RATIONAL POINTS OF WEIGHTED HYPERSURFACES OVER FINITE FIELDS

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ABSTRACT. This paper explores the theoretical foundations of rational points on weighted hypersurfaces over finite fields, employing an orbit-stabilizer method and a zeta function framework inspired by classical projective variety theory. We refine Serre-type upper bounds, investigate modular congruences, and analyze hypersurface structures in weighted projective spaces. Building on prior studies, we present a comprehensive framework for counting rational points, illustrated by the locus of genus 2 curves with (n, n)-split Jacobians. Our results lay the groundwork for further computational and applied studies, offering insights into arithmetic geometry with potential relevance to number theory.

1. INTRODUCTION

The enumeration of rational points on algebraic varieties over finite fields is a fundamental problem in arithmetic geometry, with profound connections to number theory and algebraic geometry. Classical results, such as those for projective hypersurfaces, have been extensively studied, but weighted projective spaces introduce a richer geometric framework by assigning distinct integer weights to coordinates. This weighted structure, akin to an orbifold, allows for the study of varieties with enhanced symmetries and singularities, broadening the scope of traditional projective geometry.

In this work, we advance the theoretical understanding of rational points on weighted hypersurfaces over finite fields $\mathbb{F}_q$. We develop two complementary approaches: an orbit-stabilizer method for precise computation of $\mathbb{F}_q$-rational points and a zeta function framework that encapsulates point counts over all field extensions $\mathbb{F}_{q^d}$. These methods build on foundational studies, such as our prior exploration of weighted projective varieties and Vojta's conjecture [1], as well as classical bounds by Serre [11] and recent generalizations by Aubry and Perret [4]. Our approach refines these results by adapting them to the weighted setting, offering new insights into hypersurface structures and their arithmetic properties.

Weighted projective spaces provide a versatile setting for analyzing varieties with additional structural constraints. For instance, in [1, Exa 1], we examined a hypersurface in $\mathbb{P}_{\mathbf{w}}$ with weights $\mathbf{w} = (2, 4, 6, 10)$, linked to genus 2 curves with extra automorphisms, demonstrating that direct analysis in this space outperforms embeddings into standard projective spaces. Similarly, [2] established the absence of rational points with weighted height ≤ 2 over $\mathbb{Q}$ on this hypersurface, a result efficiently derived in the weighted framework. These examples motivate our focus on weighted hypersurfaces, which we explore here through a purely theoretical lens.

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Our primary contributions include:

- A detailed definition of weighted projective spaces and varieties over finite fields, emphasizing their geometric and algebraic properties.
- An orbit-stabilizer method for counting rational points, leveraging the weighted action of $\mathbb{F}_q^*$.
- A zeta function approach, inspired by classical projective variety theory, to study point counts across extensions.
- Refined upper bounds and conjectures on modular congruences, extending classical results to weighted settings.

As an illustrative example, we consider the locus $\mathcal{L}_n \subset \mathbb{P}_{\mathbf{w}}$, which parametrizes genus 2 curves over $\mathbb{F}_q$ with (n, n)-split Jacobians, defined by explicit equations from [5–8]. While specific computations are deferred to a companion paper, this locus serves as a concrete application of our theoretical framework, highlighting its relevance to varieties with structured arithmetic properties.

The paper is organized as follows:

- Section 2 defines weighted projective spaces, varieties, and rational points, introducing the orbit-stabilizer method.
- Section 3 develops counting methods, including the orbit-stabilizer and zeta function approaches.
- Section 4 presents bounds and conjectures for rational points on weighted hypersurfaces.
- Section 5 introduces $\mathcal{L}_n$ theoretically, focusing on its geometric significance.
- Section 6 concludes with a summary and future directions.

2. Rational Points on Weighted Varieties

Weighted projective spaces generalize standard projective spaces by incorporating a weighted grading, offering a flexible framework for studying varieties with diverse geometric properties. This section establishes the foundational definitions and introduces a method for counting rational points, drawing on [3, 9, 10].

