorname{Pic}(\mathcal{X}')}$. Under this homomorphism, any locally bounded M-metrized line bundles remain locally bounded.

2.3. **Local Weil heights.** We assume that the reader is familiar with Cartier divisors for varieties in projective spaces. Given any effective Cartier divisor $D = \{(U_i, f_i)\}$ on $\mathcal{X}$, let $\mathcal{L}_D = \mathcal{O}_{\mathcal{X}}(D)$ be the line bundle of regular functions on D. It can be constructed by gluing $\mathcal{O}_{\mathcal{X}}(D)|_{U_i} = f_i^{-1}\mathcal{O}_{\mathcal{X}}(U_i)$ and the constant section 1 becomes a canonical invertible regular section on $\mathcal{L}_D$, which we denote it by g_D. We equip $\mathcal{L}_D$ with a locally bounded M-metric $\|\cdot\|$, which is possible by Thm. 2.1, and denote it by $\widehat{D} = (\mathcal{L}_D, \|\cdot\|)$. Given $\nu \in M_k$, the **local Weil height** $\lambda_{\widehat{D}}(\cdot, \nu)$ with respect to $\widehat{D}$ on $\mathcal{X}$ is defined to be

$$\lambda_{\widehat{D}}(\mathbf{x}, \nu) = -\log \|g_D(\mathbf{x})\|_v, \text{ for } \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D), \tag{6}$$

where $v \in M$ such that $v|_k = \nu$.

The following lemma provides a summary of all properties of local heights, which can be found in [7, Prop. 2.7.10 and 2.7.11] or [15, Chap. 10].

Lemma 2.2 (Local Weil heights)**.** *For each of $\nu \in M_k$, let $v \in M$ such that $\nu = v|_k$. Let $\mathcal{X} \subseteq \mathbb{P}^n_k$ be a variety defined over k, and $\widehat{D}, \widehat{D}_1, \widehat{D}_2 \in \widehat{\operatorname{Pic}(\mathcal{X})}$. Then, we have:*

(i) *For $\mathbf{x} \notin \operatorname{Supp}(D_1) \cup \operatorname{Supp}(D_2)$, we have*

$$\lambda_{\widehat{D_1+D_2}}(\mathbf{x}, \nu) = \lambda_{\widehat{D}_1}(\mathbf{x}, \nu) + \lambda_{\widehat{D}_2}(\mathbf{x}, \nu).$$

(ii) *If $\phi : \mathcal{X}' \to \mathcal{X}$ is a morphism over k such that $\phi(\mathcal{X}') \cap \operatorname{Supp}(D) = \emptyset$, then*

$$\lambda_{\phi^*(\widehat{D})}(\mathbf{x}', \nu) = \lambda_{\widehat{D}}(\phi(\mathbf{x}'), \nu), \quad \text{for } \mathbf{x}' \in \mathcal{X}' \setminus \phi^{-1}(\operatorname{Supp}(D)).$$

(iii) *If D is effective and $\mathcal{X}$ is M_k-bounded (e.g $\mathcal{X}$ is projective), then there exists an M_k-constant function γ such that $\lambda_{\widehat{D}}(\mathbf{x}, \nu) \geq \gamma(\nu)$, for $\mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D)$.*

(iv) *If $D = div(f)$ for some nonzero rational function on $\mathcal{X}$, then*

$$\lambda_{\widehat{D}}(\mathbf{x}, \nu) = -\log \frac{|f(\mathbf{x})|_v}{|\mathbf{x}|_v}, \text{ for } \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D),$$

by giving the trivial metric $\|1\|_v = |1|_v$ on $\mathcal{O}_{\mathcal{X}}(D) \cong \mathcal{O}_{\mathcal{X}}$.

(v) *If $\mathcal{X}$ is M_k-bounded, $\|\cdot\|'$ is another M_k-bounded metric on $\mathcal{O}_{\mathcal{X}}(D)$ and $\lambda'_{\widehat{D}}$ is the resulting local Weil height, then $\lambda_{\widehat{D}} = \lambda'_{\widehat{D}} + O(1)$.*

(vi) *If $K|k$ is a finite field extension and $u \in M_K$ over some $\nu \in M_k$, then*

$$\lambda_{\widehat{D}}(\mathbf{x}, \nu) = \frac{1}{[K:k]} \lambda_{\widehat{D}}(\mathbf{x}, u), \text{ for } \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D).$$

(vii) *There are $m, n \in \mathbb{Z}^{\geq 0}$, and nonzero rational functions $f_{i,j}$ on $\mathcal{X}$ for $i = 0, \cdots, n_1$, $j = 0, \cdots, n_2$ such that*

$$\lambda_{\widehat{D}}(\mathbf{x}, \nu) = \max_{0 \leq i \leq n_1} \min_{0 \leq j \leq n_2} \log |f_{ij}(\mathbf{x})|_\nu .$$

2.4. **Global Weil heights.** Let $\mathcal{X} \subset \mathbb{P}^n_{\bar{k}}$ be a variety defined over k and $\mathcal{L}$ any line bundle on $\mathcal{X}$. Consider the pair $\widehat{\mathcal{L}} = (\mathcal{L}, (\|\cdot\|_v)) \in \widehat{\mathrm{Pic}(\mathcal{X})}$, a given $\mathbf{x} \in \mathcal{X}$, and K a finite extension of k containing $k(\mathbf{x})$. For each $u \in M_K$, we choose a place $v \in M$ over u and define

$$\|\cdot\|_u := \|\cdot\|_v^{1/[K:k]}$$

on $\mathcal{L}_{\mathbf{x}}(k(\mathbf{x}))$. By the second condition of a M-metric, one can see that it is independent of the choice of $v \in M$. We let g be an invertible rational function of $\mathcal{L}$ with $\mathbf{x} \notin \mathrm{Supp}(D_g)$ where $D_g = \mathrm{div}(g)$. Note that such function exists because there is an open dense trivialization in a neighborhood of $\mathbf{x}$. Then, $\mathcal{O}_{\mathcal{X}}(D_g)$ is a locally M_K-bounded with respect to M_K-metric given above. We denote by $\widehat{\mathcal{L}_g} := (\mathcal{O}_{\mathcal{X}}(D_g), (\|\cdot\|_u))$. The **global Weil height** $h_{\widehat{\mathcal{L}}}(\mathbf{x})$ of $\mathbf{x} \in \mathcal{X}$ with respect to $\widehat{\mathcal{L}}$ is defined by

$$h_{\widehat{\mathcal{L}}}(\mathbf{x}) := \sum_{u \in M_K} \lambda_{\widehat{\mathcal{L}_g}}(\mathbf{x}, u), \tag{7}$$

where we have $\lambda_{\widehat{\mathcal{L}_g}}(\mathbf{x}, u) = -\log \|g(\mathbf{x})\|_u$, assuming $v|_k = u$. These definitions are independent of the choice of K and g. For the following see [7, Prop. 2.7.18].

Lemma 2.3 (Global Weil height machinery). *Let $\mathcal{X}$ be a variety and $\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_1$, and $\widehat{\mathcal{L}}_2 \in \widehat{\mathrm{Pic}(\mathcal{X})}$. Then:*

(i) *$h_{\widehat{\mathcal{L}}}$ depends only on the isometry class of $\widehat{\mathcal{L}}$, i.e, if $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ are isometric pairs, then $h_{\widehat{\mathcal{L}}_1} = h_{\widehat{\mathcal{L}}_2}$.*
(ii) *If $\mathcal{X}$ is a complete variety or generally M-bounded, then $h_{\widehat{\mathcal{L}}}$ does not depends on the choice of the locally bounded M-metrics up to a locally M-bounded constant function.*
(iii) *For any $\mathbf{x} \in \mathcal{X}$, we have $h_{\widehat{\mathcal{L}}_1 \otimes \widehat{\mathcal{L}}_2}(\mathbf{x}) = h_{\widehat{\mathcal{L}}_1}(\mathbf{x}) + h_{\widehat{\mathcal{L}}_2}(\mathbf{x})$.*
(iv) *If $\phi : \mathcal{X}' \to \mathcal{X}$ is a morphism over k, then $h_{\phi^*(\widehat{\mathcal{L}})}(\mathbf{x}) = h_{\widehat{\mathcal{L}}}(\phi(\mathbf{x}))$, for $\mathbf{x} \in \mathcal{X}'$.*
(v) *If $\mathcal{X} = \mathbb{P}^n_{\bar{k}}$ and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)$, then $h(\mathbf{x}) = h_{\widehat{\mathcal{L}}}(\mathbf{x}) + O(1)$.*

3. Weighted projective varieties

Let k be a field and for any integer $n \geq 1$ denote by $\mathbb{A}^n_k$ (resp. $\mathbb{P}^n_k$) the affine (resp. projective) space over k. When k is an algebraically closed field, we will drop the subscript. For any integer $\ell \geq 1$, let μ_ℓ denote the group of ℓ-th roots of unity generated by ξ_m, which is assumed to be contained in k.

A fixed tuple of positive integers $\mathfrak{q} = (q_0, \dots, q_n)$ is called **weights**. Let $\mathbb{V}^n_k := \mathbb{A}^n_k \setminus \{(0, \cdots, 0)\}$ and consider the action of $k^* = k \setminus \{0\}$ on $\mathbb{V}^{n+1}_k$ given by

$$\lambda \star (x_0, \dots, x_n) = (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n), \text{ for } \lambda \in k^*. \tag{8}$$

Define the **weighted projective space** $\mathbb{P}^n_{\mathfrak{q},k}$ to be the quotient space $\mathbb{V}^{n+1}_k/k^*$ of this action, which is a geometric quotient since k^* is a reductive group. An element $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k}$ is denoted by $\mathbf{x} = [x_0 : \cdots : x_n]$ and its i-th coordinate by $x_i(\mathbf{x})$. For each $i = 0, \ldots, n$, we define affine pieces of $\mathbb{P}^n_{\mathfrak{q},k}$ by

$$U_i = \{\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k} : \; x_i(\mathbf{x}) \neq 0\}.$$

Hence, $\mathbb{P}^n_{\mathfrak{q},k} = \cup_{i=0}^n U_i$. We assume that the field k contains a q_i-th root of unity ξ_{q_i} for every $i = 0, \cdots, n$. Then, for each $i = 0, \ldots, n$, the affine piece U_i is isomorphic to $\mathbb{V}^n_k/\mu_{q_i}$, the quotient space of the action of μ_{q_i} on $\mathbb{V}^n_k$ with coordinates $z_0, \cdots, \hat{z}_i, \cdots, z_n$, given by

$$\xi_i \cdot (z_0, \cdots, \hat{z}_i, \cdots, z_n) \mapsto (\xi_i^{q_0} z_0, \cdots, \hat{z}_i, \cdots, \xi_i^{q_n} z_n). \tag{9}$$

Here, for all $0 \leq j \neq i \leq n$, we have $z_j = \frac{x_j}{x_i^{q_j/q_i}}$, which is similar to the case of usual projective space $\mathbb{P}^n_k$.

Weighted projective space can also be defined as a finite quotient of usual projective space. For weights $\mathfrak{q} = (q_0, \ldots, q_n)$, we let $G_{\mathfrak{q}} := \mu_{q_0} \times \cdots \times \mu_{q_n}$, which is a finite group of order $|G_{\mathfrak{q}}| = q$ with $q := \prod_{i=0}^n q_i$. Then, there is an action of $G_{\mathfrak{q}}$ on $\mathbb{P}^n_k$ given by

$$(\xi_0, \cdots, \xi_n) \bullet [x_0 : \cdots : x_n] = [\xi_0 x_0 : \cdots : \xi_n x_n]\,. \tag{10}$$

Note that $G_{\mathfrak{q}} \cong \mu_m$ if and only if $m = \operatorname{lcm}(q_0, \ldots, q_n)$, that is, all of q_i's are pairwise coprime. In this case, action of G_m on $\mathbb{P}^n_k$ can be expressed as

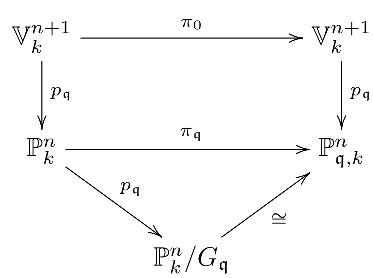
$$\xi^\alpha \cdot [x_0 : \cdots : x_n] = \left[\xi^{\alpha/q_0} x_0 : \cdots : \xi^{\alpha/q_n} x_n\right], \tag{11}$$

for $0 \leq \alpha \leq m-1$, where $\xi \in G_m$ is a m-th root of unity. The morphism $\pi_0 : \mathbb{V}^{n+1}_k \longrightarrow \mathbb{V}^{n+1}_k$ given by

$$(x_0, \cdots, x_n) \mapsto (x_0^{q_0}, \ldots, x_n^{q_n})$$

induces the following diagram

(12)



where $p_{\mathfrak{q}}$ is the canonical quotient map and $\pi_{\mathfrak{q}} : \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_{\mathfrak{q},k}$ is given by

$$[x_0 : \cdots : x_n] \mapsto [x_0^{q_0} : \cdots : x_n^{q_n}] .$$

The morphism $\pi_{\mathfrak{q}}$ is surjective, finite, and its fibers are orbits of the action of $G_{\mathfrak{q}}$ on $\mathbb{P}^n_k$, see [12, Chap. V, Props. 1.3 and 1.8].

