RATIONAL FUNCTIONS ON THE PROJECTIVE LINE FROM A COMPUTATIONAL VIEWPOINT

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Abstract. We explore the moduli space $\mathcal{M}_d^1$ of degree $d \geq 3$ rational functions on the projective line using a machine learning approach, focusing on automorphism group classification. For $d = 3$, where $\mathcal{M}_3^1 = \mathbb{P}^5_{\mathbf{w}}(\mathbb{Q})$ with weights $\mathbf{w} = (2, 2, 3, 3, 4, 6)$, we generate a dataset of 2,078,697 rational functions over $\mathbb{Q}$ with naive height ≤ 4. Initial coefficient-based models achieved high overall accuracy but struggled with minority classes due to extreme class imbalance. By using invariants $\xi_0, \ldots, \xi_5$ as features in a Random Forest classifier, we achieved approximately 99.992% accuracy, mirroring successes in genus 2 curves [9]. This highlights the transformative role of invariants in arithmetic dynamics, yet for $d > 3$, unknown generators of $\mathcal{R}_{(d+1,d-1)}$ pose scalability challenges. Our framework bridges data-driven and algebraic methods, with potential extensions to higher degrees and $\mathcal{M}_d^2$.

1. Introduction

Let k be an algebraically closed field of characteristic zero and $\mathbb{P}^1_k$ the projective line over k. A degree $d \geq 2$ rational function $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is given as the ratio of two degree d binary forms, say $\phi(x, y) = \frac{f_0(x,y)}{f_1(x,y)}$ such that the resultant between $f_0(x, y)$ and $f_1(x, y)$ is non-zero. Hence, a rational function is a pair of binary forms of the same degree with no common roots. If we denote

$$f_0(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i} \quad \text{and} \quad f_1(x, y) = \sum_{i=0}^{d} b_i x^i y^{d-i},$$

then the collection of pairs $[f_0 : f_1]$ can be parametrized via

$$[a_d : \cdots : a_0 : b_d : \cdots : b_0] \in \mathbb{P}^{2d+1},$$

such that $\mathrm{Res}(f_0, f_1) \neq 0$. So the parameter space of degree $d > 1$ rational functions on $\mathbb{P}^1$ is the complement of the resultant locus in $\mathbb{P}^{2d+1}$, say $\mathrm{Rat}_d^1 := \mathbb{P}^{2d+1} \setminus V(\mathrm{Res})$.

The group $\mathrm{SL}_2(k)$ acts on Rat_d^1 by conjugation, i.e., for some $M \in \mathrm{SL}_2(k)$,

$$\phi \to \phi^M := M^{-1} \circ \phi \circ M.$$

Two rational functions $\phi, \psi \in \mathrm{Rat}_d^1$ are called *conjugate* if there is an $M \in \mathrm{SL}_2(k)$ such that $\phi = \psi^M$. The moduli space of degree $d > 1$ rational functions (in one

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variable) is denoted by $\mathcal{M}_d^1$ and can be constructed as a quotient space of this SL_2-action. The automorphism group of ϕ is defined as

$$\mathrm{Aut}(\phi) := \{\sigma \in \mathrm{PGL}_2(k) \mid \phi^\sigma = \phi\}.$$

It is a finite subgroup of $\mathrm{PGL}_2(k)$, so it is isomorphic to one of the following: a cyclic group C_n, a dihedral group D_n, A_4, S_4, or A_5. Determining which one of these groups occur for a fixed degree $d \geq 2$ is studied in [2, 4, 6].

When the degree d gets large, determining the automorphism groups, inclusions among the loci of functions with fixed automorphism groups, minimal fields of definition, and other questions that arise in arithmetical dynamics (see [17]) becomes challenging and involved. Hence, we aim to use a machine learning approach to study such questions. The goal of this paper is to introduce such an approach and to identify some of the challenges that will be encountered along the way. Our main task in this paper is to design a machine learning model to determine properties of rational functions such as the automorphism group, inclusions among the automorphism loci, a minimal field of definition, etc. As a case study, we focus on the case of rational cubics over $\mathbb{Q}$ since this is a well-known case and gives an opportunity for training and identifying methods that would work better in higher degree cases.

We created the database of all rational cubics with height ≤ 4. There are 2,078,697 such cubics (cf. Section 7 for details). Initial machine learning models using the coefficients of these rational functions as features performed poorly in classifying the automorphism groups, achieving high overall accuracy but failing to correctly identify minority classes due to extreme class imbalance. However, by incorporating invariants of the rational functions as features, we were able to significantly improve the classification performance, achieving near-perfect accuracy across all classes.

For any $\phi(x, y) = \frac{f_0(x,y)}{f_1(x,y)}$, we define binary forms of degrees $d+1$ and $d-1$ as

$$\mathcal{I}_\phi := y f_0 - x f_1 \quad \text{and} \quad \mathcal{J}_\phi := \frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}.$$

Any two degree d rational functions ϕ and ψ are conjugate for some $M \in \mathrm{PGL}_2(k)$ via $\psi = \phi^M$ if and only if $\mathcal{I}_\psi = \mathcal{I}_\phi^M$ and $\mathcal{J}_\psi = \mathcal{J}_\phi^M$; see Lem. 1. Moreover, there is a one-to-one correspondence between degree d rational functions $\phi(x)$ and points $(f, g) \in V_{d+1} \oplus V_{d-1}$ such that

$$\mathrm{Res}\left(xg + \frac{\partial f}{\partial y}, yg - \frac{\partial f}{\partial x}\right) \neq 0.$$

Hence, determining invariants of rational functions is the same as determining generators for the ring of invariants of $V_{d+1} \oplus V_{d-1}$, which can be determined using a result of Clebsch once generators of the ring of invariants for V_{d+1} and V_{d-1} are known.

Denote by $\mathcal{R}_{(d+1,d-1)}$ the ring of invariants of $V_{d+1} \oplus V_{d-1}$ and $(\xi_0, \ldots, \xi_n)$ the tuple of generators of this ring with degrees $(q_0, \ldots, q_n)$ respectively. Since all $\xi_0, \ldots, \xi_n$ are homogeneous polynomials, then $\mathcal{R}_{(d+1,d-1)}$ is a graded ring and $\mathrm{Proj}\, \mathcal{R}_{(d+1,d-1)}$ is a weighted projective space denoted by $\mathbb{P}^n_\omega(k)$, where $\omega = (q_0, \ldots, q_n)$ is the set of weights. Thus for each $\phi \in \mathcal{P}_d$, we evaluate its invariants and have a map $\xi : \mathcal{P}_d \to \mathcal{M}_d^1$ via

$$\phi \to [\xi_0(\phi) : \cdots : \xi_n(\phi)],$$

and create a new database $\mathcal{W}_d$ where each entry represents the isomorphism class of some rational function. We can normalize points in $\mathcal{W}_d$ by "dividing" by the weighted greatest common divisor. Then using the weighted projective height we can further order them; see [7]. Obviously, the data in $\mathcal{W}_d$ doesn't necessarily have to come from $\mathcal{P}_d$, since we can randomly generate points in the weighted projective space $\mathbb{P}^n_\omega$ similarly how it was done in [9]. The benefit of training the data $\mathcal{W}_d$ instead of $\mathcal{P}_d$ is that there is no redundancy because every data point represents uniquely the equivalence class of a rational function. While the input features of the model can still be the coefficients of the rational function, we can use feature engineering and insert a symbolic layer which computes the invariants of the function and hence the corresponding point $\mathfrak{p} \in \mathcal{M}^1_d$, similar to what was used in [11].

For $d = 3$, the generators of $V_4 \oplus V_2$ are invariants of degree 2, 2, 3, 3, 4, and 6. Hence, the moduli space $\mathcal{M}^1_3$ is the weighted projective space $\mathbb{P}_\omega(k)$ for the set of weights $\omega = (2, 2, 3, 3, 4, 6)$. We explicitly determine such invariants as polynomials in $a_0, \dots, a_3, b_0, \dots, b_3$ and implement them as a symbolic layer. A complete list of automorphism groups and the corresponding loci in terms of such invariants are also computed. Adding such invariants to the database increases the classification accuracy significantly. For example, using a Random Forest classifier with invariants as features, we achieved an accuracy of approximately 99.992% in classifying the automorphism groups; see Section 7 for further details. There is no reason to believe that this accuracy will go down for higher degree d.

Our results show that machine learning, when combined with algebraic invariants, can be extremely successful in studying rational functions. In the last section, we give a brief account of our results and challenges ahead.

While our study of rational cubics over $\mathbb{Q}$ demonstrates the power of this approach, it also raises questions about its scalability and applicability to higher degrees. The explicit invariants for $d = 3$ are a solved problem, but for $d > 3$, such as $d = 4$ or beyond, the generators of $\mathcal{R}_{(d+1,d-1)}$ remain largely unknown, complicating both algebraic and computational analysis. Extending this framework requires addressing these gaps, testing its robustness across diverse fields, and refining the symbolic layer's integration, topics we explore in the subsequent work as we outline our findings and the path forward.

2. Preliminaries

Let k be an algebraically closed field, $\mathbb{P}^N(k)$ the projective N-space over k, and $k[x, y]$ be the polynomial ring in two variables. By V_d we denote the $(d + 1)$-dimensional subspace of $k[x, y]$ consisting of homogeneous polynomials

$$f(x, y) = a_d x^d + a_{d-1} x^{d-1} y + \cdots a_1 x y^{d-1} + a_0 y^d,$$

of degree d (up to multiplication by a scalar). Elements in V_d (up to multiplication by a constant) are called *binary forms* of degree d. $\mathrm{GL}_2(k)$ acts as a group of automorphisms on $k[x, y]$ as follows:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k), \ \textit{then} \quad M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \tag{1}$$

Denote by f^M the binary form

$$f^M(x, y) := f(ax + by, cx + dy).$$

It is well known that $\mathrm{SL}_2(k)$ leaves a binary form (unique up to scalar multiples) on V_d invariant.

Consider a_0, a_1, ... , a_d as parameters (coordinate functions on V_d). Then the coordinate ring of V_d can be identified with $k[a_0, \dots, a_d]$. For $I \in k[a_0, \dots, a_d]$ and $M \in \mathrm{GL}_2(k)$, define

$$I^M \in k[a_0, \dots, a_d] \quad \text{as} \quad I^M(f) := I(f^M),$$

for all $f \in V_d$. Then $I^{MN} = (I^M)^N$ and $I^M(f)$ define an action of $\mathrm{GL}_2(k)$ on $k[a_0, \dots, a_d]$.

A homogeneous polynomial $I \in k[a_0, \dots, a_d, x, y]$ is called a **covariant** of index s if $I^M(f) = \delta^s I(f)$, for all $f \in V_d$, where $\delta = \det(M)$. The homogeneous degree in $a_0, \dots, a_d$ is called the **degree** of I, and the homogeneous degree in x, y is called the **order** of I. A covariant of order zero is called **invariant**. An invariant is a $\mathrm{SL}_2(k)$-invariant on V_d.

By Hilbert's theorem the ring of invariants of binary forms is finitely generated. We denote by $\mathcal{R}_d$ the ring of invariants of the binary forms of degree d. Then, $\mathcal{R}_d$ is a finitely generated graded ring

2.1. **Change of coordinates.** Let $I_0, \dots, I_n$ be the generators of $\mathcal{R}_d$ with degrees $q_0, \dots, q_n$ respectively. For any two binary forms f and g, $f = g^M$, $M \in \mathrm{GL}_2(k)$, if and only if

$$(I_0(f), \dots I_i(f), \dots, I_n(f)) = \left(\lambda^{q_0} I_0(g), \dots, \lambda^{q_i} I_i(g), \dots, \lambda^{q_n} I_n(g)\right), \tag{2}$$

where $\lambda = (\det M)^{\frac{d}{2}}$.

2.2. **Generators of the ring of invariants.** Let V_d be the space of degree $d > 1$ binary forms defined over k, and $\mathcal{R}_d$ the ring of invariants. Below we list the generating set of $\mathcal{R}_d$ for $d \leq 10$. We assume that the binary forms are given in standard form

$$f(x, y) = \sum_{i=0}^{d} \binom{d}{i} a_i x^i y^{d-i}$$

For $f, g \in V_d$ the r-th transvectant of f and g is defined as

$$(f, g)_r := \frac{(m-r)!\,(n-r)!}{n!\,m!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \cdot \frac{\partial^r f}{\partial x^{r-k}\, \partial y^k} \cdot \frac{\partial^r g}{\partial x^k\, \partial y^{r-k}},$$

While there is no method known to determine a minimal generating set of invariants for any $\mathcal{R}_d$, we display such sets for $3 \leq d \leq 4$ as in [1].

2.2.1. *Cubics.* A generating set for $\mathcal{R}_3$ is $\xi = [\xi_0]$, where

$$\xi_0 = ((f, f)_2, (f, f)_2)_2 = -54a_0^2a_3^2 + 36a_1a_3a_0a_2 - 8a_2^3a_0 - 8a_1^3a_3 + 2a_2^2a_1^2 \tag{3}$$

2.2.2. *Quartics.* A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1]$ with $\omega = (2, 3)$, where

$$\xi_0 = (f, f)_4 \quad \text{and} \quad \xi_1 = (f, (f, f)_2)_4 \tag{4}$$

For $d = 6$ to $d = 10$, generators are known and displayed in [1]. In Table 1 are their counts and weights.

TABLE 1. Generators of $\mathcal{R}_d$ for $d = 6$ to $d = 10$

d	Weights ω	Number of Generators
6	(2, 4, 6, 10)	4
7	(4, 8, 12, 12, 20)	5
8	(2, 3, 4, 5, 6, 7)	6
9	(4, 8, 10, 12, 12, 14, 16)	7
10	(2, 4, 6, 6, 8, 9, 10, 14, 14)	9

3. Invariants of Rational Functions

This section derives the invariants of degree $d > 1$ rational functions on $\mathbb{P}^1(k)$, where k is algebraically closed. Our goal is to classify these functions up to $\mathrm{PGL}_2(k)$-equivalence, compute the invariant ring $\mathcal{R}_{(d+1,d-1)}$, and specialize to $d = 3$ for explicit invariants that enable machine learning analysis of the moduli space $\mathcal{M}_3^1$.

If one fixes homogeneous coordinates x, y on $\mathbb{P}^1$, then any rational function $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$ can be realized as

$$\phi(x, y) = \frac{f_0(x, y)}{f_1(x, y)},$$

where $d = \max\{\deg f_0, \deg f_1\}$ and f_0 and f_1 are homogeneous polynomials of degree at most d. Conversely, a pair of homogeneous polynomials of degree d in x and y determine a rational function

$$\phi(x, y) = [f_0(x, y) : f_1(x, y)] \tag{5}$$

if f_0, f_1 have no common roots in k.

For a fixed degree $d > 1$, the collection of all such pairs of homogeneous polynomials $[f_0(x, y) : f_1(x, y)]$, say

$$f_0 = \sum_{i=0}^{d} a_i x^{d-i} y^i \quad \text{and} \quad f_1 = \sum_{i=0}^{d} b_i x^{d-i} y^i,$$

can be naturally parametrized as the projective space $\mathbb{P}^{2d+1}$, via

$$[f_0 : f_1] \to [a_0 : a_1 : \cdots : a_d : b_0 : \cdots : b_d] \in \mathbb{P}^{2d+1}.$$

We denote the resultant of two binary forms f_0 and f_1 by $\mathrm{Res}(f_0, f_1)$. Notice that it's well-defined and a degree $2d$ homogeneous polynomial in

$$I_{2d}(\phi) := \mathrm{Res}(f_0, f_1) \in k[a_0, \dots, a_d, b_0, \dots, b_d]. \tag{6}$$

Hence, $I_{2d}(\phi)$ is an $\mathrm{SL}_2(k)$-invariant of degree $2d$. Moreover, ϕ is a rational function on $\mathbb{P}^1$ if and only if $I_{2d}(\phi) \neq 0$.