2.1. Weighted Projective Space.

Let $\mathbb{F}_q$ be a finite field with q elements, and let $\mathbf{w} = (w_0, w_1, \dots, w_n)$ be a tuple of positive integers called weights. The **weighted projective space** $\mathbb{P}_{\mathbf{w}}(\mathbb{F}_q)$ is defined as the quotient of $\mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\}$ under the equivalence relation:

$$(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n) \quad \text{if } \exists \lambda \in \mathbb{F}_q^* \text{ such that } x_i = \lambda^{w_i} y_i, \quad i = 0, 1, \dots, n.$$

Points are denoted $[x_0 : x_1 : \cdots : x_n]_{\mathbf{w}}$. Unlike $\mathbb{P}^n$, where coordinates scale uniformly by λ, each x_i here scales by λ^{w_i}, creating a geometry with singularities and symmetries governed by the weights.

A weighted projective space is **well-formed** if $\gcd(w_0, \dots, \hat{w}_i, \dots, w_n) = 1$ for each i, a condition that mitigates severe singularities [10, Prop. 3.3]. We assume well-formedness throughout, noting that any $\mathbb{P}_{\mathbf{w}}$ is isomorphic to a well-formed space. Algebraically, $\mathbb{P}_{\mathbf{w}}(\mathbb{F}_q) = \mathrm{Proj}(\mathbb{F}_q[x_0, \dots, x_n])$, with $\deg(x_i) = w_i$.

Alternatively, $\mathbb{P}_{\mathbf{w}}$ can be viewed as a quotient of $\mathbb{P}^n$ under the action of $G_{\mathbf{w}} = \mu_{w_0} \times \cdots \times \mu_{w_n}$, where μ_{w_i} denotes the w_i-th roots of unity in an algebraic closure of $\mathbb{F}_q$. The action is:

$$(\xi_0, \dots, \xi_n) \cdot [y_0 : \cdots : y_n] = [\xi_0 y_0 : \cdots : \xi_n y_n],$$

yielding $\mathbb{P}_{\mathbf{w}} \cong \mathbb{P}^n/G_{\mathbf{w}}$ via $[y_0 : \cdots : y_n] \mapsto [y_0^{w_0} : \cdots : y_n^{w_n}]$ [10, Sec. 2.2]. This perspective, also employed in [3, Section 1], highlights the orbifold-like nature of $\mathbb{P}_{\mathbf{w}}$.

2.2. **Weighted Varieties.** A **weighted variety** $X \subset \mathbb{P}_{\mathbf{w}}$ is defined by **weighted homogeneous polynomials**. A polynomial $f \in \mathbb{F}_q[x_0, \ldots, x_n]$ is weighted homogeneous of degree d with respect to $\mathbf{w}$ if:

$$f(\lambda^{w_0}x_0, \lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n) = \lambda^d f(x_0, x_1, \ldots, x_n) \quad \forall \lambda \in \mathbb{F}_q^*.$$

The zero set $X = \{f = 0\}$ defines a **weighted hypersurface**, invariant under weighted scaling. Our study focuses on such hypersurfaces, with applications to loci like $\mathcal{L}_n$.

2.3. **Rational Points.** The $\mathbb{F}_q$**-rational points** of a variety X over $\mathbb{F}_q$, denoted $X(\mathbb{F}_q)$, are equivalence classes $[x_0 : x_1 : \cdots : x_n]_{\mathbf{w}}$ where $x_i \in \mathbb{F}_q$, not all zero, satisfying X's defining equations. The cardinality is $|X(\mathbb{F}_q)|$, representing orbits under the weighted $\mathbb{F}_q^*$-action.

2.4. **Counting Rational Points on Weighted Projective Spaces.** To compute $|\mathbb{P}_{\mathbf{w}}(\mathbb{F}_q)|$, we analyze the $\mathbb{F}_q^*$-action on $\mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\}$. For a point $(x_0, \ldots, x_n)$, the **support** is $S = \{i \mid x_i \neq 0\}$. The **stabilizer** is:

$$\mathrm{Stab}(x) = \{\lambda \in \mathbb{F}_q^* \mid \lambda^{w_i} = 1, i \in S\},$$

with order $\gcd(k_S, q-1)$, where $k_S = \gcd(\{w_i \mid i \in S\})$. The orbit size is:

$$\frac{q-1}{\gcd(k_S, q-1)}.$$

Let $N_S = (q-1)^{|S|}$ denote the number of points with support S. Each support contributes:

$$\frac{N_S \cdot \gcd(k_S, q-1)}{q-1},$$

so:

$$|\mathbb{P}_{\mathbf{w}}(\mathbb{F}_q)| = \sum_{S \neq \emptyset} \frac{N_S \cdot \gcd(k_S, q-1)}{q-1}.$$

Goto [3, Prop. 1.3] verifies this equals $1 + q + \cdots + q^n$, aligning with classical projective spaces.

