VOJTA'S CONJECTURE ON WEIGHTED PROJECTIVE VARIETIES AND AN APPLICATION ON GREATEST COMMON DIVISORS

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ABSTRACT. We state Vojta's conjecture for smooth weighted projective varieties, weighted multiplier ideal sheaves, and weighted log pairs and prove that all three versions of the conjecture are equivalent. Moreover, we introduce generalized weighted general common divisors and express them as heights of weighted projective spaces blown-up at a point, relative to an exceptional divisor. Furthermore, we also prove that assuming Vojta's conjecture for weighted projective varieties one can bound the log $\mathrm{h}_{\mathrm{wgcd}}$ for any subvariety of codimension ≥ 2 and a finite set of places S. An analogue result is proved for weighted homogeneous polynomials with integer coefficients. As an application of our results we obtain a bound on greatest common divisors, which restricted to projective space is the same as bounds obtained by Corvaja, Zannier, et al.

1. INTRODUCTION

The theory of weighted local and global heights for weighted projective varieties and closed subvarieties was introduced in [10], where it was proved that any line bundle on a weighted variety admits a locally bounded weighted M-metric. Using these results we are able to generalize weighted general common divisors to generalized weighted **gcds** and express them as a weighted height of a blow-up relative to a weighted exceptional divisor. Moreover, we are able to state Vojta's conjecture for smooth weighted projective varieties in terms of weighted heights.

Stating results on Vojta's conjecture in terms of weighted heights is not simply a curiosity. Weighted heights provide much better bounds than classical projective bounds. This paper started from a question of J. Silverman whether weighted general common divisors (**gcds**) introduced in [1] can be extended to weighted generalized **gcds** as in [11], for example express them as a weighted height of a blown-up relative to an exceptional divisor. We answer positively this question and further explore it in terms of the Vojta's conjecture. A corollary of our main result (cf. Lem. 4) is an interpretation of well known result of Corvaja and Zannier, et al. on heights on rational functions on S-unit points; see [2–5].

This paper is organized as follows: In Sec. 2.1 is given a quick view on Vojta's conjecture on algebraic points on projective varieties; see Conj. 1. We state the conjecture using a correction term involving a multiplier ideal sheaf instead of using the normal crossing divisors; see Conj. 2 and summarize [15] on Vojta's conjecture for log pairs. This makes it possible to drop the condition that the variety be smooth in the statement of the conjecture. Vojta's conjecture for log pairs is stated

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in Conj. 3. In Sec. 3 we investigate whether it is possible to have analog statements for weighted varieties and weighted heights. In Conj. 4 we state Vojta's conjecture for $\mathcal{X}$ a smooth weighted projective variety, $K_\mathcal{X}$ a canonical divisor, $\mathcal{A}$ an ample divisor and D a normal crossings divisor on $\mathcal{X}$, all defined over $\mathbb{Q}$. An analogue of Conj. 2 for weighted projective varieties is stated in Conj. 5. The terminology and theory for weighted log pairs is developed in this section, so we are able to state Vojta's conjecture for weighted log pairs in Conj. 6. Finally, in Cor. 1 we prove that Conj. 4, Conj. 5, and Conj. 6 are equivalent.

In Sec. 4 we extend the concept of the *generalized greatest common divisor* as in [11] to that of *generalized weighted greatest common divisor*. Furthermore, we prove that generalized logarithmic weighted greatest common divisor is equal to weighted height of $\mathbf{x}$ on a blowup of $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ with respect to the exceptional divisor of the blowup. We prove that the generalized logarithmic weighted greatest common divisor $\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) > 0$ if and only if $\mathbf{x} \notin \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ (cf. Prop. 2) and analogues of Theorems 1, 2, and 6 in [11] for the weighted **gcds** that are all subject to the validity of Vojta's conjecture for weighted projective varieties.

More precisely, we prove (cf. Thm. 1) that for $\mathcal{X}$ be a smooth weighted variety defined over $\mathbb{Q}$, $\mathcal{A}$ an ample divisor on $\mathcal{X}$, $\mathcal{Y} \subset \mathcal{X}$ a smooth subvariety of codimension $r \geq 2$, and $-K_\mathcal{X}$ a normal crossing divisor whose support does not intersect $\mathcal{Y}$, assuming Conj. 4, for every finite set of places S and every $0 < \varepsilon < r-1$ there is a proper closed subvariety $\mathcal{Z} = \mathcal{Z}(\varepsilon, \mathcal{X}, \mathcal{Y}, \mathcal{A}, k, S) \not\subset \mathcal{X}$, and constants $C_\varepsilon = C_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A}, k, S)$ and $\delta_\varepsilon = \delta_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A})$, such that for all $P \in (\mathcal{X} \setminus \mathcal{Z}(\mathbb{Q}))$

$$\log \mathrm{h}_{\mathrm{wgcd}}(P; \mathcal{Y}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) + \frac{1}{r-1+\delta_\varepsilon}\, \mathfrak{s}'_{\mathcal{X},-K_\mathcal{X},S}(P) + C_\varepsilon. \tag{1}$$

Let $\mathfrak{q} = (q_0, \cdots, q_n)$ be a well-formed set of weights, $m = \mathrm{lcm}(q_0, \cdots, q_n)$, and $\mathcal{Z} \subset \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ be a closed subvariety defined by $f_1, \cdots, f_t \in \mathbb{Z}_\mathfrak{q}[x_0, \ldots, x_n]$, such that $\mathcal{Z} \cap \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \emptyset$, with codimension $r = n - \dim(\mathcal{Z}) \geq 2$ in $\mathcal{X}$. Let S be a finite set of primes and $\varepsilon > 0$. If Vojta's conjecture holds for smooth weighted varieties (see Conj. 4), then there exists a nonzero weighted polynomial $g \in \mathbb{Z}_\mathfrak{q}[x_0, \ldots, x_n]$ and a constant $\delta = \delta_{\varepsilon,\mathcal{Z}} > 0$, such that every $\tilde{\alpha} = (\alpha_0, \cdots, \alpha_n) \in \mathbb{Z}^{n+1}$ with $\mathrm{wgcd}(\alpha_0, \cdots, \alpha_n) = 1$ satisfies either $g(\tilde{\alpha}) = 0$ or

$$\gcd(f_1(\tilde{\alpha}), \cdots, f_t(\tilde{\alpha})) \leq \max\left\{ |\alpha_0|^{\frac{1}{q_0}}, \cdots, |\alpha_n|^{\frac{1}{q_n}} \right\}^{\varepsilon} \cdot \left(|\alpha_0 \cdots \alpha_n|'_S \right)^{\frac{1}{(r-1+\delta)}}, \tag{2}$$

where $|\cdot|'_S$ is the "prime-to-S" part of its origin (cf. Thm. 2).

Assuming Vojta's conjecture for weighted projective varieties, for a fixed $\varepsilon > 0$, a finite set S of prime numbers, and a triple of weights $\mathfrak{q} = (q_0, q_1, q_2)$, there exist a finite set $\mathcal{Z} = \mathcal{Z}(S, \varepsilon) \subset \mathbb{Z}^2$ such that

$$\gcd(\alpha_1^{q_0} - 1, \alpha_2^{q_0} - 1) \leq \max\{|\alpha_1|^{\frac{1}{q_1}}, |\alpha_2|^{\frac{1}{q_2}}\}^{\varepsilon} \cdot \left(|\alpha_1\alpha_2|'_S\right)^{\frac{1}{(1+\delta_\varepsilon)}}, \tag{3}$$

holds for all pairs $(\alpha_1, \alpha_2) \in \mathbb{Z}^2 \backslash Z$ (cf. Lem. 4). There is an interesting consequence of this result

$$\gcd(\alpha_1 - 1, \alpha_2 - 1) \leq \max\{|\alpha_1|^{\frac{1}{q_1}}, |\alpha_2|^{\frac{1}{q_2}}\}^{\varepsilon} \cdot \left(|\alpha_1\alpha_2|'_S\right)^{\frac{1}{(1+\delta_\varepsilon)}}, \tag{4}$$

which improves the bound on $\gcd(\alpha_1 - 1, \alpha_2 - 1)$ from results of Corvaja and Zannier; see [3]. It remains to be seen if Thm. 2 and Lem. 4 can be proved independently of Vojta's conjecture. This remains a goal of further investigation.

2. Preliminaries

Here we give a brief overview of Vojta's conjecture over projective varieties before consider the conjecture over weighted projective varieties.

2.1. **Vojta's conjecture for projective varieties.** For any finite extension $L/\mathbb{Q}$ of a number field $\mathbb{Q}$, we define the **logarithmic discriminant** $d_{\mathbb{Q}}(L)$ by

$$d_{\mathbb{Q}}(L) := \frac{1}{[L:k]} \log|\operatorname{Disc}(L)| - \log|\operatorname{Disc}(\mathbb{Q})|,$$

where $\operatorname{Disc}(\cdot)$ denotes the absolute discriminant. Given a variety $\mathcal{X}$ over $\mathbb{Q}$ and a point $\mathbf{x} \in \mathcal{X}$, we define its **logarithmic discriminant** by $d_{\mathbb{Q}}(\mathbf{x}) := d_{\mathbb{Q}}(k(\mathbf{x}))$.

Recall that a Cartier divisor D on a smooth projective variety $\mathcal{X}$ is a **normal crossing divisor** if at every point in the support of D there are local coordinates $z_0, z_1, \ldots, z_n$ such that D is given locally by an equation of the form $z_0 z_1 \ldots z_n = 0$. Furthermore, the **canonical divisor** of $\mathcal{X}$ is a divisor $K_{\mathcal{X}}$ such that $\mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}}) = \omega_{\mathcal{X}}$, where $\omega_{\mathcal{X}}$ is the canonical sheaf of regular forms on $\mathcal{X}$. Vojta made a conjecture on algebraic points on projective varieties ([12, 13]), as follows:

Conjecture 1. *Let $\mathcal{X}$ be a smooth projective variety over a number field $\mathbb{Q}$, $K_{\mathcal{X}}$ a canonical divisor, $\mathcal{A}$ an ample divisor, and D a normal crossings divisor on $\mathcal{X}$, all defined over $\mathbb{Q}$. Furthermore, let S be a finite subset of places containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and any positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $k, \mathcal{X}, D, \mathcal{A}, \varepsilon$, and r, such that*

$$h_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \lambda_D(\mathbf{x}, \nu) \leq \varepsilon h_{\mathcal{A}}(\mathbf{x}) + d_{\mathbb{Q}}(\mathbf{x}) + O(1),$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{k})$ with $[k(\mathbf{x}) : k] \leq r$.

The case $r = 1$ of the above conjecture is known in the literature as Vojta's conjecture for the rational points of algebraic varieties . In [14], Vojta restated his conjecture using a correction term involving a multiplier ideal sheaf instead of the normal crossing divisors as follows.

