WEIGHTED HEIGHTS AND GIT HEIGHTS

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ABSTRACT. This paper examines the relationship between GIT heights and weighted heights, exploring their definitions and applications to weighted projective spaces and binary forms. Drawing on the weighted height framework from [1, 2], we relate it to Zhang's GIT height via the Veronese map, showing that for a semistable cycle $\mathcal{Z} \subset \mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}}$, the GIT height $\hat{h}(\mathcal{Z})$ is given by $\hat{h}(\mathcal{Z}) = \mathfrak{L}(\mathcal{Z}) + \sum_{\nu \in M_K^\infty} \log \|s\|_{Ch,\nu}(\mathcal{Z})$, incorporating the Chow metric's Archimedean contribution. For binary forms $f \in V_d$, we define an invariant height $\hat{H}(f)$ with respect to the Chow metric and establish a connection where the moduli weighted height $\mathfrak{L}(\xi(f))$ of f's invariants satisfies $\mathfrak{L}(\xi(f)) = \hat{H}(f) + [K:\mathbb{Q}]\mathrm{h}_{Ch}(f)$, linking arithmetic and moduli properties.

1. INTRODUCTION

Height functions in algebraic geometry serve as a bridge between the arithmetic and geometric properties of varieties, providing quantitative insights into Diophantine equations, moduli spaces, and related areas. Classical heights, such as the Weil height on projective spaces or the Neron-Tate height on abelian varieties, have long been essential for measuring the "size" of algebraic points and cycles. However, as we consider weighted projective spaces—generalizations of $\mathbb{P}^n$ where coordinates carry distinct weights—these traditional tools require adaptation to address their graded structure. The Geometric Invariant Theory (GIT) height, introduced by Zhang [3], quantifies the arithmetic size of semistable cycles through a stability-sensitive lens, while our prior work on weighted projective spaces [1, 2] has developed specialized weighted heights to suit their coordinate systems. This paper investigates the interplay between GIT heights and weighted heights, with a focus on their application to binary forms, advancing the arithmetic invariant theory of weighted varieties in both theoretical and computational dimensions.

Our study is motivated by the goals of extending classical height functions to weighted projective settings and integrating them with GIT's stability framework, thereby deepening our understanding of algebraic objects like binary forms. Weighted projective spaces, denoted $\mathbb{P}^n_{\mathfrak{w},K}$ with weights $\mathfrak{w} = (q_0, \dots, q_n)$, arise naturally in contexts where coordinates have varying degrees, such as moduli spaces of hypersurfaces or forms with symmetry. The weighted height, introduced in [1] and refined in [2], adapts Weil's approach by accounting for these weights, providing a measure tailored to such spaces. In contrast, Zhang's GIT height leverages invariant theory to assess semistability, a property central to moduli constructions. Through the Veronese map and the Chow metric, we connect these heights, shedding light on the arithmetic properties of weighted varieties and offering tools for analyzing the moduli space $\mathcal{B}_d$ of degree-d binary forms. This work builds on computational

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insights from prior studies (e.g., [4,5]) and explores weighted heights' potential links to classical measures like Neron-Tate and Faltings heights.

The paper is structured as follows. Section 2 establishes foundational definitions—vector bundles, line bundles, number fields, and classical heights, culminating in the Arakelov height—familiar to arithmetic geometers. Section 3 introduces weighted varieties and their heights, proving the local boundedness of the weighted M-metric (Lem. 3) as a basis for their arithmetic study. Section 4 examines the GIT height, linking it to weighted heights via the Veronese map (Lem. 10) and presenting a theorem that includes the Chow metric's Archimedean contribution (Thm. 3). Section 5 applies these ideas to binary forms, defining and computing naive, moduli, and invariant heights (e.g., Lem. 17, Thm. 5), culminating in a key result (Thm. 6) that connects the moduli weighted height to invariant properties. Section 6 summarizes our findings and outlines future directions, considering weighted heights' broader implications.

Our contributions center on relating GIT and weighted heights for semistable cycles, as seen in $\hat{h}(\mathcal{X}) = \mathfrak{L}(\mathcal{X}) + \sum_{\nu \in M_K^\infty} \log \|s\|_{Ch,\nu}(\mathcal{X})$ (Thm. 3), which highlights the Chow metric's role at infinity, and providing computational tools for binary forms, such as $\hat{H}(f) = \frac{d}{2}\log(1+|a_0|^2)$ for $f = x^d - a_0 y^d$ (Lem. 18). A notable outcome is Thm. 6, which expresses the moduli weighted height $\mathfrak{L}(\xi(f))$ of a binary form's invariants as $\hat{H}(f) + [K:\mathbb{Q}]\mathrm{h}_{Ch}(f)$, linking it to Zhang's invariant height $\hat{H}(f)$ and the Chow height $\mathrm{h}_{Ch}(f)$. This relation offers a clear connection between the arithmetic of binary forms and their moduli space $\mathcal{B}_d$, enhancing our toolkit for weighted projective spaces with both theoretical structure and practical utility. While these results refine the arithmetic geometry of binary forms, they also suggest a path for future exploration into weighted heights' ties to classical height theories, a prospect considered in our concluding remarks.