$\mathbb{P}^n_{\mathfrak{q},k}(k)$ will denote the set of k-rational points of $\mathbb{P}^n_{\mathfrak{q},k}$. When k is algebraically closed and there is no room for confusion sometimes $\mathbb{P}^n_{\mathfrak{q}}$ is used instead of $\mathbb{P}^n_{\mathfrak{q},k}$.

3.1. **Zariski topology on weighted projective spaces.** Consider the ring of polynomials $k[x_0, \dots, x_n]$ and assign to every variable x_i the weight $\mathrm{wt}(x_i) = q_i$, for all $i = 0, \dots, n$. Every polynomial is a sum of monomials $x^d = \prod x_i^{d_i}$ with $\mathrm{wt}(x^d) = \sum d_i q_i$.

Let $f \in k[x_0, \dots, x_n]$, where $\mathrm{wt}(x_i) = q_i$, for $i = 0, \dots, n$. Then, f is called a **weighted homogeneous**[1] **polynomial of degree** d if each monomial in f is weighted of degree d, i.e.

$$f(x_0, \dots, x_n) = \sum_{i=1}^{t} a_i \prod_{j=0}^{n} x_j^{d_j}, \quad \text{for} \quad a_i \in k \quad \text{and} \quad t \in \mathbb{N}$$

and for all $0 \leq i \leq n$, we have that $\sum_{i=1}^n q_i d_j = d$. For every $\lambda \in k^*$ and any weighted homogeneous polynomial f, we have

$$f(\lambda^{q_0} x_0, \lambda^{q_1} x_1, \dots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \dots, x_n),$$

We denote by $k_{\mathfrak{q}}[x_0, \dots, x_n]$ the **set of weighted homogeneous polynomials** over k. It is a subring of $k[x_0, \dots, x_n]$ and therefore a Noetherian ring. By $k_{\mathfrak{q}}[x_0, \dots, x_n]_d$ we mean the **additive group of all weighted homogeneous polynomials of degree** d.

Let $\alpha = [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}^n_{\mathfrak{q},k}$ and $f \in k_{\mathfrak{q}}[x_0, \dots, x_n]_d$. Then, for any $\lambda \in k^*$, we have $\alpha = [\lambda^{q_0} \alpha_0 : \cdots : \lambda^{q_n} \alpha_n]$. Since

$$f\left(\lambda^{q_0} \alpha_0, \dots, \lambda^{q_n} \alpha_n\right) = \lambda^d f(\alpha_0, \dots, \alpha_n) = 0,$$

then α being a zero of f is well-defined for all $\alpha \in \mathbb{P}^n_{\mathfrak{q},k}$.

A **weighted hyperplane** in $\mathbb{P}_{\mathfrak{q},k}$ is a weighted homogeneous polynomial of degree m. Hence, it is the set of points $\mathbf{x} = [x_0 : \dots : x_n] \in \mathbb{P}_{\mathfrak{q},k}$ satisfying a polynomial of the form

$$\ell(\mathbf{x}) = a_0 x_0^{m/q_0} + a_1 x_1^{m/q_1} + \cdots + a_n x_n^{m/q_n} = \sum_{i=0}^{n} a_i x_i^{\frac{m}{q_i}} \tag{13}$$

---

[1]In some papers on weighted projective spaces, a weighted homogeneous polynomial is also called **quasihomogeneous** polynomial.

Notice that if $\mathfrak{q} = (1, \dots, 1)$ all definitions agree with those of $\mathbb{P}^n$.

An ideal $I \subset k_{\mathfrak{q}}[x_0, \dots, x_n]$ is called a **weighted homogeneous ideal** if every element of $f \in I$ can be written as $f = \sum_{i=0}^{d} f_i$ where $f_i \in k_{\mathfrak{q}}[x_0, \dots, x_n]_i \cap I$ with $\deg(f_i) = i$. The **sum** of two weighted homogeneous ideals I and J, is denoted by $I + J$ and is defined to be

$$I + J = \{f + g | f \in I, g \in J.\}$$

If I and J are weighted homogeneous ideals in $k_{\mathfrak{q}}[x_0, \dots, x_n]$, then $I + J$ is also an weighted homogeneous ideal in $k_{\mathfrak{q}}[x_0, \dots, x_n]$. The **product** of two weighted homogeneous ideals I and J is denoted by IJ and is defined to be the ideal

$$IJ = \langle \{fg \mid f \in I, g \in J\} \rangle.$$

For any given weighted homogeneous ideal I, we define **weighted projective variety of** I by

$$V(I) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k} \;\middle|\; f(\mathbf{x}) = 0 \ \text{ for all } \ f \in I \right\} \tag{14}$$

Let I and J be weighted homogeneous ideals. Then the following hold:
(i) $V(I) \cap V(J) = V(I + J)$
(ii) $V(I) \cup V(J) = V(IJ)$
(iii) $\mathbb{P}^n_{\mathfrak{q},k} = V(0)$

Conversely, given any $V \subset \mathbb{P}^n_{\mathfrak{q},k}$ the **weighted homogeneous ideal associated to** V is given by

$$I(V) = \left\{ f \in k_{\mathfrak{q}}[x_0, \dots, x_n] \;\middle|\; f(\mathbf{x}) = 0 \ \text{ for all } \ \mathbf{x} \in V \right\}$$

A weighted homogeneous ideal I is called a **radical weighted homogeneous ideal** if $f \in k_{\mathfrak{q}}[x_0, \dots, x_n]$ such that $f^r \in I$ for an integer $r \geq 1$ then $f \in I$.

Lemma 3.1. *Let $V \subset \mathbb{P}^n_{\mathfrak{q},k}$ be a weighted projective variety. Then, weighted homogeneous ideal $I(V)$ associated to V is a radical weighted homogeneous ideal.*

Proof. Let f and g be two polynomials in $I(V)$. Then, $f(P) = g(P) = 0$ for all points $P \in V$, i.e. they both vanish at all points P in the variety V then so does $f + g$ and fh where h is any polynomial in $I(V)$. Therefore, $I(V)$ is a weighted homogeneous ideal.

Since, $k_{\mathfrak{q}}[x_0, \dots, x_n]$ is Noetherian, then $I(V)$ is finitely generated, say $I(V) = \langle f_1, \dots, f_n \rangle$. However, $f_i \in k_{\mathfrak{q}}[x_0, \dots, x_n]$ for all i and therefore every f_i is weighted homogeneous polynomial. Hence $I(V)$ is weighted homogeneous ideal since it is generated by finitely many weighted homogeneous polynomials.

Finally let us prove that $I(V)$ is radical. Let $f^r \in I(V)$. Then, for all points $P \in V$ we have that $f^r(P) = 0$. But since $f \in k_{\mathfrak{q}}[x_0, \dots, x_n]$, which is an integral domain, then $f^r(P) = (f(P))^r = 0$ implies that $f(P) = 0$ for all $P \in V$. Therefore, $I(V)$ is radical. This completes the proof. $\square$

For weighted projective varieties V and W then we say that V is a **weighted subvariety** of W if $V \subset W$. It can be shown that any finite union of weighted projective varieties is a weighted projective variety. Furthermore, an arbitrary intersection of weighted projective varieties is a weighted projective variety. A weighted projective variety is said to be **irreducible** if it has no non-trivial decomposition into subvarieties. We notice that any weighted projective varieties are projective varieties too. Hence, we can define the Zariski topology for weighted projective varieties. **Zariski topology** on a weighted projective space $\mathbb{P}^n_{\mathfrak{q},k}$ is given by defining closed sets of $\mathbb{P}^n_{\mathfrak{q},k}$ to be those of the form $V(I)$ for some weighted homogeneous ideal $I \subset k_{\mathfrak{q}}[x_0, \dots, x_n]$.

Definition 1. ***Zariski closure*** *of a subset S of a weighted projective space $\mathbb{P}^n_{\mathfrak{q},k}$ is the smallest weighted projective variety that contains S.*

Remark 1. *Let $S \subset \mathbb{P}^n_{\mathfrak{q},k}$. Then, $V(I(S))$ is the Zariski closure of S. The proof is similar to the case of projective varieties.*

Example 1. *Let $\mathfrak{q} = (q_0, q_1, q_2)$ and $f \in k_{\mathfrak{q}}[x, y, z]_d$. Then, $V(f) \subset \mathbb{P}^2_{\mathfrak{q},k}$ is a degree d-plane curve in $\mathbb{P}^2_{\mathfrak{q},k}$.*

The following gives the third equivalent definition of weighted projective space in language of schemes, see [11, Subsection 1.2.2] or [4, Theorem 3A.1].

Proposition 3.2. *$\mathbb{P}^n_{\mathfrak{q},k}$ is isomorphic to* $\operatorname{Proj}(k_{\mathfrak{q}}[x_0, \dots, x_n])$.

For the rest of this paper, by a **weighted variety** we mean an integral, separated subscheme of finite type in $\operatorname{Proj}(k_{\mathfrak{q}}[x_0, \cdots, x_n])$. In other words, $\mathcal{X} \subseteq \mathbb{P}^n_{\mathfrak{q},k}$ is a weighetd variety if there are $f_1, \cdots, f_t \in k_{\mathfrak{q}}[x_0, \dots, x_n]$ such that $\mathcal{X}$ is isomorphic to the k-scheme $\operatorname{Proj}\left(\frac{k_{\mathfrak{q}}[x_0, \dots, x_n]}{\langle f_1, \cdots, f_t\rangle}\right)$.

A weighted space $\mathbb{P}^n_{\mathfrak{q},k}$ is called **reduced** if $\gcd(q_0, \cdots, q_n) = 1$. It is called **normalized** or **well-formed** if

$$\gcd(q_0, \dots, \hat{q}_i, \dots, q_n) = 1, \quad \text{for each } i = 0, \dots, n.$$

3.2. **Veronese map.** Let R be a graded ring and $d \geq 1$ be an integer. Its **d-th truncated ring** is the subring $R^{[d]} \subseteq R$ defined by

$$R^{[d]} := \bigoplus_{d|n} R_n = \bigoplus_{i \geq 0} R_{di}.$$

Clearly we have the embedding $R^{[d]} \hookrightarrow R$, which is called the d**-th Veronese embedding**, implying that $\mathrm{Proj}(R^{[d]}) \cong \mathrm{Proj}(R)$ by [13, Prop. 2.4.7]. Moreover, the sheaf $\mathcal{O}(1)$ on $\mathrm{Proj}(R^{[d]})$ corresponds via the isomorphism to $\mathcal{O}(d)$ on $\mathrm{Proj}(R)$.

Proposition 3.3. *Given any tuple of weights* $\mathfrak{q} = (q_0, \ldots, q_n)$*, the following hold:*

(i) *Any weighted projective space* $\mathbb{P}^n_{\mathfrak{q},k}$ *is isomorphic to* $\mathbb{P}^n_{\mathfrak{q}',k}$*, where* $\mathfrak{q}'$ *is a reduced tuple of weights.*

(ii) *If* $\mathbb{P}^n_{\mathfrak{q},k}$ *is reduced and* $d_i = \gcd(q_0, \cdots, \hat{q}_i, \cdots, q_n)$ *for* $0 \le i \le n$*, then* $\mathbb{P}^n_{\mathfrak{q},k} \cong \mathbb{P}^n_{\mathfrak{q}',k}$ *with* $\mathfrak{q}' = \left(\frac{q_0}{d_i}, \ldots, \frac{q_{i-1}}{d_i}, q_i, \frac{q_{i+1}}{d_i}, \ldots, \frac{q_n}{d_i}\right)$.

(iii) *Any* $\mathbb{P}^n_{\mathfrak{q},k}$ *is isomorphic to a reduced and well-formed one.*

(iv) *If* $\mathfrak{q}$ *is reduced and all of* m/q_i *are co-prime, where* $m = \mathrm{lcm}\,(q_0, \cdots, q_i)$*, then* $\mathbb{P}^n_{\mathfrak{q},k}$ *is isomorphic to* $\mathbb{P}^n_k$ *by* $\phi_m : \mathbb{P}^n_{\mathfrak{q},k} \longrightarrow \mathbb{P}^n_k$ *defined as*

$$\phi_m([x_0, \ldots, x_n]) = [x_0^{m/q_0}, x_1^{m/q_1}, \ldots, x_n^{m/q_n}]. \tag{15}$$

Proof. Let $d = \gcd(q_0, \ldots, q_n)$, $R = k_{\mathfrak{q}}[x_0, \ldots, x_n]$, and $R^{[d]}$ be the d-th truncated subring of R. Then, $R^{[d]} = k_{\mathfrak{q}}[x_0^d, \ldots, x_n^d]$ and by Thm. 3.2 we have

$$\mathbb{P}^n_{\mathfrak{q},k} = \mathrm{Proj}(R) \cong \mathrm{Proj}(R^{[d]}) = \mathbb{P}^n_{\mathfrak{q}',k}, \text{ with } \mathfrak{q}' = \left(\frac{q_0}{d}, \ldots, \frac{q_n}{d}\right),$$

under the isomorphism

$$[x_0 : \cdots : x_n] \to [y_0 : \cdots : y_n] := [x_0^d : x_1^d : \cdots : x_n^d]. \tag{16}$$

This shows that $\mathbb{P}^n_{\mathfrak{q},k}$ is isomorphic to a reduced weighted projective space $\mathbb{P}^n_{\mathfrak{q}',k}$, i.e., with $\mathfrak{q}' = (q_0', \cdots, q_n')$ such that $\gcd(q_0', \cdots, q_n') = 1$. This completes the proof of part (i).