Let $\mathcal{I}$ be a nonzero sheaf of ideals on a projective variety $\mathcal{X}$ and $c \in \mathbb{R}^{\geq 0}$ some constant. Let $f : \mathcal{X}' \to \mathcal{X}$ be a proper birational morphism such that $\mathcal{X}'$ is a smooth variety and $f^*\mathcal{I} = \mathcal{O}_{\mathcal{X}'}(-E)$, for some normal crossing divisor E on $\mathcal{X}'$. Denote by $\mathcal{R}_{\mathcal{X}'/\mathcal{X}}$ the ramification divisor of $\mathcal{X}'$ over $\mathcal{X}$ and define the **multiplier ideal sheaf** $\mathcal{I}_c$ and $\mathcal{I}_c^-$ associated to $\mathcal{I}$ and c as

$$\mathcal{I}_c := f_* \mathcal{O}_{\mathcal{X}'}(\mathcal{R}_{\mathcal{X}'/\mathcal{X}} - \lfloor cE \rfloor), \text{ and } \mathcal{I}_c^- := \lim_{\varepsilon \to 0^+} \mathcal{I}_{c-\varepsilon}.$$

We will denote $\mathcal{I}_1$ and $\mathcal{I}_1^-$ by $\mathcal{I}$ and $\mathcal{I}^-$ respectively.

Conjecture 2. *Let $\mathcal{X}$ be a smooth projective variety over a number field $\mathbb{Q}$, $K_{\mathcal{X}}$ a canonical divisor, $\mathcal{A}$ an ample divisor, and $\mathcal{I}$ a nonzero ideal sheaf $\mathcal{X}$, all defined over $\mathbb{Q}$. Let S be a finite subset of places containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $k, \mathcal{X}, \mathcal{I}, \mathcal{A}, \varepsilon, r$, such that*

$$h_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \lambda_{\mathcal{I}}(\mathbf{x}, \nu) - \sum_{\nu \in S} \lambda_{\mathcal{I}^-}(\mathbf{x}, \nu) \leq \varepsilon h_{\mathcal{A}}(\mathbf{x}) + d_{\mathbb{Q}}(\mathbf{x}) + O(1).$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{k})$ with $[k(\mathbf{x}) : k] \leq r$.

2.2. **Vojta's conjecture for log pairs.** Since Vojta's conjectures does not deal with singular varieties in [15] Yasuda formulated a generalization of it in terms of log pairs and variants of multiplier ideals. In order to state his generalization, first we need to recall some terminology. The reader can refer to [8], [7], or [15] for more details.

Let $\mathcal{X}$ be a variety defined over $\mathbb{Q}$. Then, $\mathcal{X}$ is said to be **$\mathbb{Q}$-Gorenstein** if it is Gorenstein in codimension one, satisfies Serre's condition S_2, and a canonical divisor $K_\mathcal{X}$ is $\mathbb{Q}$-Cartier. For example, if $\mathcal{X}$ is normal, then the first two conditions are true automatically and hence a canonical divisor exists unequally up to linear equivalence and is Cartier in codimension one.

A **$\mathbb{Q}$-subscheme** of $\mathcal{X}$ is a formal linear combination $\mathcal{Y} = \sum_{i=1}^m c_i \cdot \mathcal{Y}_i$ of proper closed subschemes $\mathcal{Y}_i \subset \mathcal{X}$ with all $c_i \in \mathbb{Q}$. The support of such $\mathcal{Y}$ is defined to be the closed subset $\cup_{c_i \neq 0} \mathcal{Y}_i$, and it is called **effective $\mathbb{Q}$-subscheme** if $c_i \geq 0$ for every i. By a **log pair**, we mean a pair $(\mathcal{X}, \mathcal{Y})$ of a $\mathbb{Q}$-Gorenstein variety and an effective $\mathbb{Q}$-subscheme $\mathcal{Y}$ of $\mathcal{X}$. For example, if $\mathcal{X}$ is a normal $\mathbb{Q}$-Gorenstein and D is an effective $\mathbb{Q}$-divisor, then $(\mathcal{X}, D)$ is a log pair.

A **resolution** of $\mathcal{X}$ over $\mathbb{Q}$ is a projective birational morphism $f : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is a smooth variety over $\mathbb{Q}$. By a **log resolution** of a log pair $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{Y} = \sum_{i=1}^m c_i \cdot \mathcal{Y}_i$, we mean a resolution $f : \mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}$ such that the set-theoretic inverse image $f^{-1}(\mathcal{Y}_i)$ is a Cartier divisor on $\mathcal{X}'$, and the union of exceptional divisor $\mathrm{Exc}(f)$ of f with all $f^{-1}(\mathcal{Y}_i)_{\mathrm{red}}$ is a simple normal crossing divisor of $\mathcal{X}'$. For a log resolution $f : \mathcal{X}' \to \mathcal{X}$ os a log pair $(\mathcal{X}, \mathcal{Y})$, the **relative canonical divisor** of $\mathcal{X}'$ over $(\mathcal{X}, \mathcal{Y})$ is defined to be the $\mathbb{Q}$-Weil divisor

$$K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} = K_{\mathcal{X}'/\mathcal{X}} - f^*\mathcal{Y},$$

where $K_{\mathcal{X}'/\mathcal{X}}$ is the relative canonical divisor of $\mathcal{X}'$ over $\mathcal{X}$, and $f^*\mathcal{Y}$ is the pull-back of $\mathcal{Y}$ by f over $\mathcal{X}'$.

For a log pair $(\mathcal{X}, \mathcal{Y})$ with a log resolution $f : \mathcal{X}' \to \mathcal{X}$, we define $\mathcal{I}(\mathcal{X}, \mathcal{Y})$ a **variant of multiplier sheaf** as

$$\mathcal{I}(\mathcal{X}, \mathcal{Y}) := f_* \mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil)$$

if $\mathcal{X}$ is a normal variety, otherwise, we let

$$\mathcal{I}(\mathcal{X}, \mathcal{Y}) := \bar{f}_* \mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil),$$

where $\bar{f}_* \mathcal{O}_{\mathcal{X}'}(E)$ denotes the largest ideal sheaf in $\mathcal{O}_\mathcal{X}$ for which its pull-back by f is contained in $\mathcal{O}_{\mathcal{X}'}(E)$ as an $\mathcal{O}_{\mathcal{X}'}$-submodule of (constant) function field sheaf $\mathcal{M}_{\mathcal{X}'}$. Moreover, there exist a constant $\varepsilon_0 > 0$ such that for every rational number $0 < \varepsilon \leq \varepsilon_0$, one has $\mathcal{I}(\mathcal{X}, (1-\varepsilon)\mathcal{Y}) = \mathcal{I}(\mathcal{X}, (1-\varepsilon_0)\mathcal{Y})$. Based on this fact, we let

$$\mathcal{I}^-(\mathcal{X}, \mathcal{Y}) := \mathcal{I}(\mathcal{X}, (1-\varepsilon)\mathcal{Y}), \ (0 < \varepsilon \ll 1).$$

We also define another ideal sheaf as

$$\mathcal{H}(\mathcal{X}, \mathcal{Y}) := \bar{f}_* \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor),$$

where $\bar{f}_*$ is as above. The definitions of $\mathcal{I}(\mathcal{X}, \mathcal{Y})$, and hence $\mathcal{I}^-(\mathcal{X}, \mathcal{Y})$, as well as $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ are independent of the choice of a log resolution by [15, Lem. 3.1] and [15, Prop. 3.4] respectively.

For a $\mathbb{Q}$-Gorenstein projective variety $\mathcal{X}$ with a canonical divisor $K_\mathcal{X}$, we can define a global height function $h_{K_\mathcal{X}}$ up to addition of a bounded function. For a log pair $(\mathcal{X}, \mathcal{Y})$ of a $\mathbb{Q}$-Gorenstein $\mathcal{X}$, we define the height function associated to the subscheme $K_{(\mathcal{X},\mathcal{Y})} := K_{\mathcal{X}} + \mathcal{Y}$ as

$$h_{K_{(\mathcal{X},\mathcal{Y})}} = h_{K_{\mathcal{X}}} + h_{\mathcal{Y}}, \tag{5}$$

where $h_{\mathcal{Y}}$ is the height function associated to the subscheme $\mathcal{Y}$ or its ideal sheaf. Next is Yasuda's generalization of Vojta's conjecture for algebraic points.

Conjecture 3. *Let $(\mathcal{X}, \mathcal{Y})$ a log pair with projective $\mathcal{X}$, $\mathcal{Y}$ a closed subscheme with ideal sheaf $\mathcal{I} = \mathcal{I}(\mathcal{Y})$, $K_{\mathcal{X}}$ a canonical divisor and $\mathcal{A}$ an ample divisor on $\mathcal{X}$ all defined over a number field k. Let S be a finite subset of places containing M_k^∞. Then, given any real constant $\varepsilon > 0$ and positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $k, \mathcal{X}, \mathcal{I}, \mathcal{A}, \varepsilon, r$, such that*

$$h_{K_{(\mathcal{X},\mathcal{Y})}} - \sum_{\nu \in S} \lambda_{\mathcal{H}}(\mathbf{x}, \nu) - \sum_{\nu \in S} \lambda_{\mathcal{I}^-}(\mathbf{x}, \nu) \leq \varepsilon h_{\mathcal{A}}(\mathbf{x}) + d_{\mathbb{Q}}(\mathbf{x}) + O(1).$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{k})$ with $[k(\mathbf{x}) : k] \leq r$, where $\mathcal{H} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{I}^- = \mathcal{I}^-(\mathcal{X}, \mathcal{Y})$.

One can see that the above conjecture holds for a log pair $(\mathcal{X}, \mathcal{Y})$ and a log resolution $f : \mathcal{X}' \to \mathcal{X}$, if the Vojta's conjecture Conj. 1 holds for $\mathcal{X}'$ and the reduced simple normal crossing divisor

$$\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil + \varepsilon f^* \mathcal{Y} - \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor,$$

for $0 < \varepsilon \ll 1$. Moreover, the conjectures Conj. 1, Conj. 2 and Conj. 3 are equivalent; see [15, Prop. 5.4 and Rem. 5.5] for a proof.

3. Vojta's conjecture for weighted projective varieties

We assume the reader is familiar with weighted projective varieties in the level covered in [1] and [10]. Let's recall some basic terminology.

Consider the polynomial ring $k_{\mathfrak{q}}[x_0, \ldots, x_n]$ where each x_i has weight $\operatorname{wt} x_i = q_i$. Every polynomial is a sum of monomials $x^d = \prod x_i^{d_i}$ with weight

$$\operatorname{wt}(x^d) = \sum d_i q_i.$$

A polynomial $f \in k_{\mathfrak{q}}[x_0, \ldots, x_n]$ is called a **weighted homogeneous of weight (degree)** d if every monomial of f has weight d, i.e.

$$f(x_0, \ldots, x_n) = \sum_{i=1}^{t} a_i \prod_{j=0}^{n} x_j^{d_j}, \quad \text{for} \quad a_i \in k \quad \text{and} \quad t \in \mathbb{N}$$

and for all $0 \leq i \leq n$, we have

$$\sum_{i=1}^{n} q_i d_j = d.$$

For every $\lambda \in k^*$ and any weighted homogeneous polynomial f of degree d, we have

$$f(\lambda^{q_0} x_0, \lambda^{q_1} x_1, \ldots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \ldots, x_n),$$

Let us consider a simple example of weighted homogenous polynomials.