2. Preliminaries

This section introduces the foundational concepts used throughout the paper. For further details, refer to [6] and [7].

2.1. Vector Bundles.

Let $\mathcal{X}$ be a variety over a field K. A **vector bundle** of rank r over $\mathcal{X}$ is a variety E over K equipped with a morphism $\pi_E : E \to \mathcal{X}$ satisfying the following conditions:

(1) There exists an open covering $\{U_\alpha\}_{\alpha\in I}$ of $\mathcal{X}$ and isomorphisms

$$\phi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times \mathbb{A}_K^r$$

of varieties over K, where ϕ_α is linear on each fiber $\pi_E^{-1}(x) \cong \mathbb{A}_K^r$ for $x \in U_\alpha$.

(2) For each $\alpha, \beta \in I$ and $x \in U_\alpha \cap U_\beta$, there is a transition matrix $M_{\alpha\beta}(x) \in \mathrm{GL}_r(K)$ such that

$$\phi_\alpha \circ \phi_\beta^{-1}(x, \mathbf{v}) = (x, M_{\alpha\beta}(x)\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbb{A}_K^r.$$

The maps ϕ_α are called **local trivializations**, and the matrices $M_{\alpha\beta}$ are the **transition functions**. This ensures that the bundle is locally isomorphic to a product space, with fiber transitions being linear.

2.2. **Sections.** A **section** of a vector bundle E over an open set $U \subset \mathcal{X}$ is a morphism $s : U \to E$ such that $\pi_E \circ s = \mathrm{id}_U$. If $U = \mathcal{X}$, then s is a **global section**. The set of sections over U, denoted $\Gamma(U, E)$, forms a K-vector space. The **sheaf of sections** $\mathcal{E}$ is defined by $\mathcal{E}(U) = \Gamma(U, E)$ for each open $U \subset \mathcal{X}$, providing a sheaf-theoretic framework for studying sections locally and globally.

2.2.1. *Line Bundles.* A **line bundle** $\mathcal{L}$ on $\mathcal{X}$ is a vector bundle of rank 1. For $n \in \mathbb{Z}$, the n-fold tensor power is denoted $\mathcal{L}^{\otimes n}$, with conventions $\mathcal{L}^{\otimes 0} = \mathcal{O}_{\mathcal{X}}$ (the structure sheaf) and $\mathcal{L}^{\otimes(-n)} = (\mathcal{L}^*)^{\otimes n}$, where $\mathcal{L}^*$ is the dual bundle. The **Picard group** $\mathrm{Pic}(\mathcal{X})$ is the group of isomorphism classes of line bundles under the tensor product $\otimes$, with the inverse of $[\mathcal{L}]$ given by $[\mathcal{L}^*]$.

2.2.2. *Tautological Line Bundle.* Consider the trivial bundle $E = \mathbb{P}^n_K \times \mathbb{A}^{n+1}_K$, with coordinates $[x_0 : \cdots : x_n]$ on $\mathbb{P}^n_K$ and $(y_0, \ldots, y_n)$ on $\mathbb{A}^{n+1}_K$. The **tautological line bundle** $\mathcal{O}_{\mathbb{P}^n_K}(-1)$ is the subvariety $\mathcal{L} \subset E$ defined by the equations

$$x_i y_j - x_j y_i = 0 \quad \text{for all } 0 \leq i, j \leq n.$$

The projection $\pi_{\mathcal{L}} : \mathcal{L} \to \mathbb{P}^n_K$ is a morphism, and local trivializations over $U_\alpha = \{x_\alpha \neq 0\}$ are given by

$$\phi_\alpha : \pi_{\mathcal{L}}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{A}^1_K, \quad (\mathbf{x}, \mathbf{y}) \mapsto \left(\mathbf{x}, \frac{y_\alpha}{x_\alpha}\right),$$

with transition functions $M_{\alpha\beta}(\mathbf{x}) = \frac{x_\beta}{x_\alpha}$. For $m \in \mathbb{Z}$, define $\mathcal{O}_{\mathbb{P}^n_K}(m) = \mathcal{O}_{\mathbb{P}^n_K}(-1)^{\otimes(-m)}$, whose sections correspond to homogeneous polynomials of degree m.

2.3. **Number Fields.** Let k be a number field of degree $[k : \mathbb{Q}] = m$, with ring of integers $\mathcal{O}_k$, and let $\overline{k}$ be an algebraic closure of k. A variety $\mathcal{X}$ over k is an integral separated scheme of finite type over $\mathrm{Spec}(k)$, with $\mathcal{X}(\overline{k})$ denoting its $\overline{k}$-points and $\mathcal{X}(k)$ its k-rational points. The set of places M_k of k comprises:

i) Non-Archimedean places M_k^0, corresponding to prime ideals $\mathfrak{p} \subset \mathcal{O}_k$, denoted $\nu = \nu_{\mathfrak{p}}$,
ii) Archimedean places M_k^∞, corresponding to embeddings $\sigma : k \hookrightarrow \mathbb{C}$ up to conjugation, denoted $\nu = \nu_\sigma$.

