POLYNOMIALS, GALOIS GROUPS, AND DEEP LEARNING

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Abstract. This paper introduces a novel approach to understanding Galois theory, one of the foundational areas of algebra, through the lens of machine learning. By analyzing polynomial equations with machine learning techniques, we aim to streamline the process of determining solvability by radicals and explore broader applications within Galois theory. This summary encapsulates the background, methodology, potential applications, and challenges of using data science in Galois theory.

1. Introduction

Galois theory, a cornerstone of modern algebra, provides profound insights into the solvability of polynomial equations. Since its inception by Évariste Galois, it has explained why there are no general formulas for polynomials of degree five or higher by radicals, unlike the well-known quadratic, cubic, and quartic formulas. This theory links the algebraic structure of field extensions to the symmetry of polynomial roots, encapsulated by their Galois groups. While traditional methods allow us to determine solvability for lower-degree polynomials through invariants like discriminants, the complexity escalates dramatically for higher degrees, where the Galois group might not be solvable, leading to no radical solution.

This project embarks on an innovative journey to merge the abstract realm of Galois theory with the practical capabilities of machine learning (ML). Our goal is to harness ML's pattern recognition and prediction abilities to address some of the most challenging aspects of Galois theory, potentially revolutionizing our understanding and approach to polynomial solvability and related problems. At the heart of Galois theory is the connection between a polynomial's roots and its Galois group, which describes how these roots can be permuted while preserving the field operations. A polynomial is solvable by radicals if its Galois group is solvable; this means there exists a chain of normal subgroups where each quotient is cyclic, allowing for the roots to be constructed by sequential additions, multiplications, and root extractions. However, for degrees five and above, generic polynomials often have non-solvable groups like S_n (the symmetric group), rendering them unsolvable by radicals.

We propose an approach where we compile or generate datasets of polynomials with known Galois groups. Key to our approach will be identifying or creating features from polynomials that are indicative of Galois group properties or solvability. These might include traditional invariants like discriminants or novel features derived from root distributions or algebraic properties. Using supervised learning, we aim to predict the Galois group or solvability of polynomials, potentially employing neural networks for their ability to handle complex patterns or decision trees for interpretability. Unsupervised methods could explore clustering of polynomials, perhaps revealing new mathematical insights. By learning from simpler polynomials, we hope to generalize these insights to more complex polynomials, possibly using techniques like transfer learning where models adapt knowledge from one task to another.

This integration could lead to automated solvability prediction, offering mathematicians tools to quickly assess if a polynomial can be solved by radicals, and might uncover patterns or invariants not yet recognized by traditional mathematics. The methodology could extend to other areas like field theory or algebraic geometry. However, several challenges loom, including the computational cost of handling high-degree polynomials, ensuring interpretability of ML models to enhance theoretical understanding, and balancing between providing practical tools and contributing to the theoretical body of Galois theory.

This project stands at the intersection of pure mathematics and cutting-edge computational science. By leveraging machine learning, we aim not only to solve practical problems within Galois theory but also to catalyze new theoretical advancements. This exploration could redefine how we approach some of the oldest and most

Date: January 22, 2025.

fundamental questions in algebra, potentially opening new avenues for research in both mathematics and computer science.

A neuro-symbolic network is a type of artificial intelligence system that combines the strengths of neural networks (good at pattern recognition) with symbolic reasoning (based on logic and rules) to create models that can both learn from data and reason through complex situations, essentially mimicking human-like cognitive abilities by understanding and manipulating symbols to make decisions. This approach aims to overcome the limitations of either method alone, providing better explainability and adaptability in AI systems. In this paper, we experiment with such models to study some classical questions of Galois theory.

The paper proceeds as follows. In the second section, we cover basic terminology on polynomials, including their heights and weighted polynomials. Since the intended audience of this paper includes engineers and computer scientists, we provide some basic definitions and terminology that are normally found in every basic graduate algebra book.

Since this paper primarily deals with databases of polynomials with integer coefficients, in section three, we discuss the equivalence classes of polynomials, including $\mathbb{Z}$-equivalence, $\mathrm{GL}_2(\mathbb{Z})$-equivalence, Tschirnhaus equivalence, Hermite equivalence, and Julia equivalence. A detailed account of these topics can be found in [4].

Our data is ordered by height, whether that is the height of the polynomials or the weighted moduli height. Most open questions and arithmetic considerations are related to the heights of polynomials. Section four covers the basic definitions of the theory of heights. In section five, we discuss binary forms in detail and provide the generators for the ring of invariants of binary forms for degrees up to ten.

The basic foundation of Galois groups of polynomials over $\mathbb{Q}$ is discussed in section six. We cover in detail the solution of cubics, quartics, and quintics not only to put things in proper context but also to emphasize that each degree is different. There is no universal method in Galois theory that works for every degree, which strongly suggests that AI models should be tailored specifically for each degree. This indicates that neuro-symbolic networks might be the best approach for designing models which not only predict the Galois group but also aim to derive solution formulas by radicals (when the group is solvable) and express these formulas in terms of invariants.

In section seven, we describe some general methods for determining the Galois group of a higher-degree polynomial, namely listing transitive subgroups of the symmetric group S_n, reducing polynomials modulo primes, and identifying special classes of polynomials based on the number of non-real roots.

Section eight is the core of the paper and delves into how to create databases of polynomials, providing a glimpse into how quickly computations can escalate. We detail how we build databases for cubics, quartics, and quintics and uncover some surprising trends even for such small degree polynomials where the theory is well-known. For instance, we find how rare it is for the cyclic group C_n to be the Galois group of a degree n polynomial. For example, among roughly 20^6 quintic polynomials of height ≤ 10, only three (up to $\overline{\mathbb{Q}}$-isomorphism) have a Galois group isomorphic to C_5, with a total of 20 polynomials (counting twists) corresponding to these three classes. Training an AI model to identify such rare cases might indeed be an impossible task, as noted in Section eight. Our data could serve various purposes, such as checking Malle's conjecture on Galois groups, verifying results by Bhargava et al. on the number of quartics with bounded heights, or comparing the height of polynomials with the weighted height of invariants.

In section nine, we offer a glimpse of what a neuro-symbolic network might look like for this application. This is not a fully developed product yet, as it could be refined with many symbolic layers based on theoretical knowledge. However, it shows that for small degrees, it can work relatively well. While there might not be a compelling reason to use AI models to predict the Galois group for degrees $d = 3, 4, 5$, this approach could prove very useful for higher degrees.

We hope this paper will encourage mathematicians and computer scientists to explore the use of AI in mathematical research, particularly in tackling classical problems of mathematics. Although this is a modest attempt to incorporate such methods into Galois theory, the rapid development of Artificial Intelligence promises new and innovative applications in mathematics.

2. Preliminaries

In this section we will go over some preliminary results on polynomials. Even though we will start with the general setup of polynomials defined over number fields and their rings of integers, later in the paper we will mostly focus on $\mathbb{Q}$ and its ring of integers $\mathbb{Z}$. For any field k, $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ denote the affine and projective spaces of dimension n over k, respectively.

2.1. **Polynomials.** Let R is a commutative ring with identity. An expression of the form

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{1}$$

where $a_i \in R$ and $a_n \neq 0$, is called a **polynomial over** R with **variable** x. The elements $a_0, a_1, \ldots, a_n$ are called **coefficients** of $f(x)$. The coefficient a_n is called the **leading coefficient**. A polynomial is called **monic** if its leading coefficient is 1.

If n is the largest non negative integer for which $a_n \neq 0$, then we say that the **degree** of $f(x)$ is n and write $\deg f(x) = n$. The set of all polynomials , with coefficient in a ring R is denoted by $R[x]$. It is also a commutative ring with identity. Two **polynomials are equal** if their corresponding coefficients are equal, so if we have

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ q(x) &= b_0 + b_1 x + \cdots + b_m x^m, \end{aligned} \tag{2}$$

then $p(x) = q(x)$ if and only if $a_i = b_i$ for every $i = 0, \ldots, \max\{m, n\}$.

Let $p(x)$ and $q(x)$ be polynomials in $R[x]$, where R is a integral ring. Then,

$$\deg(p \cdot q) = \deg p + \deg q.$$

Moreover, $R[x]$ is a integral ring. If $\mathbb{F}$ is a field, then $\mathbb{F}[x]$ is a Euclidean domain with norm $N : \mathbb{F}[x] \to \mathbb{Z}^{\geq 0}$, such that $N(p(x)) = \deg(p(x))$.

Lemma 1 (Division Algorithm)**.** *Let $f(x)$ and $g(x)$ be two nonzero polynomials in $\mathbb{F}[x]$, where $\mathbb{F}$ is a field and $g(x)$ is a non-constant polynomial. Then, there exist unique polynomials $q(x), r(x) \in \mathbb{F}[x]$ such that*

$$f(x) = g(x)q(x) + r(x),$$

where $\deg r(x) < \deg g(x)$ and $r(x)$ is a nonzero polynomial.

Let $p(x)$ be a polynomial in $\mathbb{F}[x]$ and $\alpha \in F$. We say that α is a **zero** or **root** of $p(x)$, if $p(x)$ is in the kernel of the homomorphism ϕ_α or we say α is a zero of $p(x)$ if $p(\alpha) = 0$.

Corollary 1. *Let $\mathbb{F}$ be a field. An element $\alpha \in \mathbb{F}$ is a zero of $p(x) \in \mathbb{F}[x]$, if and only if $(x - \alpha)$ is a factor of $p(x)$ in $\mathbb{F}[x]$. A nonzero polynomial $p(x)$ with degree n in $\mathbb{F}[x]$ has at most n distinct zeroes in $\mathbb{F}$.*

A monic polynomial $d(x)$ is called **greatest common divisor** of polynomials $p(x), q(x) \in \mathbb{F}[x]$ if $d(x)$ divides $p(x)$ and $q(x)$; and if for every other polynomial $d'(x)$ that divides $p(x)$ and $q(x)$, $d'(x) \mid d(x)$. We write

$$d(x) = \gcd(p(x), q(x)).$$

Two polynomials $p(x)$ and $q(x)$ are **relatively prime** if $\gcd(p(x), q(x)) = 1$. Similarly as for the greatest common divisor of integers, we have the following:

Lemma 2. *Let $\mathbb{F}$ be a field and assume that $d(x)$ is the greatest common divisor of two polynomials $p(x)$ and $q(x)$ in $\mathbb{F}[x]$. Then, there exist polynomials $r(x)$ and $s(x)$ such that*

$$d(x) = r(x) \cdot p(x) + s(x) \cdot q(x).$$

Moreover, the greatest common divisor of two polynomials is unique.

A polynomial $f(x) \in \mathbb{F}[x]$ is called **irreducible** if it has degree ≥ 1 and can not be written as

$$f(x) = g(x) \cdot h(x)$$

for some $g, h \in \mathbb{F}[x]$ and both $g, h \notin \mathbb{F}$. Elements of $\mathbb{F}$ are called **constant polynomials**.

Let A be a UFD and k its field of fractions. We take $a \in k$ such that $a = \frac{r}{s}$, where $(r, s) = 1$. For any prime element $p \in A$, we can write

$$a = p^m \, a'$$

where m is an integer and $a' \in k$ such that p does not divide numerator or denominator of a'. The **order of** a **in** p is defined as m, say $\text{ord}_p(a) = m$. For $f(x) \in \mathbb{F}[x]$ given as in Eq. (1) we define

$$\text{ord}_p(f) = min\{ ord_p(a_i) \mid a_i \neq 0\}.$$

The **content** of $f(x)$, which is denoted $\text{cont}(f)$, is defined as the product (up to multiplication to a unit in A)

$$\text{cont}(f) := \prod p^{\text{ord}_p(f)}, \tag{3}$$

taking all p such that $\text{ord}_p(f) \neq 0$. If $\text{cont}(f) = 1$, then $f(x)$ is called a **primitive polynomial**. Thus, every polynomial $f(x) \in \mathbb{F}[x]$ can be written as

$$f(x) = \text{cont}(f) \cdot f_1(x),$$

where $f_1(x)$ is primitive and $f_1(x) \in A[x]$. Notice that if $f \in A[x]$ then $\text{cont}(f)$ is simply

$$\text{cont}(f) = \gcd(a_0, \ldots, a_n).$$

The **height** of $f(x)$ is defined as

$$\mathfrak{h}(f) := \max\{\text{ord}_p(a_i) \,|\, a_i \neq 0\}$$

The following result is known as Gauss' lemma.

Lemma 3 (Gauss Lemma)**.** *Let A be a UFD, k its field of fractions and $f, g \in \mathbb{F}[x]$. Then,*

$$cont(fg) = cont(f) \cdot cont(g)$$

Moreover, for $f, g \in A[x]$, fg is primitive if and only if f and g are both primitive.

2.2. **Several variables.** A polynomial with n variables is denoted by

$$f(x_1, \ldots, x_n) = \sum_{i=(i_1,\ldots,i_n)\in I} a_i x_1^{i_1} \cdots x_n^{i_n}$$

where all $a_i \in K$, $I \subset \mathbb{Z}^{\geq 0}$, and I is finite. We use lexicographic ordering to order the terms in a given polynomial, and let

$$x_1 > x_2 > \cdots > x_n.$$

While the primary goal of this paper are polynomials with one variable, we will use polynomials with several variables when we discuss invariants of binary forms.

2.3. **Weighted polynomials.** Given any integer $n \geq 1$, let $\mathbf{w} = (q_0, \ldots, q_n)$ be a vector of positive integers. Consider the polynomial ring $R = k_{\mathbf{w}}[x_0, \ldots, x_n]$ where x_i has weight q_i for $i = 0, 1, \cdots, n$.

Every polynomial is a sum of monomials $x^d = \prod x_i^{d_i}$ with weight $\sum_{i=1}^n q_i d_j$. For every $\lambda \in k^*$ and any weighted homogeneous polynomial f of degree d, we have

$$f(\lambda^{q_0} x_0, \lambda^{q_1} x_1, \ldots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \ldots, x_n).$$

A degree d binary weighted form, where $w = (q_0, q_1)$ be respectively the weights of x_0 and x_1, is given by

$$f(x_0, x_1) = \sum_{d_0, d_1} a_{d_0,d_1} x_0^{d_0} x_1^{d_1}, \quad \text{such that} \quad d_0 q_0 + d_1 q_1 = d$$

and in decreasing powers of x_0 we have

$$f(x_0, x_1) = a_{d/q_0,0} x_0^{d/q_0} + \cdots + a_{d_0,d_1} x_0^{d_0} x_1^{d_1} + \cdots + a_{0,d/q_1} x_1^{d/q_1}$$

By dividing with x_1^{d/q_1} and making a change of coordinates $X = x_0^{q_1}/x_1^{q_0}$ we get

$$f(x_0, x_1) = a_{d/q_0,0} X^{d/q_0 q_1} + \cdots + a_{d_0,d_1} X^{d_0/q_1} + \cdots + a_{0,d/q_1} = f(X) \tag{4}$$

Notice that the condition $f(P) = 0$ is well defined on $\mathbb{P}^n_{\mathbf{w},k}$.

3. Equivalences of polynomials

Two polynomial $f(x)$ and $g(x)$ are called **equivalent** if there is a nonzero scalar λ such that $f(x) = \lambda g(x)$. Hence, $f(x)$ (up to multiplication by a scalar) can be conveniently thought as a point $[a_0 : \cdots : a_n]$ in the projective space $\mathbb{P}^n$.

Since we want to identify polynomials up to multiplication by a non-zero constant it is convenient sometimes to think of them in their projective form.

3.1. **Binary forms.** Let $k[x, y]$ be the polynomial ring in two variables and V_d denote the $(d+1)$-dimensional subspace of $k[x, y]$ consisting of homogeneous polynomials

$$f(x, y) = a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_0 y^d \tag{5}$$

of degree d. Elements of V_d are called **binary forms** of degree d.

To every polynomial $f(x)$ we associate a binary form $f(x, y) = y^n f\left(\frac{x}{y}\right)$ as above, which is called the *homogenization of* $f(x)$. Conversely, every binary form $f(x, y)$ can be associated to a polynomial $f(x, 1)$, called the *dehomogenization of* $f(x, y)$.

Notice that any polynomial $f \in \mathbb{Q}[x]$ can be written as $f = \lambda g(x)$ for some $g \in \mathbb{Z}[x]$. Since $f(x)$ and $g(x) = \lambda f(x)$ have the same Galois group over $\mathbb{Q}$, it is enough to consider only polynomials in $\mathbb{Z}[x]$.

Let $\mathrm{GL}_2(\mathbb{Z})$ be the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ such that matrices have integer entries. Hence every matrix $M \in \mathrm{GL}_2(\mathbb{Z})$ has determinant $\det M = \pm 1$ and entries in $\mathbb{Z}$.

Two polynomials $f, g \in \mathbb{Z}[x]$ of degree n are called $\mathbb{Z}$**-equivalent** if $f(x) = a^n\, g(ax + b)$ for some $a = \pm 1$ and $b \in \mathbb{Z}$.