Example 1. *A binary weighted form of degree d, where $w = (q_0, q_1)$ be respectively the weights of x_0 and x_1, is given by a polynomial as follows*

$$f(x_0, x_1) = \sum_{d_0, d_1} a_{d_0, d_1} x_0^{d_0} x_1^{d_1}, \quad \textit{such that} \;\; d_0 q_0 + d_1 q_1 = d$$

and in decreasing powers of x_0 *we have*

$$f(x_0, x_1) = a_{d/q_0,0} x_0^{d/q_0} + \cdots + a_{d_0,d_1} x_0^{d_0} x_1^{d_1} + \cdots + a_{0,d/q_1} x_1^{d/q_1}$$

By dividing with x_1^{d/q_1} *and making a change of coordinates* $X = x_0^{q_1}/x_1^{q_0}$ *we get*

$$f(x_0, x_1) = a_{d/q_0,0} X^{d/q_0 q_1} + \cdots + a_{d_0,d_1} X^{d_0/q_1} + \cdots + a_{0,d/q_1} = f(X) \tag{6}$$

as noted in [1]. *Notice that the condition* $f(P) = 0$ *is well defined on* $\mathbb{P}_{\mathfrak{q},k}$.

From now on we assume that $\mathbb{P}_{\mathfrak{q},k}$ is a weighted projective space with $\mathfrak{q} = (q_0, \ldots, q_n)$ and $m = \operatorname{lcm}(q_0, \ldots, q_n)$. A **hyperplane** in $\mathbb{P}_{\mathfrak{q},k}$ is a weighted homogeneous polynomial of degree m. Hence, it is the set of points $\mathbf{x} = [x_0 : \ldots : x_n] \in \mathbb{P}_{\mathfrak{q},k}$ satisfying a polynomial of the form

$$\ell(\mathbf{x}) = a_0 x_0^{m/q_0} + a_1 x_1^{m/q_1} + \cdots + a_n x_n^{m/q_n} = \sum_{i=0}^{n} a_i x_i^{\frac{m}{q_i}} \tag{7}$$

Notice that if $\mathfrak{q} = (1, \ldots, 1)$ all definitions agree with those of $\mathbb{P}^n$.

Recall from [10] that $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is regular with codimension one and if $q_i > 1$'s are mutually coprime then

$$\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \{\mathbf{x}_i = [0 : \cdots : 1 : \cdots : 0] : \ 0 \leq i \leq n\}.$$

A non-singular weighted projective space $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is called a smooth weighted projective variety. Let $\mathcal{X}$ be a smooth weighted projective variety in $\mathbb{P}^n_{\mathfrak{q},k}$, with $\mathfrak{q} = (q_0, q_1, \cdots, q_n)$, defined over $\mathbb{Q}$. Suppose that there is an open subvariety $\mathcal{U} \subset \mathcal{X}$ with complement of codimension at least two which is Gorenstein, i.e., the dualizing sheaf $\omega^0_{\mathcal{U}}$ is invertible. We let $K_{\mathcal{U}}$ be a canonical divisor on $\mathcal{U}$ such that $\omega^0_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}})$, and then define the **canonical sheaf** $K_{\mathcal{X}}$ as the closure of $K_{\mathcal{U}}$ in $\mathcal{X}$.

By a **weighted normal crossing divisor** on $\mathcal{X}$, we mean a Cartier divisor D such that at every point in the support of D such that D is given locally by an equation of the form $x_0^{1/q_0} x_1^{1/q_1} \ldots x_n^{1/q_1} = 0$. Then, we formulate the analogue of Conj. 1 for the smooth weighted projective varieties as follows:

Conjecture 4. *Let* $\mathcal{X}$ *be a smooth weighted projective variety over* $\mathbb{Q}$, $K_{\mathcal{X}}$ *a canonical divisor,* $\mathcal{A}$ *an ample divisor, and* D *a normal crossings divisor on* $\mathcal{X}$, *all defined over* $\mathbb{Q}$. *Let* S *be a finite subset of places containing* $M^\infty_{\mathbb{Q}}$. *Then, given any real constant* $\varepsilon > 0$ *and any positive integer* r, *there exists a proper weighted Zariski-closed subset* Z *of* $\mathcal{X}$, *depending only on* $k, \mathcal{X}, D, \mathcal{A}, \varepsilon, r$, *such that*

$$\mathfrak{s}_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \zeta_D(\mathbf{x}, \nu) \leq \varepsilon \cdot \mathfrak{s}_{\mathcal{A}}(\mathbf{x}) + \frac{1}{m} d_{\mathbb{Q}}(\mathbf{x}) + O(1),$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{k})$ *with* $[k(\mathbf{x}) : k] \leq r$.

3.1. **Weighted multiplier ideal sheaf.** In order to avoid using the weighted normal crossing divisors and replace it by a an error term as in [14], let $\mathcal{I}$ be a nonzero weighted ideal sheaf on a weighted projective variety $\mathcal{X}$ and $c \geq 0$ a real constant. Let $f : \mathcal{X}' \to \mathcal{X}$ be a proper birational morphism such that $\mathcal{X}'$ is smooth weighted variety and $f^*\mathcal{I} = \mathcal{O}_{\mathcal{X}'}(-E)$, for a normal crossing divisor E on $\mathcal{X}'$. Denoting by $\mathcal{R}_{\mathcal{X}'/\mathcal{X}}$ the ramification divisor of $\mathcal{X}'$ over $\mathcal{X}$, we define the **weighted multiplier ideal sheaf** $\mathcal{I}_c$ and $\mathcal{I}_c^-$ associated to $\mathcal{I}$ and c as

$$\mathcal{I}_c := f_* \mathcal{O}_{\mathcal{X}'}(\mathcal{R}_{\mathcal{X}'/\mathcal{X}} - \lfloor cE \rfloor), \text{ and } \mathcal{I}_c^- := \lim_{\varepsilon \to 0^+} \mathcal{I}_{c-\varepsilon}. \tag{8}$$

As in the case of projective varieties, we denote $\mathcal{I}_1$ and $\mathcal{I}_1^-$ by $\mathcal{I}$ and $\mathcal{I}^-$, respectively. An analogue of Conj. 2 for weighted projective varieties follows:

Conjecture 5. *Let $\mathcal{X}$ be a smooth weighted projective variety over $\mathbb{Q}$, $K_\mathcal{X}$ a canonical divisor, $\mathcal{A}$ an ample divisor and $\mathcal{I}$ a nonzero weighted ideal sheaf $\mathcal{X}$ all defined over $\mathbb{Q}$. Let S be a finite subset of places containing $M_\mathbb{Q}^\infty$. Then, given any real constant $\varepsilon > 0$ and positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $k, \mathcal{X}, \mathcal{I}, \mathcal{A}, \varepsilon, r$, such that*

$$\mathfrak{s}_{K_\mathcal{X}}(\mathbf{x}) + \sum_{\nu \in S} \zeta_\mathcal{I}(\mathbf{x}, \nu) - \sum_{\nu \in S} \zeta_{\mathcal{I}^-}(\mathbf{x}, \nu) \leq \varepsilon\, \mathfrak{s}_\mathcal{A}(\mathbf{x}) + \frac{1}{m} d_\mathbb{Q}(\mathbf{x}) + O(1).$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{k})$ with $[k(\mathbf{x}) : k] \leq r$.

3.2. **Weighted log pairs.** Next we follow closely the terminology of the log pairs for projective varieties as in Sec. 2.2. A **weighted $\mathbb{Q}$-divisor** on a weighted variety $\mathcal{X}$ is a formal finite sum

$$D = \sum c_i D_i,$$

where $c_i \in \mathbb{Q}$ and $D_i \in \operatorname{CaDiv}_\mathfrak{q}(\mathcal{X})$. A weighted $\mathbb{Q}$-divisor D is said **integral** if all coefficients $c_i's$ are integers.

By clearing the denominators of $c_i's$, we can write $D = cD'$ for some $c \in \mathbb{Q}$ and an integral weighted divisor D'. A weighted $\mathbb{Q}$-divisor D is called **effective** if $c_i \geq 0$ and D_i are weighted integral divisors on $\mathcal{X}$. The **support** of D, denoted by $\operatorname{Supp}(D)$, is

$$\operatorname{Supp}(D) = \bigcup \operatorname{Supp}(D_i)$$

as in the case of projective varieties. $D = \sum c_i D_i$ is called **ample** if $c_i \in \mathbb{Q}$, $c_i > 0$ and D_i are all ample Cartier divisors on $\mathcal{X}$. Here, by a **Weil $\mathbb{Q}$-divisor** on a weighted variety $\mathcal{X}$, we mean a $\mathbb{Q}$-linear combination of its codimension one subvarieties, i.e, an element of

$$\operatorname{WeDiv}_\mathfrak{q}(\mathcal{X}) \otimes \mathbb{Q}.$$

We use $\lceil D \rceil$ and $\lfloor D \rfloor$ to denote the round up and round down of any Weil $\mathbb{Q}$-divisor $D = \sum_i c_i \mathcal{Y}_i$, that is,

$$\lceil D \rceil = \sum_i \lceil c_i \rceil \mathcal{Y}_i \quad \text{and} \quad \lfloor D \rfloor = \sum_i \lfloor c_i \rfloor \mathcal{Y}_i.$$

A weighted projective variety $\mathcal{X}$ defined over $\mathbb{Q}$ is said to be **$\mathbb{Q}$-Gorenstein** if it is Gorenstein in codimension one, satisfies Serre's condition S_2, and a canonical divisor $K_\mathcal{X}$ is $\mathbb{Q}$-Cartier. For example, if $\mathcal{X}$ is normal, then the first two conditions are true and hence a weighted canonical divisor exists unequally up to linear equivalence and is Cartier in codimension one. A **$\mathbb{Q}$-subscheme** of $\mathcal{X}$ is a formal linear combinations

$$\mathcal{Y} = \sum_{i=1}^m c_i \cdot \mathcal{Y}_i$$

of proper closed subschemes $\mathcal{Y}_i \subset \mathcal{X}$ with all $c_i \in \mathbb{Q}$. The support of such $\mathcal{Y}$ is defined to be the closed subset $\cup_{c_i \neq 0} \mathcal{Y}_i$, and it is called **effective $\mathbb{Q}$-subscheme** if $c_i \geq 0$ for every i.

A **weighted log pair** is called a pair $(\mathcal{X}, \mathcal{Y})$ of a $\mathbb{Q}$-Gorenstein weighted variety $\mathcal{X}$ and an effective weighted $\mathbb{Q}$-subscheme $\mathcal{Y}$ of $\mathcal{X}$.

A **resolution** of $\mathcal{X}$ over $\mathbb{Q}$ is a projective birational morphism $f : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is a weighted smooth variety over $\mathbb{Q}$. A **weighted log resolution** of a weighted log pair $(\mathcal{X}, \mathcal{Y})$ with

$$\mathcal{Y} = \sum_{i=1}^{m} c_i \cdot \mathcal{Y}_i,$$

is a projective birational morphism $f : \mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}$ such that $\mathcal{X}'$ is a weighted smooth variety defined over $\mathbb{Q}$, the set-theoretic inverse image $f^{-1}(\mathcal{Y}_i)$ is a weighted Cartier divisor on $\mathcal{X}'$, and the union of $\mathrm{Exc}(f)$ of the exceptional divisor of f with all $f^{-1}(\mathcal{Y}_i)_{\mathrm{red}}$ is a simple weighted normal crossing divisor of $\mathcal{X}'$. The existence of a resolution of a weighted variety $\mathcal{X}$ and the weighted log resolution of $(\mathcal{X}, \mathcal{Y})$ is a consequence of Hironoka's theorem [8, Thm. 4.1.3].