For a place $\nu \in M_k$, the **local degree** is $n_\nu = [k_\nu : \mathbb{Q}_\nu]$, where k_ν is the completion of k at ν. The absolute value $|\cdot|_\nu$ is normalized to extend the standard absolute value on $\mathbb{Q}$. For $x \in k^*$, the **product formula** states

$$\prod_{\nu \in M_k} |x|_\nu^{n_\nu} = 1.$$

For a finite extension K/k, the places $w \in M_K$ extending $\nu \in M_k$ satisfy

$$\sum_{\substack{w \in M_K \\ w|_k = \nu}} [K_w : k_\nu] = [K : k].$$

2.4. **M-Bounded Sets and Functions.** Let $M = M_{\overline{k}}$ be the set of places of $\overline{k}$ extending M_k. An **M_k-constant** is a function $\gamma : M_k \to \mathbb{R}$ such that $\gamma(\nu) = 0$ for all but finitely many ν, extended to M by $\gamma(v) = \gamma(v|_k)$. An **M_k-function** on $\mathcal{X}$ is a map $\lambda : \mathcal{X} \times M \to \mathbb{R}$ where $\lambda(\mathbf{x}, v)$ is either an M_k-constant or infinite.

Two M_k-functions λ_1, λ_2 are **equivalent**, written $\lambda_1 \sim \lambda_2$, if there exists an M_k-constant γ such that

$$|\lambda_1(\mathbf{x}, v) - \lambda_2(\mathbf{x}, v)| \leq \gamma(v) \quad \text{for all } (\mathbf{x}, v) \in \mathcal{X} \times M.$$

An M_k-function λ is M_k**-bounded** if $\lambda \sim 0$.

For an affine variety $\mathcal{X}$, a set $E \subset \mathcal{X} \times M$ is an **affine** M_k**-bounded set** if there exist coordinates $x_1, \ldots, x_n$ on $\mathcal{X}$ and an M_k-bounded constant γ such that

$$|x_i(\mathbf{x})|_v \leq e^{\gamma(v)} \quad \text{for } 1 \leq i \leq n, \, (\mathbf{x}, v) \in E.$$

For a general variety, $E \subset \mathcal{X} \times M$ is M_k**-bounded** if it is covered by finitely many affine M_k-bounded sets.

A function $\lambda : \mathcal{X} \times M \to \mathbb{R}$ is **locally** M_k**-bounded above** if, for every M_k-bounded set $E \subset \mathcal{X} \times M$, there exists an M_k-constant γ such that $\lambda(\mathbf{x}, v) \leq \gamma(v)$ on E. It is **locally** M_k**-bounded** if both λ and $-\lambda$ are locally M_k-bounded above.

Example 1. *For $\mathcal{X} = \mathbb{P}^n_{\bar{k}}$, consider the standard affine opens $U_i = \{x_i \neq 0\}$ and sets*

$$E_i = \left\{ (\mathbf{x}, v) \in \mathcal{X} \times M \mid \left| \frac{x_j}{x_i} \right|_v \leq 1, \, 0 \leq j \leq n \right\}.$$

Each E_i is M_k-bounded, and since $\mathcal{X} = \bigcup_{i=0}^n E_i$, $\mathcal{X}$ itself is M_k-bounded.

2.5. **Metrized Line Bundles.** Let $\mathcal{T}$ be a topological space, $\mathcal{L}$ a line bundle on $\mathcal{T}$, $U \subset \mathcal{T}$ an open set, and s a section of $\mathcal{L}$ over U. A **metric** $\| \cdot \|$ on $\mathcal{L}$ is a continuous function $\|s\|_U : U \to \mathbb{R}$ satisfying:

i) $\|s\|_U|_V = \|s\|_V$ for any open $V \subset U$,
ii) $\|fs\|(z) = |f(z)| \cdot \|s\|(z)$ for $f \in \mathcal{O}_{\mathcal{T}}(U)$,
iii) If $s \neq 0$ on U, then $\|s\|_U \neq 0$.

A **metrized line bundle** is a pair $\widehat{\mathcal{L}} = (\mathcal{L}, \| \cdot \|)$.

For a number field k, an $\mathcal{O}_k$-scheme $\mathcal{X}$, and an embedding $\sigma : \mathcal{O}_k \hookrightarrow \mathbb{C}$, the induced map $\sigma^* : \operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(\mathcal{O}_k)$ equips $\mathcal{X}_\sigma(\mathbb{C})$ with a complex manifold structure. Metrics on $\mathcal{L}$ are required to be conjugation-invariant at Archimedean places.

A metric is **smooth** if $\|s\|_U$ is $\mathcal{C}^\infty$ for smooth, non-vanishing sections s. A metrized bundle $\bar{\mathcal{L}}$ is **Hermitian** if its metric is smooth.

Example 2. *For $\mathcal{X} = \mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{L} = \mathcal{O}(1)$, the* ***standard metric*** *is*

$$\|s\|(\mathbf{x}) = \frac{|s(\mathbf{x})|}{\max_i |x_i|}, \tag{1}$$

which is locally bounded (see [7, Example 2.7.4]*). The* ***Fubini-Study metric*** *is*

$$\|s\|_f(\mathbf{x}) = \frac{|s(\mathbf{x})|}{\left(\sum_{i=0}^n |x_i|^2\right)^{1/2}}, \tag{2}$$

also locally bounded (see [7, Example 2.7.4]*).*

Every line bundle on a variety over k admits a locally bounded metric (see [7, Prop. 2.7.5]).