Now, we assume that $\gcd(q_0, \ldots, q_n) = 1$ and let $d_i = \gcd(q_0, \cdots, \hat{q}_i, \cdots, q_n)$, for $0 \le i \le n$. Then, $\gcd(d_i, q_j) = 1$, for all $0 \le i \ne j \le n$. If $x_0^{p_0} \cdots x_n^{p_n}$ is a monomial of degree pd_i for an integer $p \ge 1$, then

$$p_0 q_0 + \cdots + p_n q_n = p d_i,$$

and so d_i divides $p_i q_i$, and hence $d_i | p_i$. This implies that x_i only appears in $R^{[d_i]}$ as $x_i^{d_i}$. Thus, we have $R^{[d_i]} = k[x_0, \ldots, x_{i-1}, x_i^{d_i}, x_{i+1}, \ldots, x_n]$ and hence

$$\mathbb{P}^n_{\mathfrak{q},k} = \mathrm{Proj}(R) \cong \mathrm{Proj}(R^{[d_i]}) = \mathbb{P}^n_{\mathfrak{q}',k}, \tag{17}$$

with $\mathfrak{q}' = \left(\frac{q_0}{d_i}, \ldots, \frac{q_{i-1}}{d_i}, q_i, \frac{q_{i+1}}{d_i}, \ldots, \frac{q_n}{d_i}\right)$ under the isomorphism

$$[x_0 : \cdots : x_n] \to [y_0 : \cdots : y_n] := [x_0 : \cdots : x_i^{d_i} : \cdots : x_n],$$

see [5, Prop. 3] for more details. Thus, the part (ii) is proved.

One can conclude part (iii) by repeatedly using (ii). Indeed, by defining

$$d_i = \gcd(q_0, \cdots, \hat{q}_i, \cdots, q_n), \; a_i = \operatorname{lcm}(d_0, \cdots, \hat{d}_i, \cdots, d_n), \; a = \operatorname{lcm}(d_0, \cdots, d_n),$$

for all $0 \leq i \leq n$, one can easily check the following:

(1) $a_i | q_i$, $\gcd(a_i, d_i) = 1$ and $a_i d_i = a$ for $0 \leq i \leq n$;
(2) $\gcd(d_j, d_i) = 1$, and $d_j | q_i$, for $0 \leq i \neq j \leq n$.

Then, denoting by $R^{[\mathbf{d}]} := k_{\mathfrak{q}}[x_0^{d_0}, \cdots, x_n^{d_n}]$, we have

$$\mathbb{P}^n_{\mathfrak{q},k} = \operatorname{Proj}(R) \cong \operatorname{Proj}(R^{[\mathbf{d}]}) = \mathbb{P}^n_{\mathfrak{q}',k} \text{ with } \mathfrak{q}' = (q_0', \dots, q_n').$$

where $q_i' = q_i / a_i$ for all $0 \leq i \leq n$, under the morphism

$$[x_0 : \cdots : x_n] \to [y_0 : \cdots : y_n] := [x_0^{d_0} : \cdots : x_n^{d_n}]. \tag{18}$$

Since $\gcd(q_0', \cdots, \hat{q'}_i, \cdots, q_n') = 1$ for all $0 \leq i \leq n$, then $\mathbb{P}^n_{\mathfrak{q}',k}$ is a well-formed weighted projective space; see [2, Prop. 2.3] for more details. This completes the proof of part (iii).

If $a_i = q_i$ for all $0 \leq i \leq n$ in the above discussion, then $\mathbb{P}^n_{\mathfrak{q},k} \cong \mathbb{P}^n_k$. This holds if m/q_i are all co-primes, where $m = \operatorname{lcm}(q_0, \cdots, q_n)$ The isomorphism is given by Eq. (15). ☐

We call the isomorphism ϕ_m given in Eq. (15) the **Veronese map**.

Example 2 (The space $\mathcal{M}_2$)**.** *Consider the weighted projective moduli space of genus 2 curves, say $\mathbb{P}^3_{\mathfrak{q},k}$ for $\mathfrak{q} = (2, 4, 6, 10)$.*

Let $d_0 := \gcd(4, 6, 10) = 2$, $d_1 = \gcd(2, 6, 10) = 2$, $d_2 = \gcd(2, 4, 10) = 2$, $d_3 := \gcd(2, 4, 6) = 2$ and $a_0 = \operatorname{lcm}(2, 2, 2) = 2 = a_1 = a_2 = a_3$, and $a = \operatorname{lcm}(2, 2, 2, 2) = 2$. The new set of weights is $q_i' = \frac{q_i}{a_i}$. Hence $\mathfrak{q}' = (1, 2, 3, 5)$. Thus, the morphism $\mathbb{P}^3_{(2,4,6,10),k} \to \mathbb{P}^3_{(1,2,3,5),k}$, given by

$$[x_0 : x_1 : x_2 : x_3] \to [y_0 : y_1 : y_2 : y_3] = \left[x_0^2 : x_1^2 : x_2^2 : x_3^2\right] \tag{19}$$

is an isomorphism, from Eq. (18). Then $q = 2 \cdot 3 \cdot 5 = 30$ and the Veronese embedding is

$$[J_2 : J_4 : J_6 : J_{10}] \longrightarrow \left[J_2^{30} : J_4^{15} : J_6^{10} : J_{10}^6\right].$$

Since J_{10} is the discriminant then $J_{10} \neq 0$, then

$$\left[J_2^{30} : J_4^{15} : J_6^{10} : J_{10}^6\right] = \left[\frac{J_2^{30}}{J_{10}^6} : \frac{J_4^{15}}{J_{10}^6} : \frac{J_6^{10}}{J_{10}^6} : 1\right]$$

Thus, two genus curves are isomorphic if and only if they have the same $i_1 := \frac{J_2^{30}}{J_{10}^6}$, $i_2 := \frac{J_4^{15}}{J_{10}^6}$, and $i_3 := \frac{J_6^{10}}{J_{10}^6}$ invariants. Such invariants i_1, i_2, i_3 are $\mathrm{GL}_2(k)$-invariants and sometimes are called absolute invariants. *To avoid invariants with* *such high degrees sometimes different invariants have been used, where* $i_1 = \frac{J_4}{J_2^2}$, $i_2 = \frac{J_2 J_4 - J_6}{J_2^3}$, *and* $i_3 = \frac{J_{10}}{J_2^5}$, *but then we have to define new invariants for the locus* $J_2 = 0$*; see* [6], *and many other authors.*

Example above shows the benefits of weighted projective spaces from a computational point of view, since it is much easier to compute with $[J_2 : J_4 : J_6 : J_{10}]$ because the coordinates have much smaller degrees instead of $\left[J_2^{30} : J_4^{15} : J_6^{10} : J_{10}^6\right]$. It was exactly this fact and computational efforts in [6] which led to the definition of the weighted general common divisors and weighted heights in [16] and [5]; as we will see in detail in Sec. 4. $\mathcal{M}_2$ is a very nice example of doing explicit computations, however GIT guarantees that the theory works in every genus.

3.3. **Singular locus of weighted projective varieties.** Singularities of $\mathbb{P}_{\mathfrak{q},k}$ are classified in the following proposition, see [11] or [4] for its proof.

Proposition 3.4. $\mathbb{P}^n_{\mathfrak{q},k}$ *is an irreducible, normal and Cohen-Macaulay variety having only cyclic quotients singularities. Moreover, if* $\mathbb{P}^n_{\mathfrak{q},k}$ *is non-singular, then it is isomorphic to* $\mathbb{P}^n_k$.

We let $d = \gcd(q_0, \dots, q_n)$ and denote by $\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k})$ the **singular locus** of $\mathbb{P}^n_{\mathfrak{q},k}$. Then, following the proof of [10, Prop. 7], one can show that

$$\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k}) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k} : \gcd_{i \in J(\mathbf{x})} (q_i) > d \right\}$$

For $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k}$ denote by $J(\mathbf{x}) := \{j \ : \ x_j(\mathbf{x}) \neq 0\})$, the set of indexes where $\mathbf{x}$ has non-zero coordinates. Let $m = \operatorname{lcm}(q_0, \cdots, q_n)$, p a prime dividing m, and

$$S_{\mathfrak{q}}(p) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k} : \ dp \mid q_i \text{ for all } i \in J(\mathbf{x}) \right\}.$$

The singular locus decomposes into irreducible components as

$$\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k}) = \bigcup_{\text{primes } p \mid m,} S_{\mathfrak{q}}(p),$$

where only the maximal sets are considered in the union. The proof can be easily extended from that of [9] see remark below.

Remark 2. *In most papers the weighted projective space is assumed well formed. This is not really a restriction since every weighted projective space is isomorphic to a well-formed space. Then*

$$S_{\mathfrak{q}}(p) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k} : \ p \mid q_i \textit{ for all } i \in J(\mathbf{x}) \right\}. \tag{20}$$

and the singular locus is

$$\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k}) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},k} : \gcd_{i \in J(\mathbf{x})} (q_i) > 1 \right\}$$

see [10, Prop. 7]. *Since* $\mathbb{P}^n_{\mathfrak{q},k}$ *is well-formed then* $\mathbf{x} \in \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k})$ *implies that* $x_i(\mathbf{x}) = 0$ *for at least one index* $i \in \{0, \dots, n\}$.

Example 3 ($\mathcal{M}_2$ again)**.** *Let us consider again Exa. 2.*

Consider $\mathbb{P}^3_{\mathfrak{q}}$ *for* $\mathfrak{q} = (2, 4, 6, 10)$. *Then* $m = \operatorname{lcm}(2, 4, 6, 10) = 60$. *The only primes dividing* $m = 60$ *are* $p = 2, 3, 5$. *Then*

$$\begin{aligned} S_{\mathfrak{q}}(2) &= \{[0 : t : 0 : 0] \in \mathbb{P}^3_{\mathfrak{q}}\}, \\ S_{\mathfrak{q}}(3) &= \{[0 : 0 : t : 0] \in \mathbb{P}^3_{\mathfrak{q}}\}, \\ S_{\mathfrak{q}}(5) &= \{[0 : 0 : 0 : t] \in \mathbb{P}^3_{\mathfrak{q}}\} \end{aligned}$$

Hence, $\operatorname{Sing} \mathbb{P}^3_{\mathfrak{q},\mathbb{Q}} = S_{\mathfrak{q}}(2) \cup S_{\mathfrak{q}}(3) \cup S_{\mathfrak{q}}(5)$.

One can take $\mathfrak{q}' = (1, 2, 3, 5)$ *and* $\mathbb{P}^3_{\mathfrak{q}',\mathbb{Q}}$. *Then* $m = \operatorname{lcm}(1, 2, 3, 5) = 30$. *Only primes* $p = 2, 3, 5$ *divide* m. *Then,*

$$\begin{aligned} S_{\mathfrak{q}'}(2) &= \{[0 : t : 0 : 0] \in \mathbb{P}^3_{\mathfrak{q}}\}, \\ S_{\mathfrak{q}'}(3) &= \{[0 : 0 : t : 0] \in \mathbb{P}^3_{\mathfrak{q}}\}, \\ S_{\mathfrak{q}'}(5) &= \{[0 : 0 : 0 : t] \in \mathbb{P}^3_{\mathfrak{q}}\}. \end{aligned}$$

Hence, $\operatorname{Sing} \mathbb{P}^3_{\mathfrak{q}',\mathbb{Q}} = S'_{\mathfrak{q}}(2) \cup S'_{\mathfrak{q}}(3) \cup S'_{\mathfrak{q}}(5)$. □

For a fixed prime p such that $p \nmid m$, then $S_{\mathfrak{q}}(p) = \emptyset$. If $p \mid m$ then denote

$$J(p) = \{j \mid \text{ such that } p \mid q_j\}, \quad \text{and} \quad n_p = \#J(p).$$

Then $S_{\mathfrak{q}}(p) \neq \emptyset$ is isomorphic to the weighted projective space $\mathbb{P}^{n_p}_{\mathfrak{q}',k}$, where $\mathfrak{q}' = (q_{i_1}, \cdots, q_{i_{n_p}})$ with $i_\ell \in J(p)$ for $1 \leq \ell \leq n_p$. Moreover, as a consequence of the normality of $\mathbb{P}^n_{\mathfrak{q},k}$, we have $\operatorname{Codim}_{\mathbb{P}^n_{\mathfrak{q},k}}(\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k})) \geq 2$. This means that $\mathbb{P}^n_{\mathfrak{q},k}$ is regular in codimension one. In particular, if q_i's are mutually coprime and $q_i > 1$, then

$$\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k}) = \{\mathbf{x}_i = [0 : \cdots : 1 : \cdots : 0] : \ 0 \leq i \leq n\}.$$

Next we consider the canonical quotient map $p_{\mathfrak{q}} : \mathbb{V}^{n+1}_k \to \mathbb{P}^n_{\mathfrak{q},k}$, which induces the surjective morphism $\pi_{\mathfrak{q}} : \mathbb{P}^n_k \to \mathbb{P}^n_{\mathfrak{q},k}$. Let $\mathcal{X}$ be a weighted subvariety of $\mathbb{P}^n_{\mathfrak{q},k}$. The **punctured affine cone** over $\mathcal{X}$ is $\mathcal{C}^*_{\mathcal{X}} = p_{\mathfrak{q}}^{-1}(\mathcal{X})$. The **affine cone** $\mathcal{C}_{\mathcal{X}}$ over $\mathcal{X}$ is the closure of $\mathcal{C}^*_{\mathcal{X}}$ in $\mathbb{A}^{n+1}_k$. The origin point $\mathbf{0} = (0, \cdots, 0)$ refers to the vertex of $\mathcal{C}^*_{\mathcal{X}}$. We note that k^* acts on the punctured affine cone $\mathcal{C}^*_{\mathcal{X}} = p_{\mathfrak{q}}^{-1}(\mathcal{X})$ to result $\mathcal{X} = \mathcal{C}^*_{\mathcal{X}}/k^*$. Moreover, $\mathcal{C}^*_{\mathcal{X}}$ has no isolated singularities.