For a weighted log resolution $f : \mathcal{X}' \to \mathcal{X}$ as a weighted log pair $(\mathcal{X}, \mathcal{Y})$, the **relative canonical divisor** of $\mathcal{X}'$ over $(\mathcal{X}, \mathcal{Y})$ is defined to be the weighted $\mathbb{Q}$-Weil divisor

$$K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} = K_{\mathcal{X}'/\mathcal{X}} - f^*\mathcal{Y},$$

where $f^*\mathcal{Y}$ is the pull-back of $\mathcal{Y}$ by f over $\mathcal{X}'$ and $K_{\mathcal{X}'/\mathcal{X}}$ is the relative canonical divisor of $\mathcal{X}'$ over $\mathcal{X}$, i.e.,

$$\mathcal{O}(K_{\mathcal{X}'}) \cong \mathcal{O}(K_{\mathcal{X}'/\mathcal{X}}) \otimes f^*\mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}}),$$

which is a $\mathbb{Q}$-Weil divisor on $\mathcal{X}'$. Given a weighted log pair $(\mathcal{X}, \mathcal{Y})$ and a weighted log resolution $f : \mathcal{X}' \to \mathcal{X}$, we write

$$K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} = \sum_{\mathbb{Z}} a_{\mathcal{Z}} \cdot \mathcal{Z},$$

where $\mathcal{Z}$ runs over all prime divisors of $\mathcal{X}'$. The weighted log pair $(\mathcal{X}, \mathcal{Y})$ is called **strongly canonical** (resp. **Kawamata log terminal, log canonical**) if $a_{\mathcal{Z}} \geq 0$ (resp. $a_{\mathcal{Z}} > 0$, and $a_{\mathcal{Z}} \geq -1$) for every $\mathcal{Z}$. These properties are independent of the resolution and are also local.

Define the **weighted non-SC locus** of the pair $(\mathcal{X}, \mathcal{Y})$ to be the smallest weighted closed subset $\mathcal{W} \subset \mathcal{X}$ such that the weighted pair $(\mathcal{X}\backslash\mathcal{W}, \mathcal{Y}|_{\mathcal{X}\backslash\mathcal{W}})$ is strongly canonical. The **weighted non-KLT locus** of the pair $(\mathcal{X}, \mathcal{Y})$ to be the smallest weighted closed subset $\mathcal{W} \subset \mathcal{X}$ such that the weighted pair is Kawamata log terminal. Similarly the **weighted non-LC**) of the pair $(\mathcal{X}, \mathcal{Y})$ to be the smallest weighted closed subset $\mathcal{W} \subset \mathcal{X}$ such that the weighted pair is log canonical. We denote them respectively as $\mathrm{wnsc}(\mathcal{X}, \mathcal{Y})$, $\mathrm{wnklt}(\mathcal{X}, \mathcal{Y})$, $\mathrm{wnsc}(\mathcal{X}, \mathcal{Y})$. One may check that

$$\mathrm{wnsc}(\mathcal{X}, \mathcal{Y}) \subset \mathrm{wnklt}(\mathcal{X}, \mathcal{Y}) \subset \mathrm{wnklt}(\mathcal{X}, \mathcal{Y}).$$

For a weighted log pair $(\mathcal{X}, \mathcal{Y})$ with a log resolution $f : \mathcal{X}' \to \mathcal{X}$, we define $\mathcal{I}(\mathcal{X}, \mathcal{Y})$ a **variant of multiplier sheaf** as

$$\mathcal{I}(\mathcal{X}, \mathcal{Y}) := f_*\mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil)$$

if $\mathcal{X}$ is a weighted normal variety; see [9, 9.3.56] for the definition of multiplier ideal sheaf in usual case. Otherwise, we let

$$\mathcal{I}(\mathcal{X}, \mathcal{Y}) := \bar{f}_*\mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil),$$

where $\bar{f}_*\mathcal{O}_{\mathcal{X}'}(E)$ denotes the largest ideal sheaf in $\mathcal{O}_{\mathcal{X}}$ for which its pull-back by f is contained in $\mathcal{O}_{\mathcal{X}'}(E)$ as an $\mathcal{O}_{\mathcal{X}'}$-submodule of (constant) function field sheaf

$\mathcal{M}_{\mathcal{X}'}$. Moreover, there exist a constant $\varepsilon_0 > 0$ such that for every rational number $0 < \varepsilon \leq \varepsilon_0$, one has $\mathcal{I}(\mathcal{X}, (1-\varepsilon)\mathcal{Y}) = \mathcal{I}(\mathcal{X}, (1-\varepsilon_0)\mathcal{Y})$.

Let

$$\begin{aligned} &\mathcal{I}^-(\mathcal{X}, \mathcal{Y}) := \mathcal{I}(\mathcal{X}, (1-\varepsilon)\mathcal{Y}), \ (0 < \varepsilon \ll 1) \\ &\mathcal{H}(\mathcal{X}, \mathcal{Y}) := \bar{f}_* \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor), \end{aligned} \tag{9}$$

where $\bar{f}_*$ is as above. We note that the definition of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ does not depend on the notion of "weighted simple normal crossing".

Lemma 1. *The definitions of $\mathcal{I}(\mathcal{X}, \mathcal{Y})$, $\mathcal{I}^-(\mathcal{X}, \mathcal{Y})$, and $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ are independent of the choice of a weighted log resolution.*

Proof. By adopting [15, Lem. 3.1] and [15, Prop. 3.4] respectively to the case of weighted projective schemes, one get the result for $\mathcal{I}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{I}^-(\mathcal{X}, \mathcal{Y})$. An argument similar to the proof of the Proposition 3.4 in [15] shows the assertion for $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. □

Proposition 1. *Let $(\mathcal{X}, \mathcal{Y})$ be a weighted log pair. Then, the following are true:*

(i) $\operatorname{wnlc}(\mathcal{X}, \mathcal{Y}) \subset \operatorname{Supp}(\mathcal{O}_\mathcal{X}/\mathcal{I}^-(\mathcal{X}, \mathcal{Y})) \subset \operatorname{wnklt}(\mathcal{X}, \mathcal{Y})$;

(ii) *If $(\mathcal{X} \setminus \operatorname{Supp}(\mathcal{Y}), 0)$ is a weighted Kawamata log terminal, then*

$$\operatorname{wnlc}(\mathcal{X}, \mathcal{Y}) = \operatorname{Supp}(\mathcal{O}_\mathcal{X}/\mathcal{I}^-(\mathcal{X}, \mathcal{Y}));$$

(iii) $\operatorname{Supp}(\mathcal{O}_\mathcal{X}/\mathcal{H}(\mathcal{X}, \mathcal{Y})) = \operatorname{wnsc}(\mathcal{X}, \mathcal{Y})$.

(iv) *If $(\mathcal{X}, \mathcal{Y})$ is weighted log canonical, then $\mathcal{O}_\mathcal{X}/\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is reduced, i.e., as a reduced closed subscheme, $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is the ideal sheaf of* $\operatorname{wnsc}(\mathcal{X}, \mathcal{Y})$.

Proof. Let $f : \mathcal{X}' \to \mathcal{X}$ be a weighted log resolution of $(\mathcal{X}, \mathcal{Y})$, and denote by $\operatorname{mult}_\mathcal{Z}(E)$ the multiplicity of any divisor E on $\mathcal{X}'$.

(i) Given any prime divisor $\mathcal{Z}$ of $\mathcal{X}'$ and real constant $0 < \varepsilon \ll 1$, we have

$$\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) < -1,$$

which implies that

$$\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) < 0,$$

and hence $\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) \leq -1$. This proves part (i).

(ii) It is enough to show that if the pair $(\mathcal{X} \setminus \operatorname{Supp}(\mathcal{Y}), 0)$ is a weighted Kawamata log terminal, then

$$\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) \geq -1$$

and so

$$\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) \geq 0.$$

If $\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) > -1$, then the result is trivial. If $\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) = -1$, then $\mathcal{Z}$ is contained in $\operatorname{Supp}(f^*\mathcal{Y})$ by assumption on $(\mathcal{X} \setminus \operatorname{Supp}(\mathcal{Y}), 0)$. Thus,

$$\operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) > -1 \Rightarrow \operatorname{mult}_\mathcal{Z}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) \geq 0.$$

(iii) If $(\mathcal{X}, \mathcal{Y})$ is a weighted strongly canonical, then $\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor \geq 0$. By definition of $\bar{f}$, we have

$$\bar{f}_* \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor) = \mathcal{O}_\mathcal{X}.$$

This shows that

$$\operatorname{wnsc}(\mathcal{X}, \mathcal{Y}) \subset \operatorname{Supp}(\mathcal{O}_\mathcal{X}/\mathcal{H}(\mathcal{X}, \mathcal{Y})).$$

If $(\mathcal{X}, \mathcal{Y})$ is not a weighted strongly canonical around $x \in \mathcal{X}$, then there is a prime divisor $\mathcal{Z}$ on $\mathcal{X}'$ such that $x \in f(\mathcal{Z})$ and $\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) < 0$. Thus, we have

$$\mathcal{O}_{\mathcal{X}'} \not\subset \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor).$$

Replacing $\mathcal{X}$ with any open neighborhood of x does not change the last result. Therefore,

$$\mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{H}(\mathcal{X},\mathcal{Y})) \subset \mathrm{wnsc}(\mathcal{X},\mathcal{Y}).$$

(iv) By part (iii), we have $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a subset of the ideal sheaf of $\mathrm{wnsc}(\mathcal{X},\mathcal{Y})$ denoted by $\mathcal{N}$. Now, let $\mathcal{U} \subset \mathcal{X}$ be an open set and $g \in \mathcal{N}(\mathcal{U})$. Then f^*g vanishes along the closed set $f^*(\mathrm{wnsc}(\mathcal{X},\mathcal{Y}))$ containing every prime divisor $\mathcal{Z}$ on $\mathcal{X}'$ having negative coefficient in $\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor$, which is equal to -1 since $(\mathcal{X},\mathcal{Y})$ is weighted log canonical. Therefore,

$$f^*g \in \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor)(f^{-1}\mathcal{U}),$$

which implies $g \in \mathcal{H}(\mathcal{U})$ and hence $\mathcal{H} \subset \mathcal{N}$.

□

Given a Weil $\mathbb{Q}$-divisor D on a weighted projective variety $\mathcal{X}$, such that nD is a Weil divisor on $\mathcal{X}$, we define height function

$$\mathfrak{s}_D := \frac{1}{n}\, \mathfrak{s}_{nD}\,.$$

For a weighted $\mathbb{Q}$-Gorenstein projective variety $\mathcal{X}$ with a canonical divisor $K_{\mathcal{X}}$, we can define a weighted global height function $\mathfrak{s}_{K_{\mathcal{X}}}$ up to addition of a bounded function. Given a weighted log pair $(\mathcal{X},\mathcal{Y})$ of a weighted $\mathbb{Q}$-Gorenstein $\mathcal{X}$, we define the weighted height function associated to the subscheme $K_{(\mathcal{X},\mathcal{Y})} = K_{\mathcal{X}} + \mathcal{Y}$ as

$$\mathfrak{s}_{K_{(\mathcal{X},\mathcal{Y})}} = \mathfrak{s}_{K_{\mathcal{X}}} + \mathfrak{s}_{\mathcal{Y}}, \tag{10}$$

where $\mathfrak{s}_{\mathcal{Y}}$ is the height function associated to the subscheme $\mathcal{Y}$ or its ideal sheaf. Next we are ready to state Vojta's conjecture for weighted log pairs.