2.5.1. *The Natural M-Metric.* For a line bundle $\mathcal{L}$ on a flat proper reduced scheme $\mathcal{X}$ over $\mathcal{O}_k$, with generic fiber $L = \mathcal{L}_k$ on $X = \mathcal{X}_k$, the **natural M-metric** $\|\cdot\|_{\mathcal{L}}$ is defined by setting $\|s(x)\|_{\mathcal{L},u} = 1$ for a constant section s at each place $u \in M$, using the integral model to trivialize locally.

Lemma 1. *The natural M-metric $\|\cdot\|_{\mathcal{L}}$ is well-defined and locally bounded.*

Proof. The metric is well-defined because different choices of trivializing sections differ by units, preserving the norm. Local boundedness follows from the integral model's properties and the boundedness of coordinates on affine patches (see [7, Prop. 2.7.5]). □

Example 3. *For a constant section s, the standard metric is*

$$\|s\|(\mathbf{x}) = \frac{1}{\max_i |x_i|},$$

and the Fubini-Study metric is

$$\|s\|(\mathbf{x}) = \frac{1}{\left(\sum_{i=0}^n |x_i|^2\right)^{1/2}}.$$

2.6. **Heights on Projective Spaces.** Let $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$, where k is a number field. For a place $\nu \in M_k$, define the local multiplicative and logarithmic heights relative to a metrized line bundle $\mathcal{L} = (\mathcal{O}_{\mathbb{P}^n_k}(1), \|\cdot\|)$ as

$$H_{k,\nu}(\mathbf{x}) := \|s(\mathbf{x})\|_\nu^{-1} \quad \text{and} \quad h_{k,\nu}(\mathbf{x}) := -\log \|s(\mathbf{x})\|_\nu,$$

where s is a section of $\mathcal{L}$ such that $s(\mathbf{x}) \neq 0$. Using the standard metric from Eq. (1) with a constant global section $s(\mathbf{x}) = 1$, we obtain

$$H_\nu(\mathbf{x}) = \left(\max_{0\le i\le n} \{|x_i|_\nu\}\right)^{-1} \quad \text{and} \quad h_\nu(\mathbf{x}) = -\log \max_{0\le i\le n} \{|x_i|_\nu\}.$$

The **multiplicative height** and **logarithmic height** of $\mathbf{x}$ are defined as

$$H_k(\mathbf{x}) = \prod_{\nu\in M_k} \max_{0\le i\le n} \{|x_i|_\nu\}^{n_\nu} \quad \text{and} \quad h_k(\mathbf{x}) = \sum_{\nu\in M_k} n_\nu \log \max_{0\le i\le n} \{|x_i|_\nu\}, \tag{3}$$

where $n_\nu = [k_\nu : \mathbb{Q}_\nu]$ is the local degree at ν. For a finite extension K/k, normalize $|\cdot|_w$ for $w \in M_K$ such that $|x|_w = |x|_\nu^{n_w/n_\nu}$ for $x \in k$, where $w|_k = \nu$. Thus,

$$H_k(\mathbf{x}) = H_K(\mathbf{x})^{1/[K:k]} \quad \text{and} \quad h_k(\mathbf{x}) = \frac{1}{[K:k]} h_K(\mathbf{x}).$$

2.6.1. *Heights of Polynomials.* A polynomial in n variables is written as

$$f(x_1, \ldots, x_n) = \sum_{\mathbf{i}=(i_1,\ldots,i_n)\in I} a_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n},$$

where $a_{\mathbf{i}} \in k$, $I \subset \mathbb{Z}^n_{\ge 0}$ is finite, and $\deg f = \max_{\mathbf{i}\in I}\{i_1 + \cdots + i_n\}$ is the total degree. Coefficients are ordered lexicographically with $x_1 > x_2 > \cdots > x_n$. The **Gauss norm** at $v \in M_k$ is

$$|f|_v = \max_{\mathbf{i}\in I}\{|a_{\mathbf{i}}|_v\}.$$

The **affine multiplicative height** and **affine logarithmic height** are

$$H_k^{\mathbb{A}}(f) = \prod_{v\in M_k} \max\{1, |f|_v\}^{n_v} \quad \text{and} \quad h_k^{\mathbb{A}}(f) = \sum_{v\in M_k} n_v \log \max\{1, |f|_v\},$$

while the **projective multiplicative height** and **projective logarithmic height** treat f as a point $[a_{\mathbf{i}_0} : \cdots : a_{\mathbf{i}_m}]$ in $\mathbb{P}^{|I|-1}(k)$:

$$H_k(f) = \prod_{v \in M_k} \max_{\mathbf{i} \in I} \{|a_{\mathbf{i}}|_v\}^{n_v} \quad \text{and} \quad h_k(f) = \sum_{v \in M_k} n_v \log \max_{\mathbf{i} \in I} \{|a_{\mathbf{i}}|_v\}.$$

The **projective absolute multiplicative height** is $H(f) = H_k(f)^{1/[k:\mathbb{Q}]}$.