We can construct the parameter space of rational functions on $\mathbb{P}^1$ as the complement of the vanishing locus $V(I_{2d})$ of I_{2d}. Hence the **rational space of rational functions of degree d on $\mathbb{P}^1$** is defined as

$$\mathrm{Rat}_d^1 := \mathbb{P}^{2d+1} \setminus V(I_{2d})$$

The action of $\mathrm{PGL}_2(k)$ on V_d extends naturally to an action on Rat_d^1. For each $\sigma \in \mathrm{PGL}_2(k)$, we have $\mathrm{PGL}_2(k) \times \mathrm{Rat}_d^1 \to \mathrm{Rat}_d^1$ via

$$(\sigma, \phi(x, y)) \to \phi^\sigma := \sigma^{-1} \phi \sigma,$$

Two rational functions $\phi, \psi \in \mathrm{Rat}_d^1$ are called **k-conjugate** if and only if there exists a matrix $\sigma \in \mathrm{PGL}_2(k)$ such that $\psi = \sigma^{-1}\phi\sigma$.

Remark 1. *Notice two different uses of notation ϕ^σ for rational functions and f^σ for binary forms.*

The function $\phi^\sigma = \sigma^{-1}\phi\sigma$ is explicitly given as

$$\left(\sigma^{-1}\phi\sigma\right)(x) = \frac{e\, f_0(ax+b, cx+e) - b\, f_1(ax+b, cx+e)}{-c\, f_0(ax+b, cx+e) + a\, f_1(ax+b, cx+e)} = \frac{e\, f_0^\sigma - b\, f_1^\sigma}{-c\, f_0^\sigma + a\, f_1^\sigma} \tag{7}$$

Let $\phi(x,y)$ and $\psi(x,y)$ be degree $d \geq 2$ rational functions given by

$$\phi(x,y) = \frac{f_0(x,y)}{f_1(x,y)} \quad \text{and} \quad \psi(x,y) = \frac{g_0(x,y)}{g_1(x,y)}. \tag{8}$$

By Eq. (7), ϕ and ψ are k-conjugate if and only if there is $\sigma = \begin{pmatrix} a & b \\ c & e \end{pmatrix} \in \mathrm{PGL}_2(k)$ such that

$$g_0 = ef_0^\sigma - bf_1^\sigma \quad \text{and} \quad g_1 = -cf_0^\sigma + af_1^\sigma. \tag{9}$$

Definition 1. For $\phi(x,y) = \frac{f_0(x,y)}{f_1(x,y)}$, we define its **associated pair of binary forms** as

$$\mathcal{I}_\phi := yf_0 - xf_1 \quad \text{and} \quad \mathcal{J}_\phi := \frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y} \tag{10}$$

Notice that $\mathcal{I}_\phi \in V_{d+1}$ and $\mathcal{J}_\phi \in V_{d-1}$.

Lemma 1. *Let $\phi, \psi \in \mathrm{Rat}_d^1$ and $\sigma \in \mathrm{PGL}_2(k)$. Then $\psi = \phi^\sigma$ if and only if*

$$\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma \quad \textit{and} \quad \mathcal{J}_\psi = \mathcal{J}_\phi^\sigma.$$

Proof. Let $\phi = \frac{f_0}{f_1}$ and $\psi = \frac{g_0}{g_1}$. Assume that ϕ and ψ are k-conjugate in Rat_d^1, meaning there exists $\sigma \in \mathrm{PGL}_2(k)$ such that $\psi = \phi^\sigma$.

Substituting the expressions for $\mathcal{I}_\psi = yg_0 - xg_1$ and using the values of g_0 and g_1 as in Eq. (9), we have

$$\begin{aligned}\mathcal{I}_\psi &= yg_0 - xg_1 = y(ef_0^\sigma - bf_1^\sigma) - x(-cf_0^\sigma + af_1^\sigma) \\ &= (cx+ey)f_0^\sigma - (ax+by)f_1^\sigma = (yf_0 - xf_1)^\sigma = \mathcal{I}_\phi^\sigma.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{J}_\psi &= \frac{\partial g_0}{\partial x} + \frac{\partial g_1}{\partial y} = \frac{\partial(ef_0^\sigma - bf_1^\sigma)}{\partial x} + \frac{\partial(-cf_0^\sigma + af_1^\sigma)}{\partial y} \\ &= e\frac{\partial f_0^\sigma}{\partial x} - b\frac{\partial f_1^\sigma}{\partial x} - c\frac{\partial f_0^\sigma}{\partial y} + a\frac{\partial f_1^\sigma}{\partial y} = \left(\frac{\partial f_0^\sigma}{\partial x} + \frac{\partial f_1^\sigma}{\partial y}\right)^\sigma = \mathcal{J}_\phi^\sigma.\end{aligned}$$

Thus, we conclude that $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ (and similarly, $\mathcal{J}_\phi$ and $\mathcal{J}_\psi$) are k-equivalent via σ as binary forms.

Conversely, suppose that $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ (respectively, $\mathcal{J}_\phi$ and $\mathcal{J}_\psi$) are k-equivalent via σ as binary forms. This means that $\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma$ and $\mathcal{J}_\psi = \mathcal{J}_\phi^\sigma$. In particular, we have

$$\begin{aligned}\mathcal{I}_\psi &= yg_0 - xg_1 = (yf_0 - xf_1)^\sigma = (cx+ey)f_0^\sigma - (ax+by)f_1^\sigma, \\ \mathcal{J}_\psi &= \frac{\partial g_0}{\partial x} + \frac{\partial g_1}{\partial y} = \frac{\partial f_0^\sigma}{\partial x^\sigma} + \frac{\partial f_1^\sigma}{\partial y^\sigma}.\end{aligned}$$

From the equation for $\mathcal{I}_\psi$, we obtain

$$y\left(g_0 - e f_0^\sigma + b f_1^\sigma\right) = x\left(g_1 + c f_0^\sigma - a f_1^\sigma\right),$$

which leads to

$$g_0 - e f_0^\sigma + b f_1^\sigma = x \cdot h(x,y), \quad g_1 + c f_0^\sigma - a f_1^\sigma = y \cdot h(x,y),$$

for some $h \in V_{d-1}$. Therefore, we have

$$\begin{aligned} \mathcal{J}_\psi &= 2h + \left(x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} \right) + e\frac{\partial f_0^\sigma}{\partial x} + b\frac{\partial f_1^\sigma}{\partial x} - c\frac{\partial f_0^\sigma}{\partial y} + a\frac{\partial f_1^\sigma}{\partial y} \\ &= h + \frac{d-1}{2}h + \frac{\partial g_0}{\partial x} + \frac{\partial g_1}{\partial y}. \end{aligned}$$

Thus, we must have $h = 0$, which implies that $\phi^\sigma = \psi$ as claimed. □

Since the pair of binary forms $(\mathcal{I}_\phi, \mathcal{J}_\phi)$ determines the rational function ϕ, we can use the classical theory of binary forms to determine invariants for ϕ. Define

$$\begin{aligned} \Phi : \mathrm{Rat}_d^1 &\to V_{d+1} \oplus V_{d-1}, \\ \phi &\to \left(\mathcal{I}_\phi, \mathcal{J}_\phi\right). \end{aligned} \tag{11}$$

The inverse of Φ is not well-defined since not every pair $(f,g) \in V_{d+1} \oplus V_{d-1}$ determines a rational function.

Definition 2. For any $(f,g) \in V_{d+1} \oplus V_{d-1}$, define the **modular resultant**

$$\Delta_{\mathcal{I}_f, \mathcal{J}_g} = \mathrm{Res}\left(xg + \frac{\partial f}{\partial y}, \, yg - \frac{\partial f}{\partial x} \right), \tag{12}$$

and the **moduli resultant locus** $\mathcal{N}$ as

$$\mathcal{N} := \{(f,g) \in V_{d+1} \oplus V_{d-1} \mid \Delta_{f,g} = 0\}.$$

Then, we have the following:

Lemma 2. *The map*

$$\Phi : \mathrm{Rat}_d^1 \to (V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$$

is bijective. Moreover, for any $(f,g) \in V_{d+1} \oplus V_{d-1} \setminus \mathcal{N}$*,*

$$\Phi^{-1}(f,g) = \frac{xg + \frac{\partial f}{\partial y}}{yg - \frac{\partial f}{\partial x}}.$$

Proof. The map Φ is obviously well-defined. Let $\phi, \psi \in \mathrm{Rat}_d^1$ such that $\Phi(\phi) = \Phi(\psi)$. Then, $\mathcal{I}_\phi = \mathcal{I}_\psi$ and $\mathcal{J}_\phi = \mathcal{J}_\psi$ as binary forms in V_{d+1} and V_{d-1}, respectively. Since binary forms are defined up to multiplication by a scalar, there exists a diagonal matrix $\sigma \in \mathrm{PGL}_2(k)$ such that

$$\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma \quad \text{and} \quad \mathcal{J}_\psi = \mathcal{J}_\phi^\sigma.$$

By Lem. 1, we conclude that $\psi = \phi^\sigma = \phi$. Thus, Φ is injective.

Now, assume that $(f,g) \in (V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$. The condition $\Delta_{\mathcal{I}_f, \mathcal{J}_g} \neq 0$ ensures that the preimage

$$\Phi^{-1}(f,g) = \frac{xg + \frac{\partial f}{\partial y}}{yg - \frac{\partial f}{\partial x}}$$

belongs to Rat_d^1. Consequently, the map Φ^{-1} is well-defined.

To show that Φ and Φ^{-1} are inverses of each other, we will demonstrate that $\Phi \circ \Phi^{-1} = \mathrm{id}$ on $(V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$ and $\Phi^{-1} \circ \Phi = \mathrm{id}$ on Rat_d^1.

First, compute $\Phi \circ \Phi^{-1}$ on $(V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$: Let $(f, g) \in (V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$ and $\phi = \Phi^{-1}(f, g) = \frac{xg + \frac{\partial f}{\partial y}}{yg - \frac{\partial f}{\partial x}}$. Then,

$$\mathcal{I}_\phi = y\left(xg + \frac{\partial f}{\partial y}\right) - x\left(yg - \frac{\partial f}{\partial x}\right) = y\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial x} = (d+1)f = f \in V_{d+1}$$

$$\mathcal{J}_\phi = \frac{\partial}{\partial x}\left[xg + \frac{\partial f}{\partial y}\right] + \frac{\partial}{\partial y}\left[yg - \frac{\partial f}{\partial x}\right] = 2g + x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 2g + (d-1)g = g \in V_{d-1}$$

Thus, $\mathcal{I}_\phi = f$ and $\mathcal{J}_\phi = g$, so $\Phi(\phi) = (\mathcal{I}_\phi, \mathcal{J}_\phi) = (f, g)$.

Second, compute $\Phi^{-1} \circ \Phi$ on Rat_d^1: Let $\phi = \frac{f_0}{f_1} \in \mathrm{Rat}_d^1$. Then,

$$\begin{aligned}
\left(\Phi^{-1} \circ \Phi\right)(\phi) &= \Phi^{-1}(\mathcal{I}_\phi, \mathcal{J}_\phi) = \Phi^{-1}\left(yf_0 - xf_1, \frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) \\
&= \frac{x\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) + \frac{\partial}{\partial y}\left[yf_0 - xf_1\right]}{y\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial x}\left[yf_0 - xf_1\right]} \\
&= \frac{x\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) + \left(f_0 + y\frac{\partial f_0}{\partial y} - x\frac{\partial f_1}{\partial y}\right)}{y\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) - \left(y\frac{\partial f_0}{\partial x} - f_1 - x\frac{\partial f_1}{\partial x}\right)} \\
&= \frac{x\frac{\partial f_0}{\partial x} + y\frac{\partial f_0}{\partial y} + f_0}{x\frac{\partial f_1}{\partial x} + y\frac{\partial f_1}{\partial y} + f_1} = \frac{df_0 + f_0}{df_1 + f_1} = \frac{f_0}{f_1} = \phi.
\end{aligned}$$

Since both compositions $\Phi \circ \Phi^{-1}$ and $\Phi^{-1} \circ \Phi$ are the identity maps, we conclude that Φ and Φ^{-1} are indeed inverse to each other. This completes the proof. □

3.1. Ring of invariants $\mathcal{R}_{(d+1),(d-1)}$ and a theorem of Clebsch.

The action of $\mathrm{GL}_2(k)$ on V_d induces an action of $\mathrm{GL}_2(k)$ in $V_{d+1} \oplus V_{d-1}$. To determine the isomorphism classes of degree d rational functions, we have to determine the ring of invariants of $V_{d+1} \oplus V_{d-1}$. This is a well known in classical invariant theory. We briefly describe it below; ; see [18] for details.

Let V be an SL_2-module and $\mathcal{O}(V)$ the algebra of polynomial functions on V. $\mathrm{SL}_2(k)$ acts on $\mathcal{O}(V)$ via

$$M \cdot p(f_1, \ldots, f_r) \to p(M^{-1}f_1, \ldots, M^{-1}f_r),$$

for every $M \in \mathrm{SL}_2(k)$. An invariant of V is an element $\mathcal{T} \in \mathcal{O}(V)$ such that $M\mathcal{T} = \mathcal{T}$, for all $M \in \mathrm{SL}_2(k)$. The set of invariants is denoted by $\mathcal{O}(V)^{\mathrm{SL}_2}$.

A transvectant $(\mathcal{T}, \mathcal{S})_l$ is called **irrelevant** if there exist $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2$ and l_1, l_2 such that

$$l = l_1 + l_2, \quad \mathcal{T} = \mathcal{T}_1 \cdot \mathcal{T}_2, \quad \mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2,$$

and $l_1 \leq \operatorname{ord} \mathcal{T}_1, \operatorname{ord} \mathcal{S}_1$, and $l_2 \leq \operatorname{ord} \mathcal{T}_2, \operatorname{ord} \mathcal{S}_2$. A transvectant which is not irrelevant is called **relevant**.

Let V and W be two SL_2-modules whose covariants are finitely generated, and assume

$$\begin{aligned}
&\mathcal{T}_1, \ldots, \mathcal{T}_r: \quad \text{are the generators of the covariants of } V \\
&\mathcal{S}_1, \ldots, \mathcal{S}_s: \quad \text{are the generators of the covariants of } W.
\end{aligned} \tag{13}$$

Theorem (Clebsch)**.** *The ring of covariants of $V \oplus W$ is finitely generated. Moreover, a finite generating system can be chosen from the set of all transvectants*

$$(\mathcal{T}, \mathcal{S})_l\,, \quad \textit{for} \quad l \geq 0,$$

where $\mathcal{T}$ is a monomial in the $\mathcal{T}_i$'s and $\mathcal{S}$ a monomial in the $\mathcal{S}_j$'s. In other words, by the relevant transvectants $(\mathcal{T}, \mathcal{S})_l$.

3.2. Proj $\mathcal{R}_{(d+1,d-1)}$ **as a weighted projective space.** Let $\xi_0, \dots, \xi_n$ be a generating system of $\mathcal{R}_{(d+1,d-1)}$. Since all $\xi_0, \dots, \xi_n$ are homogeneous polynomials, then $\mathcal{R}_{(d+1,d-1)}$ is a graded ring and Proj $\mathcal{R}_{(d+1,d-1)}$ is a weighted projective space.

Let $\omega := (q_0, \dots, q_n) \in \mathbb{Z}^{n+1}$ be a fixed tuple of positive integers called **weights**. Consider the action of $k^\star = k \setminus \{0\}$ on $\mathbb{A}^{n+1}(k)$ as follows:

$$\lambda \star (x_0, \dots, x_n) = (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n)$$

for $\lambda \in k^\star$. The quotient of this action is called a **weighted projective space** and denoted by $\mathbb{P}^n_\omega(k)$. It is the projective variety $\mathrm{Proj}(k[x_0, \dots, x_n])$ associated to the graded ring $k[x_0, \dots, x_n]$ where the variable x_i has degree q_i for $i = 0, \dots, n$.