Conjecture 6. *Let $\mathcal{X}$ be a weighted projective scheme, $\mathcal{Y}$ a closed weighted subscheme with ideal sheaf $\mathcal{I} = \mathcal{I}(\mathcal{Y})$, $K_{\mathcal{X}}$ a canonical divisor, and $\mathcal{A}$ an ample divisor on $\mathcal{X}$ all defined over $\mathbb{Q}$.*

Let $(\mathcal{X},\mathcal{Y})$ be a weighted log pair and S be a finite subset of places containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and a positive integer r, there exists a proper weighted Zariski-closed subset Z of $\mathcal{X}$, depending only on $k, \mathcal{X}, \mathcal{I}, \mathcal{A}, \varepsilon, r$, such that

$$\mathfrak{s}_{K_{(\mathcal{X},\mathcal{Y})}(\mathbf{x})} - \sum_{\nu \notin S} \zeta_{\mathcal{H}}(\mathbf{x},\nu) - \sum_{\nu \in S} \zeta_{\mathcal{I}^-}(\mathbf{x},\nu) \leq \varepsilon\, \mathfrak{s}_{\mathcal{A}}(\mathbf{x}) + \frac{1}{m} d_{\mathbb{Q}}(\mathbf{x}) + O(1).$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{k})$ with $[k(\mathbf{x}) : k] \leq r$, where

$$\mathcal{H} = \mathcal{H}(\mathcal{X},\mathcal{Y}) \quad \textit{and} \quad \mathcal{I}^- = \mathcal{I}^-(\mathcal{X},\mathcal{Y}).$$

We note that the terms

$$\sum_{\nu \in S} \zeta_{\mathcal{H}}(\mathbf{x},\nu) \quad \text{and} \quad \sum_{\nu \in S} \zeta_{\mathcal{I}^-}(\mathbf{x},\nu)$$

can be thought of as the contribution of $\mathrm{wnsc}(\mathcal{X},\mathcal{Y})$ and $\mathrm{wnklt}(\mathcal{X},\mathcal{Y})$, or $\mathrm{wnlc}(\mathcal{X},\mathcal{Y})$ if $(\mathcal{X} \backslash (Supp)(\mathcal{Y}), 0)$ is Kawamata log terminal.

Since a pair $(\mathcal{X}, D)$ with $\mathcal{X}$ a smooth weighted variety and D a reduced simple weighted normal crossing divisor on $\mathcal{X}$ is a weighted log canonical, by parts (i), (iii) and (iv) of Prop. 1, one can conclude that

$$\sum_{\nu \in S} \zeta_{\mathcal{I}^-}(\mathbf{x}, \nu) = 0$$

and hence the right hand side of the inequality of Conj. 6 is equal to

$$\mathfrak{s}_{K_{(\mathcal{X},D)}(\mathbf{x})} - \sum_{\nu \notin S} \zeta_{\mathcal{H}}(\mathbf{x}, \nu) = \mathfrak{s}_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \zeta_D(\mathbf{x}, \nu).$$

Thus Conj. 6 is the same as Conj. 5 and Conj. 4 in this case.

In contrast, given a weighted log pair $(\mathcal{X}, \mathcal{Y})$ and a log resolution

$$f : \mathcal{X}' \to \mathcal{X},$$

if we suppose that Conj. 4 holds for $\mathcal{X}'$ and the reduced simple normal crossing divisor

$$\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^* \mathcal{Y} \rceil - \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor,$$

for $0 < \varepsilon \ll 1$, then Conj. 5 and Conj. 6 holds for $(\mathcal{X}, \mathcal{Y})$.

Indeed, the argument is similar to those given in [14, Prop. 4.3] and [15, Prop. 5.4] as follows. By definition, we have

$$f^{-1}\mathcal{H} \subset \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor)$$

and

$$f^{-1}\mathcal{I}^- \subset \mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^* \mathcal{Y} \rceil)$$

for $0 < \varepsilon \ll 1$. Using these and the properties of weighted height functions, we get

$$\begin{aligned} &\zeta_{\mathcal{H}} \circ f \geq \zeta_{-D_1}, \text{ with } D_1 = \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor, \\ &\zeta_{\mathcal{I}^-} \circ f \geq \zeta_{-D_2} \text{ with } D_2 = \lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil + \varepsilon f^* \mathcal{Y} \rceil. \end{aligned}$$

Then, using the above inequalities, we have

$$\begin{aligned} \left(\mathfrak{s}_{K_{(\mathcal{X},\mathcal{Y})}} - \sum_{\nu \notin S} \zeta_{\mathcal{H}}(\cdot, \nu) - \sum_{\nu \in S} \zeta_{\mathcal{I}^-}(\cdot, \nu) \right) \circ f &\leq \mathfrak{s}_{K_{\mathcal{X}'}} - \mathfrak{s}_{K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}} - \sum_{\nu \notin S} \zeta_{-D_1} - \sum_{\nu \in S} \zeta_{-D_2} \\ &\leq \mathfrak{s}_{K_{\mathcal{X}'}} + \mathfrak{s}_{-D_1} - \sum_{\nu \notin S} \zeta_{-D_1} - \sum_{\nu \in S} \zeta_{-D_2} \\ &\leq \mathfrak{s}_{K_{\mathcal{X}'}} + \sum_{\nu \notin S} \zeta_{D_2 - D_1}, \end{aligned}$$

where

$$D_2 - D_1 = \lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil + \varepsilon f^* \mathcal{Y} \rceil - \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor.$$

Recall that the pullback $f^\star D$ of an ample divisor D is ample. Therefore, the above argument leads to following result.

Corollary 1. *Conjectures Conj. 4, Conj. 5, and Conj. 6 are equivalent.*

4. Weighted blowups and generalized weighted gcds

In [11] Silverman used the idea of generalized **gcds** to define a height for blowups of smooth projective varieties and then assuming Vojta's conjecture for such height function obtained some conjectural results on the generalized gcds. We generalize such results for weighted heights by defining the generalized weighted greatest common divisor and defining a height for weighted blowups as defined in [10].

As above, k is a number field, $\mathcal{O}_k$ its ring of integers, and ν_p the valuation at a prime $p \in \mathcal{O}_k$. For any two elements $\alpha, \beta \in \mathcal{O}_k$ the **greatest common divisor** is defined as

$$\gcd(\alpha, \beta) := \prod_{p \in \mathcal{O}_k} p^{\min\{\nu_p(\alpha),\, \nu_p(\beta)\}},$$

and the **logarithmic greatest common divisor** is

$$\log \gcd(\alpha, \beta) := \sum_{p \in \mathcal{O}_k} \min\{\nu(\alpha), \nu(\beta)\} \log p = \sum_{\nu \in M_k^0} \min\{\nu(\alpha), \nu(\beta)\}$$

For each place $\nu \in M_k$, we define

$$\begin{aligned} \nu^+ : k &\longrightarrow [0, \infty], \\ \alpha &\mapsto \max\{\nu(\alpha), 0\}, \end{aligned} \tag{11}$$

which ν^+ can be viewed as a height function on $\mathbb{P}^1(k) = k \cup \{\infty\}$ with respect the divisor (0), where we set $\nu^+(\infty) = 0$. The **generalized logarithmic greatest common divisor** of $\alpha, \beta \in \mathbb{Q}$ is defined as

$$\mathrm{h}_{\mathrm{gcd}}(\alpha, \beta) := \sum_{\nu \in M_k} \min\{\nu^+(\alpha), \nu^+(\beta)\}.$$

Then, given $(\alpha, \beta) \neq (0, 0)$, one may consider the following function

$$\begin{aligned} G_\nu : \mathbb{P}^1(k) \times \mathbb{P}^1(k) &\to [0, \infty], \\ (\alpha, \beta) &\mapsto \min\{\nu^+(\alpha), \nu^+(\beta)\}, \end{aligned} \tag{12}$$

as a **local height function** and the generalized logarithmic greatest common divisor, being their sum all together,

$$\mathrm{h}_{\mathrm{gcd}}(\alpha, \beta) = \sum_{\nu \in M_k} G_\nu. \tag{13}$$

as a **global height function** on $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$. In [11], it was given a theoretical interpretation of the function G_ν in terms of blowups. More precisely, for $\mathcal{X} = (\mathbb{P}^1_k)^2$ let $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blowup of the point $(0, 0)$ and $E = \pi^{-1}(0, 0)$ be the exceptional divisor of the blowup. Then, for all $(\alpha, \beta) \in \mathcal{X}(k) \backslash (0, 0)$ and $\nu \in M_k$, one has

$$\lambda_{\mathcal{X}, E}(\pi^{-1}(\alpha, \beta), \nu) = \min\{\nu^+(\alpha), \nu^+(\beta)\},$$

and adding all of these over all $\nu \in M_k$ leads to

$$\mathrm{h}_{\mathrm{gcd}}(\alpha, \beta) = h_{\tilde{\mathcal{X}}, E}(\pi^{-1}(\alpha, \beta)).$$

By this realization, in [11, Def. 2], Silverman's introduced generalized logarithmic greatest common divisors of points on smooth varieties with respect to its subvarieties.

4.1. **Weighted greatest common divisors.** Most of this material follows from [1, Section 3]. Notice that the goal of Section 3 in [1] was to extend definition of *generalized gcd* from Silverman's paper [11] to that of *generalized gcd*. In all of this $k = \mathbb{Q}$, however authors in [1] continue with the notation of their previous sections k for the field and that could be a cause for confusion.

A **weighted tuple of integers** in $\mathbb{Z}$ is a tuple $\tilde{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ such that to each coordinate x_i is assigned the weight q_i. We multiply weighted tuples by scalars $\lambda \in \mathbb{Q}$ via

$$\lambda \star (x_0, \ldots, x_n) = (\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n).$$

We assume $k = \mathbb{Q}$ for the rest of this paper. Given $\mathfrak{q} = (q_0, \cdots, q_n)$, we let $\mathfrak{q}_i = (1, q_i)$ for each $i = 0, 1, \cdots n$. The canonical inclusion

$$\mathbb{Q}_{\mathfrak{q}_i}(x_i) \hookrightarrow \mathbb{Q}_{\mathfrak{q}}(x_0, \cdots, x_n)$$

which induces the rational map $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \to \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}}$ given by

$$\mathbf{x} = [x_0 : \cdots : x_n] \mapsto [1 : x_i],$$

which is defined precisely in the complement of $V(x_i)$ in $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$. Considering all of these maps, we have the rational map

$$\begin{aligned} \phi_{n,\mathfrak{q}} : \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} &\longrightarrow \prod_{i=0}^{n} \mathbb{P}^1_{\mathfrak{q}_i,k}, \\ \mathbf{x} = [x_0 : \cdots : x_n] &\mapsto \phi_{n,\mathfrak{q}}(\mathbf{x}) := ([1 : x_0], [1 : x_1], \cdots, [1 : x_n]), \end{aligned} \tag{14}$$

which is defined in the open set $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \setminus \cup_{i=0}^{n} V(x_i)$. For each $p \in \mathbb{Z}$, we define the function F_{ν_p} as:

$$\begin{aligned} F_{\nu_p} : \prod_{i=1}^{n} \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}} &\longrightarrow \mathbb{Z}, \\ ([1 : x_0], [1 : x_1], \cdots, [1 : x_n]) &\mapsto p^{\min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\}}. \end{aligned}$$

For $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$, we define the **generalized weighted greatest common divisor** as

$$\mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) := \prod_{\nu_p \in M_{\mathbb{Q}}} F_{\nu_p}(\phi_{n,\mathfrak{q}}(\mathbf{x})) = \prod_{\nu_p \in M_{\mathbb{Q}}} p^{\min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\}} \tag{15}$$

4.2. **Generalized weighted greatest common divisors as heights for blowups.** The weighted greatest common divisor for any tuple of integers $(x_0, \cdots, x_n) \in \mathbb{Z}^{n+1}$ was defined in [1], which we are going to recall in below.