A weighted subvariety $\mathcal{X}$ of $\mathbb{P}^n_{\mathfrak{q},k}$ is called **quasi-smooth** of dimension m if its affine cone $\mathcal{C}_\mathcal{X}$ is smooth variety of dimension $m+1$ outside its vertex. The singularities of a quasi-smooth variety $\mathcal{X}$ are due to the k^*-action and hence are cyclic quotients singularities. Furthermore, by [4, Cor. 5.9], if $\mathcal{X} \subset \mathbb{P}^n_{\mathfrak{q},k}$ is subvariety such that $\mathcal{X} \cap \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},k}) = \emptyset$, then $\mathcal{X}$ is non-singular if and only if $\mathcal{X}$ is quasi-smooth.

A weighted subvariety $\mathcal{X}$ of $\mathbb{P}^n_{\mathfrak{q},k}$ of codimension c is called **well-formed** if $\mathbb{P}^n_{\mathfrak{q},k}$ itself is well-formed and $\mathcal{X}$ contains no codimension $c+1$ singular stratum of $\mathbb{P}^n_{\mathfrak{q},k}$. Hence, any codimension 1 stratum of a well-formed variety $\mathcal{X}$ is either nonsingular on $\mathbb{P}^n_{\mathfrak{q},k}$ or it is equal to $\mathcal{X} \cap \mathcal{Y}$, where $\mathcal{Y}$ is a codimension 1 stratum of $\mathbb{P}^n_{\mathfrak{q},k}$. This means that $\operatorname{Codim}_\mathcal{X}(\mathcal{X} \cap \mathbb{P}^n_{\mathfrak{q},k}) \geq 2$.

Given a weighted polynomial $f \in k_\mathfrak{q}[x_0, \cdots, x_n]$ of degree d, let $\mathcal{X}_d$ denotes the hypersurfaces defined by f. It is called a **linear cone** if $d = q_i$ for some $0 \leq i \leq n$, i.e, it is defined by $x_i + g$ with $g \in k$. A linear cone is well-formed if and only if it is isomorphic to $\mathbb{P}^{n-1}_{(q_0, \cdots, \hat{q}_i, \cdots, q_n), k}$. In the case of hypersurfaces, $\mathcal{X}_d$ is well-formed if and only if the following hold:

(i) $\gcd(q_0, \cdots, \hat{q}_i, \cdots, q_n) = 1$ for all $0 \leq i \leq n$;
(i) $\gcd(q_0, \cdots, \hat{q}_i, \cdots, \hat{q}_j, \cdots, q_n)$ divides d for $0 \leq i \neq j \leq n$.

For more on well formed subvarieties of $\mathbb{P}^n_{\mathfrak{q},k}$ of codimension ≥ 2, see [14].

3.4. **Analytic structure of weighted projective spaces.** As regular projective spaces, the weighted complex projective spaces can also be equipped with an analytic structure. We consider the decomposition of

$$\mathbb{P}^n_{\mathfrak{q},\mathbb{C}} = U_0 \cup \ldots \cup U_n,$$

where

$$U_i = \{\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},\mathbb{C}} : \ x_i(\mathbf{x}) \neq 0\} \subset \mathbb{P}^n_{\mathfrak{q},\mathbb{C}},$$

for each $0 \leq i \leq n$. Then, the map $\tilde{\psi}_i : \mathbb{C}^n \to U_i$,

$$(x_0, \ldots, x_{i-1}, x_{i+1} \ldots, x_n) \to [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]_\mathfrak{q} \tag{21}$$

is a surjective analytic map, but not a chart since it is not injective. However, it induces the isomorphism $\psi_i : \mathcal{X}(q_i : q_0, \ldots, \widehat{q}_i, \ldots q_n) \to U_i$, such as

$$[(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)] \to [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]_\mathfrak{q},$$

where $\mathcal{X}(q_i : q_0, \ldots, \widehat{q}_i, \ldots q_n)$ is the cyclic quotient space of the action of μ_{q_i} on $\mathbb{C}^n$ given by $\mu_{q_i} \times \mathbb{C}^n \to \mathbb{C}^n$ such as

$$(\xi_i, (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)) \to (\xi_i^{q_0} x_0, \ldots, \xi_i^{q_{i-1}} x_{i-1}, \xi^{q_{i+1}} x_{i+1}, \ldots, \xi_i^{q_n} x_n), \tag{22}$$

where $\xi_i \in \mu_{q_i}$. Since the changes of charts are analytic, then $\mathbb{P}^n_{\mathfrak{q},\mathbb{C}}$ is an analytic space with cyclic quotient singularities; see [2,3] for details.

3.5. **Weighted Blow-ups.** Consider $\widehat{\mathbb{C}}^{n+1}_{\mathfrak{q}} := \left\{ (\mathbf{x}, [\mathfrak{u}]_{\mathfrak{q}}) \in \mathbb{C}^{n+1} \times \mathbb{P}^n_{\mathfrak{q},\mathbb{C}} \mid \mathbf{x} \in \overline{[\mathfrak{u}]}_{\mathfrak{q}} \right\}$, where $\overline{[\mathfrak{u}]}_{\mathfrak{q}}$ denote the Zariski closure of $[\mathfrak{u}]_{\mathfrak{q}}$ and $\mathbf{x} \in \overline{[\mathfrak{u}]}_{\mathfrak{q}}$ means that there exists $t \in \mathbb{C}$ satisfying $x_i = t^{q_i} \cdot u_i$ for each $0 \leq i \leq n$. The natural projection map

$$\pi_{\mathfrak{q}} : \widehat{\mathbb{C}}^{n+1}_{\mathfrak{q}} \to \mathbb{C}^{n+1} \tag{23}$$

is an isomorphism over $\widehat{\mathbb{C}}^{n+1}_{\mathfrak{q}} \setminus \pi_{\mathfrak{q}}^{-1}(\mathbf{0})$ and the **exceptional divisor** $E := \pi_{\mathfrak{q}}^{-1}(\mathbf{0})$ is identified with $\mathbb{P}^n_{\mathfrak{q},\mathbb{C}}$. The space $\widehat{\mathbb{C}}^{n+1}_{\mathfrak{q}} = \widehat{U}_0 \cup \ldots \cup \widehat{U}_n$ can be covered with $(n+1)$ charts, where

$$\widehat{U}_i = \{(\mathbf{x}, [\mathfrak{u}]_{\mathfrak{q}}) \in \mathbb{C}^{n+1} \times \mathbb{P}^n_{\mathfrak{q},\mathbb{C}} : u_i \neq 0\} \subset \widehat{\mathbb{C}}^{n+1}(\mathfrak{q}).$$

However, $\phi^i : \mathbb{C}^{n+1} \to \widehat{U}_i$,

$$\mathbf{x} \to (x_0^{q_0}, x_0^{q_1} x_1, \ldots, x_0^{q_n} x_n), [x_1 : \ldots, x_{i-1} : 1 : x_{i+1} : \ldots : x_n]),$$

are surjective, but not injective. Indeed, we have that $\phi^i(\mathbf{x}) = \phi^i(\mathbf{y})$ is and only if there exists $\xi \in \mu_{q_i}$ such that $y_i = \xi^{-1} x_i$ and $y_j = \xi^{q_j} x_j$ for $j \neq i$. Hence, the map ϕ^i induces an isomorphism $\mathcal{X}(q_i : q_0, \ldots, q_{i-1}, -1, q_{i+1}, \ldots, q_n) \to \widehat{U}_i$.

These charts are compatible with the ones of $\mathbb{P}^n_{\mathfrak{q},\mathbb{C}}$. In $\widehat{U}_i$ the exceptional divisor is $\{x_i = 0\}$ and the i-th chart of $\mathbb{P}^n_{\mathfrak{q},\mathbb{C}}$ is the quotient space

$$\mathcal{X}(q_i : q_0, \ldots, q_{i-1}, -1, q_{i+1}, \ldots, q_n).$$

Example 4 (Case $n = 2$)**.** *Let $\mathfrak{q} = (q_0, q_1, q_2)$ be a tuple of reduced weights, i.e.,* $\gcd(q_0, q_1, q_2) = 1$ *and $\pi_{\mathfrak{q}} : \widehat{\mathbb{C}}^3_{\mathfrak{q}} \to \mathbb{C}^3$, be the weighted blow-up at the origin with respect to $\mathfrak{q}$. Then $\widehat{\mathbb{C}}^3 \cong \widehat{U}_0 \cup \widehat{U}_1 \cup \widehat{U}_2$, where*

$$\widehat{U}_0 \cong X(q_0 : -1, q_1, q_2),\ \widehat{U}_1 \cong X(q_1 : q_0, -1, q_2),\ \widehat{U}_2 \cong X(q_2 : q_0, q_1, -1),$$

and the charts are given by

$$\begin{aligned}
\psi^0 :\ & X(q_0 : -1, q_1, q_2) \to U_0, && [(x_0 : x_1 : x_2)] \mapsto ((x_0^{q_0}, x_0^{q_1} x_1, x_0^{q_2} x_2), [1 : x_1 : x_2]) \\
\psi^1 :\ & X(q_1 : q_0, -1, q_2) \to U_1, && [(x_0 : x_1 : x_2)] \mapsto ((x_1^{q_1} x_0, x_1^{q_1}, x_1^{q_2} x_2), [x_0 : 1 : x_2]) \\
\psi^2 :\ & X(q_1 : q_0, q_1, -1) \to U_2, && [(x_0 : x_1 : x_2)] \mapsto ((x_2^{q_2} x_0, x_2^{q_2} x_1, x_2^{q_2}), [x_0 : x_1 : 1]).
\end{aligned}$$

The exceptional divisor $\pi_{\mathfrak{q}}^{-1}((0,0,0))$ is isomorphic to $\mathbb{P}^2_{\mathfrak{q},\mathbb{C}}$, which can be simplified by isomorphism $\mathbb{P}^2_{\mathfrak{q},\mathbb{C}} \cong \mathbb{P}^2_{\mathfrak{q}',\mathbb{C}}$ given by

$$[x_0 : x_1 : x_2] \mapsto \left[x_0^{\gcd(q_1,q_2)} : x_1^{\gcd(q_0,q_2)} : x_2^{\gcd(q_0,q_1)} \right],$$

where

$$\mathfrak{q}' = \left(\frac{q_0}{\gcd(q_0,q_1)\cdot\gcd(q_0,q_2)}, \frac{q_1}{\gcd(q_0,q_1)\cdot\gcd(q_1,q_2)}, \frac{q_2}{\gcd(q_0,q_2)\cdot\gcd(q_1,q_2)} \right).$$

4. Weighted heights

In [5] a height function was defined for weighted projective spaces $\mathbb{P}^n_{\mathfrak{q},k}$, called weighted height. We briefly describe basic definitions here. To avoid confusion with projective heights we will use different notation than that of [5]. We will follow the parallelism with Weil heights by using $\mathcal{S}$, $\mathfrak{s}$ instead of H, h. $\mathbb{P}^n_{\mathfrak{q}}(k)$ denotes the set of k-rational points of $\mathbb{P}^n_{\mathfrak{q},k}$.

4.1. **Weighted heights on $\mathbb{P}^n_{\mathfrak{q},k}$.** Given any $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(k)$, the **multiplicative weighted height** over k is defined as

$$\mathcal{S}_k(\mathbf{x}) := \prod_{\nu \in M_k} \max\left\{ |x_0|_\nu^{\frac{1}{q_0}}, \dots, |x_n|_\nu^{\frac{1}{q_n}} \right\} \tag{24}$$

and its **logarithmic weighted height** (over k) as

$$\mathfrak{s}_k(\mathbf{x}) := \log \mathcal{S}_k(\mathbf{x}) = \sum_{\nu \in M_k} \max_{0\le j\le n} \left\{ \frac{1}{q_j}\cdot \log|x_j|_\nu \right\}. \tag{25}$$

In [5, Prop. 1] it is shown that height functions $\mathcal{S}_k(\mathbf{x})$ and hence $\mathfrak{s}_k(\mathbf{x})$ are independent of the choice of coordinates of the point $\mathbf{x}$. Moreover, in [5, Prop. 5-ii], it is proved that for any finite extension $K|k$ we have

$$\mathcal{S}_k(\mathbf{x})^{[K:k]} = \mathcal{S}_K(\mathbf{x}), \text{ and hence } [K:k]\cdot \mathfrak{s}_k(\mathbf{x}) = \mathfrak{s}_K(\mathbf{x}).$$

Weighted heights can be interpreted in terms of Weil height on projective varieties using Veronese map defined by Eq. (15). Assume that $\mathfrak{q} = (q_0, \cdots, q_n)$ is reduced, well-formed and satisfies $\gcd(m/q_0, \cdots, m/q_n) = 1$, where $m = \operatorname{lcm}(q_0, q_1 \cdots, q_n)$. Proof of the following can be found in [5].