We will denote a point $\mathfrak{p} \in \mathbb{P}^n_\omega(k)$ by $\mathfrak{p} = [x_0 : x_1 : \cdots : x_n]$. Let $\phi(x, y) \in \mathrm{Rat}^1_d$ given by $\phi(x, y) = [f_0 : f_1]$. Its associated binary forms $\mathcal{I}_\phi \in V_{d+1}$ and $\mathcal{J}_\phi \in V_{d-1}$, and $\xi_0, \xi_1, \dots, \xi_n$ the generators of the ring of invariants $\mathcal{R}_{(d+1,d-1)}$.

The **invariants** of the rational function ϕ are defined as

$$\xi(\phi) := [\xi_0(\mathcal{I}_\phi, \mathcal{J}_\phi), \xi_1(\mathcal{I}_\phi, \mathcal{J}_\phi), \dots, \xi_n(\mathcal{I}_\phi, \mathcal{J}_\phi)] \in \mathbb{P}^n_\omega(k). \tag{14}$$

Moreover, $\phi = \psi^\sigma$ for $\sigma \in \mathrm{GL}_2(k)$ if and only if $\xi(\phi) = \lambda \star \xi(\psi)$, for $\lambda = (\det \sigma)^{\frac{d}{2}}$.

Next we will determine explicitly invariants of $\mathcal{R}_{d+1,d-1}$. From now on $f \in V_{d+1}$ and $g \in V_{d-1}$ where

$$f = \sum_{i=0}^{d+1} a_i x^i y^{d+1-i} \quad \text{and} \quad g = \sum_{i=0}^{d-1} b_i x^i y^{d-1-i}. \tag{15}$$

3.3. **Invariants of $V_4 \oplus V_2$.** To apply this framework, we specialize to $d = 3$, where $\mathcal{I}_\phi \in V_4$ and $\mathcal{J}_\phi \in V_2$. We take $d = 3$, $f \in V_4$ and $g \in V_2$ as in Eq. (15)

$$\begin{aligned} f(x,y) &= a_4 x^4 + a_3 x^3 + a_2 x^2 y + a_1 x y^2 + a_0 y^3 \\ g(x,y) &= b_2 x^2 + b_1 x y + b_0 y^2 \end{aligned} \tag{16}$$

The generators of covariants of V_4 and V_2 are

$$\begin{aligned} T &= \{f, \, \mathcal{T} = (f,f)_2, \, \mathcal{T}_2 = (f,f)_4, \, \mathcal{T}_3 = ((f,f)_2, f)_4\} \\ S &= \{g, \, \mathcal{S}_2 = (g,g)_2\} \end{aligned}$$

respectively. Hence, we are considering all transvectants $\left(f^{m_1} \mathcal{T}^{m_2} \mathcal{T}_2^{m_3} \mathcal{T}_3^{m_4}, g^{s_1} S_2^{s_2}\right)_l$, for some $m_1, m_2, m_3, m_4, s_1, s_2$. Since $\mathcal{T}_2, \mathcal{T}_3$ and S_2 are invariants, their exponents must be zero, otherwise we get reducible invariants. Hence, $\mathcal{T}_2, \mathcal{T}_3, S_2$ are part of the generating set and further we only consider $\left(f^{m_1} \mathcal{T}^{m_2}, g^s\right)_l$. Then the relevant transvectants are $S_2, \mathcal{T}_2, \mathcal{T}_3$, and

$$R_3 := (\mathcal{T}, g^2)_4, \quad R_4 := (f, g^2)_4, \quad R_6 := (g^3, (f, \mathcal{T})_1)_6,$$

Hence, the set of invariants is $\xi(\phi) = (\xi_0, \dots, \xi_5)$, where

$$\xi_0 = (g,g)_2, \; \xi_1 = (f,f)_4, \; \xi_2 = (\mathcal{T}, f)_4, \; \xi_3 = R_3, \; \xi_4 = R_4, \; \xi_5 = R_6$$

with weights $(2,2,3,3,4,6)$ respectively. Let $d=3$ and f,g as in Eq. (15). We have the following expressions for invariants:

$$
\begin{aligned}
\xi_0 &= \frac{1}{2}\left(4b_0b_2-b_1^2\right)\\
\xi_1 &= \frac{1}{6}\left(a_2^2-3a_1a_3+12a_0a_4\right)\\
\xi_2 &= \frac{1}{72}\left(-2a_2^3+9(a_1a_3+8a_0a_4)a_2-27\left(a_4a_1^2+a_0a_3^2\right)\right)\\
\xi_3 &= \frac{1}{6}\left(6a_4b_0^2-3a_3b_1b_0+2a_2b_2b_0+a_2b_1^2+6a_0b_2^2-3a_1b_1b_2\right)\\
\xi_4 &= -\frac{1}{72}\left(2a_2^2b_1^2+4a_2^2b_0b_2-24a_4a_2b_0^2-24a_0a_2b_2^2-6a_3a_2b_0b_1\right.\\
&\quad -6a_1a_2b_1b_2+9a_3^2b_0^2-3a_1a_3b_1^2-24a_0a_4b_1^2+9a_1^2b_2^2+36a_1a_4b_0b_1-6a_1a_3b_0b_2\\
&\quad \left.-48a_0a_4b_0b_2+36a_0a_3b_1b_2\right)\\
\xi_5 &= -\frac{1}{32}\left(a_3^3b_0^3+8a_1a_4^2b_0^3-4a_2a_3a_4b_0^3-a_2a_3^2b_1b_0^2-8a_0a_1a_4b_1^2b_2-16a_0a_4^2b_1b_0^2\right.\\
&\quad -2a_1a_3a_4b_1b_0^2+a_1a_3^2b_2b_0^2-4a_1a_2a_4b_2b_0^2+8a_0a_3a_4b_2b_0^2+a_1a_3^2b_1^2b_0+4a_2^2a_4b_1b_0^2\\
&\quad -4a_1a_2a_4b_1^2b_0+8a_0a_3a_4b_1^2b_0-a_1^2a_3b_2^2b_0+4a_0a_2a_3b_2^2b_0-8a_0a_1a_4b_2^2b_0\\
&\quad -6a_0a_3^2b_1b_2b_0+6a_1^2a_4b_1b_2b_0-a_0a_3^2b_1^3+a_1^2a_4b_1^3-a_1^3b_2^3+4a_0a_1a_2b_2^3-8a_0^2a_3b_2^3\\
&\quad \left.-4a_0a_2^2b_1b_2^2+a_1^2a_2b_1b_2^2+2a_0a_1a_3b_1b_2^2+16a_0^2a_4b_1b_2^2-a_1^2a_3b_1^2b_2+4a_0a_2a_3b_1^2b_2\right)
\end{aligned}
$$

The ring of invariants $\mathcal{R}_{4,2}$ is generated by $\xi_0,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5$ and a relation between invariants $\xi_0,\ldots,\xi_5$ is, according to [18], satisfy the following equation

$$
\xi_5^2=\frac{1}{108}\xi_0^3\xi_1^3-18\xi_0^3\xi_2^2-\frac{1}{24}\xi_0\xi_1^2\xi_3^2-\frac{1}{6}\xi_2\xi_3^3+\frac{1}{2}\xi_0\xi_2\xi_3\xi_4+\frac{1}{4}\xi_1\xi_3^2\xi_4-\frac{1}{4}\xi_0\xi_1\xi_4^2-\frac{1}{2}\xi_4^3 \tag{17}
$$

Given a rational cubic

$$
\phi(x)=\frac{f_0(x)}{f_1(x)}=\frac{c_0x^3+c_1x^2+c_2x+c_3}{c_4x^3+c_5x^2+c_6x+c_7} \tag{18}
$$

we compute its invariants in terms of its coefficients. We have $I_6=\operatorname{Res}(f_0,f_1)$

$$
\begin{aligned}
I_6 =& c_3^3c_4^3-c_0^3c_7^3+c_3c_0^2c_6^3-c_2^3c_7c_4^2+c_1^3c_7^2c_4-c_3^2c_2c_5c_4^2-c_3^2c_0c_5^3-2c_3^2c_1c_6c_4^2\\
&+c_3c_2^2c_6c_4^2+c_3^2c_1c_5^2c_4-3c_3^2c_0c_7c_4^2-c_1^2c_0c_7^2c_5+c_1c_0^2c_7^2c_6+2c_2c_0^2c_7^2c_5\\
&+3c_3c_0^2c_7^2c_4-c_2^2c_0c_7c_5^2+3c_3^2c_0c_6c_5c_4-2c_3c_2c_0c_6^2c_4+c_3c_2c_0c_6c_5^2\\
&+3c_3c_2c_1c_7c_4^2+2c_3c_1c_0c_7c_5^2-c_3c_1c_0c_6^2c_5-c_2c_1^2c_7c_6c_4-3c_2c_1c_0c_7^2c_4\\
&+c_2^2c_1c_7c_5c_4+2c_2^2c_0c_7c_6c_4-c_3c_2c_1c_6c_5c_4-c_3c_2c_0c_7c_5c_4+c_3c_1c_0c_7c_6c_4\\
&+c_3c_1^2c_6^2c_4-c_2c_0^2c_7c_6^2+c_2c_1c_0c_7c_6c_5-3c_3c_0^2c_7c_6c_5-2c_3c_1^2c_7c_5c_4
\end{aligned}
$$

The pair of binary forms $\mathcal{I}_\phi$ and $\mathcal{J}_\phi$ associated to ϕ are

$$
\begin{aligned}
\mathcal{I}_\phi &= c_3x^3y+c_2x^2y^2+c_1xy^3+c_0y^4-c_7x^4-c_6x^3y-c_5x^2y^2-c_4xy^3\\
\mathcal{J}_\phi &= 3c_3x^2+2c_2xy+c_1y^2+c_6x^2+2c_5xy+3c_4y^2
\end{aligned} \tag{19}
$$

Next we evaluate the following transvectants

$$
\begin{aligned}
&\xi_0=(\mathcal{J}_\phi,\mathcal{J}_\phi)_2, \quad \xi_1=(\mathcal{I}_\phi,\mathcal{I}_\phi)_4, \quad \xi_2=((\mathcal{I}_\phi,\mathcal{I}_\phi)_2,\mathcal{I}_\phi)_4,\\
&\xi_3=\left(\mathcal{I}_\phi,\mathcal{J}_\phi^2\right)_4, \quad \xi_4=\left((\mathcal{I}_\phi,\mathcal{I}_\phi)_2,\mathcal{J}_\phi^2\right)_4, \quad \xi_5=\left(\mathcal{J}_\phi^3,(\mathcal{I}_\phi,(\mathcal{I}_\phi,\mathcal{I}_\phi)_2)_1\right)_6
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\xi_0 =& 2(3c_2c_0 + c_2c_5 - c_1^2 - 2c_1c_6 + 9c_0c_7 + 3c_7c_5 - c_6^2) \\
\xi_1 =& -\frac{1}{6}(12c_3c_4 + 3c_2c_0 - 3c_2c_5 - c_1^2 + 2c_1c_6 - 3c_0c_7 + 3c_7c_5 - c_6^2) \\
\xi_2 =& -\frac{1}{72}(72c_3c_1c_4 + 27c_3c_0^2 - 54c_3c_0c_5 - 72c_3c_6c_4 + 27c_3c_5^2 - 27c_2^2c_4 - 9c_2c_1c_0 \\
& + 9c_2c_1c_5 + 9c_2c_0c_6 + 54c_2c_7c_4 - 9c_2c_6c_5 + 2c_1^3 - 6c_1^2c_6 + 9c_1c_0c_7 - 9c_1c_7c_5 \\
& + 6c_1c_6^2 - 9c_0c_7c_6 - 27c_7^2c_4 + 9c_7c_6c_5 - 2c_6^3) \\
\xi_3 =& \frac{1}{3}(27c_3c_0^2 + 18c_3c_0c_5 + 3c_3c_5^2 - 3c_2^2c_4 - 9c_2c_1c_0 + c_2c_1c_5 - 15c_2c_0c_6 + 9c_7c_6c_5 \\
& - 2c_6^3 - 18c_2c_7c_4 - c_2c_6c_5 + 2c_1^3 + 2c_1^2c_6 + 9c_1c_0c_7 + 15c_1c_7c_5 - 2c_1c_6^2 \\
& - 9c_0c_7c_6 - 27c_7^2c_4) \\
\xi_4 =& -\frac{1}{9}(18c_3c_2c_0c_4 + 6c_3c_2c_5c_4 + 12c_3c_1^2c_4 - 36c_3c_1c_0c_5 + 24c_3c_1c_6c_4 - 12c_3c_1c_5^2 \\
& + 54c_3c_0^2c_6 + 54c_3c_0c_7c_4 + 18c_3c_7c_5c_4 + 12c_3c_6^2c_4 - 6c_3c_6c_5^2 - 6c_2^2c_1c_4 + 9c_2^2c_0^2 \\
& + 6c_2^2c_0c_5 - 12c_2^2c_6c_4 + 3c_2^2c_5^2 - 6c_2c_1^2c_0 + 2c_2c_1^2c_5 - 6c_2c_1c_0c_6 + 2c_2c_1c_6c_5 \\
& - 18c_2c_0^2c_7 - 24c_2c_0c_7c_5 + 6c_2c_0c_6^2 - 36c_2c_7c_6c_4 + 6c_2c_7c_5^2 + 2c_2c_6^2c_5 + c_1^4 \\
& + 6c_1^2c_0c_7 + 6c_1^2c_7c_5 - 2c_1^2c_6^2 - 6c_1c_0c_7c_6 + 54c_1c_7^2c_4 - 6c_1c_7c_6c_5 \\
& + 27c_0^2c_7^2 - 18c_0c_7^2c_5 + 6c_0c_7c_6^2 + 9c_7^2c_5^2 - 6c_7c_6^2c_5 + c_6^4) \\
\xi_5 =& -\frac{1}{4}(36c_3^2c_1c_0^2c_4 + 24c_3^2c_1c_0c_5c_4 + 4c_3^2c_1c_5^2c_4 + 27c_3^2c_0^4 + 36c_3^2c_0^2c_6c_4 - 18c_3^2c_0^2c_5^2 \\
& + 24c_3^2c_0c_6c_5c_4 - 8c_3^2c_0c_5^3 + 4c_3^2c_6c_5^2c_4 - c_3^2c_5^4 + 4c_3c_2^2c_1c_4^2 - 6c_3c_2^2c_0^2c_4 - 8c_3c_2^2c_0c_5c_4 \\
& + 4c_3c_2^2c_6c_4^2 - 2c_3c_2^2c_5^2c_4 - 8c_3c_2c_1^2c_0c_4 - 8c_3c_2c_1^2c_5c_4 - 18c_3c_2c_1c_0^3 - 18c_3c_2c_1c_0^2c_5 \\
& - 16c_3c_2c_1c_0c_6c_4 + 2c_3c_2c_1c_0c_5^2 + 24c_3c_2c_1c_7c_4^2 - 16c_3c_2c_1c_6c_5c_4 + 2c_3c_2c_1c_5^3 + 18c_3c_2c_0^3c_6 \\
& + 6c_3c_2c_0^2c_6c_5 - 24c_3c_2c_0c_7c_5c_4 - 8c_3c_2c_0c_6^2c_4 + 6c_3c_2c_0c_6c_5^2 + 24c_3c_2c_7c_6c_4^2 - 8c_3c_2c_7c_5^2c_4 \\
& - 8c_3c_2c_6^2c_5c_4 + 2c_3c_2c_6c_5^3 + 4c_3c_1^3c_0^2 + 8c_3c_1^3c_0c_5 + 4c_3c_1^3c_5^2 - 12c_3c_1^2c_0^2c_6 + 24c_3c_1^2c_0c_7c_4 \\
& - 8c_3c_1^2c_0c_6c_5 - 8c_3c_1^2c_7c_5c_4 + 4c_3c_1^2c_6c_5^2 + 18c_3c_1c_0^3c_7 - 6c_3c_1c_0^2c_7c_5 + 48c_3c_1c_0c_7c_6c_4 \\
& + 14c_3c_1c_0c_7c_5^2 - 16c_3c_1c_0c_6^2c_5 + 36c_3c_1c_7^2c_4^2 - 16c_3c_1c_7c_6c_5c_4 + 6c_3c_1c_7c_5^3 + 18c_3c_0^3c_7c_6 \\
& + 54c_3c_0^2c_7^2c_4 - 42c_3c_0^2c_7c_6c_5 + 16c_3c_0^2c_6^3 + 24c_3c_0c_7c_6^2c_4 - 10c_3c_0c_7c_6c_5^2 + 36c_3c_7^2c_6c_4^2 \\
& - 6c_3c_7^2c_5^2c_4 - 8c_3c_7c_6^2c_5c_4 + 2c_3c_7c_6c_5^3 - c_2^4c_4^2 + 2c_2^3c_1c_0c_4 + 2c_2^3c_1c_5c_4 + 4c_2^3c_0^3 + 4c_2^3c_0^2c_5 \\
& + 6c_2^3c_0c_6c_4 - 8c_2^3c_7c_4^2 + 2c_2^3c_6c_5c_4 - c_2^2c_1^2c_0^2 - 2c_2^2c_1^2c_0c_5 - c_2^2c_1^2c_5^2 + 2c_2^2c_1c_0^2c_6 \\
& - 10c_2^2c_1c_0c_7c_4 + 6c_2^2c_1c_7c_5c_4 + 4c_2^2c_1c_6^2c_4 - 2c_2^2c_1c_6c_5^2 - 12c_2^2c_0^3c_7 - 4c_2^2c_0^2c_7c_5 + 3c_2^2c_0^2c_6^2 \\
& + 14c_2^2c_0c_7c_6c_4 - 8c_2^2c_0c_7c_5^2 + 2c_2^2c_0c_6^2c_5 - 18c_2^2c_7^2c_4^2 + 2c_2^2c_7c_6c_5c_4 + 4c_2^2c_6^3c_4 - c_2^2c_6^2c_5^2 \\
& + 2c_2c_1^2c_0^2c_7 + 4c_2c_1^2c_0c_7c_5 - 16c_2c_1^2c_7c_6c_4 + 2c_2c_1^2c_7c_5^2 - 8c_2c_1c_0^2c_7c_6 - 42c_2c_1c_0c_7^2c_4 \\
& + 8c_2c_1c_0c_7c_6c_5 + 6c_2c_1c_7^2c_5c_4 - 8c_2c_1c_7c_6^2c_4 + 24c_2c_0^2c_7^2c_5 - 10c_2c_0^2c_7c_6^2 - 6c_2c_0c_7^2c_6c_4 \\
& - 4c_2c_0c_7^2c_5^2 + 4c_2c_0c_7c_6^2c_5 - 18c_2c_7^2c_6c_5c_4 + 4c_2c_7^2c_5^3 + 8c_2c_7c_6^3c_4 - 2c_2c_7c_6^2c_5^2 + 16c_1^3c_7^2c_4 \\
& + 3c_1^2c_0^2c_7^2 - 10c_1^2c_0c_7^2c_5 + 3c_1^2c_7^2c_5^2 + 6c_1c_0^2c_7^2c_6 + 18c_1c_0c_7^3c_4 - 8c_1c_0c_7^2c_6c_5 + 18c_1c_7^3c_5c_4 \\
& - 12c_1c_7^2c_6^2c_4 + 2c_1c_7^2c_6c_5^2 + 3c_0^2c_7^2c_6^2 + 18c_0c_7^3c_6c_4 - 12c_0c_7^3c_5^2 + 2c_0c_7^2c_6^2c_5 + 27c_7^4c_4^2 \\
& + 4c_7^3c_5^3 + 4c_7^2c_6^3c_4 - 18c_7^3c_6c_5c_4 - c_7^2c_6^2c_5^2)
\end{aligned}
$$