Let $\tilde{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ with $r = \gcd(x_0, \ldots, x_n)$ and

$$r = u \cdot \prod_{j=1}^{s} p_j,$$

where u is a unit in $\mathbb{Z}$ and $p_1, \cdots, p_s$ are primes. The **weighted greatest common divisor** of $\tilde{x} \in \mathbb{Z}^{n+1}$ is defined as

$$\mathrm{wgcd}(\tilde{x}) := \prod_{\substack{p \in \{p_1, \cdots, p_s\} \\ p^{q_i} \mid x_i}} p = \prod_{\nu_p \in \mathbb{Z}} p^{\min\left\{ \left\lfloor \frac{\nu_p(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p(x_n)}{q_n} \right\rfloor \right\}}, \tag{16}$$

where the last equality comes from [1, Lem. 4]. Here, the symbol $\lfloor \cdot \rfloor$ denotes the integer part function. A tuple $\tilde{x} \in \mathbb{Z}^{n+1}$ is said to be **normalized** if $\operatorname{wgcd}(\tilde{x}) = 1$. In [1, Lem. 7 and Corollary 1], it is proved that any point $\mathbf{x}$ in a well-formed space $\mathbb{P}^n_{\mathfrak{q}}$ has a unique normalization $\mathbf{y} = \frac{1}{\operatorname{wgcd}(\tilde{x})} \star \mathbf{x}$.

For the rest of this paper, a **normalized point** $\mathbf{x} \in \mathbb{P}_{\mathfrak{q},\mathbb{Q}}$ means a point $\mathbf{x} = [x_0 : \dots : x_n]$ with integer coordinates $x_i \in \mathbb{Z}$ such that $\operatorname{wgcd}(x_0, \dots, x_n) = 1$.

The **generalized weighted greatest common divisor** of a given tuple $\bar{x} = (x_0, \dots, x_n) \in \mathbb{Q}^{n+1}$ is defined as

$$\mathrm{h}_{\mathrm{wgcd}}(\bar{x}) := \prod_{p \in \mathbb{Z}} p^{\min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\}}, \tag{17}$$

We define the **logarithmic weighted greatest common divisor** of any tuple of integers $\tilde{x} = (x_0, \cdots, x_n) \in \mathbb{Z}^{n+1}$ as the sum

$$\log \operatorname{wgcd}(\tilde{x}) := \sum_{\nu \in M^0_{\mathbb{Q}}} \min\left\{ \left\lfloor \frac{\nu(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu(x_n)}{q_n} \right\rfloor \right\}, \tag{18}$$

and the **generalized logarithmic weighted greatest common divisor** of any tuple $\bar{x} = (x_0, \cdots, x_n) \in \mathbb{Q}^{n+1}$ is defined to be

$$\log \mathrm{h}_{\mathrm{wgcd}}(\bar{x}) := \sum_{\nu \in M_{\mathbb{Q}}} \min\left\{ \left\lfloor \frac{\nu^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu^+(x_n)}{q_n} \right\rfloor \right\}. \tag{19}$$

Let us consider the following positive real-valued function on $\mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$,

$$\begin{aligned} T_\nu : \prod_{i=0}^{n} \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}} &\to [0, \infty] \\ ([1 : x_0], [1 : x_n], \dots, [1 : x_n]) &\to \min\left\{ \left\lfloor \frac{\nu^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu^+(x_n)}{q_n} \right\rfloor \right\} \end{aligned}. \tag{20}$$

For any point $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$, define its **generalized logarithmic weighted greatest common divisor** as

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = \sum_{\nu \in M_{\mathbb{Q}}} T_\nu(\phi_{n,\mathfrak{q}}(\mathbf{x})) = \sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\}, \tag{21}$$

where $\phi_{n,\mathfrak{q}}$ is defined by Eq. (14).

Notice that all points $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$ with $\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = 0$ belong to the singular locus $\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})$ as shown next.

Proposition 2. *Let $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ be a well-formed weighted projective space with $\mathfrak{q} = (q_0, \cdots, q_n)$ and $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$. If $\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = 0$ then $\mathbf{x} \in \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})$.*

Proof. Let $m = \operatorname{lcm}(q_0, \cdots, q_n)$, and define $J(\mathbf{x}) = \{j : x_j(\mathbf{x}) \neq 0\}$ for any point $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$. Given any prime divisor $p \mid m$, we define

$$S_{\mathfrak{q}}(p) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} : \ p \mid q_i \text{ for all } i \in J(\mathbf{x}) \right\}.$$

where $\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \bigcup_{p|m} S_{\mathfrak{q}}(p)$. Then, for $p \in \mathbb{Z}$ we have

$$\begin{aligned}\mathbf{x} = [x_0 : \cdots : x_n] \in S_{\mathfrak{q}}(p) &\Rightarrow p \mid q_i, \text{ for all } i \in J(\mathbf{x})\\ &\Rightarrow \nu_p^+(x_i) < q_i, \text{ for all } i \in J(\mathbf{x})\\ &\Rightarrow \left\lfloor \frac{\nu_p^+(x_i)}{q_i} \right\rfloor = 0, \text{ for all } i \in J(\mathbf{x})\\ &\Rightarrow \min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\} = 0.\end{aligned}$$

If we assume $\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = 0$, then

$$\sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\} = 0. \tag{22}$$

Thus, for all $\nu_p \in M_{\mathbb{Q}}$ with $p \in \mathbb{Z}$, we have

$$\min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\} = 0. \tag{23}$$

This implies that $\mathbf{x} \in S_{\mathfrak{q}}(p)$ for any prime $p \mid m$ and hence $\mathbf{x} \in \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})$. □

Example 2. *Consider the weights $\mathfrak{q} = (1, \dots, 1)$. Then $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} = \mathbb{P}^n_{\mathbb{Q}}$ is the projective space and the weighted height $\mathcal{S}_{\mathbb{Q}}$ is simply the projective height $H_{\mathbb{Q}}$. Since $m = \operatorname{lcm}(q_0, \dots, q_n) = 1$ then there are no primes dividing m and $\operatorname{Sing} \mathbb{P}^n_{\mathbb{Q}} = \emptyset$. On the other side from Eq. (21) we have*

$$\begin{aligned}\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) &= \sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\}\\ &= \sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{ \nu_p^+(x_0), \dots, \nu_p^+(x_n) \right\}\\ &\geq \min\{\{\nu_\infty^+(x_0), \dots, \nu_\infty^+(x_n)\} > 0,\end{aligned}$$

since at least one of the coordinates $x_i \neq 0$,

Lemma 2. *Let $\mathcal{X}/\mathbb{Q}$ be a smooth weighted variety, defined over $\mathbb{Q}$, and $\mathcal{Y}/\mathbb{Q}$ a subvariety of $\mathcal{X}/\mathbb{Q}$ of codimension $r \geq 2$. Let $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blowup of $\mathcal{X}$ along $\mathcal{Y}$ and denote by $\tilde{\mathcal{Y}} := \pi^{-1}(\mathcal{Y})$ its the exceptional divisor. Then,*

(i) $\pi|_{\pi^{-1}(\mathcal{X}\setminus\mathcal{Y})} : \pi^{-1}(\mathcal{X} \setminus \mathcal{Y}) \to \mathcal{X} \setminus \mathcal{Y}$ *is an isomorphism.*
(ii) *Exceptional divisor $\tilde{\mathcal{Y}}$ is an effective Cartier divisor on $\tilde{\mathcal{X}}$.*

Proof. This is a direct consequence of [6, Prop. II.7.13]. For every $y \in \mathcal{Y}$ we have an open neighborhood O around $\pi^{-1}(y)$ and $f \in O$. The conditions from the definition of Cartier divisors are satisfied. □

Proposition 3. *Let $\mathcal{X} := \prod_{i=1}^n \mathbb{P}^1_{\mathfrak{q}_i,k}$, and consider $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$, the blowup of $\mathcal{X}$ along $\bar{0} = ([1:0], [1:0], \cdots, [1:0])$. Denote by $\tilde{\mathcal{Y}} = \pi^{-1}(\bar{0})$ the exceptional divisor of this blowup. Then, for all $\nu \in M_{\mathbb{Q}}$ and any non-singular points*

$$\mathbf{x} = [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \setminus \{[1 : 0 : \cdots : 0]\}$$

with $\bar{x} = \phi_{n,\mathfrak{q}}|_{\mathcal{X}}(\mathbf{x}) \in \mathcal{X}(\mathbb{Q}) \setminus \{\bar{0}\}$, we have

$$\zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\bar{x}), \nu) = \min\left\{ \left\lfloor \frac{\nu^+(x_0)}{q_0} \right\rfloor, \dots, \left\lfloor \frac{\nu^+(x_n)}{q_n} \right\rfloor \right\} = T_\nu(\bar{x}), \tag{24}$$

and

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = \sum_{\nu \in M_{\mathbb{Q}}} \zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\bar{x}),\nu) = \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}\left(\pi^{-1}(\bar{x}),\nu\right). \tag{25}$$

Proof. Since $\tilde{\mathcal{Y}}$ is an effective divisor on $\tilde{\mathcal{X}}$ by Lem. 2, so using the functoriality of local weighted heights, we have

$$\begin{aligned}
\zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\bar{x}),\nu) = \zeta_{\mathcal{X},\bar{0}}(\bar{x},\nu) &= \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},[1:0:\cdots:0]}\left([x_0^{\frac{1}{q_0}} : x_1^{\frac{1}{q_1}} : \cdots : x_n^{\frac{1}{q_n}}],\nu\right), \\
&= \min\left\{\nu^+(x_0^{\frac{1}{q_1}}),\cdots,\nu^+(x_n^{\frac{1}{q_n}})\right\} \\
&= \min\left\{\left\lfloor\frac{\nu^+(x_0)}{q_0}\right\rfloor,\cdots,\left\lfloor\frac{\nu^+(x_n)}{q_n}\right\rfloor\right\} = T_\nu(\bar{x}).
\end{aligned}$$

Adding these weighted local heights together we get the global formula. □

The above result leads to the following definition.

Definition 1. *Let $\mathcal{X}/\mathbb{Q}$ be a smooth weighted variety, defined over $\mathbb{Q}$, and $\mathcal{Y}/\mathbb{Q}$ a subvariety of $\mathcal{X}/\mathbb{Q}$ of codimension $r \geq 2$ and $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$, the blowup of $\mathcal{X}$ along $\mathcal{Y}$. For any $P \in (\mathcal{X} \setminus \mathcal{Y})/\mathbb{Q}$ we denote by*

$$\tilde{P} := \pi^{-1}(P) \in \tilde{\mathcal{X}} \quad \text{and} \quad \tilde{\mathcal{Y}} = \pi^{-1}(\mathcal{Y}).$$

*The **generalized logarithmic weighted greatest common divisor of the point P with respect to $\mathcal{Y}$** is defined to be*

$$\log \mathrm{h}_{\mathrm{wgcd}}(P;\mathcal{Y}) = \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}). \tag{26}$$

A point $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is called *normalized* if it has integers coordinates and $\mathrm{wgcd}(x_0, x_1, \cdots, x_n) = 1$; see [1] for details.