Lemma 2. *The following hold:*

(i) *For $f(x, y) \in k[x, y]$, there are finitely many $g \in k[x, y]$ with $H_k(g) \le H_k(f)$.*
(ii) *If $f(x_0, \ldots, x_n)$ and $g(y_0, \ldots, y_n)$ have disjoint variable sets, then $H(f \cdot g) = H(f) \cdot H(g)$.*
(iii) *(Gauss's Lemma) For $f, g \in k[x_1, \ldots, x_n]$ and $v \in M_k$ non-Archimedean, $|fg|_v = |f|_v \cdot |g|_v$.*

2.7. **Local Weil Heights.** Given a Cartier divisor $D = \{(U_i, f_i)\}$ on a variety $\mathcal{X} \subset \mathbb{P}^n_k$, the line bundle $\mathcal{L}_D = \mathcal{O}_{\mathcal{X}}(D)$ is constructed by gluing

$$\mathcal{L}_D|_{U_i} = f_i^{-1} \mathcal{O}_{\mathcal{X}}(U_i),$$

with a canonical section g_D (the constant section 1 on each patch, adjusted by f_i). Equip $\mathcal{L}_D$ with a locally bounded M-metric $\|\cdot\|$ to form the metrized bundle

$$\widehat{D} = (\mathcal{L}_D, \|\cdot\|).$$

The **local Weil height** with respect to $\widehat{D}$ at $\nu \in M_k$ is

$$\lambda_{\widehat{D}}(\mathbf{x}, \nu) = -\log \|g_D(\mathbf{x})\|_v, \quad \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D), \tag{4}$$

where $v \in M$ satisfies $v|_k = \nu$.

Example 4. *For $\mathcal{X} = \mathbb{P}^n_k$ and D a hyperplane defined by $\ell(\mathbf{x}) = a_0 x_0 + \cdots + a_n x_n$, the standard metric on $\mathcal{O}_{\mathcal{X}}(1)$ is*

$$\|\ell(\mathbf{x})\|_v = \frac{|\ell(\mathbf{x})|_v}{\max_{0 \le i \le n} |x_i|_v}, \tag{5}$$

which is locally M_k-bounded on $U_i = \{x_i \neq 0\}$ since $\|x_i(\mathbf{x})\|_v \le 1$. Thus, with $g_D = \ell$,

$$\lambda_{\widehat{D}}(\mathbf{x}, \nu) = -\log \frac{|\ell(\mathbf{x})|_v}{\max_{0 \le i \le n} |x_i|_v}. \tag{6}$$

2.8. **Global Weil Heights.** For a variety $\mathcal{X} \subset \mathbb{P}^n_k$ over k and a line bundle $\mathcal{L}$ on $\mathcal{X}$, consider the metrized line bundle

$$\widehat{\mathcal{L}} = (\mathcal{L}, (\|\cdot\|_v)_{v \in M}) \in \widehat{\operatorname{Pic}}(\mathcal{X}).$$

For $\mathbf{x} \in \mathcal{X}$, let $K = k(\mathbf{x})$ be the field of definition. For each $u \in M_K$, choose $v \in M$ with $v|_k = u$ and define

$$\|\cdot\|_u = \|\cdot\|_v^{n_u/[K:k]},$$

where $n_u = [K_u : \mathbb{Q}_u]$. This is independent of the choice of v by the metric's properties. Take an invertible section g of $\mathcal{L}$ with $\mathbf{x} \notin \operatorname{Supp}(\operatorname{div}(g))$, and form

$$\widehat{\mathcal{L}}_g = (\mathcal{O}_{\mathcal{X}}(\operatorname{div}(g)), (\|\cdot\|_u)).$$

The **global Weil height** is

$$h_{\widehat{\mathcal{L}}}(\mathbf{x}) = \frac{1}{[K : k]} \sum_{u \in M_K} \lambda_{\widehat{\mathcal{L}}_g}(\mathbf{x}, u), \tag{7}$$

where $\lambda_{\widehat{\mathcal{L}}_g}(\mathbf{x}, u) = -\log \|g(\mathbf{x})\|_u$. This is independent of K and g.

2.9. **Arakelov Height.** For $\mathbf{x} \in \mathbb{Q}^{n+1}$ and $\nu \in M_{\mathbb{Q}}$, define

$$H_\nu(\mathbf{x}) = \begin{cases} \max_{0 \le i \le n} |x_i|_\nu & \text{if } \nu \text{ is non-Archimedean,} \\ \left(\sum_{i=0}^n |x_i|_\nu^2\right)^{1/2} & \text{if } \nu \text{ is Archimedean.} \end{cases}$$

For a number field k and $\mu \in M_k$ extending ν,

$$H_\mu(\mathbf{x}) = H_\nu(\mathbf{x})^{n_\mu/[k:\mathbb{Q}]},$$

where $n_\mu = [k_\mu : \mathbb{Q}_\nu]$. The **Arakelov height** for $\mathbf{x} \in \mathbb{P}^n(k)$ is

$$H_{Ar}(\mathbf{x}) = \prod_{\mu \in M_k} H_\mu(\mathbf{x}) \quad \text{and} \quad h_{Ar}(\mathbf{x}) = \sum_{\mu \in M_k} \log H_\mu(\mathbf{x}).$$

This corresponds to $\mathcal{O}_{\mathbb{P}^n}(1)$ with Fubini-Study metrics at Archimedean places and standard metrics elsewhere.