Two degree n binary forms $f, g \in \mathbb{Z}[x, y]$ are called $\mathrm{GL}_2(\mathbb{Z})$**-equivariant** if $g(x, y) = \pm f(ax + by, cx + dy)$ for some $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. Two degree n polynomials $f, g \in \mathbb{Z}[x]$ are called $\mathrm{GL}_2(\mathbb{Z})$**-equivalent** if their homogenizations are $\mathrm{GL}_2(\mathbb{Z})$-equivalent, in other words if

$$g(x) = \pm(cx + d)^n\, f\left(\frac{ax + d}{cs + d}\right), \quad \text{for some} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

$f, g \in \mathbb{Q}[x]$ are called $\mathbb{Q}$**-equivalent** if $f(x) = g\left(\frac{ax+b}{cx+d}\right)$ for $a, b, c, d \in \mathbb{Q}$.

Lemma 4. *Let $f, g \in \mathbb{Z}[x]$. If f, g are $\mathbb{Z}$-equivalent, then they are $\mathrm{GL}_2(\mathbb{Z})$-equivalent and their homogenizations are $\mathrm{GL}_2(\mathbb{Q})$-equivalent.*

Hence the $\mathrm{GL}_2(\mathbb{Q})$ orbit, is partitioned into $\mathrm{GL}_2(\mathbb{Z})$-orbits and each $\mathrm{GL}_2(\mathbb{Z})$-orbit into $\mathbb{Z}$-orbits.

3.2. **Tschirnhaus-equivalent.** f and g (monic separable and irreducible of the same degree) are Tschirnhaus-equivalent iff they have the same splitting field E and moreover, if we let P and Q be the subgroups of $G := \mathrm{Gal}\,(E/k)$ fixing a root of f and g respectively, then P and Q are conjugate in G.

3.3. **Hermite equivalence.** Let $f(x) \in \mathbb{Z}[x]$ given as in Eq. (1) and $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ its roots. Hence

$$f(x) = \sum_{i=0}^{d} a_i x^i = a_d \prod_{i=0}^{d} (x - \alpha_i)$$

To every root α_i we associate a linear form in new variables $x_1, \ldots, x_d$ via

$$\alpha_i \to \alpha_i^{d-1} x_1 + \alpha_i^{d-2} x_2 + \cdots + \alpha_i x_{d-1} + x_d$$

Then we associate to f the d-ary form

$$f \longrightarrow a_d^{d-1} \prod_{i=1}^{d} \left(\alpha_i^{d-1} x_1 + \alpha_i^{d-2} x_2 + \cdots + \alpha_i x_{d-1} + x_d\right) =: [f]$$

The d-ary form $[f]$ is called the **Hermite form associated to** f. It is easy to show that the Hermite form is given by the resultant with respect to x of $f(x)$ and $g(x) = x_1 x^{d-1} + x_2 x^{d-2} + \cdots + x_{d-1} x + x_d$, namely

$$[f] = \mathrm{Res}(f, g, x)$$

Hence, $[f](x_1, \ldots, x_d)$ is a d-ary form with integer coefficients. Moreover

$$\text{cont}([f]) = (\text{cont}(f))^{d-1}$$

Two polynomials $f, g \in \mathbb{Z}[x]$ of degree n are called **Hermite equivalent** if their corresponding Hermite forms are $\text{GL}_n(\mathbb{Z})$-equivalent.

The discriminant of a decomposable d-ary form

$$F(x_1, \ldots, x_d) = \prod_{i=1}^{d} (\alpha_{i,1}x_1 + \cdots + \alpha_{i,d}x_d)$$

is defined as

$$\Delta(F) = \left(\det\left(\alpha_{i,j}\right)_{i,j=1,\ldots,d}\right)^2$$

Here are some properties of Hermitian forms. For proofs one can check [4].

Lemma 5. *The following are true:*

(i) *The discriminant of any polynomials is the same as the discriminant of its Hermite form. In other words,* $\Delta([f]) = \Delta_f$.

(ii) *Two polynomials which are Hermite equivalent have the same discriminants.*

(iii) *Let* $f, g \in \mathbb{Z}[x]$ *be* $\text{GL}_2(\mathbb{Z})$*-equivalent polynomials. Then* f *and* g *are Hermite equivalent. Moreover, if* f *and* g *are monic and* $\mathbb{Z}$*-equivalent then they are Hermite equivalent.*

(iv) *(Hermite) There are finitely many Hermite equivalence classes of polynomials in* $\mathbb{Z}[x]$ *of a given degree and given discriminant* $\Delta \neq 0$.

3.4. **Julia equivalence.** Hermite defined the equivalence class of polynomials to develop a reduction theory for degree $d > 2$ polynomials. A reduction theory that was developed further by Julia; see [15] and [31] for more recent treatments.

Let $f(x, y) \in \mathbb{Z}[x, y]$ be a degree n binary form

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$

and suppose that $a_0 \neq 0$. Let the real roots of $f(x, y)$ be α_i, for $1 \leq i \leq r$ and the pair of complex roots β_j, $\bar{\beta}_j$ for $1 \leq j \leq s$, where $r + 2s = n$. The form can be factored as

$$f(x, 1) = \prod_{i=1}^{r} (x - \alpha_i) \cdot \prod_{i=1}^{s} (x - \beta_i)(x - \bar{\beta}_i). \tag{6}$$

The ordered pair (r, s) of numbers r and s is called the **signature** of the form f.

We associate to f the two quadratics $T_r(x, 1)$ and $S_s(x, 1)$ of degree r and s respectively given by the formulas

$$T_r(x, 1) = \sum_{i=1}^{r} t_i^2 (x - \alpha_i)^2, \quad \textit{and} \quad S_s(x, 1) = \sum_{j=1}^{s} 2u_j^2 (x - \beta_j)(x - \bar{\beta}_j), \tag{7}$$

where t_i, u_j are to be determined. For a binary form f of signature (r, s) the quadratic Q_f is defined as

$$\boxed{Q_f(x, 1) = T_r(x, 1) + S_s(x, 1).} \tag{8}$$

Let $\beta_i = a_i + b_i \cdot I$, for $i = 1, \ldots, s$.

The discriminant of Q_f is a degree 4 homogenous polynomial in $t_1, \ldots t_r$, $u_1, \ldots, u_s$. We pick values for $t_1, \ldots t_r, u_1, \ldots, u_s$ such that this discriminant is square free and minimal. Then we can use the reduction theory of quadratics (with square free, minimal discriminant) to determine the reduced form for Q_f. Define

$$\theta_T = \frac{a_0^2 \cdot \Delta_T}{t_1^2 \cdots t_r^2}, \qquad \theta_S = \frac{a_0^2 \cdot \Delta_S}{u_1^4 \cdots u_s^4} \tag{9}$$

Proposition 1. *Let $f \in V_{n,\mathbb{Q}}$ with signature (r, s) and equation as in Eq. (6). Then Q_f is a positive definite quadratic form with discriminant $\mathfrak{D}_f$ given by the formula*

$$\mathfrak{D}_f = \Delta(T_r) + \Delta(S_s) - 8 \sum_{i,j} t_i^2 u_j^2 \left((\alpha_i - a_j)^2 + b_j^2 \right). \tag{10}$$

From the above formula it can be seen that $\mathfrak{D}_f$ is expressed in terms of the root differences. Hence, $\mathfrak{D}_f$ is fixed by all the transpositions of the roots. However, it is not an invariant of the binary form. In order to get an invariant we need to fix it by all symmetries of the roots, hence by an element of order n. Indeed $\mathfrak{D}_f^n$ is an invariant of the binary form f as we will see later. We define the θ_0 of a binary form as follows

$$\theta_0(f) = \frac{a_0^2 \cdot |\mathfrak{D}_f|^{n/2}}{\prod_{i=1}^{r} t_i^2 \prod_{j=1}^{s} u_j^4}. \tag{11}$$

Notice that in order for f to be in somewhat "simpler" or "minimal" form we would like the discriminant $\mathfrak{D}_f$ to be minimal. Hence, we would like $\theta_0(f)$ to be minimal. Consider $\theta_0(t_1, \ldots, t_r, u_1, \ldots, u_s)$ as a multivariable function in the variables $t_1, \ldots, t_r, u_1, \ldots, u_s$. We would like to pick these variables such that Q_f is a reduced quadratic, hence $\mathfrak{D}_f$ is minimal. This is equivalent to $\theta_0(t_1, \ldots, t_r, u_1, \ldots, u_s)$ obtaining a minimal value.

Proposition 2. *The function $\theta_0 : \mathbb{R}^{r+s} \to \mathbb{R}$ obtains a minimum at a unique point $(\bar{t}_1, \ldots, \bar{t}_r, \bar{u}_1, \ldots, \bar{u}_s)$.*

Choosing $(\bar{t}_1, \ldots, \bar{t}_r, \bar{u}_1, \ldots, \bar{u}_s)$ that make θ_0 minimal gives a unique positive definite quadratic $Q_f(x, z)$. We call this unique quadratic $Q_f(x, z)$ for such a choice of $(\bar{t}_1, \ldots, \bar{t}_r, \bar{u}_1, \ldots, \bar{u}_s)$ the **Julia quadratic** of $f(x, z)$, denote it by $\mathcal{J}_f(x, z)$, and the quantity $\theta_f := \theta_0(\bar{t}_1, \ldots, \bar{t}_r, \bar{u}_1, \ldots, \bar{u}_s)$ the **Julia invariant**.

Lemma 6. *Consider $\mathrm{SL}_2(\mathbb{Q})$ acting on $V_{n,\mathbb{Q}}$. Then θ is an $\mathrm{SL}_2(\mathbb{Q})$- invariant and $\mathcal{J}$ is an $\mathrm{SL}_2(\mathbb{Q})$ covariant of order 2.*

Performing Julia reduction symbolically is very difficult, but a machine learning approach is used in [17] to perform Julia reduction to higher degree polynomials. Hence, our database will have irreducible polynomials $f(x) \in \mathbb{Q}[x]$ (up to the above equivalence) which are represented as polynomials in $\mathbb{Z}[x]$. There are two main issues here:

i) identifying $\mathbb{Q}$-equivalence classes of polynomials,

ii) determining a method of listing and ordering such polynomials.

The first issue can be addressed via the classical invariant theory of binary forms, which motivates the material for the rest of this section. The second issue can be addressed via heights of polynomials which is the focus of next section.

4. Heights of polynomials

Let K be a number field, $\mathcal{O}_K$ its ring of integers, and M_K the set of absolute values of K. The **(affine) multiplicative height of a polynomial** $f(x)$ is defined as

$$H_K^{\mathbb{A}}(f) = \prod_{v \in M_K} \max\left\{ 1, |f|_v^{n_v} \right\},$$

where

$$|f|_v := \max_j \left\{ |a_j|_v \right\}$$

is the **Gauss norm** for any $v \in M_K$. The **(affine) logarithmic height of** f is defined to be

$$h_K^{\mathbb{A}}(f) = h_K([1, \ldots, a_j, \ldots]_{j \in I}).$$

Hence, the affine height of a polynomial is defined to be the height of its coefficients taken as affine coordinates. The affine height sometimes is called the **naive height**.

The **(projective) multiplicative height** is

$$H_K(f) = \prod_{v \in M_K} |f|_v^{n_v} \tag{12}$$

where n_v is the completion of K_v; see [7] among other sources. The **(projective) absolute multiplicative height** is defined as

$$\begin{aligned} H : \mathbb{P}^n(\mathbb{Q}) &\to [1, \infty) \\ H(f) &= H_K(f)^{1/[K:\mathbb{Q}]}, \end{aligned}$$

Example 1. *Let $f(x) \in \mathbb{Z}[x]$ and assume that $f(x)$ is primitive. Then the projective height of $f(x)$ is simply the maximum of the absolute values of its coefficients.*

It is a consequence of Northcott's theorem that:

Lemma 7. *There are only finitely many polynomials of bounded height. In particular, for any polynomial $f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ there are only finitely many polynomials $g(x_1, \ldots, x_n) \in K[x_1, \ldots, x_b]$ such that $H_K(g) \leq H_K(f)$.*

Let $f(x_0, \ldots, x_n)$ and $g(y_0, \ldots, y_n)$ be polynomials in different variables. Then, the projective height has the following property

$$H(f \cdot g) = H(f) \cdot H(g)$$

Before considering the height of polynomials in the same variables, we will consider $|f \cdot g|_v$. The following lemma is true for the product of a finite number of polynomials.

Lemma 8 (Gauss's lemma)**.** *Let K be a number field and $f, g \in K[x_1, \ldots, x_n]$. If v is not Archimedean, then $|fg|_v = |f|_v \, |g|_v$.*

The proof can be found in [7, pg. 22]. An analogous Archimedean estimate is given by the following lemma. Gauss's lemma and the following are used to give an estimate of $H(f_1 f_2 \cdots f_r)$ in terms of $H(f_i)$ for $1 \leq i \leq r$ and $f_1, f_2, \ldots, f_r \in K[x_1, \ldots, x_n]$.

Lemma 9. *Let $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$, $f = f_1 \cdots f_r$, and $d_i = \deg(f, x_i)$. Then,*

$$\prod_{i=1}^{r} |f_i|_v \leq e^{(d_1 + \cdots + d_n)} \, |f|_v \, . \tag{13}$$

The proof of this can be found in [14, pg. 232] and uses the concept of Mahler measure which is defined as follows. Let $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$. The **Mahler measure** is

$$M(f) := \exp\left(\int_{\mathbb{T}^n} \log \left| f(e^{i\theta_1}, \ldots, e^{i\theta_n}) \right| d\mu_1 \cdots d\mu_n \right)$$

where $\mathbb{T}$ is the unit circle $\{e^{i\theta} | 0 \leq \theta \leq 2\pi\}$ equipped with the standard measure $d\mu = \frac{1}{2\pi} d\theta$. Then

$$M(fg) \;=\; M(f) \cdot M(g),$$

see [14, pg. 230] for proof.

Lemma 10. *Let K be a number field and $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$. Denote with $\deg f_j$ the total degree of f_j. Then the following are true*

(i) $H^{\mathbb{A}}(f_1 f_2 \cdots f_r) \leq N \cdot \prod_{j=1}^{r} H^{\mathbb{A}}(f_j) \leq r \cdot \max_{1\leq j\leq r} \left\{ h(f_j) + (\deg f_j + m)\log 2 \right\}$.

(ii) $H^{\mathbb{A}}(f_1 + f_2 + \cdots + f_r) \leq r \cdot \prod_{j=1}^{r} H^{\mathbb{A}}(f_j)$.

(iii) *If $f_1, \ldots, f_r \in \mathcal{O}_K[x_1, \ldots, x_n]$, then*

$$H^{\mathbb{A}}(f_1 + f_2 + \cdots + f_r) \leq r \cdot \max_j \left\{ H^{\mathbb{A}}(f_j) \right\}^{[K:\mathbb{Q}]}.$$

The converse of part (i) is known as Gelfand's inequality.

Lemma 11 (Gelfand's inequality)**.** *Let $f_1, \ldots, f_r \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$, $d_i = \deg f_i$ such that $\deg(f_1 \cdots f_r, x_i) \leq d_i$ for each $1 \leq i \leq r$. Then*

$$\prod_{i=1}^{r} H(f_i) \leq e^{(d_i + \cdots + d_n)} \cdot H(f_1 \cdots f_r).$$

5. Binary forms

Next we want to focus in detail in the space of binary forms. $\mathrm{GL}_2(k)$ acts as a natural group of automorphisms on $k[x,y]$. Denote by $f \to f^M$ this action. It is well known that $\mathrm{SL}_2(k)$ leaves a bilinear form (unique up to scalar multiples) on V_d invariant. If k is algebraically closed then $f(x,y)$ can be factored as

$$f(x,y) = (\beta_1 x - \alpha_1 y)\cdots(\beta_d x - \alpha_d y) = \prod_{1\le i\le d} \det\left(\begin{pmatrix} x & \alpha_i \\ y & \beta_i \end{pmatrix}\right) \tag{14}$$

Points with homogeneous coordinates $(\alpha_i, \beta_i) \in \mathbb{P}^1$ are called the **projective roots** of f. For $M \in \mathrm{GL}_2(k)$ we have

$$f^M(x,y) = (\det M)^d\,(\beta_1' x - \alpha_1' y)\cdots(\beta_d' x - \alpha_d' y), \quad \text{where} \quad \begin{pmatrix} \alpha_i' \\ \beta_i' \end{pmatrix} = M^{-1}\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}.$$

Consider $a_0, a_1, \ldots, a_d$ as transcendentals over k (coordinate functions on V_d). Then the coordinate ring of V_d can be identified with $k[a_0,\ldots,a_d]$. There is an action of $\mathrm{GL}_2(k)$ on $k[a_0,\ldots,a_d]$ via

$$\mathrm{GL}_2(k) \times k[a_0,\ldots,a_d] \to k[a_0,\ldots,a_d]$$
$$(M,F) \to F^M := F(f^M), \quad \text{for all } \ f \in V_d.$$

Thus for a polynomial $F \in k[a_0,\ldots,a_d]$ and $M \in \mathrm{GL}_2(k)$, define $F^M \in k[a_0,\ldots,a_d]$ as $F^M(f) := F(f^M)$, for all $f \in V_d$. Then $F^{MN} = (F^M)^N$. The homogeneous degree in $a_0,\ldots,a_d$ is called the **degree** of F, and the homogeneous degree in x,y is called the **order** of F. An **invariant** is usually referred to an $\mathrm{SL}_2(k)$-invariant on V_d. Hilbert's theorem says that the ring of invariants $\mathcal{R}_d$ is finitely generated. Thus, $\mathcal{R}_d$ is a finitely generated graded ring.