Lemma 3. *Let $\mathfrak{q} = (q_0, \cdots, q_n)$ be a well-formed set of weights, $m = \mathrm{lcm}(q_0, \cdots, q_n)$, $\mathbf{y} = [1 : 0 : \cdots : 0]$, and $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ a smooth and normalized point. Then*

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x};\{\mathbf{y}\}) = \frac{1}{m}\log\gcd(x_1,\ldots,x_n) + O(1). \tag{27}$$

Proof. Indeed, letting $\mathfrak{q}_i = (q_0, q_i)$ for each $i = 1, \cdots, n$ and

$$\mathcal{X} = \prod_{i=0}^{n} \mathbb{P}^1_{\mathfrak{q}_i,k},$$

then considering the rational map $\pi_{n,\mathfrak{q}} : \mathbb{P}^n_{\mathfrak{q},k} \to \mathcal{X}$, we have $\bar{0} = \phi_{n,\mathfrak{q}}(\mathbf{y})$, where

$$\bar{0} = ([1:0],\cdots,[1:0]) \in \mathcal{X}.$$

Let $\bar{x} = \phi_{n,\mathfrak{q}}(\mathbf{x})$ and apply Prop. 3 to the blowup $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ along $\mathcal{Y} = \{\bar{0}\}$. Let $\tilde{\mathcal{Y}} = \pi^{-1}(\mathcal{Y})$ be the exceptional divisor of the blowup. Then

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x};\{\mathbf{y}\}) = \log \mathrm{h}_{\mathrm{wgcd}}(\bar{x};\mathcal{Y}) = \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}\left(\pi^{-1}(\bar{x})\right)$$

By definition of the global weighted height and properties of local weighted height [10, Thm. 1 (iv)], one can see that the last term is equal to the right-hand side of Eq. (27). □

One can extend the result of Lem. 3.

Proposition 4. *Let $\mathfrak{q} = (q_0, \cdots, q_n)$ be a well-formed set of weights and $m = \operatorname{lcm}(q_0, \cdots, q_n)$. Assume that $\mathcal{Z} \subset \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is a closed subvariety defined by the weighted homogeneous polynomials $f_1, \cdots, f_r$ with integer coefficients such that*

$$\mathcal{Z} \cap \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \emptyset.$$

Then

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}; \mathcal{Z}) = \frac{1}{m} \log \gcd(f_1(\mathbf{x}), \dots, f_r(\mathbf{x})) + O(1), \tag{28}$$

for $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \backslash \left\{ \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) \cup \operatorname{Supp}(\mathcal{Z}) \right\}$ *with* $x_i \in \mathbb{Z}$ *and* $\operatorname{wgcd}(x_0, x_1, \cdots, x_n) = 1$.

Proof. Let $\mathcal{Y}$ be given by

$$\mathcal{Y} = \phi_{n,\mathfrak{q}}(\mathcal{Z}) \subset \mathcal{X} = \prod_{i=1}^{n} \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}}, \tag{29}$$

where $\mathfrak{q}_i = (q_0, q_i)$ for $i = 1, \cdots, n$ and where $\phi_{n,\mathfrak{q}}$ is defined by Eq. (14).

Consider the blowup $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ along $\mathcal{Y}$ and its exceptional divisor $\tilde{\mathcal{Y}} = \pi^{-1}(\mathcal{Y})$. Let $\mathbf{y} = \phi_{n,\mathfrak{q}}(\mathbf{x})$, for any $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \backslash \left\{ \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) \cup \operatorname{Supp}(\mathcal{Z}) \right\}$. Then, we have

$$\begin{aligned}
\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}; \mathcal{Z}) &= \sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\mathbf{y}), \nu_p) \\
&= \sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\mathcal{X},\mathcal{Y}}(\mathbf{y}, \nu_p) \\
&= \sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p). \\
&= \sum_{p \in M_{\mathbb{Q}} \backslash \{\infty\}} \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p) + \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_\infty)
\end{aligned}$$

By definition (see [10, Eq. (37)]) of the local weighted height associated to $\mathcal{Z}$, for $\nu_p \in M_{\mathbb{Q}}$, we have

$$\begin{aligned}
\zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p) &= \min_{1 \le j \le r} \left\{ -\log \frac{|f_j(\mathbf{x})|_{\nu_p}^{\frac{1}{m}}}{\max_i |x_i^{1/q_i}|_{\nu_p}} \right\} \\
&= \frac{1}{m} \log \max_{1 \le j \le r} |f_j(\mathbf{x})|_{\nu_p} + \frac{1}{q_i} \log \max_i |x_i|_{\nu_p}.
\end{aligned}$$

Thus,

$$\sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p) = \frac{1}{m} \log \gcd(f_1(\mathbf{x}), \dots, f_t(\mathbf{x})) + O(1).$$

Therefore, putting all together gives the desired equality.

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}; \mathcal{Z}) = \frac{1}{m} \log \gcd(f_1(\mathbf{x}), \dots, f_t(\mathbf{x})) + O(1).$$

This completes the proof. ∎

5. Vojta's conjecture and bounds on greatest common divisors

It is a well-known fact that the canonical bundle of the blowup $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ is given by

$$K_{\tilde{\mathcal{X}}} \sim \pi^\star K_{\mathcal{X}} + (r-1)\tilde{\mathcal{Y}},$$

see Def. 1. If $\mathcal{A}$ is an ample divisor on $\mathcal{X}$, then there is an integer N such that $-\tilde{\mathcal{Y}} + N\pi^\star \mathcal{A}$ is ample on $\tilde{\mathcal{X}}$, see [6, Thm. A.5.1]. Let

$$\tilde{\mathcal{A}} = -\frac{1}{N}\tilde{\mathcal{Y}} + \pi^\star \mathcal{A}$$

be the ample cone of $\tilde{\mathcal{X}}$. We further assume that $-K_{\mathcal{X}}$ is a normal crossing and

$$\operatorname{Supp}(K_{\mathcal{X}}) \cap \mathcal{Y} = \emptyset.$$

Let S be a finite set of places of $\mathbb{Q}$ and define

$$\mathfrak{s}_{\mathcal{X},D,S}(\cdot) := \sum_{\nu \in S} \zeta_{\mathcal{X},D}(\cdot, \nu) \quad \text{and} \quad \mathfrak{s}'_{\mathcal{X},D,S}(\cdot) := \sum_{\nu \notin S} \zeta_{\mathcal{X},D}(\cdot, \nu) \tag{30}$$

Then we have the following:

Theorem 1. *Let $\mathcal{X}$ be a smooth weighted variety, $\mathcal{A}$ an ample divisor on $\mathcal{X}$, $\mathcal{Y} \subset \mathcal{X}$ a smooth subvariety of codimension $r \geq 2$, and $-K_{\mathcal{X}}$ a normal crossing divisor whose support does not intersect $\mathcal{Y}$, all defined over $\mathbb{Q}$.*

Assume Vojta's conjecture (see Conj. 4) for smooth weighted varieties. Then for every finite set of places S and every $0 < \varepsilon < r-1$ there is a proper closed subvariety

$$\mathcal{Z} = \mathcal{Z}(\varepsilon, \mathcal{X}, \mathcal{Y}, \mathcal{A}, k, S) \not\subset \mathcal{X},$$

and constants $C_\varepsilon = C_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A}, k, S)$ and $\delta_\varepsilon = \delta_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A})$, such that

$$\log \mathrm{h}_{\mathrm{wgcd}}(P; \mathcal{Y}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) + \frac{1}{r-1+\delta_\varepsilon}\, \mathfrak{s}'_{\mathcal{X},-K_{\mathcal{X}},S}(P) + C_\varepsilon, \tag{31}$$

for all $P \in (\mathcal{X} \setminus \mathcal{Z})(\mathbb{Q})$.

Proof. The proof goes similarly to [11, Thm. 6] with necessary adjustments. We apply Conj. 4 for the weighted blow-up $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ and the divisor $D = -\pi^\star K_{\mathcal{X}}$ to get

$$\mathfrak{s}_{\tilde{\mathcal{X}},-\pi^\star K_{\mathcal{X}},S}(\tilde{P}) + \mathfrak{s}_{\tilde{\mathcal{X}},K_{\mathcal{X}}}(\tilde{P}) \leq \varepsilon\, \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{A}}}(\tilde{P}) + C_\varepsilon,$$

for all $\tilde{P} \in \tilde{\mathcal{X}}(\mathbb{Q}) \setminus \tilde{\mathcal{Z}}$. Substituting $K_{\tilde{\mathcal{X}}} = \pi^\star K_{\mathcal{X}} + (r-1)\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{A}} = -\frac{1}{N}\tilde{\mathcal{Y}} + \pi^\star \mathcal{A}$ we get

$$-\,\mathfrak{s}_{\mathcal{X},K_{\mathcal{X}},S}(P) + \mathfrak{s}_{\mathcal{X},K_{\mathcal{X}}}(P) + (r-1)\,\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) - \frac{\varepsilon}{N}\, \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) + C_\varepsilon$$

for all $P \in \mathcal{X}(\mathbb{Q}) \setminus \pi(\tilde{\mathcal{Z}})$. Since $-\,\mathfrak{s}_{\mathcal{X},K_{\mathcal{X}},S}(P) + \mathfrak{s}_{\mathcal{X},K_{\mathcal{X}}}(P) = \mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}$ we have

$$\mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}(P) + \left(r - 1 + \frac{\varepsilon}{N}\right) \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) + C_\varepsilon,$$

for all $P \in \mathcal{X}(\mathbb{Q}) \setminus \mathcal{Z}$. Hence,

$$\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) \leq \frac{N}{N(r-1)+\varepsilon} \left(-\,\mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}(P) + \varepsilon\, \mathfrak{s}_{\tilde{\mathcal{X}},\mathcal{A}}(P) + C_\varepsilon \right)$$

Since $\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) = \log \mathrm{h}_{\mathrm{wgcd}}(P; \mathcal{Y})$, we have

$$\log \mathrm{h}_{\mathrm{wgcd}}(P; \mathcal{Y}) \leq \frac{N}{N(r-1)+\varepsilon} \left(-\,\mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}(P) + \varepsilon\, \mathfrak{s}_{\tilde{\mathcal{X}},\mathcal{A}}(P) + C_\varepsilon \right).$$

Finally, setting $\delta = \varepsilon/N$ gives Eq. (31) and this completes the proof.

$\square$

Let $\mathfrak{q} = (q_0, \cdots, q_n)$ and assume that $X = \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is well-formed weighted projective space. Define

$$H_i = \{\mathbf{x} \in X \mid x_i = 0\}, \quad \text{for} \quad i = 0, \dots, n,$$

$A_0 = H_0$, and $K_X = -\sum_{i=0}^n H_i$. Consider the map

$$\phi_{n,\mathfrak{q}} : \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \to \mathcal{X} := \prod_{i=0}^n \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}},$$

where $\mathfrak{q}_i = (q_0, q_i)$ for $i = 1, \cdots, n$ and denote by

$$\mathcal{H}_i = \phi_{n,\mathfrak{q}}(H_i), \quad \mathcal{A}_0 = \phi_{n,\mathfrak{q}}(H_0), \quad K_{\mathcal{X}} = -\sum_{i=0}^n \phi_{n,\mathfrak{q}}(H_i).$$

Notice that the canonical divisor $K_{\mathcal{X}}$ is a normal crossing on $\mathcal{X}$ satisfying

$$\mathcal{Y} \cap \operatorname{Supp}(-K_{\mathcal{X}}) = \emptyset, \quad \text{where } \mathcal{Y} = \phi_{n,\mathfrak{q}}(\mathcal{Z}).$$

Theorem 2. *Assume that $\mathcal{Z} \subset \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ be a closed subvariety defined by*

$$f_1, \cdots, f_t \in \mathbb{Z}_{\mathfrak{q}}[x_0, \dots, x_n],$$

such that $\mathcal{Z} \cap \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \emptyset$, and has transversal intersection with the union $\cup_{i=0}^n \mathcal{H}_i$ Let $r := n - \dim(\mathbb{Z}) \geq 2$ be the codimension of $\mathbb{Z}$ in $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$.