Lemma 4.1. *Weighted height $\mathcal{S}_k$ is given in terms of projective height H_k via*

$$\mathcal{S}_k(\mathbf{x}) = H_k\left(\phi_m(\mathbf{x})\right)^{\frac{1}{m}} \quad \textit{and} \quad \mathfrak{s}_k(\mathbf{x}) = \frac{1}{m}\cdot h_k\left(\phi_m(\mathbf{x})\right), \tag{26}$$

for all $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(k)$, where ϕ_m is the Veronese map given in Eq. (15).

The **absolute weighted height** on $\mathbb{P}^n_{\mathfrak{q}}(\bar{k})$ is defined as

$$\begin{aligned} \mathcal{S} : \mathbb{P}^n_{\mathfrak{q}}(\bar{k}) &\to [0,\infty], \\ \mathbf{x} \mapsto \mathcal{S}(\mathbf{x}) &:= \mathcal{S}_K(\mathbf{x})^{1/[K:k]}, \end{aligned} \tag{27}$$

and the **absolute logarithmic weighted height** on $\mathbb{P}^n_{\mathfrak{q}}(\bar{k})$ is given by

$$
\begin{aligned}
\mathfrak{s} : \mathbb{P}^n_{\mathfrak{q}}(\bar{k}) &\to [0, \infty], \\
\mathbf{x} \mapsto \mathfrak{s}(\mathbf{x}) &:= \frac{1}{[K:k]} \log \mathcal{S}_K(\mathbf{x}),
\end{aligned}
\tag{28}
$$

for which $K \subset \bar{k}$ is a finite extension of k containing $k(\mathbf{x})$, the **field of definition** of $\mathbf{x}$ defined by

$$
k(\mathbf{x}) := k\left(\frac{x_0^{1/q_0}}{x_i^{1/q_i}}, \cdots, 1, \cdots, \frac{x_n^{1/q_n}}{x_i^{1/q_i}} \right),
$$

for some $x_i \neq 0$. Notice that both of these height functions are independent of the choice of the field K; see [5]. For simplicity, we call $\mathfrak{s}(\mathbf{x})$ the **global weighted height** on $\mathbb{P}^n_{\mathfrak{q}}(\bar{k})$.

By Eq. (26), for a field $K \subset \bar{k}$ containing and $k(\mathbf{x})$, we have:

Lemma 4.2. *For all $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(\bar{k})$, we have*

$$
\mathcal{S}(\mathbf{x}) = H\left(\phi_m(\mathbf{x})\right)^{\frac{1}{m}}, \quad \textit{and} \quad \mathfrak{s}(\mathbf{x}) = \frac{1}{m} \cdot h\left(\phi_m(\mathbf{x})\right),
\tag{29}
$$

where ϕ_m is as in Eq. (15), $H(\cdot)$, $h(\cdot)$ as in Eq. (5), and $\mathcal{S}(\cdot)$, $\mathfrak{s}(\cdot)$ as in Eq. (27).

4.2. **Cartier and Weil divisors on weighted varieties.** Let $\mathcal{X}$ be a weighted variety in $\mathbb{P}^n_{\mathfrak{q},k}$ over the field k. The group of **Weil divisors** on $\mathcal{X}$ is a free Abelian group generated by weighted closed subvarieties of codimension one on $\mathcal{X}$. This group is denoted by $\mathrm{WeDiv}_{\mathfrak{q}}(\mathcal{X})$. The **support** of the divisor $D = \sum_Y n_{\mathcal{Y}} \cdot \mathcal{Y}$ is the union of all codimension one weighted subvarieties $\mathcal{Y}$ such that $n_{\mathcal{Y}} \neq 0$, which is denoted by $\mathrm{Supp}(D)$. A divisor is said to be **effective** if every $n_{\mathcal{Y}} \geq 0$ for all codimension one subvarieties $\mathcal{Y} \subset \mathcal{X}$. We define $\mathrm{ord}_{\mathcal{Y}} : \mathcal{O}_{\mathcal{X},\mathcal{Y}} \setminus \{0\} \to \mathbb{Z}$ to be

$$
\mathrm{ord}_{\mathcal{Y}}(f) = \mathrm{length}_{\mathcal{O}_{\mathcal{X},\mathcal{Y}}} \left(\frac{\mathcal{O}_{\mathcal{X},\mathcal{Y}}}{\langle f \rangle} \right),
$$

which is well defined since $\mathcal{O}_{\mathcal{X},\mathcal{Y}}$ is a local ring. Then, one can extend $\mathrm{ord}_{\mathcal{Y}}$ to the fraction field $k_{\mathfrak{q}}(\mathcal{X})^*$ in the usual way. The order function $\mathrm{ord}_{\mathcal{Y}} : k_{\mathfrak{q}}(\mathcal{X})^* \to \mathbb{Z}$ has the following properties:

(1) $\mathrm{ord}_{\mathcal{Y}}(f \cdot g) = \mathrm{ord}_{\mathcal{Y}}(f) + \mathrm{ord}_{\mathcal{Y}}(g)$
(2) For a fixed $f \in k_{\mathfrak{q}}(\mathcal{X})^*$ there are only finitely many $\mathcal{Y}$ such that $\mathrm{ord}_{\mathcal{Y}} \neq 0$.
(3) Let $f \in k_{\mathfrak{q}}(\mathcal{X})^*$. Then, $f \in \mathcal{O}_{\mathcal{X},\mathcal{Y}}$ if and only if $\mathrm{ord}_{\mathcal{Y}}(f) \geq 0$. Similarly, $f \in \mathcal{O}^*_{\mathcal{X},\mathcal{Y}}$ if and only if $\mathrm{ord}_{\mathcal{Y}}(f) = 0$.
(4) If $\mathcal{X}$ is weighted projective variety and $f \in k_{\mathfrak{q}}(\mathcal{X})^*$, then $f \in k^*$ if and only if $\mathrm{ord}_{\mathcal{Y}}(f) \geq 0$ for all $\mathcal{Y}$; if and only if $\mathrm{ord}_{\mathcal{Y}}(f) = 0$ for all $\mathcal{Y}$.

The divisor of any $f \in k_{\mathfrak{q}}(\mathcal{X})^*$ is defined as

$$\operatorname{div}(f) = \sum_{\mathcal{Y} \subset \mathcal{X}} \operatorname{ord}_{\mathcal{Y}}(f) \cdot \mathcal{Y}$$

which is called a **principal divisor**. Two divisors D and D' are said to be **linearly equivalent** if their difference is a principal divisor. The divisor of zeros and divisor of poles of f, denoted by $(f)_0$ and $(f)_\infty$ respectively, are

$$(f)_0 = \sum_{\operatorname{ord}_{\mathcal{Y}}(f) > 0} \operatorname{ord}_{\mathcal{Y}}(f) \cdot \mathcal{Y}, \; (f)_\infty = - \sum_{\operatorname{ord}_{\mathcal{Y}} < 0} \operatorname{ord}_{\mathcal{Y}}(f) \cdot \mathcal{Y}$$

The **divisor class group** of $\mathcal{X}$ is the group of divisor classes modulo linear equivalence. This group is denoted by $\operatorname{Cl}_{\mathfrak{q}}(\mathcal{X})$, and $\operatorname{Cl}(\mathbb{P}^n_{\mathfrak{q},k})$ for $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},k}$.

A **Cartier divisor** on a weighted variety $\mathcal{X}$ is an equivalence class of collection of pairs $(U_i, f_i)_{i \in I}$ satisfying the following conditions:

(i) The U_i are affine weighted open sets that cover $\mathcal{X}$.
(ii) The f_i are non zero rational functions, $f_i \in k_{\mathfrak{q}}(U_i)^* = k_{\mathfrak{q}}(\mathcal{X})^*$.
(iii) $\frac{f_i}{f_j} \in \mathcal{O}_{\mathcal{X}}(U_i \cap U_j)^*$, so $\frac{f_i}{f_j}$ has no poles or zeros on $U_i \cap U_j$.

Two Cartier divisors $\{(U_i, f_i) | i \in I\}$ and $\{(V_j, g_j) | j \in J\}$ are equivalent if for all $i \in I$ and $j \in J$ we have

$$\frac{f_i}{g_j} \in \mathcal{O}_{\mathcal{X}}(U_i \cap V_j)^*.$$

The **sum of two Cartier divisors** is

$$\{(U_i, f_i) | i \in I\} + \{V_j, g_j) | j \in J\} = \{(U_i \cap V_j, f_i \, g_j) | (i, j) \in I \times J\}.$$

The Cartier divisors with this operation on a weighted variety $\mathcal{X}$ form a group that we denote it by $\operatorname{CaDiv}_{\mathfrak{q}}(\mathcal{X})$. The **support** of a Cartier divisor is the set of zeros and poles of f_i, which is denoted by $\operatorname{Supp}(D)$. A Cartier divisor is said to be **effective** or **positive** if it can be defined by a collection $\{(U_i, f_i) | i \in I\}$ such that every $f_i \in \mathcal{O}_{\mathcal{X}}(U_i)$. For a given $f \in k_{\mathfrak{q}}(\mathcal{X})^*$, the divisor $\operatorname{div}(f) = \{(\mathcal{X}, f)\}$ is called a **principal Cartier divisor**. Two Cartier divisors are **linearly equivalent** if their difference is a principal divisor. The group of Cartier divisors classes modulo linear equivalence is called **Picard group** of a weighted variety $\mathcal{X}$ and is denoted by $\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X})$. In the case $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},k}$, we write $\operatorname{Pic}(\mathbb{P}^n_{\mathfrak{q},k})$. A Cartier divisor D on a weighted variety $\mathcal{X}$ is said to be **ample** or **big** if the corresponding line bundle $\mathcal{O}(D)$ is ample or big, respectively.

For $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},k}$ with reduced weights $\mathfrak{q}$, in [1, Sections 5, 6], it is proved that the following maps

$$\begin{aligned} &\mathbb{Z} \to \mathrm{Cl}(\mathcal{X}), && \mathbb{Z} \to \mathrm{Pic}(\mathcal{X}), \\ &1 \mapsto \mathcal{O}_{\mathcal{X}}(1), && 1 \mapsto \mathcal{O}_{\mathcal{X}}(m), && m = \mathrm{lcm}(q_0, \cdots, q_n), \end{aligned} \tag{30}$$

induce the following isomorphism $\mathrm{Cl}(\mathcal{X}) \cong \mathbb{Z}$, and $\mathrm{Pic}(\mathcal{X}) \cong \mathbb{Z}$, respectively. Furthermore, $\mathcal{O}_{\mathcal{X}}(a)$ is not necessarily an invertible sheaf for any given integer $a \in \mathbb{Z}$. However, by [17, Lem. 1.3], the sheaf $\mathcal{O}_{\mathcal{X}}(m)$ with $m = \mathrm{lcm}(q_0, \cdots, q_n)$ is ample and invertible, and for $a, b \in \mathbb{Z}$ we have

$$\mathcal{O}_{\mathcal{X}}(a) \otimes \mathcal{O}_{\mathcal{X}}(m)^{\otimes b} \cong \mathcal{O}_{\mathcal{X}}(a + bm).$$

In [4, Thm. 4B. 7], it is proved that $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(m)$ is ample and there is $c \in \mathbb{Z}$ such that $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(cm)$ is very ample. Furthermore, the sheaf $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(a)$ is coherent and Cohen-Macaulay for any $a \in \mathbb{Z}$. If $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(a) \neq 0$, then it is reflexive of rank 1 by [4, Cor. 5.8].

Following [17], we define the *weak projective space* over any field k as follows:

Definition 2. *The complement of* $\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},k})$ *in* $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},k}$ *is called the* ***weak projective space*** *over* k*, which is a smooth weighted subvariety, denoted by*

$$\mathbb{WP}^n_{\mathfrak{q},k} := \mathbb{P}^n_{\mathfrak{q},k} \setminus \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},k}). \tag{31}$$

By [17, Prop. 1.1], the sheaf $\mathcal{O}_{\mathcal{X}}(1)$ is locally free on $\mathbb{WP}^n_{\mathfrak{q},k}$. Hence, defining

$$\mathcal{O}_{\mathbb{WP}^n_{\mathfrak{q},k}}(1) := \mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(1)|_{\mathbb{WP}^n_{\mathfrak{q},k}},$$

one can see that $\mathbb{WP}^n_{\mathfrak{q},k}$ is the largest open set $U \subset \mathbb{P}^n_{\mathfrak{q},k}$ such that $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(1)|_U$ is an invertible sheaf on U and

$$\left(\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(1)|_U\right)^{\otimes a} \cong \mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(a)|_U$$

for any $a \in \mathbb{Z}$ by [17, Thm. 1.7]. Furthermore, we have $\mathrm{Pic}_{\mathfrak{q}}(\mathbb{WP}^n_{\mathfrak{q},k}) \cong \mathbb{Z}$ and it is generated by $\mathcal{O}_{\mathbb{WP}^n_{\mathfrak{q},k}}(1)$.