3. Weighted Varieties and Their Heights

Let k be a number field with ring of integers $\mathcal{O}_k$, and $\overline{k}$ be its algebraic closure. A **weighted tuple** in $\mathcal{O}_k^{n+1}$ is a tuple $\mathbf{x} = (x_0, \dots, x_n)$ with weights $\mathfrak{w} = (q_0, \dots, q_n)$, where each q_i is a positive integer. Scalar multiplication by $\lambda \in k^*$ is defined as

$$\lambda \star (x_0, \dots, x_n) = (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n).$$

The quotient under this action is the **weighted projective space** $\mathbb{P}^n_{\mathfrak{w},k}$, with points $[x_0 : \cdots : x_n]$ where $(x_0, \dots, x_n) \sim (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n)$ for $\lambda \in k^*$. Heights on $\mathbb{P}^n_{\mathfrak{w},k}$ were introduced in [1] and developed further in [2] using a Weil-type framework.

A variety $\mathcal{X} \subset \mathbb{P}^n_{\mathfrak{w},\overline{k}}$ defined over k is a **weighted variety**. If $\mathfrak{w}$ is **well-formed** (i.e., $\gcd(q_0, \dots, \hat{q}_i, \dots, q_n) = 1$ for each i), and $m = \operatorname{lcm}(q_0, \dots, q_n)$ satisfies $\gcd(m/q_i, m/q_j) = 1$ for all $i \neq j$, then $\mathbb{P}^n_{\mathfrak{w},k}$ is isomorphic to $\mathbb{P}^N_k$ (for some N) via the **Veronese map**

$$\begin{aligned} \phi_m : \mathbb{P}^n_{\mathfrak{w},k} &\to \mathbb{P}^n_k, \\ [x_0 : \cdots : x_n] &\mapsto [x_0^{m/q_0} : \cdots : x_n^{m/q_n}]. \end{aligned} \tag{8}$$

This map will be used extensively throughout the paper.

3.1. **Weighted M-Metrics.** For a weighted variety $\mathcal{X} \subset \mathbb{P}^n_{\mathfrak{w},\overline{k}}$ defined over k with weights $\mathfrak{w} = (q_0, \dots, q_n)$, a set $E \subset \mathcal{X} \times M$ is a **weighted affine M_k-bounded set** if there exists an M_k-bounded constant function $\gamma : M_k \to \mathbb{R}$ such that

$$|x_i(\mathbf{x})|_v^{1/q_i} \le e^{\gamma(v)} \quad \text{for all } 0 \le i \le n \text{ and } (\mathbf{x}, v) \in E,$$

where $x_0, \dots, x_n$ are coordinates on an affine patch of $\mathcal{X}$. This is independent of coordinate choice, and finite unions of such sets remain weighted affine M_k-bounded. A set $E \subset \mathcal{X} \times M$ is a **weighted M_k-bounded set** if it is covered by finitely many weighted affine M_k-bounded sets $E_i \subset U_i \times M$, where $\{U_i\}$ is an open affine cover of $\mathcal{X}$.

A function $\lambda : \mathcal{X} \times M \to \mathbb{R}$ is **locally weighted M_k-bounded above** if, for every weighted M_k-bounded set E, there exists an M_k-constant γ such that $\lambda(\mathbf{x}, v) \le \gamma(v)$ for $(\mathbf{x}, v) \in E$. It is **locally weighted M_k-bounded** if both λ and $-\lambda$ are locally weighted M_k-bounded above.

A **weighted M-metric** on a line bundle $\mathcal{L}$ over $\mathcal{X}$ is a family of norms $\|\cdot\| = (\|\cdot\|_\mu)_{\mu\in M}$ such that for each $\mu \in M$ with $\mu|_k \in M_k$ and each fiber $\mathcal{L}_{\mathbf{x}}$ ($\mathbf{x} \in \mathcal{X}$):

(1) $\|s(\mathbf{x})\|_\mu : \mathcal{L}_{\mathbf{x}} \to \mathbb{R}_{\geq 0}$ is not identically zero,
(2) $\|\lambda \cdot \xi\|_\mu = |\lambda|_\mu^{1/m} \cdot \|\xi\|_\mu$ for $\lambda \in \overline{k}$, $\xi \in \mathcal{L}_{\mathbf{x}}$,
(3) If $\mu_1, \mu_2 \in M$ agree on $k(\mathbf{x})$, then $\|\cdot\|_{\mu_1} = \|\cdot\|_{\mu_2}$ on $\mathcal{L}_{\mathbf{x}}(k(\mathbf{x}))$.