The modular resultant $\Delta_{\mathcal{I}_\phi,\mathcal{J}_\phi} = \mathrm{Res}(\mathcal{I}_\phi, \mathcal{J}_\phi)$ is a homogeneous polynomial of degree six in terms of $c_0, \dots, c_7$. Hence, we will denote it by $J_6 := \mathrm{Res}(\mathcal{I}_\phi, \mathcal{J}_\phi)$.

Thus, there are three invariants of degree 6 in the case of cubics, namely I_6, J_6, and ξ_5. We would like to express I_6 and J_6 in terms of $\xi_0, \dots, \xi_5$. The expression of I_6 was computed in [18, pg. 38]

$$I_6 = \frac{1}{8}\xi_1^3 + \frac{1}{384}\xi_1\xi_0^2 - \frac{3}{4}\xi_2^2 - \frac{3}{16}\xi_2\xi_4 + \frac{1}{256}\xi_4^2 + \frac{3}{16}\xi_1\xi_3 - \frac{1}{64}\xi_0\xi_3 - \frac{1}{8}\xi_5 \tag{21}$$

It seems there are a couple of typos in the printed version of [18, pg. 38], and we could not verify it directly. It can be easily noticeable that it is incorrect since it is not a homogeneous polynomial of degree 6; see for example the monomials ξ_4^2, ξ_1, ξ_3, $\xi_0\xi_3$ which are not of degree 6.

The correct expression can be derived using computational algebra. This involves expressing I_6 and J_6 as linear combinations of the 10 degree 6 invariants: ξ_5, $\xi_4\xi_0$, $\xi_4\xi_1$, $\xi_2^{2-j}\xi_3^j$, $\xi_0^{3-i}\xi_1^i$, for $i = 0, 1, 2, 3$ and $j = 0, 1, 2$. We have

$$\begin{aligned} I_6 &= -\frac{1}{8}\xi_1^3 - \frac{1}{384}\xi_0^2\xi_1 + \frac{3}{4}\xi_2^2 - \frac{3}{16}\xi_1\xi_4 - \frac{1}{256}\xi_3^2 + \frac{3}{16}\xi_2\xi_3 + \frac{1}{64}\xi_0\xi_4 - \frac{1}{8}\xi_5 \\ J_6 &= \xi_3^2 - 4\xi_4\xi_0 + \frac{2}{3}\xi_0^2\xi_1 \end{aligned}$$

The space $\mathbb{P}_{(2,2,3,3,4,6)}$ can be embedded into $\mathbb{P}^5$ via Veronese embedding as

$$[\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, I_6] \to [\xi_0^6, \xi_1^6, \xi_2^4, \xi_3^4, \xi_4^3, I_6^2] \tag{22}$$

Since $I_6 \neq 0$, we can divide by I_6^2 and represent each point as

$$\left[\frac{\xi_0^6}{I_6^2} : \frac{\xi_1^6}{I_6^2} : \frac{\xi_2^4}{I_6^2} : \frac{\xi_3^4}{I_6^2} : \frac{\xi_4^3}{I_6^2} : 1\right] \in \mathbb{P}^5$$

This motivates the definition of the following invariants

$$i_1 = \frac{\xi_0^6}{I_6^2},\ i_2 = \frac{\xi_1^6}{I_6^2},\ i_3 = \frac{\xi_2^4}{I_6^2},\ i_4 = \frac{\xi_3^4}{I_6^2},\ i_5 = \frac{\xi_4^3}{I_6^2}, \tag{23}$$

which are $\mathrm{GL}_2(k)$-invariants and are defined everywhere in the moduli space Rat_3^1. We call such invariants $i_1, \dots, i_5$ **absolute invariants** of $\phi(x, y)$. Hence, we have:

Lemma 3. *Two degree three rational functions are conjugate if and only if they have the same absolute invariants.*

Proof. Let $\phi, \psi \in \mathrm{Rat}_3^1$ be degree 3 rational functions over k given by $\phi = \frac{f_0}{f_1}$, $\psi = \frac{g_0}{g_1}$ and $\xi(\phi) = [\xi_0(\phi), \dots, \xi_5(\phi)]$, $\xi(\psi) = [\xi_0(\psi), \dots, \xi_5(\psi)]$ as in Eq. (14) and absolute invariants as in Eq. (23), where $I_6(\phi) \neq 0$ and $I_6(\psi) \neq 0$.

Suppose $\phi \simeq \psi$, i.e., $\psi = \phi^\sigma = \sigma^{-1} \circ \phi \circ \sigma$ for some $\sigma \in \mathrm{PGL}_2(k)$. By Lem. 1, $\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma$, $\mathcal{J}_\psi = \mathcal{J}_\phi^\sigma$. Since ξ_i are $\mathrm{SL}_2(k)$-invariants of $V_4 \oplus V_2$ (Subsection 3.3), $\xi_i(\psi) = \xi_i(\mathcal{I}_\psi, \mathcal{J}_\psi) = \xi_i(\mathcal{I}_\phi^\sigma, \mathcal{J}_\phi^\sigma) = \xi_i(\phi)$. For $\sigma \in \mathrm{GL}_2(k)$ with $\det \sigma = \lambda$, we have $\xi_i(\phi^\sigma) = \lambda^{w_i/2}\xi_i(\phi)$, $I_6(\phi^\sigma) = \lambda^3 I_6(\phi)$, so:

$$i_j(\psi) = \frac{\xi_{j-1}(\psi)^{12/w_{j-1}}}{I_6(\psi)^2} = \frac{(\lambda^{w_{j-1}/2}\xi_{j-1}(\phi))^{12/w_{j-1}}}{(\lambda^3 I_6(\phi))^2} = \frac{\lambda^6 \xi_{j-1}(\phi)^{12/w_{j-1}}}{\lambda^6 I_6(\phi)^2} = i_j(\phi).$$

Thus, $\phi \simeq \psi$ implies $i_j(\phi) = i_j(\psi)$.

Suppose $i_j(\phi) = i_j(\psi)$ for $j = 1, \dots, 5$. Then $\frac{\xi_{j-1}(\phi)^{12/w_{j-1}}}{I_6(\phi)^2} = \frac{\xi_{j-1}(\psi)^{12/w_{j-1}}}{I_6(\psi)^2}$, so $\left(\frac{\xi_{j-1}(\phi)}{\xi_{j-1}(\psi)}\right)^{12/w_{j-1}} = \left(\frac{I_6(\phi)}{I_6(\psi)}\right)^2$. With $\omega = (2, 2, 3, 3, 4, 6)$, $\gcd(w_i) = 1$, there exists $\lambda \in k^\star$ such that $\xi_{j-1}(\phi) = \lambda^{w_{j-1}}\xi_{j-1}(\psi)$, $I_6(\phi) = \lambda^3 I_6(\psi)$. Hence, $\xi(\phi) = \lambda \star \xi(\psi)$ and $\exists \sigma \in \mathrm{GL}_2(k)$, $\det \sigma = \lambda^2$, such that $\phi^\sigma = \psi$. Thus, $\phi \simeq \psi$. ∎

We now define the map $\Phi : \mathbb{C}^8 \setminus \{I_6 = 0\} \to \mathbb{C}^5$ as

$$(c_0, \dots, c_7) \to (i_1(c_0, \dots, c_7), \dots, i_5(c_0, \dots, c_7)), \tag{24}$$

where $i_1, \dots, i_5$ are defined in Eq. (23). The condition $I_6 \neq 0$, ensures that $i_1, \dots, i_5$ are defined everywhere in the moduli space.

To study the image $\operatorname{Img}(\Phi)$, we examine the Jacobian matrix $J(\Phi)$, a 5×8 matrix with entries $\frac{\partial i_j}{\partial c_k}$ for $j = 1, \dots, 5$ and $k = 0, \dots, 7$. The rank of $J(\Phi)$, which is at most 5, governs the local geometry of Φ:

If $\operatorname{rank}(J(\Phi)) = 5$, then Φ is a submersion at that point. By the Implicit Function Theorem, $\operatorname{Img}(\Phi)$ is locally a smooth 5-dimensional manifold near $\Phi(c_0, \dots, c_7)$.

If $\operatorname{rank}(J(\Phi)) < 5$, i.e., all 5×5 minors vanish, Φ is not a submersion, and $\operatorname{Img}(\Phi)$ may have singularities at such points. These singularities typically correspond to rational functions with extra automorphisms.

We hypothesize that the singular locus of $\operatorname{Img}(\Phi)$, where $\operatorname{rank}(J(\Phi)) < 5$, aligns with the set $\{J_6 = 0\} \subset \mathbb{C}^8$, as defined in Eq. (25). It would be challenging to verify this computationally.

(25)

$$\begin{aligned}
J_6 = {} & 81c_3^4c_0^2 - 54c_3^3c_2c_1c_0 + 54c_3^3c_2c_0c_4 + 12c_3^3c_1^3 - 36c_3^3c_1^2c_4 + 54c_3^3c_1c_0c_5 - 108c_3^3c_1c_4^2 \\
& + 108c_3^3c_0^2c_6 + 378c_3^3c_0c_5c_4 + 324c_3^3c_4^3 + 12c_3^2c_2^3c_0 - 3c_3^2c_2^2c_1^2 + 6c_3^2c_2^2c_1c_4 - 36c_3^2c_2^2c_0c_5 \\
& + 45c_3^2c_2^2c_4^2 + 12c_7c_5^3c_4^2 + 6c_3^2c_2c_1^2c_5 - 126c_3^2c_2c_1c_0c_6 - 60c_3^2c_2c_1c_5c_4 - 162c_3^2c_2c_0c_6c_4 \\
& - 108c_3^2c_2c_0c_5^2 - 234c_3^2c_2c_5c_4^2 + 28c_3^2c_1^3c_6 - 18c_3^2c_1^2c_0c_7 + 12c_3^2c_1^2c_6c_4 + 45c_3^2c_1^2c_5^2 \\
& - 108c_3^2c_1c_0c_7c_4 - 18c_3^2c_1c_0c_6c_5 - 252c_3^2c_1c_6c_4^2 + 150c_3^2c_1c_5^2c_4 + 54c_3^2c_0^2c_6^2 - 162c_3^2c_0c_7c_4^2 \\
& + 162c_3^2c_0c_6c_5c_4 - 60c_3^2c_0c_5^3 - 108c_3^2c_6c_4^3 + 45c_3^2c_5^2c_4^2 + 40c_3c_2^3c_0c_6 - 10c_3c_2^2c_1^2c_6 \\
& + 20c_3c_2^2c_1c_6c_4 + 144c_3c_2^2c_0c_7c_4 + 72c_3c_2^2c_0c_6c_5 + 150c_3c_2^2c_6c_4^2 - 10c_3c_2c_1^3c_7 \\
& - 28c_3c_2c_1^2c_6c_5 + 96c_3c_2c_1c_0c_7c_5 - 66c_3c_2c_1c_0c_6^2 + 162c_3c_2c_1c_7c_4^2 - 54c_7c_6c_5c_4^3 \\
& + 288c_3c_2c_0c_7c_5c_4 - 126c_3c_2c_0c_6^2c_4 + 24c_3c_2c_0c_6c_5^2 + 378c_3c_2c_7c_4^3 + 81c_7^2c_4^4 \\
& - 22c_3c_1^3c_7c_5 + 20c_3c_1^3c_6^2 - 12c_3c_1^2c_0c_7c_6 - 126c_3c_1^2c_7c_5c_4 + 68c_3c_1^2c_6^2c_4 + 4c_0c_6^2c_5^3 \\
& - 72c_3c_1c_0c_7c_6c_4 + 48c_3c_1c_0c_7c_5^2 - 30c_3c_1c_0c_6^2c_5 - 162c_3c_1c_7c_5c_4^2 + 48c_0c_7c_6c_5^2c_4 \\
& + 20c_3c_1c_6c_5^2c_4 + 12c_3c_0^2c_6^3 - 108c_3c_0c_7c_6c_4^2 + 144c_3c_0c_7c_5^2c_4 - 18c_3c_0c_6^2c_5c_4 - 10c_0c_6^3c_5c_4 \\
& - 8c_3c_0c_6c_5^3 + 54c_3c_7c_5c_4^3 - 36c_3c_6^2c_4^3 + 6c_3c_6c_5^2c_4^2 - 16c_2^4c_0c_7 + 4c_2^3c_1^2c_7 - 8c_2^3c_1c_7c_4 \\
& + 12c_2^3c_0c_6^2 - 60c_2^3c_7c_4^2 + 20c_2^2c_1^2c_7c_5 - 3c_2^2c_1^2c_6^2 + 16c_2^2c_1c_0c_7c_6 + 12c_6^3c_4^3 - 3c_6^2c_5^2c_4^2 \\
& + 24c_2^2c_1c_7c_5c_4 + 6c_2^2c_1c_6^2c_4 + 48c_2^2c_0c_7c_6c_4 - 96c_2^2c_0c_7c_5^2 + 28c_2^2c_0c_6^2c_5 - 108c_2^2c_7c_5c_4^2 \\
& - 6c_2c_1^3c_7c_6 - 30c_2c_1^2c_7c_6c_4 + 28c_2c_1^2c_7c_5^2 - 10c_2c_1^2c_6^2c_5 + 32c_2c_1c_0c_7c_6c_5 - 10c_2c_1c_0c_6^3 \\
& + 72c_2c_1c_7c_5^2c_4 - 28c_2c_1c_6^2c_5c_4 + 96c_2c_0c_7c_6c_5c_4 - 64c_2c_0c_7c_5^3 - 22c_2c_0c_6^3c_4 + 20c_2c_0c_6^2c_5^2 \\
& - 36c_2c_7c_5^2c_4^2 + 6c_2c_6^2c_5c_4^2 + c_1^4c_7^2 + 12c_1^3c_7^2c_4 - 10c_1^3c_7c_6c_5 + 4c_1^3c_6^3 - 2c_1^2c_0c_7c_6^2 + 54c_1^2c_7^2c_4^2 \\
& - 66c_1^2c_7c_6c_5c_4 + 12c_1^2c_7c_5^3 + 20c_1^2c_6^3c_4 - 3c_1^2c_6^2c_5^2 - 12c_1c_0c_7c_6^2c_4 + 16c_1c_0c_7c_6c_5^2 - 6c_1c_0c_6^3c_5 \\
& + 108c_1c_7^2c_4^3 - 126c_1c_7c_6c_5c_4^2 + 40c_1c_7c_5^3c_4 + 28c_1c_6^3c_4^2 - 10c_1c_6^2c_5^2c_4 + c_0^2c_6^4 - 18c_0c_7c_6^2c_4^2 \\
& + 48c_3c_2^2c_1c_0c_7 - 64c_2^3c_0c_7c_5 - 16c_0c_7c_5^4 + 45c_2^2c_6^2c_4^2 - 18c_2c_1c_7c_6c_4^2 + 54c_2c_7c_6c_4^3 \\
& - 104c_3c_2c_1c_6c_5c_4 - 18c_3c_2c_1^2c_7c_4 + 6c_3c_1^2c_6c_5^2 + 12c_3c_1c_6^2c_4^2 - 60c_3c_2c_6c_5c_4^2
\end{aligned}$$