Let $\xi_0,\ldots,\xi_n$ be a minimal set of generators of $\mathcal{R}_d$ and $\deg \xi_i = q_i$. The set of degrees $(q_0,\ldots,q_n)$ is often called the **set of weights**.

Lemma 12. *Let $f, g \in V_d, M \in \mathrm{GL}_2(k), \lambda = (\det M)^{\frac{d}{2}}$. Then $f = g^M$ if and only if*

$$(\xi_0(f),\ldots\xi_i(f),\ldots,\xi_n(f)) = (\lambda^{q_0}\,\xi_0(g),\ldots,\lambda^{q_i}\,\xi_i(g),\ldots,\lambda^{q_n}\,\xi_n(g)).$$

If $k = \mathbb{Q}$ we can choose $\xi_0,\ldots,\xi_n \in \mathbb{Z}[a_0,\ldots,a_d]$ and primitive.

The theory of binary forms is quite extensive and well understood; see [18, 21] among many other places. However, the main goal of this paper is to construct a database of irreducible polynomials $f \in \mathbb{Q}[x]$ so we can study their Galois groups. Hence, we have to consider some other equivalences of polynomials.

5.1. **Proj $\mathcal{R}_d$ as a weighted projective space.** Let $\xi_0,\ldots,\xi_n$ be the generators of $\mathcal{R}_d$ with degrees $q_0,\ldots,q_n$ respectively. Since all $\xi_0,\ldots,\xi_i,\ldots,\xi_n$ are homogenous polynomials then $\mathcal{R}_d$ is a graded ring and Proj $\mathcal{R}_d$ as a weighted projective space.

Let $\mathbf{w} := (q_0,\ldots,q_n) \in \mathbb{Z}^{n+1}$ be a fixed tuple of positive integers called **weights**. Consider the action of $k^\star = k \setminus \{0\}$ on $\mathbb{A}^{n+1}(k)$ as follows

$$\lambda \star (x_0,\ldots,x_n) = (\lambda^{q_0}x_0,\ldots,\lambda^{q_n}x_n)$$

for $\lambda \in k^*$. The quotient of this action is called a **weighted projective space** and denoted by $\mathbb{WP}^n_{(q_0,\ldots,q_n)}(k)$. It is the projective variety $Proj\,(k[x_0,...,x_n])$ associated to the graded ring $k[x_0,\ldots,x_n]$ where the variable x_i has degree q_i for $i = 0,\ldots,n$. We denote greatest common divisor of $q_0,\ldots,q_n$ by $\gcd(q_0,\ldots,q_n)$. The space $\mathbb{WP}^n_w$ is called **well-formed** if

$$\gcd(q_0,\ldots,\hat{q_i},\ldots,q_n) = 1, \quad \text{for each } \ i = 0,\ldots,n.$$

We l denote a point $\mathfrak{p} \in \mathbb{WP}^n_w(k)$ by $\mathfrak{p} = [x_0 : x_1 : \cdots : x_n]$.

Let $\xi_0,\xi_1,\ldots,\xi_n$ be the generators of the ring of invariants $\mathcal{R}_d$ of degree d binary forms. A k-isomorphism class of a binary form f is determined by the point

$$\xi(f) := [\xi_0(f),\xi_1(f),\ldots,\xi_n(f)] \in \mathbb{WP}^n_{\mathbf{w}}(k).$$

Moreover, for any two forms f, and g we have that $f = g^M$ for some $M \in \mathrm{GL}_2(k)$ if and only if $\xi(f) = \lambda \star \xi(g)$, for $\lambda = (\det A)^{\frac{d}{2}}$.

5.2. **Generators of the ring of invariants.** Finding generators for the ring of invariants R_d is a classical problem of the XIX-century. Such generators are obtained in terms of transvections or root differences. Below we list the generating set of $\mathcal{R}_d$ for $d \leq 10$. We refer the reader to many classical works in the subject [8, 11, 18, 19, 21, 22, 24, 26].

While there is no easy method to determine a generating set of invariants for any $\mathcal{R}_d$, we display a minimal generating set for all $3 \leq d \leq 10$. For the rest of this section $f(x, y)$ is given as in Eq. (5) and for given binary invariants $f, g \in V_d$ the r-th transvection of f and g is denoted by $(f, g)_r$. Most of the details for each degree can be found in [11] pr [21].

5.2.1. *Cubics.* A generating set for $\mathcal{R}_3$ is $\xi = \{\xi_0\}$, where

$$\xi_0 = ((f,f)_2,(f,f)_2)_2 = -54a_0^2a_3^2 + 36a_1a_3a_0a_2 - 8a_2^3a_0 - 8a_1^3a_3 + 2a_2^2a_1^2 = 2\cdot\Delta$$

where Δ is the discriminant of the cubic.

5.2.2. *Quartics.* A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1]$ with $\mathbf{w} = (2, 3)$, where

$$\begin{aligned} \xi_0 =& (f,f)_4 = a_2^2 - 3a_1a_3 + 12a_0a_4 \\ \xi_1 =& (f,(f,f)_2)_4 = -2a_2^3 + 9a_1a_2a_3 - 27a_0a_3^2 - 27a_1^2a_4 + 72a_0a_2a_4 \end{aligned} \tag{15}$$

We discuss the case of quartics further in section 8.4. There is another set of invariants

$$\begin{aligned} T =& a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3 \\ S =& a_0a_4 - 4a_1a_3 + 3a_2^2 \end{aligned} \tag{16}$$

T is called the *catalecticant*. See [11, pg. 150] or [26] for their bracket expression. One can easily check that the discriminant Δ of the quartic is given by $\Delta = S^3 - 27T^2$

5.2.3. *Quintics.* A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1, \xi_2, \xi_3]$ with $\mathbf{w} = (4, 8, 12, 18)$, where

$$\xi_0 = (c_1,c_1)_2, \quad \xi_1 = (c_4,c_1)_2, \quad \xi_2 = (c_4,c_4)_2, \quad \xi_{18} = \tag{17}$$

for $c_1 = (f, f)_4$, $c_2 = (f, f)_2$, $c_3 = (f, c_1)_2$, $c_4 = (c_3, c_3)_2$.

5.2.4. *Sextics.* The case of sextics was studied in detail by XIX-century mathematicians (Bolza, Clebsch, et al.) when char $k = 0$ and by Igusa for char $k > 0$. Let $c_1 = (f, f)_4$, $c_3 = (f, c_1)_4$, $c_4 = (c_1, c_1)_2$. A generating set for $\mathcal{R}_6$ is $\xi = [\xi_0, \xi_1, \xi_2, \xi_3]$ with weights $\mathbf{w} = (2, 4, 6, 10)$, where

$$\xi_0 = (f,f)_6,\ \xi_1 = (c_1,c_1)_4,\ \xi_2 = (c_4,c_1)_4,\ \xi_3 = (c_4,c_3^2)_4 \tag{18}$$

Usually the invariants of binary sextics are denoted by $[J_2, J_4, J_6, J_{10}]$ with J_{10} being the discriminant of the sextic, but that is not the case here.

5.2.5. *Septics.* A generating set of $\mathcal{R}_7$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4]$ with weights $\mathbf{w} = (4, 8, 12, 12, 20)$. We define them as follows. Let

$$c_1 = (f,f)_6, \quad c_2 = (f,f)_4, \quad c_4 = (f,c_1)_2, \quad c_5 = (c_2,c_2)_4, \quad c_7 = (c_4,c_4)_4$$

$$\begin{aligned} &\xi_0 = (c_1,c_1)_2, && \xi_1 = (c_7,c_1)_2, && \xi_2 = ((c_5,c_5)_2,c_5)_4, \\ &\xi_3 = \left((c_4,c_4)_2,c_1^3\right)_6, && \xi_4 = \left([(c_2,c_5)_4]^2,(c_5,c_5)_2\right)_4. \end{aligned}$$

5.2.6. *Octavics.* A generating set of $\mathcal{R}_8$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ with weights $\mathbf{w} = (2, 3, 4, 5, 6, 7)$. We define them as follows. Let

$$c_1 = (f,f)_6, \quad c_2 = (f,c_1)_4, \quad c_3 = (f,f)_4, \quad c_5 = (c_1,c_1)_2.$$

Then the invariants are:

$$\begin{aligned} &\xi_0 = (f,f)_8, && \xi_1 = (f,c_3)_8, && \xi_2 = (c_1,c_1)_4, && \xi_3 = (c_1,c_2)_4, \\ &\xi_4 = (c_5,c_1)_4, && \xi_5 = ((c_1,c_2)_2,c_1)_4. \end{aligned}$$

5.2.7. *Nonics.* A generating set of $\mathcal{R}_9$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]$ with weights $\mathbf{w} = (4, 8, 10, 12, 12, 14, 16)$. Let

$$c_1 = (f, f)_8,\ c_2 = (f, f)_6,\ c_4 = (f, f)_2,\ c_5 = (f, c_1)_2,\ c_6 = (f, c_2)_6,$$
$$c_7 = (c_2, c_2)_4,\ c_9 = (c_5, c_5)_4,\ c_{21} = (f, c_2)_2,\ c_{25} = (c_4, c_4)_{10},\ c_{27} = (c_6^3, c_6)_3$$

$$\xi_0 = (c_1, c_1)_2, \qquad \xi_1 = (c_2, c_6^2)_6, \qquad \xi_2 = (((c_{25}, f)_6, c_{21})_5, c_2)_6\,,$$
$$\xi_3 = ((c_7, c_7)_2, c_7)_4\,, \qquad \xi_4 = (c_9, c_1^3)_6, \qquad \xi_5 = ((c_2, c_{27})_3)_6\,,$$
$$\xi_6 = \left((c_5, c_5)_2, c_1^5\right)_{10}.$$

5.2.8. *Decimics.* A generating set of $\mathcal{R}_8$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8]$ with weights $\mathbf{w} = (2, 4, 6, 6, 8, 9, 10, 14, 14)$. Let

$$c_1 = (f, f)_8, \qquad c_2 = (f, f)_6, \qquad c_5 = (f, c_1)_4, \qquad c_6 = (f, c_2)_8,$$
$$c_7 = (c_2, c_2)_6, \qquad c_8 = (c_5, c_5)_4, \qquad c_9 = (c_2, c_7)_4, \qquad c_{10} = (c_1, c_1)_2,$$
$$c_{16} = (c_5, c_5)_2, \qquad c_{19} = (c_5, c_1)_1, \qquad c_{25} = (c_7, c_7)_2$$

$$\xi_0 = (f, f)_{10}, \qquad \xi_1 = (c_1, c_1)_4, \qquad \xi_2 = (c_5, c_5)_6,$$
$$\xi_3 = (c_6, c_6)_2, \qquad \xi_4 = (c_1, c_8)_4, \qquad \xi_5 = (c_{19}, c_1^2)_8),$$
$$\xi_6 = (c_{16}, c_1^2)_8, \qquad \xi_7 = (c_{25}, c_9)_4, \qquad \xi_8 = (c_{10}^2, c_{16})_8.$$

5.3. **Root differences.** Invariants can also be expressed in terms of root differences. For example the discriminant is given by

$$\Delta(f) = \prod_{i \neq j} (\alpha_i - \alpha_j).$$

An excellent article on invariants including root differences is [19]. Multiplicities of the roots determine the stability of the binary forms via the Hilbert-Mumford criterion; see [9].

(i) If f has a root of multiplicity $r > \frac{d}{2}$ then $\xi(f) = (\xi_0, \ldots, \xi_n) = (0, \ldots, 0)$.
(ii) If d is even, then all binary forms with a root of multiplicity $\frac{d}{2}$ have the same invariants.

5.4. **Heights and moduli heights.** Next we focus on heights of binary forms and their invariants. Let K be a number field, $f \in K[x_0, \ldots, x_n]$ a homogenous polynomial of degree d We define

$$|c(d,n)|_v := \begin{cases} \binom{n+d}{n} & \text{if } v \text{ is Archimedean} \\ 1 & \text{if } v \text{ is non-Archimedean} \end{cases}$$

Lemma 13. *Let K be a number field, $f \in K[x_0, \ldots, x_n]$ a homogenous polynomial of degree d, and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \overline{K}^{n+1}$. Then,*

$$|f(\alpha)|_v \leq |c(d,n)|_v \cdot \max_j \left\{ |\alpha_j|_v \right\}^d \cdot |f|_v\,,$$

where $|c(d,n)|_v$ is $\binom{n+d}{d}$ is v is non-Archimedean and 1 otherwise. Moreover,

$$H(f(\alpha)) \leq c_0 \cdot H(\alpha)^d \cdot H(f).$$

Lemma 13 can be used to determine the height of invariants of binary forms.

Corollary 2. *Let $f \in K[x, y]$ as in Eq. (5) and $\alpha = (\alpha_0, \alpha_1) \in \overline{K}^2$. Then,*

$$H(f(\alpha)) \leq \min\left\{d + 1, 2^{d+1}\right\} \cdot H(\alpha)^d \cdot H(f).$$

5.5. **Minimal and moduli heights of forms.** Let $f(x,y)$ be a binary form and $Orb(f)$ its $GL_2(K)$-orbit in V_d. As a consequence of Northcott's theorem, there are only finitely many $f' \in Orb(f)$ such that $H(f') \leq H(f)$. Define the height of the binary form $f(x,y)$ as follows

$$\tilde{H}(f) := \min\Big\{ H(f') | f' \in Orb(f),\, H(f') \leq H(f) \Big\}$$

we want to consider the following problem. For every f let f' be the binary form such that $f' \in Orb(f)$ and $\tilde{H}(f) = H(f')$. Determine a matrix $M \in GL_2(K)$ such that $f' = f^M$.

Let $\mathcal{B}_d$ be the moduli space of degree d binary forms defined over an algebraically closed field k. Then $\mathcal{B}_d$ is a quasi-projective variety with dimension $d-3$. We denote the equivalence class of f by $\mathfrak{f} \in B_d$. The **moduli height** of $f(x,z)$ is defined as

$$\mathfrak{H}(f) = H(\mathfrak{f})$$

where $\mathfrak{f}$ is considered as a point in the projective space $\mathbb{P}^{d-3}$. A natural question would be to investigate if the minimal height $\tilde{H}(f)$ has any relation to the moduli height $\mathfrak{H}(f)$.

Let $\{I_{i,j}\}_{j=1}^{j=s}$ be a basis of $\mathcal{R}_d$. Here the subscript i denotes the degree of the homogenous polynomial $I_{i,j}$. The fixed field of invariants is the space $V_d^{GL_2(K)}$ and is generated by rational functions $t_1, \dots t_r$ where each of them is a ratio of polynomials in $I_{i,j}$ such that the combined degree of the numerator is the same as that of the denominator.

Theorem 5.1 ([32]). *Let f be a binary form. Then, For any $SL_2(k)$-invariant I_i of degree i we have that*

$$H\left(I_i(f)\right) \leq c \cdot H(f)^d \cdot H(I_i)$$

Moreover, $\mathfrak{H}(f) \leq c \cdot \tilde{H}(f)$, for some constant c.

For a given degree d the constant c of the theorem can be explicitly computed. For binary sextics i this constant is $c = 2^{28} \cdot 3^9 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 43$; see [32].

5.6. **Weighted moduli height.** For any point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathbf{w},k}$ we can assume, without loss of generality, that $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathbf{w},k}(\mathcal{O}_k)$. Let $\mathbf{w} = (q_0, \dots, q_n)$ be a set of weights and $\mathbb{P}^n_{\mathbf{w},k}$ the weighted projective space over a number field k. Let $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$ a point such that $\mathfrak{p} = [x_0, \dots, x_n]$. We define the **weighted multiplicative height** of $\mathfrak{p}$ as

$$\mathfrak{H}_{\mathrm{k}}(\mathfrak{p}) := \prod_{v \in M_k} \max\left\{ |x_0|_v^{\frac{n_v}{q_0}}, \dots, |x_n|_v^{\frac{n_v}{q_n}} \right\}. \tag{19}$$

The **absolute weighted height** of $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$ is the function $\mathfrak{H} : \mathbb{P}^n_{\mathbf{w},\overline{\mathbb{Q}}} \to [1, \infty)$,

$$\mathfrak{H}(\mathfrak{p}) = \mathfrak{H}_{\mathrm{k}}(\mathfrak{p})^{1/[k:\mathbb{Q}]}, \tag{20}$$

where $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$, for any k which contains $\mathbb{Q}(\overline{wgcd}\,(\mathfrak{p}))$. The **absolute (logarithmic) weighted height** on $\mathbb{P}^n_{\mathbf{w},\overline{\mathbb{Q}}}$ is the function $\mathfrak{s} : \mathbb{P}^n_{\mathbf{w},\overline{\mathbb{Q}}} \to [0, \infty)$

$$\mathfrak{s}(\mathfrak{p}) = \log \mathfrak{H}_{\mathrm{k}}(\mathfrak{p}) = \frac{1}{[k:\mathbb{Q}]} \mathfrak{H}_k(\mathfrak{p}).$$

where again $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$, for any k which contains $\mathbb{Q}(\overline{wgcd}\,(\mathfrak{p}))$.