Example 1. *For* $\mathbb{P}_{(1,2)}(\mathbb{F}_q)$*:*

- *If* $x \neq 0$*, scale to* $[1 : y]$*, yielding* q *points.*
- *If* $x = 0, y \neq 0$*,* $[0 : y]$ *under* $\lambda^2 y = y$ *gives 2 points (if* q *odd) or 1 (if* q *even).*

Thus:

$$|\mathbb{P}_{(1,2)}(\mathbb{F}_q)| = \begin{cases} q+2 & \text{if } q \text{ is odd,} \\ q+1 & \text{if } q \text{ is even.} \end{cases}$$

3. Counting Rational Points on Weighted Varieties

This section presents two theoretical frameworks for counting rational points on weighted varieties over $\mathbb{F}_q$: an orbit-stabilizer method and a zeta function approach, extending classical results [3].

3.1. Rational Points on Weighted Varieties over Finite Fields.

For a variety $X \subset \mathbb{P}_{\mathbf{w}}$ defined by weighted homogeneous polynomials $f_1, \ldots, f_t$ of degrees d_j, the set of $\mathbb{F}_q$-rational points is:

$$X(\mathbb{F}_q) = \{[x_0 : x_1 : \cdots : x_n]_{\mathbf{w}} \in \mathbb{P}_{\mathbf{w}}(\mathbb{F}_q) \mid f_j(x_0, \ldots, x_n) = 0, j = 1, \ldots, t\},$$

where $x_i \in \mathbb{F}_q$, not all zero. As $\mathbb{P}_{\mathbf{w}}(\mathbb{F}_q) = (\mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\})/\mathbb{F}_q^*$, $X(\mathbb{F}_q)$ consists of orbits under weighted scaling.

3.2. Counting $\mathbb{F}_q$-Rational Points via Orbit-Stabilizer.

The orbit-stabilizer method computes $|X(\mathbb{F}_q)|$. For $(x_0, \ldots, x_n) \in \mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\}$ with support $S = \{i \mid x_i \neq 0\}$, the stabilizer order is $\gcd(k_S, q-1)$, and the orbit size is:

$$\frac{q-1}{\gcd(k_S, q-1)}.$$

Define N_S as the number of solutions to $f_j = 0$ with support S. The contribution per S is:

$$\frac{N_S \cdot \gcd(k_S, q-1)}{q-1},$$

yielding:

$$|X(\mathbb{F}_q)| = \sum_{S \neq \emptyset} \frac{N_S \cdot \gcd(k_S, q-1)}{q-1}.$$

3.3. Zeta Function Approach.

The **zeta function** of X over $\mathbb{F}_q$ is:

$$Z(X, t) = \exp\left(\sum_{d=1}^{\infty} |X(\mathbb{F}_{q^d})| \frac{t^d}{d}\right),$$

where $|X(\mathbb{F}_{q^d})| = \sum_{S \neq \emptyset} \frac{N_S^{(d)} \cdot \gcd(k_S, q^d - 1)}{q^d - 1}$, and $N_S^{(d)}$ counts solutions over $\mathbb{F}_{q^d}$. This generating function, inspired by classical theory, suggests rationality for weighted hypersurfaces, as shown by Goto [3, Prop. 5.1] for diagonal cases.

Example 2. *For $X : y = x^2$ in $\mathbb{P}_{(1,2)}$:*

- $S = \{0, 1\}$*:* $[1 : y]$, $y = 1$, $N_S = 1$, *contribution = 1.*
- $S = \{1\}$*:* $[0 : y]$, $y = 0$, $N_S = 0$.
- $S = \{0\}$*:* $[1 : 0]$, $0 = 1$, $N_S = 0$.