Let S be a finite set of primes and $\varepsilon > 0$. If Conj. 4 holds for the weighted blow-up $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$, then there exists a nonzero weighted polynomial $g \in \mathbb{Z}_{\mathfrak{q}}[x_0, \dots, x_n]$ and a constant $\delta = \delta_{\varepsilon,\mathcal{Z}} > 0$, such that every $\tilde{\alpha} = (\alpha_0, \cdots, \alpha_n) \in \mathbb{Z}^{n+1}$ with $\operatorname{wgcd}(\alpha_0, \cdots, \alpha_n) = 1$ *satisfies either $g(\tilde{\alpha}) = 0$ or*

$$\gcd(f_1(\tilde{\alpha}), \cdots, f_t(\tilde{\alpha})) \leq \max\left\{|\alpha_0|^{\frac{1}{q_0}}, \cdots, |\alpha_n|^{\frac{1}{q_n}}\right\}^{\varepsilon} \cdot \left(|\alpha_0 \cdots \alpha_n|'_S\right)^{\frac{1}{(r-1+\delta)}}, \tag{32}$$

where $|\cdot|'_S$ is the "prime-to-S" part of its origin.

Proof. By definition of the global weighted height for

$$\mathbf{x} = [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \backslash \{\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) \cup \operatorname{Supp}(\mathcal{Z})\}(\mathbb{Q})$$

with $\operatorname{wgcd}(\alpha_0, \cdots, \alpha_n) = 1$, we have

$$\mathfrak{s}_{X,A}(\mathbf{x}) = \log \max\{|\alpha_0|^{\frac{1}{q_0}}, \cdots, |\alpha_n|^{\frac{1}{q_n}}\} + O(1) \tag{33}$$

By Prop. 4,

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}; \mathcal{Z}) = \frac{1}{m} \log \gcd(f_1(\mathbf{x}), \cdots, f_t(\mathbf{x})) + O(1). \tag{34}$$

Let $\mathbf{y} = \phi_{n,\mathfrak{q}}(\mathbf{x})$ for $\mathbf{x} = [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$. Then, by definition of S-part of the weighted heights and functoriality of the weighted heights, we have

$$\mathfrak{s}'_{\mathcal{X},\mathcal{H}_i,S}(\mathbf{y}) = \mathfrak{s}'_{X,H_i,S}(\mathbf{x}) = \sum_{\nu \in S} \nu^+(\alpha_i) = \frac{1}{q_i} \log |\alpha_i|'_S,$$

which implies that

$$\mathfrak{s}'_{\mathcal{X},-K_{\mathcal{X}},S}(\mathbf{y}) = \mathfrak{s}'_{X,-K_X,S}(\mathbf{x}) = \sum_{i=0}^n \mathfrak{s}'_{X,H_i,S} = \frac{1}{m} \log |\alpha_0 \alpha_1 \cdots \alpha_n|'_S. \tag{35}$$

By substituting Eq. (33), Eq. (34), Eq. (35), into the Eq. (31) we obtain

$$\begin{aligned}\frac{1}{m}\log\gcd(f_1(\tilde{\alpha}),\cdots,f_t(\tilde{\alpha})) &\leq \frac{1}{m}\log \mathrm{h}_{\mathrm{wgcd}}\ (\mathbf{x};\mathcal{Z}) = \log \mathrm{h}_{\mathrm{wgcd}}\ (\mathbf{y};\mathcal{Y})\\ &\leq \varepsilon\frac{1}{m}\,\mathfrak{s}_{\mathcal{X},\mathcal{A}}(\mathbf{y}) + \frac{1}{r-1+\delta}\,\mathfrak{s}'_{\mathcal{X},-K_{\mathcal{X}},S}(\mathbf{y}) + C_\varepsilon,\\ &\leq \varepsilon\cdot\log\max\{|\alpha_0|^{\frac{1}{q_0}},\cdots,|\alpha_n|^{\frac{1}{q_n}}\}\\ &\qquad\cdot\frac{1}{m(r-1+\delta)}\log\left(|\alpha_0\alpha_1\cdots\alpha_n|'_S\right) + C_\varepsilon,\end{aligned}$$

where $\delta=\delta_{\varepsilon,\mathcal{Z}}$. Replacing ε with ε/m, then multiplying the both sides by m and exponentiating, we obtain the desired inequality Eq. (2). □

5.1. **An application on gcds and a result of Corvaja/Zannier.** Let S be a finite set of rational primes. The "prime-to-S" part $|x|'_S$ of any nonzero integer x is defined to be the largest divisor of x that is not divisible by any of the primes in S, in other words,

$$|x|'_S = |x|\cdot\prod_{p\in S}|x|_p.$$

In particular, x is an S-unit if and only if $|x|'_S=1$.

The following result can be though as a weighted version of the main result of [2] for weighted projective varieties.

Lemma 4. *Assume Conj. 4. Fix $\varepsilon>0$, a finite set S of prime numbers, and a triple of weights $\mathfrak{q}=(q_0,q_1,q_2)$. Then, there exist a finite set $Z=Z(S,\varepsilon)\subset\mathbb{Z}^2$ such that*

$$\gcd(\alpha_1^{q_0}-1,\alpha_2^{q_0}-1)\leq\max\{|\alpha_1|^{\frac{1}{q_1}},|\alpha_2|^{\frac{1}{q_2}}\}^{\varepsilon}\cdot\left(|\alpha_1\alpha_2|'_S\right)^{\frac{1}{(1+\delta_\varepsilon)}},\tag{36}$$

holds for all pairs $(\alpha_1,\alpha_2)\in\mathbb{Z}^2\backslash Z$.

Proof. Let $X=\mathbb{P}^2_{\mathfrak{q}}(\mathbb{Q})$ with well-formed $\mathfrak{q}=(q_0,q_1,q_2)$ and $m=q_0q_1q_2$. Take

$$\mathcal{Z}=\{f_1=f_2=0\}\subset X,\quad \text{where}\quad f_1=x_1^{q_0}-x_0^{q_1},\ f_2=x_2^{q_0}-x_0^{q_2}.$$

are weighted homogeneous polynomials of degree q_0q_1 and q_0q_2 respectively.

Points in $\mathcal{Z}$ look like

$$\mathbf{x}=[x_0 : x_0^{q_1/q_0} : x_0^{q_2/q_0}].$$

Multiplying by $\frac{1}{x_0^{1/q_0}}$ we get $\mathbf{x}=[1:1:1]$. Since $\mathcal{Z}$ has only one point $\mathbf{x}=[1:1:1]$, it is of codimension $r=2$ in X. By Thm. 2 there exists a 1-dimensional exceptional set $Z\subset X$, depending on f_1,f_2 and $\varepsilon>0$, such that

$$\gcd(\alpha_1^{q_0}-\alpha_0^{q_1},\alpha_2^{q_0}-\alpha_0^{q_2})\leq\max\left\{|\alpha_0|^{\frac{1}{q_0}},|\alpha_1|^{\frac{1}{q_1}},|\alpha_2|^{\frac{1}{q_2}}\right\}^{\varepsilon}\cdot\left(|\ \alpha_0\alpha_1\alpha_2\ |'_S\right)^{\frac{1}{m(1+\delta_\varepsilon)}}.$$

Now, we assume $x_0=1$ and $x_1=\alpha_1$, $x_2=\alpha_2$ for any given $(\alpha_1,\alpha_2)\in\mathbb{Z}^2\backslash Z$. This completes the proof. □

Corollary 2. *Assume Conj. 4. Fix $\varepsilon>0$, a finite set S of prime numbers, and a triple of weights $\mathfrak{q}=(q_0,q_1,q_2)$. Then, there exist a finite set $Z=Z(S,\varepsilon)\subset\mathbb{Z}^2$ such that*

$$\gcd(\alpha_1-1,\alpha_2-1)\leq\max\{|\alpha_1|^{\frac{1}{q_1}},|\alpha_2|^{\frac{1}{q_2}}\}^{\varepsilon}\cdot\left(|\alpha_1\alpha_2|'_S\right)^{\frac{1}{(1+\delta_\varepsilon)}},\tag{37}$$

holds for all pairs $(\alpha_1,\alpha_2)\in\mathbb{Z}^2\backslash Z$.

Proof. Since $\gcd(\alpha_1 - 1, \alpha_2 - 1)$ always divides $\gcd(\alpha_1^{q_0} - 1, \alpha_2^{q_0} - 1)$, so by last inequality one has Eq. (37). □

In particular, for $q_0 = q_1 = q_2 = 1$ if we assume $x_0 = 1$ and $x_1 = \alpha_1$, $x_2 = \alpha_2$ are S-units, then $|\alpha_1\alpha_2|'_S = 1$ and hence we recover conditionally the main result of [3], as stated in [11, Thm. 1].

Thus, Vojta's conjecture implies a natural weighted generalization of [11, Thm. 1] in which we remove the restriction that α_1 and α_2 are S-units and replace it's third condition with Eq. (37), which is more general.

Let $\mathbb{P}_{\mathfrak{q},k}$ be a well-formed weighted projective space and $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}_{\mathfrak{q},k}(k)$. Assume $\mathbf{x}$ normalized (i.e. $\operatorname{wgcd}_k(\mathbf{x}) = 1$). The above lemma suggests that there might be a way to bound $\gcd(x_0, \ldots, x_n)$ with $\mathcal{S}(\mathbf{x})$.

If $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}_{\mathfrak{q},k}(k)$ s is absolutely normalized then the situation is simpler as it can be seen below.

Remark 1. *Let $\mathbb{P}_{\mathfrak{q},k}$ be a well-formed weighted projective space and $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}_{\mathfrak{q},k}(k)$ such that $\mathbf{x}$ is absolutely normalized. Then $\gcd(x_0, \ldots, x_n) = 1$.*

Proof. Let $\mathfrak{q} = (q_0, \ldots, q_n)$ be well-formed, $m = \operatorname{lcm}(q_0, \ldots, q_n)$, and let $\phi : \mathbb{P}_{\mathfrak{q},k} \to \mathbb{P}^n$ be Veronese embedding. Assume $\gcd(x_0, \ldots, x_n) = d > 1$. Then

$$\gcd(\phi(x_0), \ldots, \phi(x_n)) = d,$$

and hence $\overline{\operatorname{wgcd}}(\mathbf{x}) = d^{\frac{1}{m}} > 1$, which is a contradiction. □

It remains to be further investigated the relation between $\gcd(\phi(x_0), \ldots, \phi(x_n))$ and $\mathcal{S}(\mathbf{x})$ when $\mathbf{x}$ is just normalized, and whether Lem. 4 can be proved independently of Vojta's conjecture.

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