Let $\mathbb{P}_{\mathbf{w},k}$ be a well-formed weighted projective space and $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}_{\mathbf{w},k}(k)$. Assume $\mathbf{x}$ normalized (i.e. $\mathrm{wgcd}_k(\mathbf{x}) = 1$). Clearly $\mathrm{wgcd}(\mathbf{x}) | \gcd(x_0, \dots, x_n)$ and therefore $\mathrm{wgcd}(\mathbf{x}) \leq \gcd(x_0, \dots, x_n)$. Let $\mathbf{x}$ be absolutely normalized. Then $\gcd(x_0, \dots, x_n) = 1$. If $\mathbf{x} = [x_0 : \dots, x_n]$ is a normalized point then by definition of the height

$$\mathfrak{H}_{\mathrm{k}}(\mathbf{x}) = \max_{i=0}^{n} \{ |x_i|^{\frac{1}{q_i}} \}$$

6. Galois groups of polynomials

Let $\mathbb{F}$ be a perfect field. For simplicity we only consider the case when char$\mathbb{F} = 0$. Let $f(x)$ be a degree $n = \deg f$ irreducible polynomial in $\mathbb{F}[x]$ which is factored as follows:

$$f(x) = (x - \alpha_1) \dots (x - \alpha_n) \tag{21}$$

in a splitting field E_f. Then, $E_f/\mathbb{F}$ is Galois because is a normal extension and separable. The group Gal $(E_f/\mathbb{F})$ is called **the Galois group** of $f(x)$ over $\mathbb{F}$ and denoted by Gal $_{\mathbb{F}}(f)$. The elements of Gal $_{\mathbb{F}}(f)$ permute roots of $f(x)$. Thus, the Galois group of polynomial has an isomorphic copy embedded in S_n, determined up to conjugacy by f. The main goal of this section is to determine Gal $_{\mathbb{F}}(f)$.

Proposition 3. *The following are true:*

(i) $\deg f \mid |G|$
(ii) *Let* $G = Gal\,_{\mathbb{F}}(f)$ *and* $H = G \cap A_n$. *Then* $H = Gal\,(E_f/\mathbb{F}(\sqrt{\Delta_f}))$. *In particular,* G *is contained in the alternating group* A_n *if and only if the discriminant* Δ_f *is a square in* $\mathbb{F}$.
(iii) *The irreducible factors of* f *in* $\mathbb{F}[x]$ *correspond to the orbits of* G. *In particular,* G *is a transitive subgroup of* S_n *if and only if* f *is irreducible.*

Proof. The first part is a basic property of the splitting field E_f. (ii) We have $\Delta_f = d_f^2$, where $d_f = \prod_{i>j}(\alpha_i - \alpha_j)$. For $g \in G$ we have $g(d_f) = \text{sgn}(g) d_f$. Thus $H = G \cap A_n$ is the stabilizer of d_f in G. But this stabilizer equals Gal $(E_f/\mathbb{F}(d_f))$. Hence the claim.
(iii) G acts transitively on the roots of each irreducible factor of f. □

Lemma 14. *The following are true:*

(1) If $\sigma \in Gal\,(E_f/\mathbb{F})$ *is a transposition then* $\sigma(\Delta_f) = -\Delta_f$.
(2) If $\sigma \in Gal\,(E_f/\mathbb{F})$ *is an even permutation then* $\sigma(\Delta_f) = \Delta_f$.
(3) $Gal\,(E_f/\mathbb{F})$ *is isomorphic to a subgroup of* A_n *if and only if* $\Delta_f \in \mathbb{F}$.

When $n = 2$ then $f(x) = a_2x^2 + a_1x + a_0$. Thus, $\Delta_f = a_1^2 - 4a_0a_2$. Hence Gal $(f) \cong A_2 = \{1\}$ if and only if Δ_f is a square.

Lemma 15. *Let* $f(x) \in \mathbb{F}[x]$ *be an irreducible polynomial of degree* $\deg f = n$. *Then* $Gal\,_{\mathbb{F}}(x)$ *is an affine invariant of* $f(x)$. *In other words,* $Gal\,(f) \cong Gal\,(g)$ *for any* $g(x) = f(ax + b)$, *for* $a, b \in \mathbb{F}$ *and* $a \neq 0$.

Let $f(x, y) \in \mathbb{F}[x, y]$ be a binary form of degree $\deg f = n$. Let $g(x) = f(x, 1)$. Can Gal (g) be characterized in terms of invariants of the binary form $f(x, y)$? From section 5.2 we know that invariants of binary forms do not change under linear substitutions. Also from lemma 15 is invariant under such substitutions. Hence, we must be able to determine Gal (g) in terms of invariants of $f(x, y)$. For the rest of this section we will see how this can be done explicitly for cubics, quartics, and quintics.

6.1. Cubics.

Let $f(x)$ be an irreducible cubic polynomial in $\mathbb{F}[x]$. From **??** we know that $[E_f : \mathbb{F}] = 3$ or 6. Hence, the Galois group Gal $_{\mathbb{F}}(f)$ is a subgroup of S_3 with order 3 or 6. Thus, Gal $_{\mathbb{F}}(f) \cong A_3$ if and only if Δ_f is a square in $\mathbb{F}$, otherwise Gal $_{\mathbb{F}}(f) \cong S_3$.

Lemma 16. *Let* $f(x) \in \mathbb{F}[x]$ *be an irreducible cubic. Then* $G = A_3$ *if and only if* $\xi_0(f) = \Delta_f$ *is a square in* $\mathbb{F}$. *Moreover, the following hold:*

(i) $\Delta_f > 0$ *if and only if* f *has three distinct real roots.*
(ii) $\Delta_f < 0$ *iff* f *has one real root and two non-real complex conjugate roots.*

Since both A_3 and S_3 are solvable, we should be able to determine formulas to give the roots of $f(x)$ in terms of radicals. These formulas are known as Cardano's formulas and we will skip them here.

Remark 1. *What we notice from the cubics is that we can determine the Galois group simply by condition on invariants. We will see next if that can be done for higher degree polynomials.*

6.2. **Quartics.** Let $f(x) \in \mathbb{F}[x]$ be an irreducible polynomial of degree 4. Then $G := \text{Gal}\,(f)$ is a transitive subgroup of S_4. Further $4 \mid |G|$, see Prop. 3. So the order of G is 4, 8, 12, or 24. It can be easily checked that transitive subgroups of S_4 of order 4, 8, 12, or 24 are isomorphic to one of the following groups

$$C_4,\ D_4,\ V_4,\ A_4,\ S_4. \tag{22}$$

Consider the normalized polynomial

$$f(x) = x^4 + ax^2 + bx + c = (x-\alpha_1)\dots(x-\alpha_4) \tag{23}$$

with $a, b, c \in \mathbb{F}$. Let $E_f = \mathbb{F}(\alpha_1, \dots, \alpha_4)$ be the splitting field of f over $\mathbb{F}$. Since f has no x^3-term, we have $\alpha_1 + \dots + \alpha_4 = 0$. We assume $\Delta_f \neq 0$, so $\alpha_1, \dots, \alpha_4$ are distinct. Let $G = \text{Gal}_{\mathbb{F}}(f)$, viewed as a subgroup of S_4 via permuting $\alpha_1, \dots, \alpha_4$.

There are 3 partitions of $\{1, \dots, 4\}$ into two pairs. S_4 permutes these 3 partitions, with kernel

$$V_4 = \{(12)(34), (13)(24), (14)(23), id\}. \tag{24}$$

Thus $S_4/V_4 \cong S_3$, the full symmetric group on these 3 partitions. Associate with these partitions the elements

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \quad \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \quad \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3 \tag{25}$$

of E_f. If $\beta_1 = \beta_2$ then $\alpha_1(\alpha_2 - \alpha_3) = \alpha_4(\alpha_2 - \alpha_3)$, a contradiction. Similarly, $\beta_1, \beta_2, \beta_3$ are 3 distinct elements. Then G acts as a subgroup of S_4 on $\alpha_1, \dots, \alpha_4$, and as the corresponding subgroup of $S_3 \cong S_4/V_4$ on $\beta_1, \dots, \beta_3$. Thus the subgroup of G fixing all β_i is $G \cap V_4$. This proves the following:

$$\begin{array}{c} E_f := \mathbb{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ \Big| \bar{G} = G \cap V_4 \\ E := \mathbb{F}(\beta_1, \beta_2, \beta_3) \\ \Big| d \\ \mathbb{F} \end{array}$$

Lemma 17. *The subgroup $G \cap V_4 \leq G$ corresponds to the subfield $\mathbb{F}(\beta_1, \beta_2, \beta_3)$, which is the splitting field over $\mathbb{F}$ of the cubic polynomial (*cubic resolvent*)*

$$g(x) = (x-\beta_1)(x-\beta_2)(x-\beta_3) = x^3 - ax^2 - 4cx + -b^2 + 4ac \tag{26}$$

The roots β_i of the cubic resolvent can be found by Cardano's formulas. The extension

$$\mathbb{F}(\alpha_1, \dots, \alpha_4)/k(\beta_1, \beta_2, \beta_3)$$

has Galois group $\leq V_4$, hence is obtained by adjoining at most two square roots to $\mathbb{F}(\beta_1, \beta_2, \beta_3)$. Moreover, $\Delta(f, x) = \Delta(g, x)$.

In general, for an irreducible quartic

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

we can first eliminate the coefficient of x^3 by the substituting x with $x - \frac{a}{4}$. In terms of the binary forms this corresponds to the transformation

$$(x, y) \to \left(x - \frac{a}{4}y, y\right)$$

and the new quartic is f^M for $M = \begin{bmatrix} 1 & -a/4 \\ 0 & 1 \end{bmatrix}$. Since $M \in \text{SL}_2(\mathbb{Q})$ then $\det M = 1$ and the invariants of f^M are the same as those of f, namely

$$\begin{aligned} \xi_0(f) &= 2a_0a_4 - \frac{a_1a_3}{2} + \frac{a_2^2}{6} \\ \xi_1(f) &= a_0a_2a_4 - \frac{3a_0a_3^2}{8} - \frac{3a_1^2a_4}{8} + \frac{a_1a_2a_3}{8} - \frac{a_2^3}{36} \end{aligned} \tag{27}$$

Moreover $g(x)$ is

$$g(x) := x^3 - bx^2 + (ac - 4d)x - a^2d + 4bd - c^2. \tag{28}$$

The discriminant of $f(x)$ is the same as the discriminant of $g(x)$ and is given below:

$$\begin{aligned}\Delta_f = &-27a^4d^2 + 18a^3bcd - 4a^3c^3 - 4a^2b^3d + a^2b^2c^2 + 144a^2bd^2 - 6a^2c^2d - 80ab^2cd \\ &+ 18abc^3 + 16b^4d - 4b^3c^2 - 192acd^2 - 128b^2d^2 + 144bc^2d - 27c^4 + 256d^3\end{aligned} \tag{29}$$

We denote by $d := [\mathbb{F}(\beta_1, \beta_2, \beta_3) : \mathbb{F}]$. Then we have the following:

Lemma 18. *The Galois group of $f(x)$ is one of the following:*

(i) $d = 1 \iff G \cong V_4$.
(ii) $d = 3 \iff G \cong A_4$.
(iii) $d = 6 \iff G \cong S_4$.
(iv) *If $d = 2$ then we have*
  a) $f(x)$ is irreducible over $F \iff G \cong D_4$
  b) $f(x)$ is reducible over $F \iff G \cong C_4$

6.2.1. *Solving quartics.* The element $(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$ is fixed by $G \cap V_4$, hence lies in $K(\beta_1, \beta_2, \beta_3)$. We find

$$-(\alpha_1 + \alpha_2)^2 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) = \beta_2 + \beta_3 \tag{30}$$

By this and symmetry we get **Ferrari's formulas**

$$\begin{aligned}\alpha_1 + \alpha_2 &= \sqrt{-\beta_2 - \beta_3} \\ \alpha_1 + \alpha_3 &= \sqrt{-\beta_1 - \beta_3} \\ \alpha_1 + \alpha_4 &= \sqrt{-\beta_1 - \beta_2}\end{aligned} \tag{31}$$

or

$$\begin{aligned}\alpha_1 &= \frac{\sqrt{-\beta_1 - \beta_2} + \sqrt{-\beta_1 - \beta_3} + \sqrt{-\beta_2 - \beta_3}}{2} \\ \alpha_2 &= \frac{-\sqrt{-\beta_1 - \beta_2} - \sqrt{-\beta_1 - \beta_3} + \sqrt{-\beta_2 - \beta_3}}{2} \\ \alpha_3 &= \frac{-\sqrt{-\beta_1 - \beta_2} + \sqrt{-\beta_1 - \beta_3} - \sqrt{-\beta_2 - \beta_3}}{2} \\ \alpha_4 &= \frac{\sqrt{-\beta_1 - \beta_2} - \sqrt{-\beta_1 - \beta_3} - \sqrt{-\beta_2 - \beta_3}}{2}\end{aligned} \tag{32}$$

This completes the case for the quartics.

6.3. **Quintics.** Now we are ready to handle quintics which has such a special case in the history of Galois theory.

Lemma 19. *Let $f(x) \in \mathbb{F}[x]$ be an irreducible quintic. Then its Galois group is one of the following C_5, D_5, $F_5 = AGL(1, 5)$, A_5, S_5.*

Proof. G is transitive, hence its 5-Sylow subgroup is isomorphic to C_5 (generated by a 5-cycle). If C_5 is not normal, then G has at least 6 of 5-Sylow subgroups; then $|G| \geq 6 \cdot 5 = 30$, hence $[S_5 : G] \leq 4$ which implies $G = S_5, A_5$. If C_5 is normal in G then G is conjugate either C_5, D_5 (dihedral group of order 10) or $F_5 = AGL(1, 5)$, the full normalizer of C_5 in S_5, of order 20 (called also the Frobenius group of order 20). □

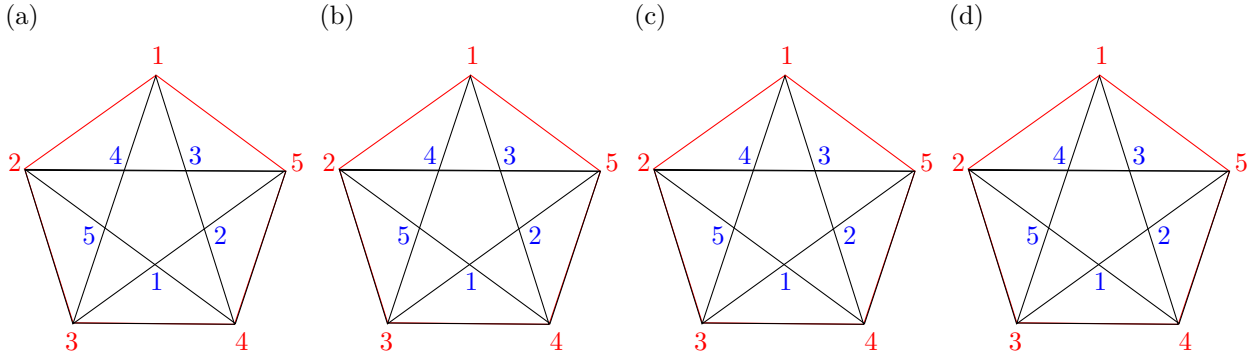
Remark 2. *If the discriminant of the quintic is a square in $\mathbb{F}$ then Gal (f) is contained in A_5. Hence, it is C_5, D_5, or A_5.*

6.3.1. *Solvable quintics.* If $G = S_5, A_5$ then the equation $f(x) = 0$ is not solvable by radicals. We want to investigate here the case G is not isomorphic to S_5 or A_5. Let $f(x)$ be an irreducible quintic in $\mathbb{F}[x]$ given by

$$f(x) = x^5 + c_4x^4 + \cdots + c_0 = (x-\alpha_1)\cdots(x-\alpha_5) \tag{33}$$

Let $G = \text{Gal}\,(f)$, viewed as a (transitive) subgroup of S_5 via permuting the (distinct) roots $\alpha_1, \cdots, \alpha_5$. As before $E_f = \mathbb{F}(\alpha_1, \cdots, \alpha_5)$ denotes the splitting field.

A 5-cycle in $S_5 = \text{Sym}(\{1, \ldots, 5\})$ corresponds to an oriented pentagon with vertices $1, \ldots, 5$. A 5-cycle and its inverse correspond to a (non-oriented) pentagon, and the full C_5 corresponds to a pentagon together with its "opposite".