4. Automorphisms

In this section we will define and study the automorphism groups of rational functions. The approach will be similar to studying the automorphism groups of hyperelliptic curves; see [12, 14]. Such groups have long been the focus of many authors in arithmetic dynamics; see [2, 4, 6, 18]. First we recall some preliminaries.

Let G be a finite subgroup of $\mathrm{PGL}_2(k)$. Therefore G is isomorphic to one of the following: C_n, D_n, A_4, S_4, or A_5. Since each is embedded in $\mathrm{PGL}_2(k)$, we can represent their generators as matrices, informing the forms of $\sigma \in \mathrm{Aut}(\phi)$ later. Below we display all the cases:

$$
\begin{aligned}
&\text{i)} \quad C_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \\
&\text{ii)} \quad D_n \cong \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \zeta_n & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \\
&\text{iii)} \quad A_4 \cong \left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \zeta_4 \\ 1 & -\zeta_4 \end{bmatrix} \right\rangle \\
&\text{iv)} \quad S_4 \cong \left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right\rangle \\
&\text{v)} \quad A_5 \cong \left\langle \begin{bmatrix} \omega & 1 \\ 1 & -\omega \end{bmatrix}, \begin{bmatrix} \omega & \zeta_5^4 \\ 1 & -\zeta_5^4\omega \end{bmatrix} \right\rangle
\end{aligned} \tag{26}
$$

where $\omega = \frac{-1+\sqrt{5}}{2}$, ζ_n is a primitive n^{th} root of unity and i is a primitive 4^{th} root of unity.

Remark 2. *In each case above, there is $\sigma \in G$ which fixes 0 and ∞. The proof is elementary. In the first two cases, the Möbius transformation $\sigma(x) = \zeta_m x$ will fix 0 and ∞. In the next two cases, $\sigma(x) = -x$ will do that, and in the last case, $\sigma(x) = \omega x$ is in the group and will fix 0 and ∞. This property constrains $\mathrm{Aut}(\phi)$ later.*

Let $\phi \in \mathrm{Rat}_d^1$. A point $[x_0 : x_1] \in \mathbb{P}^1$ is called a **fixed point** for ϕ if $\phi(x_0, x_1) = (x_0, x_1)$. Let $t = x_1/x_0$. Hence, ϕ can be taken as a rational function in t, say $\phi(t) = \frac{F(t)}{G(t)}$. Then t is a fixed point if $\phi(t) = t$, which implies that

$$S(t) := F(t) - t\,G(t) = 0$$

which is at most a degree $(d+1)$ equation in t. Hence, a degree d rational function has at most $(d+1)$ fixed points.

We denote the set of fixed points of ϕ by $\mathbf{Fix}(\phi)$. Notice that if $\mathbf{Fix}(\phi) = \{w_1, \dots, w_s\}$ with $s \leq d+1$ is known, then we can uniquely determine the rational function ϕ by solving the linear system $F(w_i) - w_i\,G(w_i) = 0$, for $i = 1, \dots, s$, assuming $s = d+1$. A function ϕ has less than $d+1$ fixed points exactly when the discriminant $\Delta(S, t) = 0$.

An **automorphism** of ϕ is called a $\sigma \in \mathrm{PGL}_2(k)$ such that $\phi \circ \sigma = \sigma \circ \phi$.

$$
\begin{array}{ccc}
\mathbb{P}^1_x & \xrightarrow{\ \sigma\ } & \mathbb{P}^1_x \\
\phi \downarrow & & \downarrow \phi \\
\mathbb{P}^1_z & \xrightarrow[\ \sigma\ ]{} & \mathbb{P}^1_z
\end{array}
$$

The set of automorphisms of ϕ is denoted by

$$\mathrm{Aut}(\phi) := \{\sigma \in \mathrm{PGL}_2(k) : \phi^\sigma = \phi\}.$$

It forms a group. For any $\sigma \in \mathrm{Aut}(\phi)$, by $\mathbf{Fix}(\sigma)$ we denote its set of fixed points.

Remark 3. *As in the case of curves, there is some confusion in the literature on what is called the* automorphism group *of ϕ. Throughout this paper, by $G := \mathrm{Aut}(\phi)$, we mean the **full automorphism group** of $\phi(x)$ over the algebraic closure of k and not simply some $G \hookrightarrow \mathrm{Aut}(\phi)$ as it is used frequently by many authors.*

Lemma 4. *Let $\phi \in \mathrm{Rat}_d^1$ and $\sigma \in \mathrm{Aut}(\phi)$. Then*

(i) *If $p \in \mathbf{Fix}(\sigma)$, then $\phi(p) \in \mathbf{Fix}(\sigma)$.*
(ii) *If $w \in \mathbf{Fix}(\phi)$, then $\sigma(w) \in \mathbf{Fix}(\phi)$.*

Hence, $\mathrm{Aut}(\phi)$ acts on $\mathbf{Fix}(\phi)$ by permutation. Moreover, if $\sigma \in \mathrm{Aut}(\phi)$ is an automorphism of order m, then m divides the cardinality of $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$. The dimension of the corresponding locus $\delta = s - 1$, where s is the number of orbits on $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$.

Proof. For any $p \in \mathbf{Fix}(\phi)$, $\sigma(p) \in \mathbf{Fix}(\phi)$ since

$$\phi(\sigma(p)) = \sigma(\phi(p)) = \sigma(p),$$

which implies that $\sigma(p) \in \mathbf{Fix}(\phi)$. If $w \in \mathbf{Fix}(\phi)$ then

$$\sigma(w) = \sigma(\phi(w)) = \phi(\sigma(w)),$$

which implies that $\sigma(w) \in \mathbf{Fix}(\phi)$.

Since $\langle\sigma\rangle$ has no fixed points in $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$, then it acts transitively on $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$. Hence, $|\sigma|$ divides its cardinality. We have fixed 0 and ∞ on $\mathbb{P}_x^1$. Hence, the dimension of the family of rational functions $\phi(x)$ is one less than the number of roots of F and G. Hence, this number is exactly $s - 1$. □

Proposition 1. *Let $\sigma \in \mathrm{Aut}(\phi)$ such that $|\sigma| = m$. Then $H := \langle\sigma\rangle$ acts on $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$. Hence, $\phi(x)$ can be written as*

$$\phi(x) = x\,\psi(x^m) \quad \text{or} \quad \phi(x) = \frac{1}{x}\,\psi(x^m),$$

where $\psi(x)$ is a rational function. Moreover, for $G \cong A_4, S_4, A_5$ then $m = 2, 4, 5$.

Proof. Let $\sigma \in G$ and $t \in \mathbf{Fix}(\sigma)$. For each $\alpha \in \phi^{-1}(t)$ we have

$$\phi(\sigma(\alpha)) = \sigma(\phi(\alpha)) = \sigma(t) = t.$$

Then $\langle\sigma\rangle$ acts on the fiber $\phi^{-1}(t)$. From Remark 2 there is $\sigma \in G$ which fixes 0 and ∞. Then $\langle\sigma\rangle$ acts on $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$. Then points in $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$ are

$$\alpha_1, \xi\alpha_1, \dots, \xi^{m-1}\alpha_1,\ \alpha_2, \xi\alpha_2, \dots, \xi^{m-1}\alpha_2,\ \dots, \alpha_r, \xi\alpha_r, \dots, \xi^{m-1}\alpha_r,$$
$$\beta_1, \xi\beta_1, \dots, \xi^{m-1}\beta_1,\ \beta_2, \xi\beta_2, \dots, \xi^{m-1}\beta_2,\ \dots, \beta_r, \xi\beta_r, \dots, \xi^{m-1}\beta_r,$$

where $r = \frac{d-1}{m}$ and $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in k \setminus \{0, 1, \infty\}$. Therefore,

$$\mathbf{F}(x) = \prod_{j=1}^{r} \prod_{i=0}^{m-1} \left(x - \xi_m^i \alpha_j\right) = \prod_{j=1}^{r} (x^m - \alpha_j^m),$$
$$\mathbf{G}(x) = \prod_{j=1}^{r} \prod_{i=0}^{m-1} \left(x - \xi_m^i \beta_j\right) = \prod_{j=1}^{r} (x^m - \beta_j^m)$$

Thus, $\phi(x)$ can be written as

$$\phi(x) = x\,\frac{\mathbf{F}(x^m)}{\mathbf{G}(x^m)} = x\,\frac{x^{rm} + a_{r-1}x^{(r-1)m} + \cdots a_1 x^m + a_0}{x^{rm} + b_{r-1}x^{(r-1)m} + \cdots b_1 x^m + b_0}$$

or $\phi(x) = \frac{\mathbf{G}(x^m)}{x\,\mathbf{F}(x^m)}$. This completes the proof. ☐

The above Lemmas give us a way to determine $\phi(x)$ once an automorphism $\sigma \in \mathrm{Aut}(\phi)$ is given. Since the goal of this paper is to investigate computational and machine learning techniques for $d = 3$ for the rest of the paper we focus solely on this case.

5. Automorphism Groups of Rational Cubics

The automorphism groups for cubic rational functions and their parametric families were determined in [4] and in [18]. Here we give a brief treatment to sort out some mistakes and furthermore determine explicitly families in terms of invariants $\xi_0, \ldots, \xi_5$ and in each case give a formula for the rational cubic defined over its field of moduli.

Lemma 5. *Let $\phi \in \mathrm{Rat}_d^1$ with $d = 3$. Then the following hold:*
 (i) *Elements of $\mathrm{Aut}(\phi)$ have orders at most 4.*
 (ii) *$\mathrm{Aut}(\phi)$ is isomorphic to one of the following: $\{e\}, C_2, C_3, C_4, V_4, D_4$, or A_4.*

Proof. Suppose that $\sigma \in \mathrm{Aut}(\phi)$ is of order n. There is no loss of generality to assume that $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_n \end{pmatrix}$, where ζ_n denotes a fixed primitive n-th root of unity in k. Then, $\phi^\sigma = \phi$ if and only if ϕ commutes with σ. If so, we can construct the pair (F, G), defining ϕ, from the set of pairs of monomials $(x^{r_0}y^{3-r_0}, x^{r_1}y^{3-r_1})$ such that $n | r_0 - r_1 - 1$. Since $|r_0 - r_1 - 1| \leq 4$, then $n \leq 4$.

As a result, we only need to characterize cubic rational functions that have an automorphism of order 2, 3, or 4 in terms of their invariants (recall that if $\sigma \in \mathrm{Aut}(\phi)$ has order 4, then $\sigma^2 \in \mathrm{Aut}(\phi)$ has order 2). From (i), we eliminate C_n and D_{2n} for $n \geq 5$, as well as A_5, from consideration. Thus, possible groups are $\{1\}$, C_2, C_3, C_4, V_4, D_4, and A_4, as evidenced by the families below. We exclude S_4 as shown in the rest of this section. ☐

Assume $\phi \in \mathrm{Rat}_3^1$ as in Eq. (18). Let $W := \mathbf{Fix}(\phi)$ be the set of fixed points, and $\sigma \in \mathrm{Aut}(\phi)$ is a non-trivial automorphism of order n. By Lem. 5 we have $n = 2, 3$, or 4. The case of $n = 4$ is a subcase of $n = 2$. Hence, there are two main cases $n = 2$ and $n = 3$.

5.1. Involutions.

Let $\sigma \in \mathrm{Aut}(\phi)$ such that $|\sigma| = 2$. We can pick a coordinate in $\mathbb{P}^1$ as in Eq. (26) such that $\sigma(z) = -z$. Then $\mathbf{Fix}(\sigma) = \{0, \infty\}$. From Lem. 4, we have that $\phi(0), \phi(\infty) \in \mathbf{Fix}(\sigma)$. Hence, there are two cases: ϕ fixes points of $\mathbf{Fix}(\sigma)$ or ϕ permutes points of $\mathbf{Fix}(\sigma)$ and each of them corresponds to an irreducible surface as we will see next.

5.1.1. *ϕ fixes points of* $\mathbf{Fix}(\sigma) = \{0, \infty\}$. In this case, $\phi(z)$ fixes both points in $\mathbf{Fix}(\sigma)$, so $\phi(0) = 0$ and $\phi(\infty) = \infty$. By Prop. 1, $\phi(z) = z\psi(z^2)$, where $\psi(z^2)$ is a rational function ensuring ϕ is degree 3 with $\sigma(z) = -z$ as an automorphism. Hence,

$$\phi(z) = z\psi(z^2) = \frac{z(z^2 + a)}{z^2 + b},$$

for $a, b \in k$. Assuming $b \neq 0$ (to avoid a pole at $z = 0$), rewrite $\phi(z)$ as

$$\phi(z) = \frac{1}{b} \cdot \frac{z(z^2+a)}{\frac{1}{b}z^2+1} = \frac{1}{b} \cdot \frac{z^3+az}{\frac{1}{b}z^2+1}.$$

Define $t = a$ and $s = \frac{1}{b}$ and we have

$$\phi(z) = \frac{1}{b} \cdot \frac{z^3+tz}{sz^2+1}.$$

Since $\phi = \frac{F(z)}{G(z)}$ is defined up to a scalar multiple, we can scale by b, yielding:

$$\phi(z) = z\frac{z^2+t}{sz^2+1},$$

for $t, s \in k$, with $s \neq 0$. This matches the family in [4]. Then

$$I_6(\phi) = t^2s^2 - 2ts + 1 = (ts-1)^2 \neq 0.$$

Hence, in this case ϕ is conjugate to a rational function written in the form

$$\boxed{\phi(z) = \frac{z^3+tz}{sz^2+1}, \qquad \mathfrak{p} = [1:0:t:0:0:s:0:1]} \tag{27}$$

for some $t, s \in k$ such that $ts \neq 1$.