For any (weighted) projective variety $\mathcal{X}$ of dimension $\dim(\mathcal{X}) = d$ over k, we denote by $\Omega^i_{\mathcal{X}}$ the sheaf of i-th regular differential forms on $\mathcal{X}$, and $\omega_{\mathcal{X}} = \Omega^d_{\mathcal{X}}$ the **canonical sheaf** of $\mathcal{X}$. By [17, Prop. 2.3], the canonical sheaf of $\mathbb{WP}^n_{\mathfrak{q},k}$ is

$$\omega_{\mathbb{WP}^n_{\mathfrak{q},k}} \cong \mathcal{O}_{\mathbb{WP}^n_{\mathfrak{q},k}}(-\tilde{q}),$$

where $\tilde{q} = q_0 + q_1 + \cdots + q_n$, by [17, Prop. 2.3].

We also denote by $\omega^0_{\mathcal{X}}$ the **dualizing sheaf** of $\mathcal{X}$. If $\mathcal{X}$ is a nonsingular or more generally normal (weighted) projective variety, then $\omega^0_{\mathcal{X}} = \omega_{\mathcal{X}}$. Otherwise, we let $\mathcal{W} = \mathcal{X} \setminus \mathrm{Sing}(\mathcal{X})$ and consider the canonical embedding $j : \mathcal{W} \to \mathcal{X}$. Then, if $\mathrm{Codim}_{\mathcal{X}}(\mathcal{X} - \mathcal{W}) \geq 2$, then

$$\omega^0_{\mathcal{X}} = j_* \, \omega^0_{\mathcal{W}} = j_* \, \omega_{\mathcal{W}}.$$

In the case $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},k}$, since it is normal and Cohen-Macaualy and $\mathcal{W} = \mathbb{WP}^n_{\mathfrak{q},k}$, so by [4, Cor. 6B.8] one has $\omega^0_{\mathbb{P}^n_{\mathfrak{q},k}} \cong \mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(-\tilde{q})$.

4.3. **Local weighted heights.** We assume that $\mathcal{X}$ is a weighted variety defined over k in $\mathbb{P}^n_{\mathfrak{q},\bar{k}}$, where $k \subset \bar{k}$ and $\mathfrak{q} = (q_0, \cdots, q_n)$. If $\mathcal{X}$ is a weighted affine variety with coordinates $x_0, x_1, \cdots, x_n$, then a set $E \subset \mathcal{X} \times M$ is called a **weighted affine M_k-bounded set** if there is an M_k-bounded constant function γ such that

$$|x_i(\mathbf{x})|_v^{\frac{m}{q_i}} \leq e^{\gamma(v)}, \; 0 \leq i \leq n \text{ and } (\mathbf{x}, v) \in E.$$

We note that this definition is independent of choice of the coordinates x_i's on $\mathcal{X}$. Moreover, any finite union of weighted affine M-bounded sets is again a weighted affine M-bounded.

For an arbitrary variety $\mathcal{X}$, we say that $E \subset \mathcal{X} \times M$ is a **weighted M_k-bounded set** if there exists a finite cover $U_i' s$ of weighted affine open subsets of $\mathcal{X}$ and the weighted M_k-bounded sets $E_i \subset U_i \times M$ such that $E = \bigcup E_i$. A function

$$\lambda : \mathcal{X} \times M \to \mathbb{R}$$

is called a **locally weighted M_k-bounded above** if for every weighted M_k bounded subset $E \subset \mathcal{X} \times M$, there exists an M_k-constant γ such that $\lambda(\mathbf{x}, v) \geq \gamma(v)$ holds for $(\mathbf{x}, v) \in E$. The **locally weighted M_k-bounded below** and **locally weighted M_k-bounded** functions are defined similarly.

Example 5. *For example, let $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},\bar{k}}$ and consider the finite cover of affine open sets $\{(U_i, x_i)\}$ and $\gamma \equiv 0$. Moreover, for $0 \leq i \leq n$, the following sets are weighted M_k-bounded:*

$$\widetilde{E}_i = \left\{ (\mathbf{x}, v) \in \mathcal{X} \times M : \textit{and} \left| \frac{x_0^{\frac{m}{q_0}}}{x_i^{\frac{m}{q_i}}} \right|_v \leq 1, \cdots, \left| \frac{x_n^{\frac{m}{q_n}}}{x_i^{\frac{m}{q_i}}} \right|_v \leq 1 \right\}. \tag{32}$$

Thus $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},\bar{k}}$ is a weighted M_k-bounded set, since it is covered by $\widetilde{E}_i' s$.

Let $\mathcal{L}$ be a line bundle on a weighted variety $\mathcal{X}$ defined over k. A **weighted M-metric** on $\mathcal{L}$ is a norm $\| \cdot \| = (\| \cdot \|_v)$ such that for each $v \in M$, extending $v|_k \in M_k$, and each fiber $\mathcal{L}_{\mathbf{x}}$ with $\mathbf{x} \in \mathcal{X}$ assigns a function $\| \cdot \|_v : \mathcal{L}_{\mathbf{x}} \to \mathbb{R}_{\geq 0}$, not identically equal to zero, satisfying the following:

- $\|\lambda \cdot \xi\|_v = |\lambda|_v \cdot \|\xi\|_v$ for $\lambda \in \bar{k}$ and $\xi \in \mathcal{L}_{\mathbf{x}}$.
- If $w_1, w_2 \in M$ agree on the residue field $k(\mathbf{x})$, then $\| \cdot \|_{w_1} = \| \cdot \|_{w_2}$ on $\mathcal{L}_{\mathbf{x}}(k(\mathbf{x}))$.

A weighted M-metric on $\mathcal{L}$ is called **locally weighted M-bounded** if for section $g \in \mathcal{O}_{\mathcal{X}}(U)$ on an open set $U \subseteq \mathcal{X}$, the function

$$(\mathbf{x}, v) \mapsto \log \|g(\mathbf{x})\|_v$$

on $U \times M$ is locally weighted M_k-bounded. We say that $\mathcal{L}$ is a **weighted M-metrized line bundle** on $\mathcal{X}$ if $\mathcal{L}$ is equipped with a weighted M-metric $\| \cdot \| = (\| \cdot \|_v)$.

Next we show that there exist a locally bounded weighted M-metric on any line bundle on the weighted variety $\mathcal{X}$.

Proposition 4.3. *Any line bundle $\mathcal{L}$ on a weighted variety $\mathcal{X} \subseteq \mathbb{P}^n_{\mathfrak{q}, \bar{k}}$ defined over k admits a locally bounded weighted M-metric.*[2]

Proof. First we assume that $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q}, \bar{k}}$ and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(m)$, where $m = \operatorname{lcm}(q_0, q_1, \cdots, q_n)$. Then, one can define an M-metric by letting

$$\|\ell(\mathbf{x})\|_v = \frac{|\ell(\mathbf{x})|_v}{\max_i |x_i|_v^{\frac{m}{q_i}}}, \tag{33}$$

for each $v \in M$, $\mathbf{x} \in \mathcal{X}$ and a global section $\ell \in \mathcal{O}_{\mathcal{X}}(m)$ given by

$$\ell = a_0 x_0^{\frac{m}{q_0}} + a_1 x_1^{\frac{m}{q_1}} + \cdots + a_1 x_1^{\frac{m}{q_1}}.$$

It is well-defined on $\mathcal{L}$, and on the set $U_i = \{x_i \neq 0\}$ we have

$$\left\| x_i^{\frac{m}{q_i}}(\mathbf{x}) \right\|_v = \frac{\left| x_i^{\frac{m}{q_i}}(\mathbf{x}) \right|_v}{\max_i |x_i|_v^{\frac{m}{q_i}}} \leq 1.$$

Moreover, the functions $\left| \frac{x_j^{m/q_j}}{x_i^{m/q_i}} \right|_v$ are bounded by an M_k-constant on the bounded sets $\widetilde{E}_i$ defined by Eq. (32). Thus, $\log \left\| x_i^{\frac{m}{q_i}}(\mathbf{x}) \right\|_v$ are bounded below for all indexes, and hence Eq. (33) gives the desired locally bounded weighted M-metric.

Next, we assume that $\mathcal{X} \subseteq \mathbb{P}^n_{\mathfrak{q}, \bar{k}}$ is a weighted projective variety and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(D)$, where D is an effective Cartier divisor on $\mathcal{X}$ both defined over k. In this case, $\mathcal{L}$ can be written as $\mathcal{L} = \mathcal{M}_1 \otimes \mathcal{M}_2^{-1}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are base point free line bundles on $\mathcal{X}$. Now, we choose generating global functions $s_1, \cdots, s_{n_1}$ of

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[2] We thank Min Ru for clarifying some details in the proof of Thm. 4.3 by indicating [18, B2.2.10 and B2.2.11].

$\mathcal{M}_1$, and $t_1, \cdots, t_{n_2}$ of $\mathcal{M}_2$. Then, for $\mathbf{x} \notin \mathrm{Supp}(D)$, the desired locally bounded weighted M-metric on $\mathcal{L}$ is given by

$$\|g_D(\mathbf{x})\|_v = \max_{1\leq i\leq n_1} \min_{1\leq j\leq n_2} \left\| \frac{s_i g_D}{t_j}(\mathbf{x}) \right\|_v, \tag{34}$$

where $v \in M_k$ and g_D is a section of $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(D)$ with $D = div(g_D)$. One can show this metric is uniquely determined and independent of choices $\mathcal{M}_1$, $\mathcal{M}_2$, and their generating sections as [18, B2.2.10 and B2.2.11]. We notice that if $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},\bar{k}}$ and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(m)$, then Eq. (34) will be same as Eq. (33) by considering $\mathcal{M}_1 = \mathcal{L}$ and $\mathcal{M}_2$ trivial line bundle and $t_i = x_i^{m/q_i}$ for $0 \leq i \leq n$ and $g_D \in \mathcal{O}_{\mathcal{X}}(m)$.

Finally, for an arbitrary weighted variety $\mathcal{X}$, first we cover it by finitely many open affine sets U_i such that on each U_i the line bundle $\mathcal{L}$ is trivialized with a non-vanishing section g_i. Letting $p_{j,t}$ be the coordinates on U_j with $p_{j0} = 1$, one can find constants C and γ (not depending on i and j) such that

$$\left| \frac{g_i(\mathbf{x})}{g_j(\mathbf{x})} \right|_v \leq C \cdot \max_t |p_{jt}|_v^{\gamma},$$

and hence for $\mathbf{x} \in U_i \cap U_j$ we have

$$|g_{ji}(\mathbf{x})|_v = \left| \frac{g_j(\mathbf{x})}{g_i(\mathbf{x})} \right|_v \geq \frac{1}{C \cdot \max_t |p_{jt}|_v^{\gamma}}.$$

Thus, for $\mathbf{x} \in U_i$, defining

$$\|g_i(\mathbf{x})\|_v = \max_t \min_{\{j:\ \mathbf{x} \in U_j\}} |p_{jt}|_v^{\gamma} \cdot \left| \frac{g_i(\mathbf{x})}{g_j(\mathbf{x})} \right|_v \tag{35}$$

we obtain the desired locally bounded weighted M-metric of $\mathcal{L}$ on U_i, which is independent of the choice of transition functions $g_{ji} = g_j/g_i$ over $U_i \cap U_j$. Using the cocycle rule $g_{ej} = g_{ei} g_{ij}$, for every $\mathbf{x} \in U_e \cap U_i$, we have

$$\|g_e(\mathbf{x})\|_v = |g_{ei}(\mathbf{x})|_v \cdot \|g_i(\mathbf{x})\|_v.$$

Therefore, Eq. (35) provides a well-defined M-metric of $\mathcal{L}$ on $\mathcal{X}$. By a similar argument as in the end of proof of [7, Prop. 2.7.5] or [18, B2.2.10], one can see that this is a locally bounded weightd metric. ☐

We denote by $\widehat{\mathrm{Pic}_{\mathfrak{q}}(\mathcal{X})}$ the group of isometric classes of pairs $\tilde{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$. As in the usual case, given any morphisms $\phi : \mathcal{X}' \to \mathcal{X}$ of weighted varieties over k, and $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}_{\mathfrak{q}}(\mathcal{X})}$, the pull-back of $\widehat{\mathcal{L}}$ by ϕ is defined as $\widehat{\phi^*(\mathcal{L})} = (\phi^*(\mathcal{L}), \|\cdot\|')$, such that

$$\|\phi^*(g)(\mathbf{x})\|' = \|g(\phi(\mathbf{x}))\| \ (\mathbf{x} \in \mathcal{X}'), \tag{36}$$

for any open subset U of $\mathcal{X}$ containing $\phi(\mathbf{x})$ and $g \in \mathcal{O}_{\mathcal{X}}(U)$.