Thus F_5, the normalizer of C_5 in S_5, is the subgroup permuting the pentagon and its opposite. D_5 is the subgroup of F_5 fixing the pentagon (symmetry group of the pentagon), and C_5 is the subgroup of rotations. For example, F_5 is generated by

$$F_5 = \langle \sigma, \tau \mid \sigma^5 = \tau^4 = (\sigma\tau)^4 = \sigma\sigma\tau\sigma^{-1}\tau^{-1} \rangle, \tag{34}$$

where $\sigma = (12345)$ and $\tau = (2453)$. Thus if $G \leq F_5$ then G fixes

$$\begin{aligned} \delta_1 = &(\alpha_1-\alpha_2)^2(\alpha_2-\alpha_3)^2(\alpha_3-\alpha_4)^2(\alpha_4-\alpha_5)^2(\alpha_5-\alpha_1)^2 \\ &-(\alpha_1-\alpha_3)^2(\alpha_3-\alpha_5)^2(\alpha_5-\alpha_2)^2(\alpha_2-\alpha_4)^2(\alpha_4-\alpha_1)^2 \end{aligned} \tag{35}$$

where the first (resp., second) term corresponds to the edges of the pentagon (resp., its opposite). There are six 5-Sylow subgroups of S_5 given by

$$\begin{aligned} H_1 &= \langle (1,2,3,4,5) \rangle = \{(), (1,2,3,4,5), (1,3,5,2,4), (1,4,2,5,3), (1,5,4,3,2)\} \\ H_2 &= \langle (1,2,3,5,4) \rangle = \{(), (1,2,3,5,4), (1,3,4,2,5), (1,5,2,4,3), (1,4,5,3,2)\} \\ H_3 &= \langle (1,2,4,5,3) \rangle = \{(), (1,2,4,5,3), (1,4,3,2,5), (1,5,2,3,4), (1,3,5,4,2)\} \\ H_4 &= \langle (1,2,4,3,5) \rangle = \{(), (1,2,4,3,5), (1,4,5,2,3), (1,3,2,5,4), (1,5,3,4,2)\} \\ H_5 &= \langle (1,2,5,3,4) \rangle = \{(), (1,2,5,3,4), (1,5,4,2,3), (1,3,2,4,5), (1,4,3,5,2)\} \\ H_6 &= \langle (1,3,4,5,2) \rangle = \{(), (1,3,4,5,2), (1,4,2,3,5), (1,5,2,4,3), (1,5,3,2,4)\} \end{aligned}$$

To see the full invariance properties, we need to "projectivize" and use the invariants of binary forms; see section 5.2. Let $y = 1 = \beta_i$. The generalized version of the δ_1's is $\tilde{\delta}_1$, formed by replacing $\alpha_i - \alpha_j$ by $D_{ij} = \det \begin{bmatrix} \gamma_i & \beta_i \\ \gamma_j & \beta_j \end{bmatrix}$ in the formulas defining the δ_i's. In particular,

$$\tilde{\delta}_1 = D_{12}^2D_{23}^2D_{34}^2D_{45}^2D_{51}^2 - D_{13}^2D_{35}^2D_{52}^2D_{24}^2D_{41}^2 \tag{36}$$

Since S_5 has six 5-Sylow subgroups let $\delta_1, \dots, \delta_6$ be the elements associated in this way to the six 5-Sylow's of S_5, i.e., to the six pentagon-opposite pentagon pairs on five given letters. We can write them all explicitly as

$$\begin{aligned}
\tilde{\delta}_2 &= D_{12}^2 D_{23}^2 D_{35}^2 D_{54}^2 D_{41}^2 - D_{13}^2 D_{34}^2 D_{42}^2 D_{25}^2 D_{51}^2 \\
\tilde{\delta}_3 &= D_{12}^2 D_{24}^2 D_{45}^2 D_{53}^2 D_{31}^2 - D_{14}^2 D_{43}^2 D_{32}^2 D_{25}^2 D_{51}^2 \\
\tilde{\delta}_4 &= D_{12}^2 D_{24}^2 D_{43}^2 D_{35}^2 D_{51}^2 - D_{14}^2 D_{45}^2 D_{52}^2 D_{23}^2 D_{31}^2 \\
\tilde{\delta}_5 &= D_{12}^2 D_{25}^2 D_{53}^2 D_{34}^2 D_{41}^2 - D_{15}^2 D_{54}^2 D_{42}^2 D_{23}^2 D_{31}^2 \\
\tilde{\delta}_6 &= D_{13}^2 D_{34}^2 D_{45}^2 D_{52}^2 D_{21}^2 - D_{14}^2 D_{42}^2 D_{23}^2 D_{35}^2 D_{51}^2
\end{aligned} \tag{37}$$

Lemma 20. *$\delta_i^\sigma = \delta_i$ dhe $\delta_i^\tau = \delta_i$ për $i = 1, \dots, 6$.*

Clearly, G permutes $\delta_1, \dots, \delta_6$. If G is conjugate to a subgroup of F_5, it fixes one of $\delta_1, \dots, \delta_6$; this fixed δ_i must then lie in $\mathbb{F}$. Let δ_1 as in Eq. (35) and $\delta_2, \dots, \delta_6$ as follows:

$$\begin{aligned}
\delta_2 =&(\alpha_1-\alpha_2)^2(\alpha_2-\alpha_3)^2(\alpha_3-\alpha_5)^2(\alpha_5-\alpha_4)^2(\alpha_4-\alpha_1)^2 \\
&-(\alpha_1-\alpha_3)^2(\alpha_3-\alpha_4)^2(\alpha_4-\alpha_2)^2(\alpha_2-\alpha_5)^2(\alpha_5-\alpha_1)^2 \\
\delta_3 =&(\alpha_1-\alpha_2)^2(\alpha_2-\alpha_4)^2(\alpha_4-\alpha_5)^2(\alpha_5-\alpha_3)^2(\alpha_3-\alpha_1)^2 \\
&-(\alpha_1-\alpha_4)^2(\alpha_4-\alpha_3)^2(\alpha_3-\alpha_2)^2(\alpha_2-\alpha_5)^2(\alpha_5-\alpha_1)^2 \\
\delta_4 =&(\alpha_1-\alpha_2)^2(\alpha_2-\alpha_4)^2(\alpha_4-\alpha_3)^2(\alpha_3-\alpha_5)^2(\alpha_5-\alpha_1)^2 \\
&-(\alpha_1-\alpha_4)^2(\alpha_4-\alpha_5)^2(\alpha_5-\alpha_2)^2(\alpha_2-\alpha_3)^2(\alpha_3-\alpha_1)^2 \\
\delta_5 =&(\alpha_1-\alpha_2)^2(\alpha_2-\alpha_5)^2(\alpha_5-\alpha_3)^2(\alpha_3-\alpha_4)^2(\alpha_4-\alpha_1)^2 \\
&-(\alpha_1-\alpha_5)^2(\alpha_5-\alpha_4)^2(\alpha_4-\alpha_2)^2(\alpha_2-\alpha_3)^2(\alpha_3-\alpha_1)^2 \\
\delta_6 =&(\alpha_1-\alpha_3)^2(\alpha_3-\alpha_4)^2(\alpha_4-\alpha_5)^2(\alpha_5-\alpha_2)^2(\alpha_2-\alpha_1)^2 \\
&-(\alpha_1-\alpha_4)^2(\alpha_4-\alpha_2)^2(\alpha_2-\alpha_3)^2(\alpha_3-\alpha_5)^2(\alpha_5-\alpha_1)^2
\end{aligned} \tag{38}$$

Thus, a necessary condition for the (irreducible) polynomial $f(x)$ to be solvable by radicals is that one δ_i lies in $\mathbb{F}$, i.e., that the polynomial

$$g(x) = (x-\delta_1)\cdots(x-\delta_6) \in \mathbb{F}[x] \tag{39}$$

has a root in $\mathbb{F}$. It is also sufficient:

Lemma 21. *If G fixes one δ_i then G is conjugate to a subgroup of F_5, provided that $\delta_1, \dots, \delta_6$ are all distinct.*

Proof. To check this it is enough to show that $\delta_1, \dots, \delta_6$ are mutually distinct (under the hypothesis $\Delta_f \neq 0$). hence, we have to show that $\Delta_f \neq 0 \implies \Delta_g \neq 0$. Using computational algebra we find Δ_g and verify that

$$\Delta_g = ((\alpha_1-\alpha_2)(\alpha_3-\alpha_4)(\alpha_4-\alpha_5)(\alpha_3-\alpha_5))^4 \cdot \Delta_f \cdot I_2^2 \cdot I_3 \cdot I_4^2 \cdot I_6^2$$

where I_2, I_3, I_4, and I_6 are given in [9]. Obviously $\Delta_f \neq 0$ implies that $\alpha_i - \alpha_j \neq 0$ for each $i \neq j$. This completes the proof. □

The coefficients of $g(x)$ are symmetric functions in $\alpha_1, \dots, \alpha_5$, hence are polynomial expressions in $c_0, \dots, c_4$. The goal is to find these expressions explicitly. This gives an explicit criterion to check whether $f(x) = 0$ is solvable by radicals.

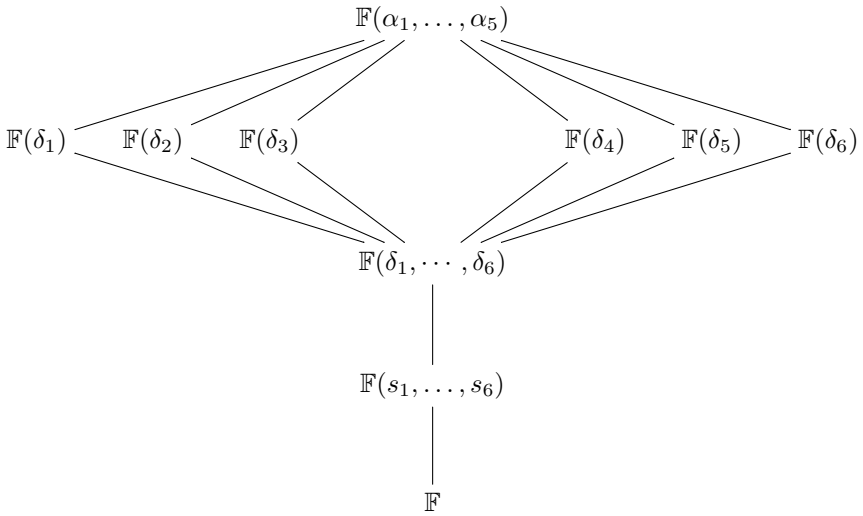
Lemma 22. *Let $s_r(x_1, \dots, x_6)$, $r = 1, \dots, 6$, be the elementary symmetric polynomials*

$$s_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \dots x_{i_r}. \tag{40}$$

Then $d_r := s_r(\tilde{\delta}_1, \dots, \tilde{\delta}_6)$ is a homogeneous polynomial expression in $b_0, \dots, b_5$ of degree $4r$. These polynomials are invariant under the action of $\mathrm{SL}_2(\mathbb{F})$ on binary quintics: For any $M \in \mathrm{SL}_2(\mathbb{F})$ the quintic f^M has the same associated d_r's.

Proof. For $\alpha_j := \gamma_j/\beta_j$ we have $\tilde{\delta}_i = (\beta_1 \cdots \beta_5)^4 \delta_i = b_5^4 \delta_i$. Thus $d_r = b_5^{4r}\, s_r(\delta_1, , \ldots, \delta_6)$. But the $s_r(\delta_1, , \ldots, \delta_6)$ are polynomial expressions in the $c_j = b_j/b_5$, for $j = 0, \ldots, 4$. Thus d_r is a rational function in $b_0, \ldots, b_5$, where the denominator is a power of b_5. Switching the roles of x and y yields that the denominator is also a power of b_0. Thus it is constant, i.e., d_r is a polynomial in $b_0, \ldots, b_5$. If we replace each β_j by $c\beta_j$ for a scalar λ then each $\tilde{\delta}_i$ gets multiplied by λ^4, so d_r gets multiplied by λ^{4r}. Thus d_r is homogeneous of degree $4r$. The rest of the claim is clear. □

(41)



There are four basic invariants of quintics, denoted by J_4, J_8, J_{12}, J_{18}, of degrees 4,8,12 and 18, such that every $\mathrm{SL}(2, \mathbb{F})$-invariant polynomial in $b_0, \ldots, b_5$ is a polynomial in J_4, J_8, J_{12}, J_{18}; see [27].

To define J_4, J_8, J_{12}, we need auxiliary quantities

$$A = \frac{1}{100}\left(20b_4 - 8b_1b_3 + 3b_2^2\right),\ B = \frac{1}{100}\left(100b_5 - 12b_1b_4 + 2b_2b_3\right),\ C = \frac{1}{100}\left(20b_1b_5 - 8b_2b_4 + 3b_3^2\right)$$

and D, E, F, G defined by

$$\begin{vmatrix} 10u + 2b_1v & 2b_1u + b_2v & b_2u + b_3v \\ 2b_1u + b_2v & b_2u + b_3v & b_3u + 2b_4v \\ b_2u + b_3v & b_3u + 2b_4v & 2b_4u + 10b_5v \end{vmatrix} = 10^3(Du^3 + Eu^2v + Fuv^2 + Gv^3)$$

Then J_2, J_8, and J_{12} are given by

$$\begin{aligned} J_4 &= 5^3(B^2 - 4AC) \\ J_8 &= 2^5 \cdot 5^6\left[2A(3EG - F^2) - B(9DG - EF) + 2C(3FD - E^2)\right] \\ J_{12} &= -2^{10} \cdot 5^9 \cdot 3^{-1}\left[4(3EG - F^2)(3FD - E^2) - (9DG - EF)^2\right] \end{aligned} \tag{42}$$

By using special quintics one gets linear equations for the coefficients expressing the d_r's in terms of J_4, J_8, J_{12}. The result is due to Berwick; see [16].

$$\begin{aligned} d_1 &= -10J_4 \\ d_2 &= 35J_4^2 + 10J_8 \\ d_3 &= -60J_4^3 - 30J_4J_8 - 10J_{12} \\ d_4 &= 55J_4^4 + 30J_4^2J_8 + 25J_8^2 + 50J_4J_{12} \\ d_5 &= -26J_4^5 - 10J_4^3J_8 - 44J_4J_8^2 - 59J_4^2J_{12} - 14J_8J_{12} \\ d_6 &= 5J_4^6 + 20J_4^2J_8^2 + 20J_4^3J_{12} + 20J_4J_8J_{12} + 25J_{12}^2 \end{aligned}$$

Lemma 23. *Let $f(x)$ be a irreducible quintic over $\mathbb{F}$ and $d_1, \ldots, d_6$ defined in terms of the coefficients of $f(x)$ as above. Then $f(x)$ is solvable by radicals if and only if $g(x) = x^6 + d_1x^5 + \cdots d_5x + d_6$ has a root in $\mathbb{F}$.*

$$
\begin{aligned}
\xi_0(f) &= -\frac{2}{625}\cdot\big(625a_0^2a_5^2 - 250a_0a_1a_4a_5 + 25a_0a_2a_3a_5 + 40a_0a_2a_4^2 - 15a_0a_3^2a_4 + 40a_1^2a_3a_5 + 9a_1^2a_4^2 \\
&\quad -15a_1a_2^2a_5 - 19a_1a_2a_3a_4 + 6a_1a_3^3 + 6a_2^3a_4 - 2a_2^2a_3^2\big) \\
\xi_1(f) &= \frac{1}{1562500}\big(125000a_0^3a_2a_3a_5^3 - 50000a_0^3a_2a_4^2a_5^2 - 75000a_0^3a_3^2a_4a_5^2 + 50000a_0^3a_3a_4^3a_5 - 8000a_0^3a_4^5 \\
&\quad -50000a_0^2a_1^2a_3a_5^3 + 20000a_0^2a_1^2a_4^2a_5^2 - 75000a_0^2a_1a_2^2a_5^3 + 55000a_0^2a_1a_2a_3a_4a_5^2 - 10000a_0^2a_1a_2a_4^3a_5 \\
&\quad +30000a_0^2a_1a_3^3a_5^2 - 30000a_0^2a_1a_3^2a_4^2a_5 + 6000a_0^2a_1a_3a_4^4 + 30000a_0^2a_2^3a_4a_5^2 - 26250a_0^2a_2^2a_3^2a_5^2 \\
&\quad - 17000a_0^2a_2^2a_3a_4^2a_5 + 4400a_0^2a_2^2a_4^4 + 19500a_0^2a_2a_3^3a_4a_5 - 4800a_0^2a_2a_3^2a_4^3 - 3375a_0^2a_3^5a_5 + 900a_0^2a_3^4a_4^2 \\
&\quad + 50000a_0a_1^3a_2a_5^3 - 10000a_0a_1^3a_3a_4a_5^2 - 30000a_0a_1^2a_2^2a_4a_5^2 - 17000a_0a_1^2a_2a_3^2a_5^2 + 26600a_0a_1^2a_2a_3a_4^2a_5 \\
&\quad - 4320a_0a_1^2a_2a_4^4 - 1800a_0a_1^2a_3^3a_4a_5 + 120a_0a_1^2a_3^2a_4^3 + 19500a_0a_1a_2^3a_3a_5^2 - 1800a_0a_1a_2^3a_4^2a_5 \\
&\quad - 11000a_0a_1a_2^2a_3^2a_4a_5 + 2120a_0a_1a_2^2a_3a_4^3 + 2325a_0a_1a_2a_3^4a_5 - 300a_0a_1a_2a_3^3a_4^2 - 45a_0a_1a_3^5a_4 \\
&\quad - 3375a_0a_2^5a_5^2 + 2325a_0a_2^4a_3a_4a_5 - 380a_0a_2^4a_4^3 - 525a_0a_2^3a_3^3a_5 + 40a_0a_2^3a_3^2a_4^2 + 15a_0a_2^2a_3^4a_4 \\
&\quad - 8000a_1^5a_5^3 + 6000a_1^4a_2a_4a_5^2 + 4400a_1^4a_3^2a_5^2 - 4320a_1^4a_3a_4^2a_5 + 864a_1^4a_4^4 - 4800a_1^3a_2^2a_3a_5^2 + 120a_1^3a_2^2a_4^2a_5 \\
&\quad + 2120a_1^3a_2a_3^2a_4a_5 - 648a_1^3a_2a_3^3a_4^3 - 380a_1^3a_3^4a_5 + 152a_1^3a_3^3a_4^2 + 900a_1^2a_2^4a_5^2 - 300a_1^2a_2^3a_3a_4a_5 \\
&\quad + 152a_1^2a_2^3a_4^3 + 40a_1^2a_2^2a_3^3a_5 + 50a_1^2a_2^2a_3^2a_4^2 - 57a_1^2a_2a_3^4a_4 + 9a_1^2a_3^6 - 45a_1a_2^5a_4a_5 + 15a_1a_2^4a_3^2a_5 \\
&\quad -57a_1a_2^4a_3a_4^2 + 37a_1a_2^3a_3^3a_4 - 6a_1a_2^2a_3^5 + 9a_2^6a_4^2 - 6a_2^5a_3^2a_4 + a_2^4a_3^4\big)
\end{aligned}
\tag{43}
$$

Extending the method of invariants becomes harder for higher degree equations. For degree six equations see [3] and [13]. We are not aware of explicit computations for degree $d \geq 6$.