5.1.2. *ϕ permutes points of* $\mathbf{Fix}(\sigma)$. We are still under assumption that there is an involution $\sigma \in \mathrm{Aut}(\phi)$ such that $\sigma(z) = -z$. Assume that ϕ permutes 0 and ∞. From Prop. 1, we have that $\phi(z) = \frac{1}{z}\psi(z^2)$, for some degree two rational function $\psi(z)$. Lem. 4 implies that σ permutes points in $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$ which are the numerator and denominator of ψ. Thus

$$\phi(z) = \frac{1}{z}\frac{z^2+b}{z^2+a}.$$

Similarly as the previous case, we can normalize $b = 1$ and get

$$\boxed{\phi(z) = \frac{sz^2+1}{z^3+tz}, \qquad \mathfrak{p} = [0:s:0:1:1:0:t:0]} \tag{28}$$

for some $t, s \in k$. The resultant in this case is

$$I_6(\phi) = -(ts-1)^2 \neq 0$$

which implies that $ts \neq 1$.

Remark 4. *Notice that $\phi(z)$ in Eq. (28) is the reciprocal of the function ϕ given in Eq. (27).*

5.2. **Extra involutions.** Assume that there is another involution $\tau \in \mathrm{Aut}(\phi)$. Since σ fixes 0 and ∞, then τ must permute. Hence $\tau(z) = \frac{1}{z}$. Thus we have the Klein 4-group V_4 embedded in the automorphism group $\mathrm{Aut}(\phi)$.

Proposition 2. *Let $\phi(z)$ be a cubic rational function with an involution, written as*

$$\phi(z) = \frac{az^2+b}{z(cz^2+d)},$$

where $a, b, c, d \in k$. If $\phi(z)$ has another involution, then $\phi(z)$ is conjugate under $\mathrm{PGL}(2, k)$ *to one of:*

$$\phi(z) = \frac{tz^2+1}{z^3+tz} \quad or \quad \phi(z) = \frac{sz^2-1}{z^3-sz},$$

for some $t, s \in k$.

Proof. Let us see what condition on coefficients a, b, c, d are enforced by $\tau(z) = \frac{1}{z}$.

$$\phi(\tau(z)) = \phi\left(\frac{1}{z}\right) = \frac{a\left(\frac{1}{z}\right)^2 + b}{\frac{1}{z}\left(c\left(\frac{1}{z}\right)^2 + d\right)} = \frac{\frac{a}{z^2}+b}{\frac{c}{z^2}+\frac{d}{z}} = \frac{\frac{a+bz^2}{z^2}}{\frac{c+dz^2}{z^3}} = \frac{z(a+bz^2)}{c+dz^2},$$

$$\tau(\phi(z)) = \frac{1}{\frac{az^2+b}{z(cz^2+d)}} = \frac{z(cz^2+d)}{az^2+b}.$$

Then $\phi(\tau(z)) = \tau(\phi(z))$ implies

$$\frac{a+bz^2}{c+dz^2} = \frac{cz^2+d}{az^2+b},$$

which implies

$$abz^4 + (a^2+b^2)z^2 + ab = cdz^4 + (c^2+d^2)z^2 + cd.$$

Hence we must have $ab = cd$ and $a^2 + b^2 = c^2 + d^2$. We solve this system of equations for b, d in terms of a, c. Recall that $c \neq 0$, otherwise $\deg \phi < 3$. Then we have $(b, d) = (c, a)$ or $(b, d) = (-c, -a)$. If $(b, d) = (c, a)$ we have

$$\phi(z) = \frac{az^2+c}{cz^3+az} = \frac{tz^2+1}{z^3+tz},$$

for $t = a/c$. If $(b, d) = (-c, -a)$ we have

$$\phi(z) = \frac{az^2-c}{cz^3-az} = \frac{tz^2-1}{z^3-tz},$$

for $t = a/c$. Thus, $\phi(z)$ is conjugate to $\frac{tz^2+1}{z^3+tz}$ or $\frac{sz^2-1}{z^3-sz}$. □

5.2.1. *First case.* Let us denote the locus of all $\phi(z)$ is in the first case of the Prop. 2 by $\mathcal{L}_4$. Notice that if $\phi \in \mathcal{L}_2$ this case implies that $t = s$. Thus,

$$\boxed{\phi(z) = \frac{tz^2+1}{z^3+tz}, \quad \mathfrak{p} = [0, t, 0, 1, 1, 0, t, 0]} \tag{29}$$

Similarly one can show that if $\phi \in \mathcal{L}_1$ then we still get $t = s$, which is a confirmation that this locus will be the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$. We will denote it by $\mathcal{L}_4$.

We denote by

$$u := t^2$$

Then I_6 invariant is

$$I_6(\phi) = -(t^2-1) = -(u-1)^2 \neq 0.$$

5.2.2. *Second case.* Let us denote the locus of all $\phi(z)$ is in the first case of the Proposition by $\mathcal{L}_4$. Notice that if $\phi \in \mathcal{L}_2$ this case implies that $t = -s$.

$$\boxed{\phi(z) = \frac{sz^2-1}{z^3-sz}, \quad \mathfrak{p} = [0, s, 0, -1, 1, 0, -s, 0]} \tag{30}$$

The resultant here is $I_6(\phi) = -s(s^2-1)^2 \neq 0$ for $s \neq 0, \pm 1$.

5.2.3. *Alternating group:* $\mathrm{Aut}(\phi) \cong A_4$. Suppose we can extend such a V_4 subgroup within $\mathrm{Aut}(\phi)$ to an A_4 subgroup. In this case, from Eq. (26), we can take $\sigma(z) = -z$ and $\tau(z) = \frac{z+i}{z-i}$. Then

$$\boxed{\phi(z) = \frac{z^3-3}{-3z^2}, \quad \mathfrak{p} = [1:0:0:-3:0:-3:0:0]} \tag{31}$$

5.2.4. *Dihedral group* D_4. Here, we can assume that σ is an automorphism of the form $\sigma(z) = \zeta_4 z$. where ζ_4 is a fixed primitive 4th root of unity in k, is an automorphism.

Since σ^2 in $\mathrm{Aut}(\phi)$ acts as $\sigma(z) = -z$, a situation we discussed earlier, we can begin with a rational function of the form in $\mathcal{L}_1$ or $\mathcal{L}_2$, which corresponds to

$$\mathfrak{p} = [1:0:t:0:0:s:0:1] \quad \text{or } \mathfrak{p} = [0:t:0:1:1:0:s:0]$$

Verifying that $\sigma \in \mathrm{Aut}(\phi)$, we can confirm that the first case does not yield any possible rational cubic functions, while in the second case, it must be the case that $t = s = 0$. Hence

$$\boxed{\phi = \frac{1}{z^3}, \qquad \mathfrak{p} = [0,0,0,1,1,0,0,0]} \tag{32}$$

5.3. **An automorphism of order 3.** Let σ be of order 3 taken as in Eq. (26), so $\sigma(z) = \zeta_3 z$. Then, by Prop. 1, we have $\langle\sigma\rangle$ acts on the fiber $\phi^{-1}(0)$, which implies that the numerator of $\phi(z)$ is a polynomial $p(z) = z^3 - t$, for some $t \in k^\star$. We can pick 0 and ∞ in $\phi^{-1}(\infty)$. Since $\phi^{-1}(\infty)$ is an orbit of $\langle\sigma\rangle$ and σ fixes them, then one of them must have multiplicity 2. Thus, we can take

$$\boxed{\phi(z) = \frac{z^3-t}{z}, \quad \mathfrak{p} = [1:0:0:-t:0:0:1:0]} \tag{33}$$

for some $t \in k$. The resultant is $I_6(\phi) = -t^3$, which gives the condition that $t \neq 0$.

	G	$\phi(z)$	$\mathfrak{p} \in \mathbb{P}^7$	dim	Eq. $\mathcal{L}_i$	$\xi(\phi)$
$\mathcal{L}_1$	C_2	$\frac{z^3+tz}{sz^2+1}$	$[1,0,t,0,0,s,0,1]$	2	(35)	$\xi_2 = \xi_3 = 0$
$\mathcal{L}_2$	C_2	$\frac{sz^2+1}{z^3+tz}$	$[0,s,0,1,1,0,t,0]$	2	(36)	$\xi_5 = 0$
$\mathcal{L}_3$	C_3	$\frac{z^3-t}{z}$	$[1,0,0,-t,0,0,1,0]$	1	(43)	
$\mathcal{L}_4$	V_4	$\frac{tz^2+1}{z^3+tz}$	$[0,t,0,1,1,0,t,0]$	1	(38)	$\xi_2 = \xi_3 = \xi_5 = 0$
$\mathcal{L}_5$	V_4	$\frac{tz^2-1}{z^3-tz}$	$[0,t,0,-1,1,0,-t,0]$	1	(39)	$\xi_0 = \xi_3 = \xi_4 = \xi_5 = 0$
$\mathcal{L}_6$	A_4	$\frac{z^3-3}{-3z^2}$	$[1,0,0,-3,0,-3,0,0]$	0	(41)	$[0:0:18:0:0:0]$
$\mathcal{L}_7$	D_4	$\frac{1}{z^3}$	$[0,0,0,1,1,0,0,0]$	1	(40)	$[0:-2:0:0:0:0]$

TABLE 2. Automorphism loci of degree 3 rational functions

This completes all the cases. We summarize all cases in Table 2. These results agree with results in [4]. Justifying the last two columns of the table will be the focus of the next section.

Inclusions among the loci are described in diagram Fig. 1. Notice that in each node of the diagram Fig. 1 we also put the invariants which vanish in that locus. Justifying these inclusions , computing the invariants, and describing each locus in terms of these invariants will be done in the next section.

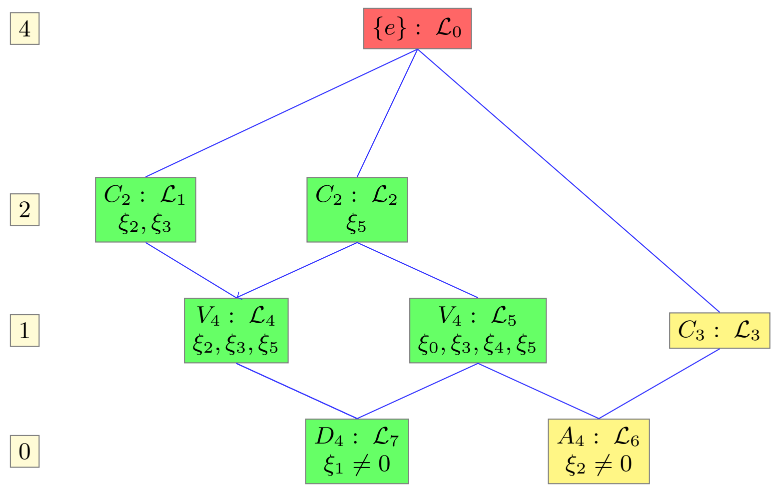

FIGURE 1. The inclusions among the loci for degree 3 rational functions

Remark 5. *Those familiar with automorphisms groups of hyperelliptic curves notice the similarity between the two problems. The zeroes and poles of $\phi(z)$ now play the role of the Weierstrass points of the hyperelliptic curve. While the reduced automorphism group of a hyperelliptic curve acts on the set of Weierstrass points, in our case here $\mathrm{Aut}(\phi)$ acts on the set of zeroes of $\phi(z)$ and on the set of poles. A general treatment of automorphism group of $\phi(z)$ can be done similarly to that of hyperelliptic curves; see [12].*

6. Computation of the Loci with Fixed Automorphism Group

In this section, we compute the loci of rational cubics with a fixed automorphism group in terms of invariants of rational cubics. We follow an analogy with algebraic curves and build on previous work by the third author in computing similar loci in the moduli space of genus 2 curves; see [15, Thm 3] or [13].

The first question is where this locus lies and what type of invariants we should use to compute it. Let $\mathcal{L}$ denote the image in $\mathbb{P}_{(2,2,3,3,4,6)}$ of the map

$$\Phi : (c_0, \dots, c_8) \to (\xi_0(c_0, \dots, c_8), \dots, \xi_5(c_0, \dots, c_8)) .$$

This image is a weighted projective variety in $\mathbb{P}_{(2,2,3,3,4,6)}$. Alternatively, one can use the Veronese embedding in Eq. (22) and compute the locus of such rational functions as a projective variety, or even as an affine variety in terms of absolute invariants $i_1, \dots, i_5$.

Computing these loci in terms of $i_1, \dots, i_5$ is straightforward but highly inefficient. We could express $i_1, \dots, i_5$ in terms of the parameters described for each family in the previous section and then use a Gröbner basis approach to eliminate those parameters. However, the degrees of the resulting equations are very large, making this approach computationally challenging even for the relatively simple families considered here.

A more efficient method is to compute the loci as subvarieties of weighted projective spaces. This requires a modification of Buchberger's algorithm to handle computations in a weighted projective space.

6.1. **Gröbner Bases for Weighted Homogeneous Ideals.** Consider a subvariety in $\mathbb{P}(w_0, \dots, w_n)$ defined parametrically by a set of polynomials over m parameters. The coordinates are given by:

$$x_i = \xi_i, \quad i = 0, \dots, n,$$

where each ξ_i is a polynomial in $k[t_1, \dots, t_m]$, and $t_1, \dots, t_m$ are parameters with assigned degrees (typically positive integers). The image of this parameterization is a variety whose dimension depends on m; for m independent parameters, the dimension is typically $m-1$. To define this variety explicitly, we seek polynomial relations among $x_0, \dots, x_n$ that hold for all parameter values, and these relations must be weighted homogeneous with respect to the weights $(w_0, \dots, w_n)$. In other words, the relations must be satisfied not only by tuples $(\xi_0, \dots, \xi_n)$ but also by all $(\lambda^{w_0}\xi_0, \dots, \lambda^{w_n}\xi_n)$.

To compute these relations, we construct the ideal

$$I = \langle x_0 - \xi_0, x_1 - \xi_1, \dots, x_n - \xi_n \rangle$$

in the polynomial ring $k[t_1, \dots, t_m, x_0, \dots, x_n]$. The defining equations of the parameterized variety's image in $\mathbb{P}(w_0, \dots, w_n)$ are the elements of the elimination ideal:

$$I \cap k[x_0, \dots, x_n],$$

which consists of all polynomials in $x_0, \dots, x_n$ satisfied by the parameterization. By choosing a monomial order that eliminates the parameters $t_1, \dots, t_m$, the Gröbner basis of I includes polynomials free of t_j, representing the desired relations.

To facilitate elimination while respecting the weighted structure, we assign degrees to both parameters and coordinates, say $\deg(t_j) = d_j$ and $\deg(x_i) = w_i$. Typically, we set $d_j = 1$.

The weighted degree lexicographic order (`wdeglex`) compares monomials first by their total weighted degree, then lexicographically, with parameters ordered above coordinates (e.g., $t_1 > \cdots > t_m > x_0 > \cdots > x_n$). This ensures that the Gröbner basis computation prioritizes elimination of the parameters, yielding a set of generators for $I \cap k[x_0, \dots, x_n]$ that define the variety's image in the coordinate ring of $\mathbb{P}(w_0, \dots, w_n)$.

The relations obtained from the elimination ideal are algebraically correct but may not be weighted homogeneous, as their terms can have differing weighted degrees. To make them valid defining equations in $\mathbb{P}(w_0, \dots, w_n)$, a homogenization step is required:

(1) **Degree Computation**: For a relation $F = \sum c_\alpha x^\alpha$, compute the weighted degree of each term as $\deg(x^\alpha) = \sum_{i=0}^{n} w_i \alpha_i$.