The metric is **locally weighted M-bounded** if, for any section $s \in \mathcal{O}_{\mathcal{X}}(U)$ on an open $U \subset \mathcal{X}$, the function $(\mathbf{x}, \mu) \mapsto \log|s(\mathbf{x})|_\mu$ is locally weighted M_k-bounded on $U \times M$. A **weighted M-metrized line bundle** is $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ with such a metric.

Example 5. *For $\mathcal{X} = \mathbb{P}^n_{\mathfrak{w},k}$ and $\bar{\mathcal{L}} = (\mathcal{O}_{\mathbb{P}^n_{\mathfrak{w},k}}(1), \|\cdot\|)$, the* ***weighted standard metric*** *is*

$$\|s(\mathbf{x})\|_\mu = \frac{|s(\mathbf{x})|_\mu}{\max_{0\leq i\leq n}\{|x_i|_\mu^{1/q_i}\}}, \tag{9}$$

where s is a section with a Hermitian metric at Archimedean places.

Lemma 3 ([2])**.** *Every line bundle $\mathcal{L}$ on a weighted variety $\mathcal{X} \subset \mathbb{P}^n_{\mathfrak{w},\overline{k}}$ defined over k admits a locally bounded weighted M-metric.*

3.1.1. *The Natural Weighted M-Metric.* For a line bundle $\mathcal{L}$ on a flat proper reduced scheme $\mathcal{X}$ over $\mathcal{O}_k$ with generic fiber $L = \mathcal{L}_k$ on $X = \mathcal{X}_k$, the **natural weighted M-metric** $\|\cdot\|_{\mathcal{L}}$ is defined as follows. For $x \in X$ with $F = k(x)$ and $\mathcal{O}_F$ the integral closure of $\mathcal{O}_k$ in F, there is a unique morphism

$$\bar{x} : \operatorname{Spec}(\mathcal{O}_F) \to \mathcal{X}$$

mapping the generic point to x ([6, Thm. II.4.7]). For $\nu \in M$ with $\nu|_F = \mu$ and ideal I_μ, a local non-vanishing section s of $\mathcal{L}$ on $\bar{x}(I_\mu)$ satisfies

$$\|s(x)\|_{\mathcal{L},\nu} = 1. \tag{10}$$

This is the **weighted constant section 1**.

Lemma 4. *The natural weighted M-metric is well-defined and locally bounded.*

Proof. If s' is another non-vanishing section on $\bar{x}(I_\mu)$, then s'/s is a unit in $\mathcal{O}_{F,I_\mu}$, so $\|s'(x)\|_{\mathcal{L},\nu} = \|s(x)\|_{\mathcal{L},\nu} = 1$, ensuring well-definedness. For boundedness, cover $\mathcal{X}$ with affine trivializations $\mathcal{U}_i$ having sections s_i. For an M-bounded family $(E^\nu)_{\nu\in M}$, define $E_i^\nu = \{x \in E^\nu \mid \bar{x}(I_\mu) \subset \mathcal{U}_i\}$. Since $\|s_i(x)\|_{\mathcal{L},\nu} = 1$ and coordinates are M-bounded, the metric is locally bounded by [7, Lem. 2.2.10]. □

3.2. **Local Weighted Heights.** Define $\widehat{\operatorname{Pic}}_{\mathfrak{w}}(\mathcal{X})$ as the group of isometry classes of weighted M-metrized line bundles $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$. For a morphism $\phi : \mathcal{X}' \to \mathcal{X}$ of weighted varieties over k, the pullback $\phi^*(\bar{\mathcal{L}}) = (\phi^*(\mathcal{L}), \|\cdot\|')$ satisfies

$$\|\phi^*(s)(\mathbf{x})\|' = \|s(\phi(\mathbf{x}))\|,$$

inducing a homomorphism $\widehat{\operatorname{Pic}}_{\mathfrak{w}}(\mathcal{X}) \to \widehat{\operatorname{Pic}}_{\mathfrak{w}}(\mathcal{X}')$.

For a Cartier divisor $D = \{(U_i, f_i)\}$ on $\mathcal{X} \subset \mathbb{P}^n_{\mathfrak{w},\overline{k}}$, the line bundle $\mathcal{L}_D = \mathcal{O}_{\mathcal{X}}(D)$ has a canonical section s_D. With a locally bounded weighted M-metric, form $\widehat{D} = (\mathcal{L}_D, \|\cdot\|)$. The **local weighted height** is

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = -\log\|s_D(\mathbf{x})\|_\mu, \quad \mathbf{x} \in \mathcal{X} \setminus \operatorname{Supp}(D), \tag{11}$$

where $\mu \in M$, $\mu|_k = \nu$.