7. General rules for higher degree polynomials

In this section we want to compile some general rules for computing the Galois group of a degree n irreducible polynomial. We will focus mostly on transitive subgroups of the symmetric group, which provide the candidates for the Galois groups, and the signature of each group which in most cases will determine the group.

7.1. **Transitive groups.** From the previous discussion we know that if $f(x)$ is a degree n irreducible polynomial then its Galois group Gal (f) is a transitive subgroup of S_n. Using computational group theory and GAP, we can compute list of transitive subgroups for relatively large n. These precompiled lists for every n will be our candidates for Galois groups

Here is the number of transitive subgroups for $n \leq 47$

n	# Subgroups	n	# Subgroups	n	# Subgroups	n	# Subgroups
5	5	6	16	7	7	8	50
9	34	10	45	11	8	12	301
13	9	14	63	15	104	16	1954
17	10	18	983	19	8	20	1117
21	164	22	59	23	7	24	25000
25	211	26	96	27	2392	28	1854
29	8	30	5712	31	12	33	162
34	115	35	407	36	121279	37	11
38	76	39	306	40	315842	41	10
42	9491	43	10	44	2113	45	10923

Table 1. Number of transitive subgroups of S_n for select values of n

Below we list all possible transitive subgroups for $n \leq 19$. To avoid confusion, in our databases we use the GAP Identity for every group.

Table 2. Transitive Subgroups of S_n for $n = 5, 6, 7, 11, 13, 17, 19$

n	Subgroups
5	C_5, D_5, F_5, A_5, S_5
6	$C(6) = 6 = 3[x]^2, D_6(6) = [3]^2, D(6) = S(3)[x]^2, A_4(6) = [2^2]3, F_{18}(6) = [3^2]2 = 3 \wr 2,$ $2A_4(6) = [2^3]3 = 2 \wr 3, S_4(6d) = [2^2]S(3), S_4(6c) = \frac{1}{2}[2^3]S(3), F_{18}(6) : 2 = \left[\frac{1}{2}S(3)^2\right] 2,$ $F_{36}(6) = \frac{1}{2}[S(3)^2]2, 2S_4(6) = [2^3]S(3) = 2 \wr S(3), L(6) = PSL(2,5) = A_5(6),$ $F_{36}(6) : 2 = [S(3)^2]2 = S(3) \wr 2, L(6) : 2 = PGL(2,5) = S_5(6), A_6, S_6$
7	$C(7) = 7, D(7) = 7 : 2, F_{21}(7) = 7 : 3, F_{42}(7) = 7 : 6, L(7) = L(3,2), A_7, S_7$
11	$C(11) = 11, D(11) = 11 : 2, F_{55}(11) = 11 : 5, F_{110}(11) = 11 : 10, L(11) = PSL(2,11)(11),$ $M(11), A_{11}, S_{11}$
13	$C(13) = 13, D(13) = 13 : 2, F_{39}(13) = 13 : 3, F_{52}(13) = 13 : 4, F_{78}(13) = 13 : 6,$ $F_{156}(13) = 13 : 12,$ $L(13) = PSL(3,3), A_{13}, S_{13}$
17	$C(17) = 17, D(17) = 17 : 2, F_{68}(17) = 17 : 4, F_{136}(17) = 17 : 8, F_{272}(17) = 17 : 16,$ $L(17) = PSL(2,16),\ L(17) : 2 = \mathrm{PZL}(2,16), L(17) : 4 = \mathrm{PYL}(2,16), A_{17}, S_{17}$
19	$C(19) = 19, D(19) = 19 : 2, F_{57}(19) = 19 : 3, F_{114}(19) = 19 : 6, F_{171}(19) = 19 : 9,$ $F_{342}(19) = 19 : 18, A_{19}, S_{19}$

7.2. **Reduction modulo p.** The reduction method uses the fact that every polynomial with rational coefficients can be transformed into a monic polynomial with integer coefficients without changing the splitting field. Let $f(x) \in \mathbb{Q}[x]$ be given by

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{44}$$

Let d be the common denominator of all coefficients $a_0, \cdots, a_{n-1}$. Then $g(x) := d \cdot f(\frac{x}{d})$ is a monic polynomial with integer coefficients. Clearly the splitting field of $f(x)$ is the same as the splitting field of $g(x)$. Thus, without loss of generality we can assume that $f(x) \in \mathbb{Z}[x]$ is a monic polynomial with integer coefficients.

Theorem 7.1. ***(Dedekind)*** *Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial such that $\deg f = n$, $Gal_{\mathbb{Q}}(f) = G$, and p a prime such that $p \nmid \Delta_f$. If $f_p := f(x) \mod p$ factors in $\mathbb{Z}_p[x]$ as a product of irreducible factors of degree $n_1, n_2, n_3, \cdots, n_k$, then G contains a permutation of type $(n_1)(n_2)\cdots(n_k)$*

Proof. van der Warden section 8.10 □

The Dedekind theorem can be used to determine the Galois group in many cases since the *type* of permutation in S_n determines the conjugacy class in S_n. Consider for example polynomials of degree 5. The cycle types for all groups that occur as Galois groups of quintics are given below.

	(2)	$(2)^2$	(3)	(4)	(3)(2)	(5)
S_5	10	15	20	30	20	24
A_5		15	20			24
F_5		5		10		4
D_5		5				4
C_5						4

TABLE 3. Cycle types for Galois groups of quintics

In table 4 we display the table for the type of elements in S_6. As it can be seen from the tables this method works well for degree 5 and 6. Unfortunately it does not always work for degree $d > 6$.

	()	(2)	(2)(2)	(2)(2)(2)	(3)	(3)(2)	(3)(3)	(4)	(4)(2)	(5)	(6)	$\lvert G\rvert$
S_6	1	15	45	15	40	120	40	90	90	144	120	720
A_6	1	-	45	-	40	-	40	-	90	144	-	360
S_5	1	-	15	10	-	-	20	30	-	24	20	120
$(S_3 \times S_3) \rtimes C_2$	1	6	9	6	4	12	4	-	18	-	12	72
A_5	1	-	15	-	-	-	20	-	-	24	-	60
$C_2 \times S_4$	1	3	9	7	-	-	8	6	6	-	8	48
$(C_3 \times C_3) \rtimes C_4$	1	-	9	-	4	-	4	-	18	-	-	36
$S_3 \times S_3$	1	-	9	6	4	-	4	-	-	-	12	36
S_4	1	-	3	6	-	-	8	6	-	-	-	24
S_4	1	-	9	-	-	-	8	-	6	-	-	24
$C_2 \times A_4$	1	3	3	1	-	-	8	-	-	-	8	24
$C_3 \times S_3$	1	-	-	3	4	-	4	-	-	-	6	18
A_4	1	-	3	-	-	-	8	-	-	-	-	12
D_{12}	1	-	3	4	-	-	2	-	-	-	2	12
S_3	1	-	-	3	-	-	2	-	-	-	-	6
C_6	1	-	-	1	-	-	2	-	-	-	2	6

TABLE 4. Cycle types for Galois groups of sextics

The main question here is how quickly can we find the primes which determine the signature of the group and hopefully the uniquely determine the group.

Theorem 7.2 (Chebotarev Density Theorem)**.** *Let L/K be a Galois extension and C a conjugacy class of $G = Gal\,(L/K)$. Then*

$$\{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime of } K, \mathfrak{p} \nmid \Delta_{L/K}, \sigma_{\mathfrak{p}} \in C\}$$

has density $\#C/\$G$. In particular, there always exist such primes.

7.3. **Polynomials with non-real roots.** Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n > 5$. Denote by r the number of non-real roots of $f(x)$. Since the complex conjugation permutes the roots then r is even, say $r = 2s$. By a reordering of the roots we may assume that if $f(x)$ has r non-real roots then

$$\alpha := (1,2)(3,4)\cdots(r-1,r) \in \text{Gal }(f). \tag{45}$$

Since determining the number of non-real roots can be very fast, we would like to know to what extent the number of non-real roots of $f(x)$ determines Gal (f). The complex conjugation assures that $m(G) \leq r$. The existence of α can narrow down the list of candidates for Gal (f). However, it is unlikely that the group can be determined only on this information unless p is "large" enough. In this case the number of non-real roots of $f(x)$ can almost determine the Galois group of $f(x)$, as we will see in the next section. Nevertheless, the test is worth running for all p since it is very fast and improves the algorithm overall.

7.3.1. *Polynomials of prime degree.* Next theorem determines the Galois group of a prime degree polynomial $f(x)$ with r non-real roots when the degree of $f(x)$ is large enough with respect to r; see [6]

Theorem 7.3. *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of prime degree $p \geq 5$ and $r = 2s$ be the number of non-real roots of $f(x)$. If s satisfies $s\,(s \log s + 2 \log s + 3) \leq p$, then Gal $(f) = A_p, S_p$.*

For a fixed p the above bound is not sharp as we will see below. However, the above theorem can be used successfully if s is fixed. We denote the above bound on p by

$$N(r) := [s\,(s \log s + 2 \log s + 3)] \tag{46}$$

for $r = 2s$. Hence, for a fixed number of non-real roots, for $p \geq N(r)$ the Galois group is always A_p or S_p.

Corollary 3. *Let a polynomial of prime degree p have r non-real roots. Then Gal $(f) = A_p$ or S_p if one of the following holds:*

(i) $r = 4$ *and* $p > 7$,
(ii) $r = 6$ *and* $p > 13$,
(iii) $r = 8$ *and* $p > 23$,
(iv) $r = 10$ *and* $p > 37$,

7.4. **Resolvents.** We used the *cubic resolvent* to determine the Galois group of irreducible quartics. here we will generalize this method, since it is the most efficient method in determining the Galois groups.

8. Databases of irreducible polynomials

8.1. **Datasets of irreducible polynomials.** In this section we want to create a database of irreducible polynomials $f \in \mathbb{Z}[x]$ of degree $\deg f = n$. Data will be stored in a Python dictionary. A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ will be represented by its corresponding binary form $f(x, y) = \sum_{i=0}^{n} a_i x^i y^{n-i}$. Hence our points will be points in the projective space $\mathbb{P}^n_{\mathbb{Q}}$, i.e. points with integer coordinates

$$\mathfrak{p} = [a_n : \cdots : a_0] \in \mathbb{P}^n_{\mathbb{Q}},$$

such that $\gcd(a_0, \ldots, a_n)$. Since $f(x)$ is irreducible over $\mathbb{Q}$ and of degree $\deg f = n$, then $a_n \neq 0$ and $a_0 \neq 0$. Moreover, $\Delta_f \neq 0$.

8.2. **Datasets with bounded height.** Let us now trying to generate a dataset with a bounded height h as defined in Eq. (12). We will denote the set of such polynomials by $\mathcal{P}^h_n$. In other words

$$\mathcal{P}^h_n := \left\{ [a_n : \cdots : a_0] \in \mathbb{P}^n_{\mathbb{Q}} \mid a_0 a_n \neq 0, \Delta_f \neq 0, H_{\mathbb{Q}}([a_n : \cdots : a_0]) \leq h \right\}$$

where H_Q is defined as in Eq. (12).

To ensure that the points in the database are not repeated we key the dictionary by the tuples $(a_0, \ldots, a_n)$. A dictionary in Python does not allow key duplicates, which ensures that there are no duplicates in our data. For given h, n the cardinality of $\mathcal{P}^h_n$ is bounded by

$$\#\mathcal{P}^h_n \leq 4h^2(2h+1)^{n-2}$$

The proof is a straightforward counting argument. There are more sophisticated methods to count algebraic points of bounded height on projective spaces; see for example [12] but we will work only over $\mathbb{Q}$ and our heights will be relatively small which does not allow for much redundant data.

For a degree $d \geq 3$ and height h one can use *Sagemath* and count such points as follows:

```
PP = ProjectiveSpace(d, QQ)
rational_points = PP.rational_points(h)
```

We then *normalize* the data by clearing denominators. Hence, all our data has integer coordinates. Furthermore, we keep only those polynomials which are irreducible over $\mathbb{Q}$. For every point $\mathfrak{p} = [a_n : \cdots : a_0]$ we will compute the following attributes

$$(a_0, \ldots, a_n) : [H(f), [\xi_0, \ldots, \xi_n, \Delta_f], \mathfrak{H}_{\mathrm{k}}(\mathfrak{p}), \mathrm{sig}, \mathrm{Gal}_{\mathbb{Q}}(f),]$$

where

$H(f)$	Height of $f(x)$defined in *Eq.* (12)
$[\xi_0, \ldots, \xi_n]$	Invariants defined in *section* 5.2
Δ_f	Discriminant of $f(x)$
$\mathfrak{H}_{\mathrm{k}}(\mathfrak{p})$	Weighted moduli height as in *Eq.* (19)
sig	Signature
$\mathrm{Gal}_{\mathbb{Q}}(f)$	Gap Identity of the Galois group of $f(x)$

Some of the datasets differ for different degrees. For example for quartics, we also compute the invariants T and S as defined in Eq. (15) and the j-invariant. For sextics we compute absolute invariants t_1, t_2, t_3; see [30] for details. We give a slice of the corresponding dictionary for each $d = 3, 4, 5$ which we discuss in the rest of this paper and make all datasets available at [28].

8.3. **Cubics.** As a simple first exercise we start with irreducible cubics. We create a database of all rational points $[c_0 : c_1 : c_2 : c_3]$ in $\mathbb{P}^3$ with projective height $h \leq 20$ such that

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

is an irreducible polynomial in $\mathbb{Q}[x]$. Since training a model for determining $\mathrm{Gal}\,(f)$ is trivial in this case we will focus mostly on comparing the naive height with the weighted moduli height and determining how the occurrence of A_3 happens with the increase of h.

A slice of five random elements of our Python dictionary looks like:

Key	Value
(-1, -9, -20, 1)	[20, 98, 3.1463462836, 'A3']
(20, -9, -20, 1)	[20, 1458632, 34.752530588, 'A3']
(8, 12, -20, 1)	[20, 540800, 13.5590472788, 'A3']
(1, 17, -20, 1)	[20, 243602, 22.2162222997, 'A3']
(19, -9, -19, 1)	[19, 1204352, 16.5637384397, 'A3']

where the 'key' has the coefficients of the cubic and the entries in 'values' are respectively: naive height, J_4 invariant, weighted heigh, and the Galois group.

Lemma 24. *The total number of rational points of heights in (0 , 20] is = 1 299 200. From those there are 1 178 856 irreducible polynomials and only 1328 of them have Galois group C_3. Moreover, the distribution of polynomials with Galois group C_3 with respect to their naive height is given in fig. 1.*

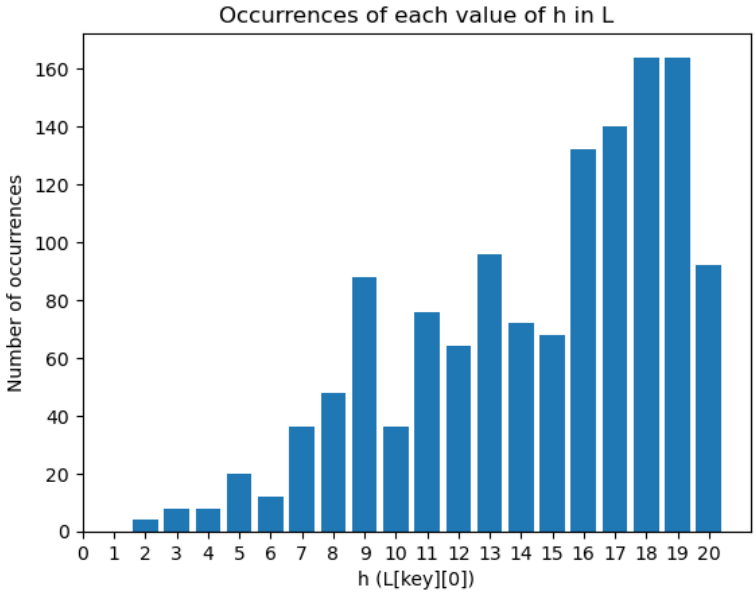


FIGURE 1. This distribution is only for cubics with Galois group C_3.

In [32] we give an estimate on the ratio of the moduli height over the naive height for binary sextics. Such bounds can be given for every degree d polynomial. In out case of cubics the minimum ratio is 0.074 for polynomial $f(x) = 7x^3 - 5x^2 - 16x + 7$ and the maximum ratio is 2.008 for $f(x) = 13x^3 - 19x^2 - 20x + 13$

Lemma 25. *There are only 40 cubics in the database with height ≤ 5 and Galois group of order 3. The discriminant Δ_f of those forty polynomials has values $\Delta_f = 7^2, 3^4, 13^2, 19^2, 31^2$, and 61^2 as shown in the table 5*

Below is the distribution of points in the database versus the invariant of cubics.