(2) **Target Degree**: Determine a common degree d, typically the least common multiple (LCM) of the degrees of all terms in F, i.e., $d = \mathrm{LCM}(\deg(x^\alpha))$.
(3) **Exponent Adjustment**: Transform each term $c_\alpha x^\alpha$ into $c_\alpha x^{\alpha k_\alpha}$, where $k_\alpha = d/\deg(x^\alpha)$, ensuring $\deg(x^{\alpha k_\alpha}) = k_\alpha \cdot \deg(x^\alpha) = d$. Adjust coefficients as needed to preserve the relation's zero set.

This process yields a weighted homogeneous polynomial that defines the same variety but adheres to the grading of $\mathbb{P}(w_0, \ldots, w_n)$.

6.1.1. *General Algorithm.* For a weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ with weights $w = (w_0, \ldots, w_n)$ and a parameterization $x_i = \xi_i(t_1, \ldots, t_m)$:

(1) Construct the ideal $I = \langle x_0 - \xi_0, x_1 - \xi_1, \ldots, x_n - \xi_n \rangle$ in $k[t_1, \ldots, t_m, x_0, \ldots, x_n]$.
(2) Compute the Gröbner basis B of I using a weighted degree lexicographic order with weights $(d_1, \ldots, d_m, w_0, \ldots, w_n)$ and order $t_1 > \cdots > t_m > x_0 > \cdots > x_n$.
(3) Extract the set $R = \{f \in B \mid f \in k[x_0, \ldots, x_n]\}$, the generators of the elimination ideal.
(4) For each polynomial $f \in R$:
   - If f is single-term or all terms have the same weighted degree, retain f as-is.
   - Otherwise, compute $d = \mathrm{LCM}(\deg(x^\alpha))$ over all terms in f, and adjust f to $F = \sum c_\alpha x^{\alpha k_\alpha}$, where $k_\alpha = d/\deg(x^\alpha)$, ensuring F is weighted homogeneous of degree d.
(5) Define the ideal $J = \langle R_{\text{homog}} \rangle$ in $k[x_0, \ldots, x_n]$, which specifies the subvariety in $\mathbb{P}(w_0, \ldots, w_n)$.

This method uses Gröbner bases to eliminate parameters and enforces weighted homogeneity, providing a computational framework for studying subvarieties in weighted projective spaces with arbitrary weights $(w_0, \ldots, w_n)$. The method is described in [3] and related works by its authors.

In our situation, however, there is an additional consideration. Our invariants $\xi_0, \ldots, \xi_5$ are homogeneous polynomials when applied to the definition of ϕ in Eq. (5), meaning they are homogeneous in $c_0, \ldots, c_7$. However, they are not necessarily homogeneous with respect to our specializations for t and s from the previous section. Thus, we first homogenize these defining equations by introducing another variable, say t_0, and then apply the above procedure.

Next, we explicitly compute the loci as weighted projective varieties for each case, illustrating the approach described above.

6.2. **Locus $\mathcal{L}_1$.** We assume that $\phi(z)$ is given as in Eq. (27). Computing the invariants $\xi(\phi)$, we obtain:

$$
\begin{aligned}
&\xi_0 = 2(t+3)(s+3), \quad \xi_1 = \frac{1}{2}(t-1)(s-1), \quad \xi_2 = \xi_3 = 0, \\
&\xi_4 = -\frac{1}{3}\left(t^2s^2 + 2t^2s + 3t^2 + 2ts^2 - 8ts - 6t + 3s^2 - 6s + 9\right), \\
&\xi_5 = -(t-s)^2(ts + t + s - 3).
\end{aligned}
$$

There is an involution permuting t and s. Let u and v denote invariants of that involution, defined as:

$$\boxed{u = t + s \quad \text{and} \quad v = ts.}$$

We can express all invariants in terms of u and v as follows:

$$\xi_0 = 2(v+3u+9), \quad \xi_1 = \frac{1}{2}(v-u+1), \quad \xi_2 = \xi_3 = 0,$$
$$\xi_4 = -\frac{1}{3}\left(v^2+3u^2+2uv-14v-6u+9\right),$$
$$\xi_5 = (u^2-4v)(3-u-v).$$

Also, $I_6 = (ts-1)^2 = (v-1)^2 \neq 0$, so $v \neq 1$. Moreover, J_6 is given by:

$$J_6 = 4(s+3)(t+3)(st+s+t-3)^2 = 4(3u+v+9)(u+v-3)^2. \tag{34}$$

By computing a Gröbner basis for the weighted homogeneous system as described above, we obtain the following degree 24 weighted hypersurface:

(35)

$$\begin{aligned}\mathcal{L}_1: \quad & \xi_0^6\xi_1^6 - 54\xi_0^4\xi_1^4\xi_4^2 - \frac{27}{4}\xi_0^4\xi_1^3\xi_5^2 - 27\xi_0^3\xi_1^4\xi_5^2 - 108\xi_0^3\xi_1^3\xi_4^3 + 729\xi_0^2\xi_1^2\xi_4^4 + \frac{729}{4}\xi_0^2\xi_1\xi_4^2\xi_5^2 \\ & + \frac{729}{64}\xi_0^2\xi_5^4 + 729\xi_0\xi_1^2\xi_4^2\xi_5^2 + 2916\xi_0\xi_1\xi_4^5 + \frac{243}{8}\xi_0\xi_1\xi_5^4 + \frac{729}{2}\xi_0\xi_4^3\xi_5^2 + \frac{729}{4}\xi_1^2\xi_5^4 \\ & + 1458\xi_1\xi_4^3\xi_5^2 + 2916\xi_4^6 + \frac{729}{2}\xi_4\xi_5^4 = 0\end{aligned}$$

We can also compute $\mathcal{L}_1$ in terms of the absolute invariants by eliminating u and v. These computations are lengthy but express u and v as rational functions of the absolute invariants. Thus, the map in Eq. (24) becomes:

$$\begin{aligned}\Phi_1 : k^2 \setminus \{v=1\} &\to k^3 \\ (u,v) &\to (i_1, i_2, i_5),\end{aligned}$$

which is invertible when $J_6 \neq 0$, i.e., $(u+v-3)(v+3u+9) \neq 0$. The map Φ_1 provides a birational parametrization of $\mathcal{L}_1$.

Remark 6. *The invariants $u = t+s$ and $v = ts$ mirror dihedral invariants in genus 2 curves [5,15], symmetric under swapping t and s. Here, they are invariant under parameter rescaling and conjugation adjusting t and s, simplifying computations akin to symmetric polynomials in root permutations—a pattern likely extending to higher-degree rational functions.*

If $\xi_0 = 0$, then $t = -3$ or $s = -3$, and the moduli point is:

$$\xi(\phi) = [0:8:0:0:0:0],$$

a singular point in $\mathcal{L}_1$ with automorphism group isomorphic to D_4 (cf. Eq. (40)).

Remark 7. *Note that the factor $u^2 = 4v$ (or equivalently $t = s$) makes $\xi_5 = 0$. This locus corresponds to $\mathcal{L}_4$ (cf. Eq. (38)), where $\phi(z)$ has an extra involution. Since $\mathcal{L}_4$ lies in the intersection of $\mathcal{L}_1$ and the next locus, we will discuss it in detail later.*

6.3. **Locus $\mathcal{L}_2$.** Assume that ϕ is given as in Eq. (28). We follow the same approach as above. Computing $\xi(\phi)$, we get:

$$\xi_0 = -2(t+s)^2, \quad \xi_1 = \frac{1}{6}\left((t-s)^2-12\right), \quad \xi_2 = -\frac{1}{36}(s-t)\left((t-s)^2+36\right),$$
$$\xi_3 = \frac{2}{3}(s-t)(t+s)^2, \quad \xi_4 = -\frac{1}{9}(t+s)^2\left((s-t)^2+12\right), \quad \xi_5 = 0.$$

We define invariants:

$$u := (t+s)^2 \quad \text{and} \quad v := s-t,$$

and express the invariants in terms of u and v as follows:

$$\xi_0 = -2u, \quad \xi_1 = \frac{1}{6}(v^2-12), \quad \xi_2 = -\frac{1}{36}v(v^2+36),$$
$$\xi_3 = \frac{2}{3}uv, \quad \xi_4 = -\frac{1}{9}u(v^2+12), \quad \xi_5 = 0.$$

Here, we have:

$$J_6(\phi) = -16(s+t)^4 = -16u^2 \quad \text{and} \quad I_6 = \frac{v^2-u+4}{4}.$$

Using a Gröbner basis to eliminate u and v, we determine the equations for $\mathcal{L}_2$ in $\mathbb{P}(2,2,3,3,4,5)$ as follows:

$$\mathcal{L}_2 : \quad \begin{cases} \xi_5 = 0, \\ \xi_0^2\xi_1 + 3\xi_0\xi_4 - 3\xi_3^2 = 0, \\ \xi_0^2\xi_2 + \frac{1}{2}\xi_0\xi_1\xi_3 - 3\xi_3\xi_4 = 0, \\ \xi_0\xi_1\xi_4 - \xi_0\xi_2\xi_3 - \frac{1}{2}\xi_1\xi_3^2 + 3\xi_4^2 = 0 \end{cases} \tag{36}$$

The absolute invariants are:

$$i_1 = \frac{1024u^6}{(-v^2+u-4)^2}, \quad i_2 = \frac{(v^2-12)^6}{2916(-v^2+u-4)^2}, \quad i_3 = \frac{v^4(v^2+36)^4}{104976(-v^2+u-4)^2},$$
$$i_4 = \frac{256v^4u^4}{81(-v^2+u-4)^2}, \quad i_5 = -\frac{16u^3(v^2+12)^3}{729(-v^2+u-4)^2}.$$

Assuming $J_6 \neq 0$ (i.e., $u \neq 0$), we eliminate u and v, finding:

$$u = -\frac{1}{2}\xi_0, \quad v = 3\frac{\xi_3}{\xi_0}. \tag{37}$$

Similarly to the previous case, we define:

$$\begin{aligned} \Phi_2 : k^2 \setminus \{u - v^2 \neq 4\} &\to k^5 \\ (u,v) &\to (i_1, \dots, i_5). \end{aligned}$$

$\mathcal{L}_2$ is an irreducible 2-dimensional variety birationally parametrized by the map:

$$\Phi_2 : (u,v) \to (\xi_0, \xi_3).$$

Moreover, the field of moduli of $\phi(z)$ is $\mathbb{F}(u,v)$.

Consider now the case when:

$$J_6 = -16(s+t)^4 = -16u^2 = 0,$$

implying $t = -s$. The function becomes:

$$\phi(z) = \frac{sz^2+1}{z^3 - sz},$$

and then $\xi_0 = \xi_3 = \xi_4 = \xi_5 = 0$, $\xi_1 = \frac{v^2}{6} - 1$, and $\xi_2 = -v$. This corresponds to $\mathcal{L}_5$, as we will see later.

This concludes the loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for rational functions with involutions as described in Section 5.1. The remaining loci are of dimension one or zero, making their computations simpler. We will address each in detail below.

6.4. **Locus $\mathcal{L}_4$.** Assume ϕ is as in Eq. (29), with homogeneous parameterization in $\mathbb{P}_{(2,2,3,3,4,6)}$:

$$\xi(\phi) = \left[-8t^2,\ -2u^2,\ 0,\ 0,\ -\frac{16t^2u^2}{3},\ 0\right].$$

In the affine patch $(u = 1)$:

$$\xi(\phi) = \left[-8t^2,\ -2,\ 0,\ 0,\ -\frac{16t^2}{3},\ 0\right].$$

The Gröbner basis with weighted degree lexicographic order is:

$$\mathcal{L}_4: \quad \begin{cases} \xi_0\xi_1 + 3\xi_4 = 0, \\ \xi_2 = \xi_3 = \xi_5 = 0 \end{cases} \tag{38}$$

This defines a 1-dimensional variety in $\mathbb{P}_{(2,2,3,3,4,6)}$.

6.5. **Locus $\mathcal{L}_5$.** Assume ϕ as in Eq. (30). Its invariants are

$$\xi(\phi) = \left[0,\ \frac{2(s^2+3)}{3},\ -\frac{2s(s-3)(s+3)}{9},\ 0,\ 0,\ 0\right]$$

and $I_6 = (s-1)^2(s+1)^2$. Here we use the absolute invariants i_2 and i_3 and have the following system

$$\begin{cases} i_2\, t^4 - 2i_2\, t^2 + i_2 - \dfrac{4}{9}t^4 - \dfrac{8}{3}t^2 - 4 = 0 \\ i_3\, t^4 - 2i_3\, t^2 + i_3 - \dfrac{4}{81}t^6 + \dfrac{8}{9}t^4 - 4t^2 = 0 \end{cases}$$

By taking the resultant of these two polynomials with respect to t we get the affine version of this 1-dimensional variety

$$16i_2^3 - 72i_2^2 i_3 + 81 i_2\, i_3^2 - 96i_2^2 + 216 i_2 i_3 - 36 i_3^2 + 144 i_2 - 96 i_3 - 64 = 0$$

By replacing for i_2 and i_3 their definitions we get a degree 18 weighted hypersurface

$$\begin{aligned} \mathcal{L}_5: \quad & 72\xi_1^6\xi_2^2 - 16\xi_1^9 + 96\xi_1^6 I_6 - 81\xi_1^3\xi_2^4 - 216\xi_1^3\xi_2^2 I_6 - 144\xi_1^3 I_6^2 + 36\xi_2^4 I_6 \\ & + 96\xi_2^2 I_6^2 + 64 I_6^3 = 0 \end{aligned} \tag{39}$$

6.6. **Locus $\mathcal{L}_7$.** Assume ϕ as in Eq. (32).. Its invariants are

$$\xi(\phi) = [0, -2, 0, 0, 0, 0] \equiv [0, 1, 0, 0, 0, 0] \tag{40}$$

Notably, the involution $\sigma'(z) = \frac{1}{z}$ acts as an additional automorphism for $\phi = \frac{y^3}{x^3}$, confirming that the loci $\mathcal{L}(C_4)$ and $\mathcal{L}(D_4)$ are identical. Additionally, $\sigma_0 \notin \mathrm{Aut}(\phi)$, implying that the locus of $\mathcal{L}(S_4)$ is empty.

6.7. **Locus $\mathcal{L}_6$.** Assume ϕ as in Eq. (29). The moduli point is

$$\mathcal{L}_6: \quad \xi(\phi) = [0:0:18:0:0:0] \equiv [0:0:1:0:0:0] \tag{41}$$

Furthermore, it should be noticed that in this case $J_6(\phi) = 0$.

6.8. **Locus $\mathcal{L}_3$.** Assume ϕ as in Eq. (33). Next, we compute its invariants

$$\xi(\phi) = \left[-2,\, \frac{1}{6},\, \frac{27t+2}{72},\, -\frac{27t+2}{3},\, \frac{54t-1}{9},\, -\frac{t(27t-16)}{4} \right]$$

Notice that $I_6 = t \neq 0$. The absolute invariants are

$$i_1 = \frac{64}{t^6},\; i_2 = \frac{1}{46656t^6},\; i_3 = \frac{(27t+2)^4}{26873856t^2},\; i_4 = \frac{(27t+2)^4}{81t^2},\; i_5 = \frac{(54t-1)^3}{729t^2}$$

By eliminating t from these equations, we can express t as a rational function in terms of i_1, i_5 and get the following affine curve

(42)
$$\begin{aligned}
&6147876261765083996 16 i_1 i_5^6 + 44264709084708604772352 i_1 i_5^5 + 49589822592 i_1^2 i_5^3 \\
&+ 115088243620242372408 1152 i_1 i_5^4 - 6248317646592 i_1^2 i_5^2 + 129843146648478573998 89920 i_1 i_5^3 \\
&+ i_1^3 + 35704672266240 i_1^2 i_5 + 594917690098483648140410 88 i_1 i_5^2 - 4760622968832 i_1^2 \\
&+ 793223586797978197520547 84 i_1 i_5 + 755451035045693521448140 8 i_1 \\
&- 399601949918492974316981858135860 8384 = 0
\end{aligned}$$

We can express this as a weighted projective curve by substituting for i_1 and i_5:

(43)
$$\begin{aligned}
&I_6^4 \xi_0^9 - 2834352 I_6^3 \xi_0^6 \xi_4^3 + 24794911296 \xi_0^3 \xi_4^9 + 3779136 I_6^5 \xi_0^6 + 892616806656 I_6^2 \xi_0^3 \xi_4^6 \\
&+ 7140934453248 I_6^4 \xi_0^3 \xi_4^3 + 4760622968832 I_6^6 \xi_0^3 + 1999004627104432128 I_6^7 = 0
\end{aligned}$$

This completes all the cases.