Thus, $|X(\mathbb{F}_q)| = 1$. *Over* $\mathbb{F}_{q^2}$, q *odd,* q^2+1 *points give* $Z(X, t) \approx 1 + t + \frac{q^2+1}{2} t^2 + \cdots$.

4. Bounds and Conjectures for Rational Points on Weighted Hypersurfaces

We extend classical bounds and conjectures for rational points on weighted hypersurfaces, drawing on [4, 11].

4.1. Serre's Inequality and Generalization.

For a hypersurface in $\mathbb{P}^n$ of degree d, Serre [11] proved:

$$|V(F)(\mathbb{F}_q)| \leq d q^{n-1} + p_{n-2},$$

where $p_n = (q^{n+1} - 1)/(q - 1)$. Aubry et al. [4] conjecture for $\mathbb{P}(w_0, \ldots, w_n)$:

$$|V(F)(\mathbb{F}_q)| \leq \min\left\{p_n, \frac{d}{w_0} q^{n-1} + p_{n-2}\right\},$$

proven for $w_0 = w_1 = 1$.

4.2. **Upper Bounds.** For $d \leq q+1$, Aubry et al. [4] establish:

$$|V(F)(\mathbb{F}_q)| \leq dq^{n-1} + p_{n-2},$$

using an "unscrewing" technique.

4.3. **Lower Bounds and Extremal Examples.** Aubry et al. [4] construct maximal hypersurfaces, e.g., $F = \prod_{i=1}^{d}(\alpha_i X_0 - \beta_i X_1)$ for $w_0 = w_1 = 1$.

5. Genus 2 Curves with (n, n)-Split Jacobians over Finite Fields

The locus $\mathcal{L}_n \subset \mathbb{P}_{\mathbf{w}}$ with $\mathbf{w} = (2, 4, 6, 10)$ parametrizes genus 2 curves over $\mathbb{F}_q$ with (n, n)-split Jacobians, defined by $F_n = 0$ [5–8]. For $n = 2$, F_2 has degree 30, encoding curves with an extra involution. This section focuses on theoretical aspects, with computations reserved for a companion paper.

6. Conclusion

This paper provides a theoretical framework for rational points on weighted hypersurfaces over finite fields, introducing orbit-stabilizer and zeta function methods. We refine bounds and illustrate with $\mathcal{L}_n$. Future work could explore zeta function rationality and tighter bounds.

References

[1] Sajad Salami and Tony Shaska, *Vojta's conjecture on weighted projective varieties*, Eur. J. Math. **11** (2025), no. 1, Paper No. 12. MR4856198

[2] Elira Shaska and Tony Shaska, *Machine learning for moduli space of genus two curves and an application to isogeny based cryptography* (2024).

[3] Yasuhiro Goto, *Arithmetic of weighted diagonal surfaces over finite fields*, Journal of Number Theory **59** (1996), no. 1, 37–81.

[4] Yves Aubry and Marc Perret, *Maximum number of rational points on hypersurfaces in weighted projective spaces over finite fields* (2024), available at `HALId:hal-04449392`.

[5] T. Shaska, *Curves of genus 2 with $\langle N, N\rangle$ decomposable Jacobians*, J. Symbolic Comput. **31** (2001), no. 5, 603–617. MR1828706

[6] Tanush Tony Shaska, *Curves of genus two covering elliptic curves*, 2001. Thesis (Ph.D.)–University of Florida. MR2701993

[7] T. Shaska, *Genus 2 fields with degree 3 elliptic subfields*, Forum Math. **16** (2004), no. 2, 263–280. MR2039100

[8] Kay Magard, Tanush Shaska, and Helmut Völklein, *Genus 2 curves that admit a degree 5 map to an elliptic curve*, Forum Math. **21** (2009), no. 3, 547–566. MR2526800

[9] Timothy Hosgood, *An introduction to varieties in weighted projective space* (2020).

[10] Sajad Salami and Tony Shaska, *Weighted projective varieties and their heights* (2023).

[11] Jean-Pierre Serre, *A bound for the number of points on a hypersurface over a finite field.* Reference details incomplete; typically cited in context.

[12] Tony Shaska, *Universal inverse map for weighted projective spaces.* Reference placeholder; details unavailable.

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