8.4. **Quartics.** We create a database of all rational points $[c_0 : c_1 : c_2 : c_3 : c_4]$ in $\mathbb{P}^3$ with projective height $h \leq 20$ such that

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

is an irreducible polynomial in $\mathbb{Q}[x]$. Other than S_4 the other possible Galois groups are C_4, D_4, V_4, and A_4 as explained in Eq. (22). We refer to Eq. (27) for its invariants. However, to avoid denominators we define

$$J_2 = 36 \cdot \xi_0, \qquad J_3 = 216 \cdot \xi_1, \qquad J_6 = \Delta(f, x)$$

TABLE 5. Irreducible degree 3 polynomials of height ≤ 5 and Galois group C_3

#	f	Δ	#	f	Δ	#	f	Δ
1	(1, 3, -4, 1)	7^2	15	(-1, -3, 0, 3)	3^4	29	(1, 2, -5, 1)	19^2
2	(-1, -4, -3, 1)	7^2	16	(1, -3, 0, 3)	3^4	30	(-1, -5, -2, 1)	19^2
3	(1, -1, -2, 1)	7^2	17	(5, 4, -5, 1)	13^2	31	(1, -5, 2, 1)	19^2
4	(1, -2, -1, 1)	7^2	18	(1, 1, -4, 1)	13^2	32	(-1, 2, 5, 1)	19^2
5	(-1, -2, 1, 1)	7^2	19	(5, -3, -2, 1)	13^2	33	(2, -1, -5, 2)	31^2
6	(-1, -1, 2, 1)	7^2	20	(-1, -4, -1, 1)	13^2	32	(2, -5, -1, 2)	31^2
7	(1, -4, 3, 1)	7^2	21	(1, -4, 1, 1)	13^2	35	(-2, -5, 1, 2)	31^2
8	(-1, 3, 4, 1)	7^2	22	(-5, -3, 2, 1)	13^2	36	(-2, -1, 5, 2)	31^2
9	(1, 0, -3, 1)	3^4	23	(-1, 1, 4, 1)	13^2	37	(3, -4, -5, 3)	61^2
10	(3, 0, -3, 1)	3^4	24	(-5, 4, 5, 1)	13^2	38	(3, -5, -4, 3)	61^2
11	(-1, -3, 0, 1)	3^4	25	(-1, -5, -4, 5)	13^2	39	(-3, -5, 4, 3)	61^2
12	(1, -3, 0, 1)	3^4	26	(1, -2, -3, 5)	13^2	40	(-3, -4, 5, 3)	61^2
13	(-3, 0, 3, 1)	3^4	27	(-1, -2, 3, 5)	13^2			
14	(-1, 0, 3, 1)	3^4	28	(1, -5, 4, 5)	13^2			

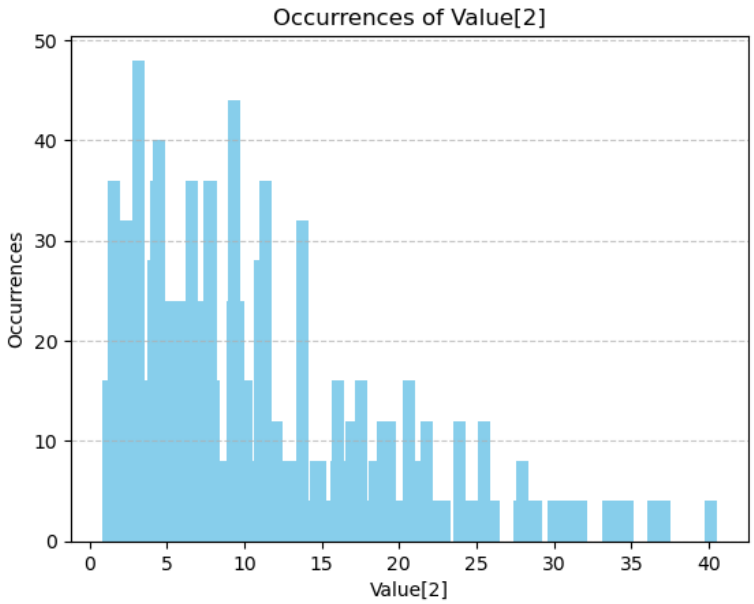


FIGURE 2. The number of occurrences versus the invariants

Hence, for a polynomial $f = [a_0, \ldots, a_4]$ we get

$$\begin{aligned}
J_2 =& 12c_0c_4 - 3c_1c_3 + c_2^2 \\
J_3 =& 72c_0c_2c_4 - 27c_0c_3^2 - 27c_1^2c_4 + 9c_1c_2c_3 - 2c_2^3 \\
J_6 =& 256c_0^3c_4^3 - 192c_0^2c_1c_3c_4^2 - 128c_0^2c_2^2c_4^2 + 144c_0^2c_2c_3^2c_4 - 27c_0^2c_3^4 + 144c_0c_1^2c_2c_4^2 - 6c_0c_1^2c_3^2c_4 - 80c_0c_1c_2^2c_3c_4 \\
& + 18c_0c_1c_2c_3^3 + 16c_0c_2^4c_4 - 4c_0c_2^3c_3^2 - 27c_1^4c_4^2 + 18c_1^3c_2c_3c_4 - 4c_1^3c_3^3 - 4c_1^2c_2^3c_4 + c_1^2c_2^2c_3^2
\end{aligned}$$

One can verify that $J_6 = \frac{1}{27}(4J_2^3 - J_3^2)$. Notice that since J_6 is the discriminant then $J_6 \neq 0$ so we also define the $\mathrm{GL}_2(\mathbb{Q})$-invariant or *j-invariant*

$$j = \frac{J_2^3}{4J_2^3 - J_3^2}$$

A slice of the database for quartics looks as follows:

Key	Value
(1, -2, -2, -2, 1)	[2, [4, -416], 4.5162, 'D(4)', -6400, -1/2700]
(-1, 2, -1, -2, 1)	[2, [1, 110], 3.23853, 'D(4)', -448, -1/12096]

TABLE 6. A slice of the database for quartics

The increase of the number of polynomials with respect to height seems very comparable to degree 3 and 4. We present this graphically in fig. 3.

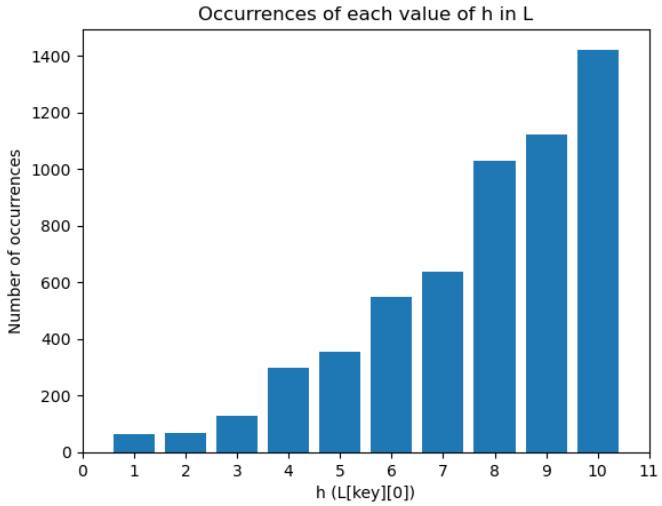


FIGURE 3. This distribution is for quartics with Galois group not isomorphic to S_4.

In [32] we give an estimate on the ratio of the moduli height over the naive height for binary sextics. Such bounds can be given for every degree d polynomial. In the case of quartics the minimum ratio is 0.2236 for the polynomial $f(x) = x^4 - 5x^3 + 10x^2 - 10x + 5$ and the maximum ratio is 3.3959 for $f(x) = x^4 - x^3 - x^2 - x + 1$. The first quartic has Galois group C_4 and the second F_5. We present the ration of the weighted height over the naive height in fig. 4

There are 5676 irreducible quartics of naive height $h \leq 10$ with Galois group not isomorphic to S_4. From those D_4: 5162 polynomials, A_4: 184 polynomials, V_4: 222 polynomials, and C_4: 108 polynomials. In fig. 3 we display how the number of such polynomials grows according to the height. The 5676 irreducible quartics are up to $\mathbb{Z}$-equivalence. However, there are only 1231 irreducible quartics up to $\mathbb{Q}$-equivalence, counted by their j-invariant.

In [5], being unaware of the weighted height, the authors define the height of a binary quartic as

$$h(f) = \max\{|J_2|^3, |J_3|^2\}$$

Of course this is what we have called the *moduli height* and it is simply the six power $\mathfrak{H}_{\mathrm{k}}(f)^6$ of the weighted height. One of the problems considered in [5] is the number of binary quadratic with bounded height. The authors give necessary and sufficient conditions for (J_2, J_3) to be invariants of an integral quartic. We verify such conditions in our database.

The case of quartics is very interesting in its own due to many connections to number theory and elliptic curves and will be the focus of a more detailed investigation in a later stage.

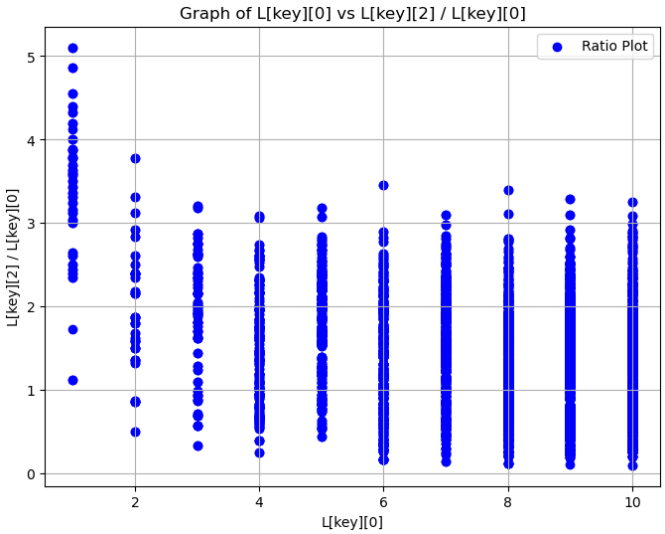


FIGURE 4. The ratio of weighted height with naive height

8.5. **Quintics.** Next we consider the irreducible quintics over $\mathbb{Q}$. Again polynomial will be identified with points $[c_0 : c_1 : c_2 : c_3 : c_4 : c_5]$ in $\mathbb{P}^4$. By lemma 19 the Galois group of an irreducible quintic is one of the following C_5, D_5, $F_5 = AGL(1,5)$, A_5, S_5. By Eq. (17) the invariants are ξ_0, ξ_1, ξ_2 of order 4, 8, 12 respectively. The expressions of such invariants in Eq. (43) suggest we use instead

$$
\begin{aligned}
J_4 =& -\frac{625}{2}\cdot\xi_0 = -625c_0^2c_5^2 + 250c_0c_1c_4c_5 - 25c_0c_2c_3c_5 - 40c_0c_2c_4^2 \\
&+ 15c_0c_3^2c_4 - 40c_1^2c_3c_5 - 9c_1^2c_4^2 + 15c_1c_2^2c_5 + 19c_1c_2c_3c_4 - 6c_1c_3^3 - 6c_2^3c_4 + 2c_2^2c_3^2 \\
J_8 =& 1562500\cdot\xi_1
\end{aligned}
\tag{47}
$$

There are two other invariants J_{12} and J_{18} whch we don't display here and there is a degree 36 homogenous polynomial $F(J_4, J_8, J_{12}, J_{18}) = 0$. This is a homogenous polynomial of degree 36 in terms of coefficients. Hence, a degree two polynomial in J_{18}. According to Dolgachev [11, pg. 152] the discriminant of the quintic is $\Delta = J_4^2 - 128J_8$.

A slice of the dictionary for quintics is given below:

Key	Value
(-2, -1, 0, -2, -2, 1)	[2, [-3264, -8152576, -29726998528], 7.55853, 'F(5) = 5:4']
(1, 0, -1, 2, -2, 1)	[2, [-539, 3599, 116197], 4.81834, 'D(5) = 5:2']
(2, -2, 2, 0, -1, 1)	[2, [-1768, 203456, 379094016], 6.48441, 'A5']

The increase of the number of polynomials with respect to height seems very comparable to degree 3 and 4. We present this graphically in fig. 5.

In [32] we give an estimate on the ratio of the moduli height over the naive height for binary sextics. Such bounds can be given for every degree d polynomial. In the case of quintics the minimum ratio is 0.5353 for the polynomial

$$f(x) = x^5 - 5x^4 + 9x^3 - 9x^2 + 4x - 1$$

and the maximum ratio is 3.7792 for

$$f(x) = x^5 - 2x^4 - 2x^3 - x - 2.$$

The first quintic has Galois group D_5 and the second F_5. We present the ration of the weighted height over the naive height in fig. 6

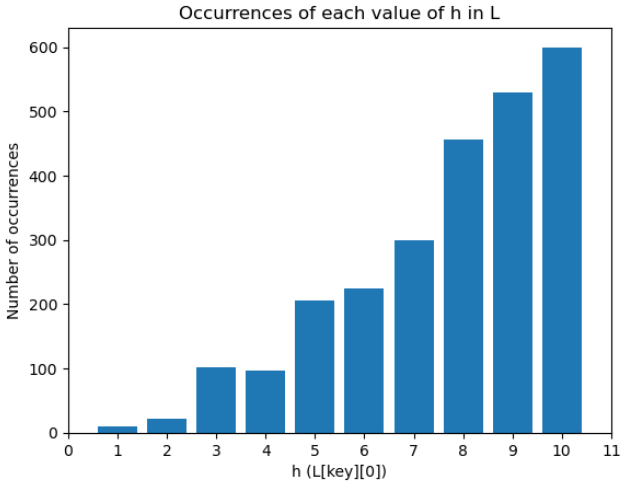


FIGURE 5. This distribution is for quintics with Galois group not isomorphic to S_5.

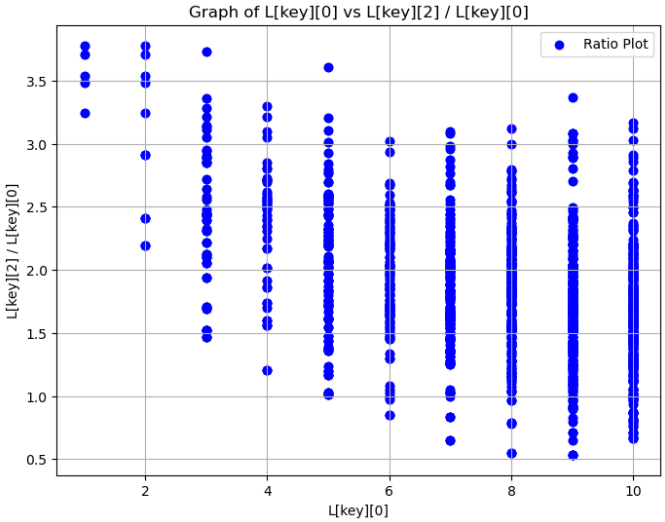


FIGURE 6. The ratio of weighted height with naive height

Lemma 26. *From all irreducible quintics in $\mathbb{Z}[x]$ with naive height ≤ 10 there are exactly 20 of them with Galois group C_5, 480 with group F_5, 900 with group D_5, and 1146 with group A_5. Moreover, all polynomials with Galois group C_5 and their invariants are listed in table 7.*

Data in table 7 shows some very interesting trends. First, There are really only 3 quintics with Galois group C_5 up to $\bar{\mathbb{Q}}$-isomorphism since they obviously have the same invariants. This once more stresses the point that the absolute invariants are really the most effective way of dealing with such databases since they considerable decrease the size of the database. Furthermore, by decreasing redundancy the learning process of any AI model becomes more efficient. Some of these issues are further illustrated and discussed in [30].

Second, the polynomials in [29] provide interesting examples of how the height of the binary form can change even for polynomials of such small height. These are very interesting examples in reduction theory; see [31] and more recently [17]

Finally, the above data emphasizes how rare such cases are. There are roughly 20^6 quintic polynomials of height ≤ 10 and from those only three (up to $\bar{\mathbb{Q}}$-isomorphism) have Galois group isomorphic to C_5. Training

Key	h	p	wh	
-1, 1, 4, -3, -3, 1	4	[4235, 4026275, -16076916075]	8.06702	C_5
-1, 3, 3, -4, -1, 1	4	[4235, 4026275, -16076916075]	8.06702	C_5
1, 3, -3, -4, 1, 1	4	[4235, 4026275, -16076916075]	8.06702	C_5
1, 1, -4, -3, 3, 1	4	[4235, 4026275, -16076916075]	8.06702	C_5
-1, -2, 5, 2, -4, 1	5	[4235, 4026275, -16076916075]	8.06702	C_5
1, 4, 2, -5, -2, 1	5	[4235, 4026275, -16076916075]	8.06702	C_5
-1, 4, -2, -5, 2, 1	5	[4235, 4026275, -16076916075]	8.06702	C_5
1, -2, -5, 2, 4, 1	5	[4235, 4026275, -16076916075]	8.06702	C_5
1, -6, 10, -1, -6, 1	10	[4235, 4026275, -16076916075]	8.06702	C_5
1, -6, -1, 10, -6, 1	10	[4235, 4026275, -16076916075]	8.06702	C_5
-1, -6, -10, -1, 6, 1	10	[4235, 4026275, -16076916075]	8.06702	C_5
-1, -6, 1, 10, 6, 1	10	[4235, 4026275, -16076916075]	8.06702	C_5
-1, 4, 9, -5, -9, 1	9	[113377, 2971552001, -47471703427379]	18.3498	C_5
-1, 9, 5, -9, -4, 1	9	[113377, 2971552001, -47471703427379]	18.3498	C_5
1, 9, -5, -9, 4, 1	9	[113377, 2971552001, -47471703427379]	18.3498	C_5
1, 4, -9, -5, 9, 1	9	[113377, 2971552001, -47471703427379]	18.3498	C_5
-1, 0, 10, 5, -10, 1	10	[109375, 2392578125, -96893310546875]	18.18568	C_5
-1, 10, -5, -10, 0, 1	10	[109375, 2392578125, -96893310546875]	18.18568	C_5
1, 10, 5, -10, 0, 1	10	[109375, 2392578125, -96893310546875]	18.18568	C_5
1, 0, -10, 5, 10, 1	10	[109375, 2392578125, -96893310546875]	18.18568	C_5

TABLE 7. The only quintics of height ≤ 10 and Galois group isomorphic to C_5

an AI model to pick such very rare cases might be an impossible task indeed. We will explore that in the next section.