Remark 8. *We computed each locus as a weighted projective variety in the weighted projective space $\mathbb{P}_{(2,2,3,3,4,6)}$. From the arithmetic point of view we are interested on rational points on these weighted projective varieties. Rational points on weighted varieties are discussed in [8] based on weighted heights which give in general a more efficient approach then projective heights.*

7. A Database of Cubic Rational Functions

In this section, we construct a comprehensive database of cubic rational functions over the rational numbers $\mathbb{Q}$, denoted Rat_3^1, leveraging the weighted projective space $\mathbb{P}_\omega^5(\mathbb{Q})$ with weights $\omega = (2,2,3,3,4,6)$ as a parametrization framework. This database, denoted $\mathcal{P}_3^h$, catalogs rational functions $\phi(x) = \frac{f_0(x)}{f_1(x)} \in \mathbb{Q}(x)$ of degree 3, where $f_0(x)$ and $f_1(x)$ are polynomials of degrees 3 and 2, respectively, ensuring the degree of the rational function is $\deg(\phi) = \deg(f_0) - \deg(f_1) = 3$. Each function is represented as a projective point in $\mathbb{P}_\mathbb{Q}^7$, and we impose constraints on height and coprimality to define the dataset systematically.

A cubic rational function $\phi(x)$ is given as in Eq. (18) and corresponds to a point $P_\phi = [c_0 : c_1 : \cdots : c_7] \in \mathbb{P}_\mathbb{Q}^7$. We define the naive height of ϕ as $H(\phi) = \max\{|c_i| \mid i = 0, \ldots, 7\}$, and restrict our dataset to functions with $H(\phi) \leq h$, where h is a specified height bound. To ensure well-definedness, we require that the coefficients are coprime, i.e., $\gcd(c_0, c_1, \ldots, c_7) = 1$, and that the resultant $I_6(\phi) = \mathrm{Res}(f_0, f_1) \neq 0$, guaranteeing that ϕ has no common roots between numerator and denominator. Thus, we define:

$$\mathcal{P}_3^h := \{ P_\phi \in \mathbb{P}_\mathbb{Q}^7 \mid H_\mathbb{Q}(P_\phi) \leq h,\, \gcd(c_0, \ldots, c_7) = 1,\, I_6(\phi) \neq 0 \}.$$

For each $\phi \in \mathcal{P}_3^h$, we compute several invariants and properties to enrich the database. These include the automorphism group $\mathrm{Aut}(\phi)$, determined by checking against the classification of possible groups for cubic rational functions (e.g., $\{e\}$, C_2, D_4, etc.) as outlined in Table 2, the invariants $\mathbf{p} = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ in $\mathbb{P}^5_\omega(\mathbb{Q})$, and the absolute invariants $(i_1, i_2, i_3, i_4, i_5)$. Additionally, we calculate the invariant $J_6(\phi)$ and the weighted moduli height $\hat{h}(\phi)$, which measures the height of the invariants in the weighted projective space. The data is stored in a Python dictionary, with keys given by the coefficient tuples $(c_0, c_1, \dots, c_7)$ and values as lists containing:

- $H(\phi)$: the naive height,
- $\mathbf{p} = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$: the projective invariants,
- $\hat{h}(\phi)$: the weighted moduli height,
- $J_6(\phi)$: an additional invariant,
- $\mathrm{Aut}(\phi)$: the automorphism group (e.g., '{e}'),
- $(i_1, i_2, i_3, i_4, i_5)$: the absolute invariants.

An example entry from the database is:

$$(2, 3, -1, -3, 1, 2, -3, 1) \mapsto [3, (32, 12, 13, -164, -424, 2572), 5.66, 89360, '\{e\}',$$
$$\left(\frac{1073741824}{44521}, \frac{2985984}{44521}, \frac{531441}{712336}, \frac{723394816}{44521}, -\frac{76225024}{44521} \right) \Big].$$

Here, the key $(2, 3, -1, -3, 1, 2, -3, 1)$ represents the coefficients of

$$\phi(x) = \frac{2 + 3x - x^2 - 3x^3}{1 + 2x - 3x^2 + x^3},$$

with $H(\phi) = 3$, invariants ξ_i mapping to a point in $\mathbb{P}^5_\omega(\mathbb{Q})$, a weighted height of 5.66, $J_6 = 89360$, the trivial automorphism group $\{e\}$, and the corresponding absolute invariants.

To illustrate the dataset's scope, we present the distribution of automorphism groups for rational cubics with height $H(\phi) \leq 4$ in Table 3. This table categorizes points by their associated group labels ($\mathcal{L}_0$ to $\mathcal{L}_7$), which correspond to $\{e\}$, C_2 variants, D_4, etc., as determined by prior classification efforts.

Table 3. Distribution of automorphism groups for rational cubics of height $H(\phi) \leq 4$

h	$\mathcal{L}_0$	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$	$\mathcal{L}_5$	$\mathcal{L}_6$	$\mathcal{L}_7$	Total
1	2223	9	8	6	0	0	0	2	2248
2	84267	34	12	17	0	0	0	2	84332
3	814126	81	66	44	1	22	18	50	814408
Total	900616	124	86	67	1	22	18	54	900988

The table reveals a significant skew toward the trivial group $\{e\}$ (labeled $\mathcal{L}_0$), reflecting that 900616 out of 900988 points possess this automorphism group up to height 3. Minority groups such as D_4 ($\mathcal{L}_3$) and A_4 ($\mathcal{L}_7$) appear less frequently, consistent with their rarity in cubic rational functions. Notably, the data for height 4 is incomplete in this summary, but earlier computations indicate a total of 350679 distinct moduli points across all heights considered, suggesting substantial conjugacy among the functions. For instance, multiple entries with $\mathrm{Aut}(\phi) \cong D_4$ are isomorphic to the canonical form $\phi(z) = \frac{1}{z^3}$, reducing the number of unique equivalence classes.

This database provides a robust foundation for subsequent analysis, including the machine learning classification of automorphism groups discussed in Section 8. The inclusion of both projective and absolute invariants, alongside height metrics and group labels, enables a detailed exploration of the geometric and arithmetic properties of Rat_3^1.

8. Classification of Automorphism Groups of Rational Functions Using Machine Learning

In this study, we aim to classify the automorphism groups of rational functions using machine learning techniques, drawing inspiration from graded neural networks as detailed in [16] and [10]. Our investigation proceeds in two phases: an initial model employing the coefficients of rational functions as input features, followed by a refined approach using invariants to enhance classification performance, particularly for minority classes. The dataset exhibits extreme class imbalance, with the distribution of automorphism groups shown in Table 4, where the dominant class, denoted '{e}' (previously group 6), constitutes 99.84% of the samples, while minority classes are significantly underrepresented.

Automorphism Group	Proportion of Data	Samples in Test Set
{e} (6)	99.84%	179,910
C2-2 (2)	0.12%	214
C2-1 (1)	0.02%	29
D4 (3)	0.01%	16
V4-1 (4)	0.0075%	17
A4 (0)	0.0067%	12
5	0.0002%	2

Table 4. Class distribution in the dataset, with group labels mapped to notation used in subsequent tables.

8.1. Initial Model: Using Coefficients as Features.

8.1. Initial Model: Using Coefficients as Features. We formulate the classification of automorphism groups as a supervised learning problem, where the input features are the coefficients of rational functions, and the target variable is the corresponding automorphism group. A Random Forest classifier with 100 estimators is trained on these features, achieving an overall accuracy of 99.97%. However, this high accuracy is deceptive due to the dataset's skewness, which favors the majority class '{e}'. The performance metrics, detailed in Table 5, reveal that while '{e}' is perfectly classified (precision, recall, and F1-score of 1.00), minority classes suffer from low recall—e.g., 0.14 for 'C2-1' and 0.41 for 'V4-1'—indicating frequent misclassification into the dominant class.

8.1.1. *Addressing Class Imbalance.* To counteract the pronounced class imbalance, we apply class weighting to the Random Forest classifier, a technique designed to prioritize minority classes during training. The weight for each class i is computed as

Class	Precision	Recall	F1-score
A4 (0)	1.00	0.58	0.74
C2-1 (1)	1.00	0.14	0.24
C2-2 (2)	0.97	0.91	0.94
D4 (3)	1.00	0.94	0.97
V4-1 (4)	0.78	0.41	0.54
{e} (6)	1.00	1.00	1.00

Table 5. Performance metrics for the initial model using coefficients as features.

$$w_i = \frac{N}{C \times n_i},$$

where N is the total number of samples, C is the number of classes, and n_i is the number of samples in class i. This adjustment assigns higher weights to underrepresented classes, inversely proportional to their frequency, aiming to balance their influence in the model. After retraining with these weights, the overall accuracy remains high at 99.96%, and the updated performance metrics are presented in Table 6. Notably, recall improves for some minority classes—e.g., 'A4' increases from 0.58 to 0.83—but remains inadequate for others, with 'C2-1' dropping to 0.10 and 'V4-1' to 0.29. These results suggest that while class weighting mitigates some effects of imbalance, it alone cannot fully compensate for the limitations of using coefficients as features.

Class	Precision	Recall	F1-score
A4 (0)	1.00	0.83	0.91
C2-1 (1)	1.00	0.10	0.19
C2-2 (2)	0.98	0.90	0.94
D4 (3)	1.00	0.94	0.97
V4-1 (4)	0.71	0.29	0.42
{e} (6)	1.00	1.00	1.00

Table 6. Performance metrics after applying class weighting to the coefficient-based model.

8.2. **Using Invariants as Input Features.** Given the shortcomings of the coefficient-based approach, particularly its struggle with minority classes despite class weighting, we propose a refined model using invariants—denoted `value[1]` in the dataset—as input features. These invariants, derived quantities unchanged under specific transformations, are hypothesized to encapsulate the structural properties of rational functions more effectively, potentially enhancing classification accuracy across all classes, with a pronounced benefit for minority groups. To test this, we replace the coefficients with invariants and train a new Random Forest classifier, configured with 100 estimators and employing the same class weighting strategy as above.

The dataset is partitioned into training and testing sets consistently with the initial model, and performance is evaluated using precision, recall, and F1-score, focusing on minority classes such as 'A4', 'C2-1', 'C2-2', 'D4', and 'V4-1'. The results, shown in Table 7, demonstrate a substantial improvement, with an overall accuracy of

$$\text{Accuracy} = 0.9999223076837701 \approx 99.992\%.$$

Class	Precision	Recall	F1-score	Support
A4 (0)	1.00	1.00	1.00	12
C2-1 (1)	1.00	0.93	0.96	29
C2-2 (2)	1.00	0.95	0.98	214
D4 (3)	1.00	1.00	1.00	16
V4-1 (4)	0.89	1.00	0.94	17
{e} (6)	1.00	1.00	1.00	179,910
Accuracy			1.00	180,198
Macro avg	0.98	0.98	0.98	180,198
Weighted avg	1.00	1.00	1.00	180,198

TABLE 7. Performance metrics using invariants as input features with class weighting.

The recall for minority classes improves dramatically: 'C2-1' rises from 0.10 to 0.93, 'C2-2' from 0.90 to 0.95, and 'A4', 'D4', and 'V4-1' achieve perfect recall of 1.00. The majority class '{e}' retains perfect classification, with 'V4-1' showing a slight precision drop to 0.89 due to minor false positives. These outcomes validate our hypothesis, demonstrating that invariants offer a superior feature set for distinguishing rare automorphism groups, significantly outperforming the coefficient-based model.

In conclusion, while the initial model using coefficients excels for the majority class, its performance on minority classes remains limited, even with class weighting. By contrast, the invariant-based model, bolstered by the same weighting strategy, achieves near-perfect accuracy and robust recall across all classes, highlighting the efficacy of invariants in capturing essential properties of rational functions and addressing class imbalance effectively.

9. Conclusions and Further Directions

In this paper, we have explored the application of machine learning techniques to study degree 3 rational functions on the projective line, with a focus on their moduli space $\mathcal{M}_3^1$, which we identify as the weighted projective space $\mathbb{P}^5_\omega(\mathbb{Q})$ with weights $\omega = (2, 2, 3, 3, 4, 6)$. We constructed a dataset $\mathcal{P}_3^4$ comprising 2,078,697 rational functions over $\mathbb{Q}$ with naive height bounded by 4, and employed supervised learning methods to classify their automorphism groups. This endeavor has provided a computational lens through which to examine the symmetry properties inherent in these functions.

The theoretical framework, developed in Sections 2 through 4, rests on the invariants $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5$, which serve as coordinates in $\mathbb{P}^5_\omega(\mathbb{Q})$, and the identification of automorphism loci $\mathcal{L}_1, \ldots, \mathcal{L}_7$, corresponding to the finite subgroups C_2, C_3, C_4, V_4, D_4, and A_4 of $\mathrm{PGL}_2(\mathbb{Q})$. These loci, defined by explicit equations in the invariants, parameterize rational functions with non-trivial automorphisms. This study demonstrates the application of machine learning techniques to classify these automorphism groups, leveraging the weighted projective space structure.

In Section 8, we translated this algebraic structure into a practical computational approach. Initial attempts using a Random Forest classifier directly on the coefficients of rational functions achieved high overall accuracy but performed poorly on minority classes due to extreme class imbalance. By incorporating class weighting, we improved the recall for some minority classes, but limitations persisted. However, when we switched to using the invariants $\xi_0, \ldots, \xi_5$ as input features, the Random Forest classifier achieved a substantial improvement, with an overall accuracy of approximately 99.992% and near-perfect classification across all classes, including the minority ones. This highlights the efficacy of using invariants in capturing the essential properties of rational functions for classification tasks.

A critical observation emerges from our experiments: machine learning models applied directly to the coefficients of rational functions perform inadequately in classifying $\mathrm{Aut}(\phi)$, whereas models operating on the invariants excel. This disparity underscores the importance of the invariant space $\mathcal{W}_3$, where each point uniquely represents an isomorphism class, compared to the raw coefficients under the $\mathrm{PGL}_2(\mathbb{Q})$-action. The success of the invariant-based approach motivates further exploration of neuro-symbolic methods, where a symbolic layer computes invariants before classification, potentially enhancing performance while retaining interpretability. For higher degrees, where the generators of the ring of invariants $\mathcal{R}_{(d+1,d-1)}$ are not fully known, this approach could be adapted once such invariants are determined or approximated.

This work opens several avenues for future research. First, extending the dataset to include rational functions of higher heights or over different fields could provide deeper insights into the distribution of automorphism groups and the geometry of $\mathcal{M}_d^1$. Second, developing neuro-symbolic models that integrate symbolic computation of invariants with machine learning could offer a powerful tool for studying rational functions of higher degrees. Finally, addressing the challenge of determining invariants for $d > 3$ remains a crucial step for scaling this framework, potentially enabling the classification of automorphism groups where traditional algebraic methods are intractable.

In summary, this integration of machine learning with arithmetic dynamics provides a practical means to classify rational functions and enhances our understanding of their dynamical properties across $\mathcal{M}_d^1$. By offering a computationally efficient alternative to traditional methods, particularly for higher degrees where automorphism groups defy complete classification, this work suggests a pathway to combine data-driven insights in arithmetic dynamics and could possibly even be adopted for $\mathcal{M}_d^2$.

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