The pull-back induces a group homomorphism between $\widehat{\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X})}$ and $\widehat{\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X}')}$. Under this homomorphism, any locally bounded weighted M-metrized line bundles remain locally bounded weighted M-metrized. Now we can define the weighted local Weil heights on a variety $\mathcal{X}$ in $\mathbb{P}^n_{\mathfrak{q},\bar{k}}$ as follows: Given any Cartier divisor $D = \{(U_i, f_i)\}$ on $\mathcal{X}$, we let $\mathcal{L}_D = \mathcal{O}_{\mathcal{X}}(D)$ be the line bundle of regular functions on D. It can be constructed by gluing

$$\mathcal{O}_{\mathcal{X}}(D)|_{U_i} = f_i^{-1}\mathcal{O}_{\mathcal{X}}(U_i)$$

and 1 becomes a canonical invertible meromorphic section of $\mathcal{L}_D$, which is denoted by g_D. Thus, by Thm. 4.3, we can equip $\mathcal{L}_D$ with a weighted locally bounded M-metric $\|\cdot\|$, determined by the max-min method in proof of Thm. 4.3, and denote it by $\widehat{D} = (\mathcal{L}_D, \|\cdot\|)$.

Definition 3. *Given $\nu \in M_k$, we define the* ***local weighted height*** $\zeta_{\widehat{D}}(-,\nu)$ ***with respect to*** $\widehat{D}$ *on the weighted variety $\mathcal{X}$ as*

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) := -\log \|g_D(\mathbf{x})\|_v \tag{37}$$

for $\mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D)$, where $v \in M$ such that $\nu = v|_k$.

We note that the local weighted height $\zeta_{\widehat{D}}(-,\nu)$ is well defined because the norm $\|\cdot\|$ is well-defined by its construction as it explained in proof of Thm. 4.3.

Here, we have the fundamental properties of the local weighted heights.

Theorem 4.4 (Weighted local Weil height machinery)**.** *For each of $\nu \in M_k$, fix $v \in M$ such that $\nu = v|_k$. Suppose that $\mathcal{X}$ is a weighted variety defined over k and $\widehat{D}, \widehat{D}_1, \widehat{D}_2 \in \widehat{\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X})}$. Then:*

(i) ***Additivity:*** *For $\mathbf{x} \notin \operatorname{Supp}(D_1) \cup \operatorname{Supp}(D_2)$, we have*

$$\zeta_{\widehat{D_1+D_2}}(\mathbf{x},\nu) = \zeta_{\widehat{D}_1}(\mathbf{x},\nu) + \zeta_{\widehat{D}_2}(\mathbf{x},\nu).$$

(ii) ***Functoriality:*** *If $\phi : \mathcal{X}' \to \mathcal{X}$ is a morphism of weighted varieties defined over k such that $\phi(\mathcal{X}') \cap \operatorname{Supp}(D) =$, then*

$$\zeta_{\phi^*(\widehat{D})}(\mathbf{x}',\nu) = \zeta_{\widehat{D}}(\phi(\mathbf{x}'),\nu) \text{ for } \mathbf{x}' \in \mathcal{X}' \backslash \phi^*(D).$$

(iii) ***Boundedness from below:*** *If D is effective and $\mathcal{X}$ is weighted M_k-bounded projective variety, then there exists an M_k-constant function γ such that*

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) \geq \gamma(\nu) \text{ for } \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D).$$

(iv) ***Normalization:*** *If $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},\bar{k}}$ and D is a hyperplane defined by $\ell \in \mathcal{O}_{\mathcal{X}}(m)$, with $m = \operatorname{lcm}(q_0, q_1, \cdots, q_n)$, then*

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) = -\log \frac{|\ell(\mathbf{x})|_v}{\max_i |x_i|_v^{\frac{m}{q_i}}} \quad \textit{for } \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D). \tag{38}$$

(v) ***Principal divisor:*** *If $D = div(f)$ for some nonzero $f \in \mathcal{O}_{\mathcal{X}}(D)$ with $\deg(f) = d$, then*

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) = -\log \frac{|f(\mathbf{x})|_v}{\max_i |x_i|_v^{\frac{d}{q_i}}}, \quad \textit{for } \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D), \tag{39}$$

by letting $\|1\|_v = |1|_v$ on $\mathcal{O}_{\mathcal{X}}(D)$ for $v \in M$ over $\nu \in M_k$.

(vi) ***Uniqueness:*** *If $\mathcal{X}$ is weighted M_k-bounded, $\|\cdot\|_v'$ is another weighted M_k-bounded metric on $\mathcal{L}_D$ and $\zeta'_{\widehat{D}}$ is the resulting local weighted Weil height respect to $(\mathcal{L}_D, \|\cdot\|')$, then*

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) = \zeta'_{\widehat{D}}(\mathbf{x},\nu) + O(1).$$

(vii) ***Base change:*** *If $K|k$ is a finite field extension and $u \in M_K$ over some $v \in M_k$, then*

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) = \frac{1}{[K_u : k_\nu]} \zeta_{\widehat{D'}}(\mathbf{x}', u), \quad \textit{for } \mathbf{x}' \in \mathcal{X}' \setminus \operatorname{Supp}(D'),$$

where $\mathcal{X}' = \mathcal{X} \otimes_k K$ and $\mathbf{x}' \in \mathcal{X}'$ corresponds to $\mathbf{x} \in \mathcal{X}(k)$, and D' $\operatorname{CaDiv}(\mathcal{X}')$ correspond to D.

(viii) ***Max-Min:*** *There are positive integers n_1 and n_2, and nonzero rational functions f_{ij} on $\mathcal{X}$ for $i = 0, \cdots, n_1$ and $j = 0, \cdots, n_2$ such that*

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) = \max_{0 \le i \le n_1} \min_{0 \le j \le n_2} \log |f_{ij}(\mathbf{x})|_\nu .$$

PROOF. The proofs are almost straightforward and similar to proof of the Weil local heights on projective heights.

(i) Using the product of weighted M-metrics from $\mathcal{O}_{\mathcal{X}}(D_1)$ and $\mathcal{O}_{\mathcal{X}}(D_2)$ on $\mathcal{O}_{\mathcal{X}}(D_1 + D_2)$, and $g_{D_1+D_2} = g_{D_1} \otimes g_{D_2}$, we have

$$\|g_{D_1+D_2}\|_\nu = \|g_{D_1} \otimes g_{D_2}\|_\nu = \|g_{D_1}\|_\nu \cdot \|g_{D_2}\|_\nu,$$

which implies the desired equality by taking logarithm from both sides.

(ii) The functoriality is a direct consequence of the functoriality of the weighted M-metrics $\|\cdot\| = (\|\cdot\|_v)$, i.e., $\|\phi^*(g_D)(\mathbf{x})\| = \|g_D(\phi(\mathbf{x}))\|$ for all $v \in M$.

(iii) Note that the rational function g_D is defined everywhere for any effective divisor D. Then, on bounded sets inside an affine open set U of $\mathcal{X}$ where $\mathcal{O}_{\mathcal{X}}(D)$ is trivial and so all global sections can be identified non-canonically as regular functions, $|g_D(\mathbf{x})|_v$ and is bounded above by an

M_k-constant. This implies that $\zeta_D(\mathbf{x}, \nu)$ is bounded below by an M_k-constant.

(iv) A locally M_k-bounded metric on $\mathcal{O}_{\mathcal{X}}(D) \cong \mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},\bar{k}}}(m)$ is given by Eq. (33) and hence $g_D = \ell$ is defined away from the hyperplane D. Given any $\nu \in M_k$ and fixing $v \in M$ such that $\nu = v|_k$, one can get (38) by taking logarithm.

(v) For a divisor $D = \operatorname{div}(f)$ with $\deg(f) = d$, we have $\mathcal{O}_{\mathcal{X}}(D) = f^{-1}\mathcal{O}_{\mathcal{X}}$ and $g_D = f$ whenever f is defined. Hence, for any v over ν, we have

$$\|f(\mathbf{x})\|_v = -\frac{|f(\mathbf{x})|_v}{\max_i |x_i|_v^{\frac{d}{q_i}}},$$

By taking logarithm, this implies Eq. (39) as desired,

(vi) Using (i) with $\widehat{D} = \widehat{D} + (0)$ where $\widehat{D}$ on the left hand side is endowed with $\|\cdot\|'$, then

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) - \zeta'_{\widehat{D}}(\mathbf{x}, \nu)$$

is the logarithm of norm of 1 with the locally bounded metric $\|\cdot\|_v / \|\cdot\|'_v$ on $\mathcal{O}(\mathcal{X})$. Since 1 is a global nowhere-vanishing section, by the definition, we have $\zeta_{\widehat{D}}(\mathbf{x}, \nu) = \zeta'_{\widehat{D}}(\mathbf{x}, \nu) + O(1)$.

(vii) Since $|\cdot|_v = |\cdot|_u^{1/[K_u:k_v]}$ for $u \in M_K$ over $v \in M_k$, so $\|\cdot\|_\nu = \|\cdot\|_u^{1/[K_u:k_v]}$ and hence the desired equality.

(viii) By linearity of the both sides of equality,

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = \max_{0 \le i \le n_1} \min_{0 \le j \le n_2} \log |f_{ij}(\mathbf{x})|_\nu$$

and the proof of Thm. 4.3, it is enough to consider $\widehat{D}$ such that $\mathcal{O}_{\mathcal{X}}(D) \cong \mathcal{O}_{\mathcal{X}}(m)$. In this case, the existence of f_{ij}'s is clear by the proof of Eq. (33). □

4.4. **Global weighted heights.** Now, we assume $\mathcal{X} \subseteq \mathbb{P}^n_{\mathfrak{q}}(\bar{k})$ is a weighted variety and consider $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X})}$. Given $\mathbf{x} \in \mathcal{X}$, let K be a finite extension of k containing $k(\mathbf{x})$. For each $u \in M_K$, we choose a place $v \in M$ over u and define $\|\cdot\|_u := \|\cdot\|_v^{1/[K:k]}$ on $\mathcal{L}_{\mathbf{x}}(k(\mathbf{x}))$. By the second condition of a weighted M-metric, one can see that it is independent of the choice of $v \in M$. We let g be an invertible regular function of $\mathcal{L}$ with $\mathbf{x} \notin \operatorname{Supp}(\mathcal{L}_g)$ where $\mathcal{L}_g = \operatorname{div}(g)$. Note that such function exists because there is an open dense trivialization in a neighborhood of the point $\mathbf{x}$. Then, we have the weighted M-metrized line bundle $\widehat{\mathcal{L}_g} = (\mathcal{O}_{\mathcal{X}}(L_g), (\|\cdot\|_u)) \in \widehat{\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X})}$.

The **global weighted height** $\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x})$ with respect to $\widehat{\mathcal{L}}$ is defined by

$$\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x}) := \sum_{u \in M_K} \zeta_{\widehat{\mathcal{L}_g}}(\mathbf{x}, u), \tag{40}$$

where $\zeta_{\widehat{\mathcal{L}_g}}(\mathbf{x}, u) = -\log \|g(\mathbf{x})\|_u$ assuming $v|_k = u$. It is easy to check that these definitions are independent of the choice of field K and regular function g.

Example 6. *Let $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},\bar{k}}$, $D = div(x_0^{1/q_i})$, and $\mathcal{L} = \mathcal{O}(D)$. Then, one has $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x})$, where $\mathfrak{s}(\mathbf{x})$ is the global weighted height on $\mathbb{P}^n_{\mathfrak{q},\bar{k}}$ given by Eq. (27).*

Indeed, if $K = k(\mathbf{x})$ and $u \in M_K$ over $\nu \in M_k$, Eq. (38) becomes

$$\zeta_{\widehat{D}}(\mathbf{x}, u) = -\log \frac{\left| x_0^{\frac{1}{q_0}} \right|_u}{\max_i \left| x_i^{\frac{1}{q_i}} \right|_u}, \quad \text{for } \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D). \tag{41}$$

Since $\zeta_{\widehat{\mathcal{L}_{x_0}}}(\mathbf{x}, u)$ and $\zeta_{\widehat{D}}(\mathbf{x}, u)$ are same local height, we have

$$\begin{aligned}
\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x}) &= \sum_{u \in M_K} \zeta_{\widehat{\mathcal{L}_{x_0}}}(\mathbf{x}, u) = \sum_{u \in M_K} -\log \frac{\left| x_0^{\frac{1}{q_0}} \right|_u}{\max_i \left| x_i^{\frac{1}{q_i}} \right|_u} \\
&= \sum_{u \in M_K} \frac{1}{q_i} \log \max_i |x_i|_u - \frac{1}{q_0} \sum_{u \in M_K} \log |x_0|_u .
\end{aligned}$$

The last term vanishes by product formula and using Thm. 4.4 (vi), we have

$$\begin{aligned}
\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x}) &= \sum_{u \in M_K} \frac{1}{q_i} \log \max_i |x_i|_v = \sum_{v \in M_k} \sum_{u|v} \frac{1}{[K_u : k_v] q_i} \log \max_i |x_i|_v \\
&= \frac{1}{[K:k]} \cdot \sum_{v \in M_k} \max_i \left\{ \frac{1}{q_i} \cdot \log |x_i|_v \right\} = \mathfrak{s}(\mathbf{x}).
\end{aligned}$$

The above example shows the normalization property of the weighted global Weil height function, and their other essential properties are given by the following theorem.