9. Neuro-symbolic networks

A neuro-symbolic network is a type of artificial intelligence system that combines the strengths of neural networks (good at pattern recognition) with symbolic reasoning (based on logic and rules) to create models that can both learn from data and reason through complex situations, essentially mimicking human-like cognitive abilities by understanding and manipulating symbols to make decisions; this approach aims to overcome limitations of either method alone, providing better explainability and adaptability in AI systems. They seem to be the most reasonable choice for our approach since we can use all the theoretical knowledge that we have about polynomials and their Galois groups and somehow incorporate this into some machine learning model. The area of research on deep learning for symbolic mathematics is very active and has had a lot of activity in the ast few years; [1, 2, 10, 20, 23, 25]

9.1. **Architecture of Neuro-Symbolic Networks.** Let x represent the raw input data. The architecture of a neuro-symbolic network consists of the following key components:

9.1.1. *Input Layer and Preprocessing.* The input x is mapped to a higher-dimensional feature space z_0 through a preprocessing function $f_0 : X \to Z_0$, where X is the input space and Z_0 is the processed feature space. This step typically involves convolutional, recurrent, or embedding layers to transform raw data into structured representations.

9.1.2. *Neural Feature Extraction.* A sequence of neural transformations $f_i : Z_{i-1} \to Z_i$ for $i = 1, \dots, n$ is applied to extract features. The final output of this stage is a feature representation z_n. These transformations may include convolutional layers for spatial data, recurrent layers for temporal data, or feedforward layers for general patterns:

$$z_n = f_n \circ f_{n-1} \circ \cdots \circ f_1(z_0).$$

9.1.3. *Interface Layer.* The feature representation z_n is mapped to a symbolic representation s through an interface function $\varphi : Z_n \to S$, where S is the symbolic space. Mechanisms such as attention models or learned symbolic encoding are employed. Feedback processes can also map symbolic insights s back into neural spaces Z_n to refine feature extraction:

$$s = \varphi(z_n), \quad z_n' = \psi(s, z_n).$$

9.1.4. *Symbolic Reasoning Layer.* The symbolic representation s undergoes logical or algebraic reasoning. This layer uses symbolic inference mechanisms $R : S \to S'$, such as rule-based systems, constraint solvers, or formal logic:

$$s' = R(s),$$

where S' is the transformed symbolic space.

9.1.5. *Output Integration.* The final output y is derived by integrating the outputs from both neural and symbolic pathways. Let $\rho : S' \times Z_n \to Y$ denote the integration function, mapping the refined symbolic reasoning s' and neural feature representation z_n to the output space Y:

$$y = \rho(s', z_n).$$

9.1.6. *Dynamic Feedback Loops.* Throughout the architecture, feedback loops dynamically adjust both neural and symbolic pathways. Symbolic reasoning s' can guide neural updates, and neural features z_n may suggest new symbolic rules or hypotheses:

$$z_n \to \varphi(s) \to R(s') \to \psi(z_n').$$

This architecture integrates the strengths of neural networks for learning from high-dimensional data and symbolic methods for reasoning, interpretability, and leveraging explicit rules. Neuro-symbolic networks are particularly effective for tasks requiring both data-driven insights and rule-based decision-making. We will try to incorporate some of the above for our particular data with the main focus of determining the Galois group of polynomials. There are many other open questions that one could ask on the data as comparing height of polynomials with the weighted moduli height, classifying equivalence classes of polynomials as described in section 3, investigating Malle's conjecture, and others and for each one of these questions a neuro-symbolic network tailored to the specific question has to be designed.

9.2. **Precomputed data for every degree d.** For each degree d we precompute two lists:

- " d-grps" which is the list of transitive subgroups of S_d as explained in section 7.1
- "d-sig" which is the list of the signature for every group in " d-grps"

Such data can be computed using GAP and methods from group theory.

9.3. **Signature layer.** The first symbolic reasoning layer that we apply to our data is the *signature layer*. This layer for every point $key = (a_0, \ldots, a_d)$ creates the polynomial $f(x)$ and computes the factorization $f_p(x)$ for a list of primes p. Normally we use $p = 2, 3, 5, 7$. This signature $\text{sig}(key)$ is compared with the list of possible signatures for the degree d. The field of *groups* for this entry is updated with the list of all groups which admit this signature. If length of $L[key][groups] = 1$ then Gal (f) is uniquely determined and the training is done.

```python
from sympy import symbols, Poly, factor_list
def sig_layer(p):
    x = symbols('x')
    f = sum(a * x**i for i, a in enumerate(p))
    signature = [5]
    primes = [2, 3, 5, 7]
    for prime in primes:
        poly_mod = Poly(f, x, modulus=prime)
        factors = factor_list(poly_mod)[1]  # Get the list of factors modulo the prime
        for factor_poly, multiplicity in factors:
            degree = factor_poly.degree()
            if degree > 1 and degree not in signature:  # Avoid linear factors and duplicates
                signature.append(degree)
    return signature
```

LISTING 1. Python implementation of the `sig_layer` function.

9.4. **Real roots layer.** If the polynomial has enough real roots then from section 7.3 the group is A_d or S_d. Computing the real roots is usually easy since it can be done with numerical methods. Hence, for high enough degree d it is usually an efficient method to compute the number of the real roots of $f(x)$.

The algorithm for finding the number of real roots of a polynomial using Sturm's theorem involves constructing a Sturm sequence, which starts with the polynomial $f(x)$ and its derivative, followed by successive remainders from polynomial division, with signs reversed. The number of real roots in a given interval is determined by evaluating the sequence at the interval endpoints and counting sign changes in the resulting values. By substituting large finite values ($\pm 10^{10}$) for infinity, the method can approximate the count of real roots over the entire real line. This approach works efficiently for polynomials with integer or rational coefficients.

```python
from sympy import symbols, diff, Poly, sign

def sturm_sequence(P, x):
    P = Poly(P, x)  # Ensure P is treated as a polynomial
    sequence = [P, P.diff(x)]  # Start with P and its derivative
    while True:
        remainder = -sequence[-2].rem(sequence[-1])  # Polynomial remainder
        if remainder.is_zero:
            break
        sequence.append(remainder)
    return sequence

def count_sign_changes(sequence, value):
    evaluations = []
    for poly in sequence:
        eval_value = poly.eval(value)
        if eval_value == 0:
            evaluations.append(0)  # Consider zero explicitly
        else:
            evaluations.append(sign(eval_value))
    evaluations = [s for i, s in enumerate(evaluations) if i == 0 or s != evaluations[
        i - 1]]
    return len(evaluations) - 1

def real_root_count(P, x, interval=(-1e10, 1e10)):
    a, b = interval
    P = Poly(P.expand(), x)  # Fully expand the polynomial
    sturm_seq = sturm_sequence(P, x)
    sign_changes_a=count_sign_changes(sturm_seq,a)
    sign_changes_b=count_sign_changes(sturm_seq,b)
    return sign_changes_a - sign_changes_b
```

LISTING 2. Real Root Counting Algorithm

9.5. **Discriminant layer.** The discriminant is computed for all polynomials in the precomputed data stage, but it is not factored. This layer is activated only if the entry has as Galois group candidates which are contained or not in the alternating group A_d. Since this layer can slow down considerably the model, we only activate it as a last resort.

9.6. **Implementation and efficiency.** We implement this approach and test it for quartics and quintics databases that we created for this paper. The case of cubics is quite trivial from the point of view of Galois theory and we ignore it here. While both quartics and quintics are well understood and we don't need any AI model to find out the Galois group, they do provide nice test cases which can tell us how reasonable and efficient such approach is. We study sextics in more detail in [29].

9.6.1. *Quartics.* Let us consider first the case of quartics. We have

#	Group	TrGrId	signature
1	C4	[4, 1]	[(), (1,2,3,4), (1,3)(2,4), (1,4,3,2)]
2	C2 x C2	[4, 2]	[(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)]
3	D8	[4, 3]	[(), (2,4), (1,2)(3,4), (1,2,3,4), (1,3)(2,4)]
4	A4	[4, 4]	[(), (1,2)(3,4), (1,2,3), (1,2,4)]
5	S4	[4, 5]	[(), (1,2), (1,2)(3,4), (1,2,3), (1,2,3,4)]

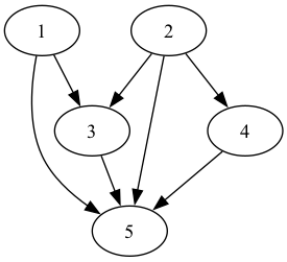


FIGURE 7. Lattice of transitive subgroups of S_4

9.6.2. *Quintics.*

#	Group	TrGrId	signature
1	C5	[5, 1]	[(), (1,2,3,4,5), (1,3,5,2,4), (1,4,2,5,3), (1,5,4,3,2)]
2	D5	[5, 2]	[(), (2,5)(3,4), (1,2,3,4,5), (1,3,5,2,4)]
3	F5	[5, 3]	[(), (2,3,5,4), (2,4,5,3), (2,5)(3,4), (1,2,3,4,5)]
4	A5	[5, 4]	[(), (1,2)(3,4), (1,2,3), (1,2,3,4,5), (1,2,3,5,4)]
5	S5	[5, 5]	[(), (1,2), (1,2)(3,4), (1,2,3), (1,2,3)(4,5), (1,2,3,4), (1,2,3,4,5)]

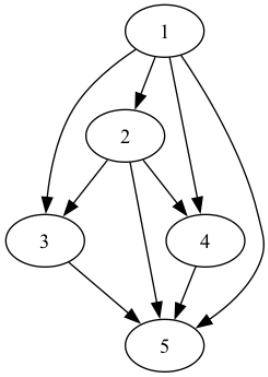


FIGURE 8. Lattice of transitive subgroups of S_5

10. Galois Network

We design a network that integrates numerical learning with symbolic reasoning to classify polynomials based on their Galois group properties. The core of this system, which we call the GaloisNetwork, processes polynomial coefficients and leverages mathematical insights to predict the corresponding group labels. This hybrid approach combines the power of deep learning with domain-specific rules, ensuring both accuracy and interpretability.

The input to the network consists of feature vectors derived from polynomial coefficients. We compute these features using mathematical invariants, such as root counts and other Galois group characteristics, creating a robust representation of each polynomial. The features are standardized to improve model performance and are then split into training and validation datasets. Labels representing Galois groups are mapped to numeric values for compatibility with the learning process.

The GaloisNetwork itself is a fully connected feedforward neural network. It begins with an input layer that matches the size of the feature vectors. The network includes three hidden layers, each with 64 neurons and ReLU activation functions, providing the capacity to learn complex patterns in the data. Finally, an output layer produces a probability distribution over all possible Galois group labels using a softmax activation. This architecture allows the network to effectively capture the relationships between features and group classifications.

Training the network involves minimizing a cross-entropy loss function using the Adam optimizer. Over 100 epochs, the network iteratively updates its weights through backpropagation, ensuring that it learns to align its predictions with the true labels. To monitor its progress, we periodically evaluate the model on a validation set, tracking the loss and refining the learning process.

To enhance the model's predictions, we implement a post-processing step that applies domain-specific rules. For example, if the number of real roots of a polynomial exceeds a certain threshold, the prediction is adjusted to align with known Galois group properties. This rule-based layer ensures that the network respects established mathematical principles, making its outputs both reliable and interpretable.

Finally, we evaluate the system using accuracy metrics, confusion matrices, and detailed classification reports. These evaluations demonstrate the effectiveness of combining numerical learning with symbolic reasoning. By integrating these two paradigms, our design not only achieves high accuracy but also maintains alignment with the underlying mathematical structure of the problem, providing a powerful tool for analyzing polynomials through the lens of their Galois groups.

```
coefficients_list = [list(key) for key in L.keys()]
labels = [L[key][3] for key in L.keys()]
features = [compute_galois_features(coeffs) for coeffs in coefficients_list]

label_mapping = {label: idx for idx, label in enumerate(set(labels))}
labels_numeric = [label_mapping[label] for label in labels]

scaler = StandardScaler()
features_scaled = scaler.fit_transform(features)
X_train, X_val, y_train, y_val = train_test_split(features_scaled, labels_numeric, test_size
    =0.2, random_state=42)

features_tensor = torch.tensor(X_train, dtype=torch.float32)
features_tensor_validation = torch.tensor(X_val, dtype=torch.float32)
labels_tensor = torch.tensor(y_train, dtype=torch.long)
labels_tensor_validation = torch.tensor(y_val, dtype=torch.long)

class GaloisNetwork(nn.Module):
    def __init__(self, input_size, hidden_size, output_size):
        super(GaloisNetwork, self).__init__()
        self.layers = nn.Sequential(
            nn.Linear(input_size, hidden_size),
            nn.ReLU(),
            nn.Linear(hidden_size, hidden_size),
            nn.ReLU(),
            nn.Linear(hidden_size, hidden_size),
            nn.ReLU(),
            nn.Linear(hidden_size, output_size)
        )
```

```
        self.softmax = nn.Softmax(dim=1)

    def forward(self, x):
        return self.softmax(self.layers(x))

input_size = len(features[0])
hidden_size = 64  # Increased for complexity
output_size = len(label_mapping)

model = GaloisNetwork(input_size, hidden_size, output_size)
criterion = nn.CrossEntropyLoss()
optimizer = optim.Adam(model.parameters(), lr=0.001)

for epoch in range(100):  # Increased epochs for better training
    model.train()
    optimizer.zero_grad()
    outputs = model(features_tensor)
    loss = criterion(outputs, labels_tensor)
    loss.backward()
    optimizer.step()

    if (epoch + 1) % 10 == 0:  # Print every 10 epochs
        model.eval()
        with torch.no_grad():
            val_outputs = model(features_tensor_validation)
            val_loss = criterion(val_outputs, labels_tensor_validation)
        print(f"Epoch {epoch+1}, Loss: {loss.item()}, Validation Loss: {val_loss.item()}")

# Evaluate model on validation set
model.eval()
with torch.no_grad():
    predictions = model(features_tensor_validation)
    predicted_classes = torch.argmax(predictions, dim=1)
    accuracy = accuracy_score(labels_tensor_validation.cpu().numpy(), predicted_classes.cpu().
        numpy())
    print(f"Validation Accuracy: {accuracy}")

    # Confusion Matrix
    cm = confusion_matrix(labels_tensor_validation.cpu().numpy(), predicted_classes.cpu().numpy
        ())
    sns.heatmap(cm, annot=True, fmt='d')
    plt.title('Confusion Matrix')
    plt.ylabel('True label')
    plt.xlabel('Predicted label')
    plt.show()

    # Classification Report
    print(classification_report(labels_tensor_validation.cpu().numpy(), predicted_classes.cpu().
        numpy(), target_names=list(label_mapping.keys())))
```

LISTING 3. Python Code for Training a Neural Network

11. Concluding remarks

This paper introduces an innovative approach to Galois theory by leveraging machine learning techniques to address challenges in understanding polynomial properties and their Galois groups. Combining classical algebraic structures with computational tools opens new avenues for exploring the connections between mathematics and data science.

We have demonstrated the potential of supervised learning to predict Galois groups and polynomial solvability, while unsupervised learning reveals latent structures in polynomial datasets. A comprehensive database of irreducible polynomials with known Galois groups has been compiled, and classical invariants such as discriminants, root differences, and moduli heights have been explored as features for machine learning models. Reduction theories, including Julia and Hermite equivalence, were employed to streamline classification, and the role of polynomial heights in minimal forms and equivalence classes was investigated. The geometric interpretation of polynomial transformations within weighted projective spaces further enhances this framework.

Future work could extend the polynomial database to higher degrees, incorporate multivariable polynomials, and develop novel invariants derived from machine learning. Advanced models, such as graph neural networks, could refine the analysis of root interactions and symmetries, while transfer learning may generalize insights to more complex cases. Automation of reduction methods and interactive visualization tools could make these techniques accessible to a broader audience. Additionally, extending this framework to analyze field extensions and connections with algebraic geometry or physics could broaden its impact.

This work demonstrates the feasibility of integrating machine learning with classical mathematics, offering new tools for algebraists while uncovering deeper theoretical insights. By bridging abstract mathematics and computational science, this approach paves the way for a more interdisciplinary perspective in mathematical research.

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