Theorem 4.5 (Global weighted height machinery)**.** *Let $\mathcal{X}$ be a weighted variety and consider $\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_1$, and $\widehat{\mathcal{L}}_2 \in \widehat{\operatorname{Pic}(\mathcal{X})}$.*

(i) ***Independence (a):*** *$\mathfrak{s}_{\widehat{\mathcal{L}}}$ depends only on the isometry class of $\widehat{\mathcal{L}}$, i.e, if $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ are isometric pairs, then $\mathfrak{s}_{\widehat{\mathcal{L}}_1} = \mathfrak{s}_{\widehat{\mathcal{L}}_2}$.*

(ii) ***Independence (b):*** *If $\mathcal{X}$ is a complete weighted variety or generally M-bounded, then $\mathfrak{s}_{\widehat{\mathcal{L}}}$ does not depend on the choice of weighted locally bounded M-metrics up to a locally M-bounded constant function.*

(iii) ***Additivity:*** *For any $\mathbf{x} \in \mathcal{X}$, we have $\mathfrak{s}_{\widehat{\mathcal{L}}_1 \otimes \widehat{\mathcal{L}}_2}(\mathbf{x}) = \mathfrak{s}_{\widehat{\mathcal{L}}_1}(\mathbf{x}) + \mathfrak{s}_{\widehat{\mathcal{L}}_2}(\mathbf{x})$.*

(iv) ***Functoriality:*** *If $\phi : \mathcal{X}' \to \mathcal{X}$ is a morphism of weighted varieties over k, then*

$$\mathfrak{s}_{\phi^*(\widehat{\mathcal{L}})}(\mathbf{x}) = \mathfrak{s}_{\widehat{\mathcal{L}}}(\phi(\mathbf{x})) \text{ for } \mathbf{x} \in \mathcal{X}.$$

(v) ***Base change:*** *If $K|k$ is a finite field extension, then*

$$\mathfrak{s}_{\widehat{D}}(\mathbf{x}) = \frac{1}{[K:k]} \mathfrak{s}_{\widehat{D'}}(\mathbf{x}'), \text{ for } \mathbf{x}' \in \mathcal{X}' \backslash \operatorname{Supp}(D').$$

where $\mathcal{X}' = \mathcal{X} \otimes_k K$ and $\mathbf{x}' \in \mathcal{X}'$ corresponds to $\mathbf{x} \in \mathcal{X}(k)$, and D' $\operatorname{CaDiv}(\mathcal{X}')$ correspond to D.

(vi) *If $\widehat{\mathcal{L}}$ is a line bundle on $\mathcal{X}$, generated by its global sections, then $\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x})$ is bounded from below for all $\mathbf{x} \in \mathcal{X}(\bar{k})$, by a constant depending on $\widehat{\mathcal{L}}$.*

Proof. The proof is essentially similar to the proof of Thm. 2.3. The part (i) is obvious by definitions. One may conclude part (ii) using (iii) of Thm. 4.4 and the definitions. The part (iii) comes from (i) of Thm. 4.4, and (iv) is a consequence of (ii) of Thm. 4.4. The part (v) comes by (vii) of Thm. 4.4, and (vi) is a result of (iii) of Thm. 4.4.

∎

4.5. **Weighted local and global heights for closed subschemes.** The local and global heights for closed subschemes of projective varieties are introduced in [20]. Here, we develop them to the closed subschems of weighted projective varieties.

Fix a weighted projective variety $\mathcal{X}$ in $\mathbb{P}^n_{\mathfrak{q},\bar{k}}$ defined over k, i.e., an integral and separated subscheme of finite type. Given any closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ over k, we let $\mathcal{I}_\mathcal{Y}$ denotes the corresponding sheaf of ideals generated by $f_1, \cdots, f_r \in k_\mathfrak{q}[x_0, x_1, \cdots, x_n]$ with $\deg(f_j) = d_j$ for $j = 1, \cdots r$. Letting $D_j := \operatorname{div}(f_j)$ for $j = 1, \cdots, r$ and $\nu \in M_k$, we define

$$\zeta_\mathcal{Y}(\cdot, \nu) : (\mathcal{X} \backslash \mathcal{Y})(k) \to \mathbb{R},$$

the **local weighted height associated to** $\mathcal{Y}$, by

$$\zeta_\mathcal{Y}(\mathbf{x}, \nu) := \min_{1 \le j \le r} \{\zeta_{\widehat{D_j}}(\mathbf{x}, \nu)\} = \min_{1 \le j \le r} \left\{ -\log \frac{|f_j(\mathbf{x})|_\nu}{\max_i |x_i|_\nu^{\frac{d_j}{q_i}}} \right\}. \tag{42}$$

By convention, we define $\zeta_{\mathcal{Y}}(\mathbf{x}, \nu) = \infty$ for $\mathbf{x} \in \mathcal{Y}(k)$. One can show that this is unique up to a weighted M_k-bounded function by a similar argument for the projective varieties.

Recall that for closed subschemes $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of $\mathcal{X}$ defined over k with corresponding ideal sheaves $\mathcal{I}_{\mathcal{Y}_1}$, $\mathcal{I}_{\mathcal{Y}_2}$, the closed subschemes $\mathcal{Y}_1 \cap \mathcal{Y}_2$, $\mathcal{Y}_1 + \mathcal{Y}_2$, and $\mathcal{Y}_1 \cup \mathcal{Y}_2$ are defined by ideal sheaves $\mathcal{I}_{\mathcal{Y}_1} + \mathcal{I}_{\mathcal{Y}_2}$, $\mathcal{I}_{\mathcal{Y}_1}\mathcal{I}_{\mathcal{Y}_2}$, and $\mathcal{I}_{\mathcal{Y}_1} \cap \mathcal{I}_{\mathcal{Y}_2}$ respectively. Note that $\mathcal{Y}_1 \cup \mathcal{Y}_2 \subset \mathcal{Y}_1 + \mathcal{Y}_2 \subset \mathcal{X}$ as schemes, since $\mathcal{I}_{\mathcal{Y}_1}\mathcal{I}_{\mathcal{Y}_2} \subset \mathcal{I}_{\mathcal{Y}_1} \cap \mathcal{I}_{\mathcal{Y}_2}$

The basic properties of weighted local heights associated to closed subschemes are given in the following proposition.

Proposition 4.6. *For any $\nu \in M_k$, and a closed subscheme $\mathcal{Y}$ of a weighted projective variety $\mathcal{X}$, the following hold:*

(1) $\zeta_{\mathcal{Y}_1 \cap \mathcal{Y}_2}(\cdot, \nu) = \min\{\zeta_{\mathcal{Y}_1}(\cdot, \nu), \zeta_{\mathcal{Y}_2}(\cdot, \nu)\}$;
(2) $\zeta_{\mathcal{Y}_1 + \mathcal{Y}_2}(\cdot, \nu) = \zeta_{\mathcal{Y}_1}(\cdot, \nu) + \zeta_{\mathcal{Y}_2}(\cdot, \nu)$;
(3) $\zeta_{\mathcal{Y}_1}(\cdot, \nu) \leq \zeta_{\mathcal{Y}_2}(\cdot, \nu)$ *if* $\mathcal{Y}_1 \subset \mathcal{Y}_2$;
(4) $\max\{\zeta_{\mathcal{Y}_1}(\cdot, \nu), \zeta_{\mathcal{Y}_2}(\cdot, \nu)\} \leq \zeta_{\mathcal{Y}_1 \cup \mathcal{Y}_2}(\cdot, \nu) \leq \zeta_{\mathcal{Y}_1}(\cdot, \nu) + \zeta_{\mathcal{Y}_2}(\cdot, \nu)$;
(5) $\zeta_{\mathcal{Y}_1}(\cdot, \nu) \leq c \cdot \zeta_{\mathcal{Y}_2}(\cdot, \nu)$ *if* $\operatorname{Supp}(\mathcal{Y}_1) \subset \operatorname{Supp}(\mathcal{Y}_2)$ *for some constant* $c > 0$, *where* $\operatorname{Supp}(\mathcal{Y})$ *denotes the support of* $\mathcal{Y}$;
(6) *If* $\mathcal{Y} = D$ *is an effective divisor, then* $\zeta_{\mathcal{Y}}(\cdot, \nu)$ *is equal* $\zeta_{\widehat{D}}(\cdot, \nu)$ *defined by Eq. (37), where* $\widehat{D} = (\mathcal{O}_{\mathcal{X}}(D), \|\cdot\|) \in \widehat{\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X})}$;
(7) *If* $\phi : \mathcal{X}' \to \mathcal{X}$ *is a morphism of weighted projective varieties,* $\mathcal{Y} \subset \mathcal{X}$ *a closed subscheme over* k, *and* $\phi^*(\mathcal{Y})$ *denotes the closed subscheme of* $\mathcal{X}'$ *associated to ideal sheaf* $\phi^{-1}\mathcal{I}_{\mathcal{Y}} \cdot \mathcal{O}_{\mathcal{X}'}$, *then* $\zeta_{\phi^*(\mathcal{Y})}(\mathbf{x}, \nu) = \zeta_{\mathcal{Y}}(\phi(\mathbf{x}), \nu)$ *for* $\mathbf{x} \in (\mathcal{X}' \backslash \phi^*(\mathcal{Y}))(k)$.

The **global weighted height associated to** $\mathcal{Y}$, can be defined up to a bounded function by summing all local weighted heights. More precisely, given $\mathbf{x} \in \mathcal{X}$, we let K be a finite extension of k containing $k(\mathbf{x})$ and define:

$$\mathfrak{s}_{\mathcal{Y}}(\mathbf{x}) := \sum_{u \in M_K} \zeta_{\mathcal{Y}}(\mathbf{x}, u), \tag{43}$$

which is independent of the choice of the field K. The weighted global heights satisfy similar properties, except the first one, as given in Thm. 4.6 for the weighted local heights.

Proposition 4.7. *For any $\nu \in M_k$, and a closed subscheme $\mathcal{Y}$ of a weighted projective variety $\mathcal{X}$ the following hold:*

(1) $\mathfrak{s}_{\mathcal{Y}_1 \cap \mathcal{Y}_2} \leq \min\{\mathfrak{s}_{\mathcal{Y}_1}, \mathfrak{s}_{\mathcal{Y}_2}\}$;
(2) $\mathfrak{s}_{\mathcal{Y}_1 + \mathcal{Y}_2} = \mathfrak{s}_{\mathcal{Y}_1} + \mathfrak{s}_{\mathcal{Y}_2}$;

(3) $\mathfrak{s}_{\mathcal{Y}_1} \leq \mathfrak{s}_{\mathcal{Y}_2}$ *if* $\mathcal{Y}_1 \subset \mathcal{Y}_2$*;*
(4) $\max\{\mathfrak{s}_{\mathcal{Y}_1}, \mathfrak{s}_{\mathcal{Y}_2}\} \leq \mathfrak{s}_{\mathcal{Y}_1 \cup \mathcal{Y}_2} \leq \mathfrak{s}_{\mathcal{Y}_1} + \mathfrak{s}_{\mathcal{Y}_2}$;
(5) $\mathfrak{s}_{\mathcal{Y}_1} \leq c \cdot \mathfrak{s}_{\mathcal{Y}_2}$ *if* $\operatorname{Supp}(\mathcal{Y}_1) \subset \operatorname{Supp}(\mathcal{Y}_2)$ *for some constant* $c > 0$*;*
(6) *If* $\mathcal{Y} = D$ *is an effective divisor, then* $\mathfrak{s}_{\mathcal{Y}}$ *is equal to* $\mathfrak{s}_{\widehat{D}}$ *defined by Eq.* (40)*, where* $\widehat{D} = (\mathcal{O}_{\mathcal{X}}(D), (\| \cdot \|_u)) \in \widehat{\operatorname{Pic}_{\mathfrak{q}}(\mathcal{X})}$*;*
(7) *If* $\phi : \mathcal{X}' \to \mathcal{X}$ *is a morphism of weighted projective varieties,* $\mathcal{Y} \subset \mathcal{X}$ *a closed subscheme over* k*, then* $\mathfrak{s}_{\phi^*(\mathcal{Y})} = \mathfrak{s}_{\mathcal{Y}} \circ \phi$*.*

All of the above assertions follow by summing from the corresponding properties for the local weighted heights associated to subschemes. When we want to emphasize on the base weighted variety $\mathcal{X}$ in any of the previously defined global weighted heights, we will put it as a subscript on them for example $\mathfrak{s}_{\mathcal{X},D}$ and $\mathfrak{s}_{\mathcal{X},\mathcal{Y}}$.

5. Conclusion

This work is devoted to develop a detailed theory of Cartier divisors, analytic structure of weighted varieties, weighted blow-ups. While it was believed that these results could be recovered from the Veronese embedding it is the first time that a direct approach is presented.

Weighted projective spaces are very natural objects which makes the theory of weighted heights a powerful tool of arithmetic geometry. However, connections of weighted heights with other heights such as GIT height, Neron-Tate height, Faltings height, etc are not well understood. Some glimpses of the connection between weighted heights and GIT height can be seen in [8], but overall this is an area that offers many open questions. Vojta's conjecture for weighted varieties in terms of weighted heights is studied in [19].

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