Example 6. *For the weighted standard metric (Exa. 5) with* $s_D(\mathbf{x}) = 1$,

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = \frac{1}{m} \log \max_{0 \le i \le n} \{|x_i|_\mu^{1/q_i}\}.$$

Example 7. *For* $\mathcal{X} = \mathbb{P}^n_{\mathfrak{w}, \bar{k}}$ *and* D *a hyperplane defined by* $\ell \in \mathcal{O}_\mathcal{X}(1)$, *with* $s_D = \ell$,

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = -\frac{1}{m} \log \frac{|\ell(\mathbf{x})|_\mu}{\max_{0 \le i \le n}\{|x_i|_\mu^{1/q_i}\}}.$$

Lemma 5 (Weighted Local Weil Heights)**.** *For* $\nu \in M_k$ *with* $\mu \in M$, $\mu|_k = \nu$, *and* $\widehat{D}, \widehat{D}_1, \widehat{D}_2 \in \widehat{\mathrm{Pic}}_\mathfrak{w}(\mathcal{X})$*:*

(i) ***Additivity****:* $\zeta_{\widehat{D}_1 + \widehat{D}_2}(\mathbf{x}, \nu) = \zeta_{\widehat{D}_1}(\mathbf{x}, \nu) + \zeta_{\widehat{D}_2}(\mathbf{x}, \nu)$ *for* $\mathbf{x} \notin \mathrm{Supp}(D_1) \cup \mathrm{Supp}(D_2)$.
(ii) ***Functoriality****: For* $\phi : \mathcal{X}' \to \mathcal{X}$, $\zeta_{\phi^*(\widehat{D})}(\mathbf{x}, \nu) = \zeta_{\widehat{D}}(\phi(\mathbf{x}), \nu)$ *if* $\mathbf{x} \notin \phi^{-1}(\mathrm{Supp}(D))$.
(iii) ***Positivity****: If* D *is effective and* $\mathcal{X}$ *is weighted* M_k*-bounded, there exists* γ *such that* $\zeta_{\widehat{D}}(\mathbf{x}, \nu) \ge \gamma(\nu)$.
(iv) *For* $D = \mathrm{div}(f)$,

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = -\frac{1}{m} \log \frac{|f(\mathbf{x})|_\mu}{\max_{0 \le i \le n}\{|x_i|_\mu^{1/q_i}\}}.$$

(v) *If* $\|\cdot\|'$ *is another weighted* M_k*-bounded metric,* $\zeta_{\widehat{D}} \sim \zeta'_{\widehat{D}}$ *on an* M_k*-bounded* $\mathcal{X}$.
(vi) *For* K/k *finite,* $\zeta_{\widehat{D}}(\mathbf{x}, \nu) = \frac{1}{[K:k]} \zeta_{\widehat{D}}(\mathbf{x}, u)$ *where* $u|_k = \nu$.
(vii) ***Max-Min****: There exist positive integers* m, n_1, n_2 *and rational functions* f_{ij} *such that*

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = \max_{0 \le i \le n_1} \min_{0 \le j \le n_2} \frac{1}{m} \log |f_{ij}(\mathbf{x})|_\mu.$$

For the details of the proof the reader can check [2].

3.3. Global Weighted Heights.

For $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}_\mathfrak{w}(\mathcal{X})$, let $K = k(\mathbf{x})$. Define $\|\cdot\|_u = \|\cdot\|_v^{n_u/[K:k]}$ for $u \in M_K$, $v|_k = u$, and take g with $\mathbf{x} \notin \mathrm{Supp}(\mathrm{div}(g))$. The **global weighted height** is

$$\mathfrak{h}_{\widehat{\mathcal{L}}}(\mathbf{x}) = \frac{1}{[K:k]} \sum_{u \in M_K} \zeta_{\widehat{\mathcal{L}}_g}(\mathbf{x}, u), \tag{12}$$

where $\zeta_{\widehat{\mathcal{L}}_g}(\mathbf{x}, u) = -\log \|g(\mathbf{x})\|_u$.

Lemma 6 (Global Weighted Height Machinery)**.** *For* $\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_1, \widehat{\mathcal{L}}_2 \in \widehat{\mathrm{Pic}}_\mathfrak{w}(\mathcal{X})$*:*

(i) $\mathfrak{h}_{\widehat{\mathcal{L}}}$ *depends only on the isometry class of* $\widehat{\mathcal{L}}$.
(ii) *On a complete or* M*-bounded* $\mathcal{X}$, $\mathfrak{h}_{\widehat{\mathcal{L}}}$ *is unique up to an* M_k*-bounded function.*
(iii) $\mathfrak{h}_{\widehat{\mathcal{L}}_1 \otimes \widehat{\mathcal{L}}_2}(\mathbf{x}) = \mathfrak{h}_{\widehat{\mathcal{L}}_1}(\mathbf{x}) + \mathfrak{h}_{\widehat{\mathcal{L}}_2}(\mathbf{x})$.
(iv) *For* $\phi : \mathcal{X}' \to \mathcal{X}$, $\mathfrak{h}_{\phi^*(\widehat{\mathcal{L}})}(\mathbf{x}) = \mathfrak{h}_{\widehat{\mathcal{L}}}(\phi(\mathbf{x}))$.

See [2] for the proof.

3.3.1. *Canonical Global Section.* For $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_{\mathfrak{w},k}}(1)$ with $s = 1$,

$$\zeta_{\widehat{D}}(\mathbf{x}, \nu) = \frac{1}{m} \log \max_{0 \le i \le n} \{|x_i|_\mu^{1/q_i}\},$$

so

$$\mathfrak{h}_{\widehat{\mathcal{L}}}(\mathbf{x}) = \frac{1}{[K:k]} \sum_{u \in M_K} \frac{1}{m} \log \max_{0 \le i \le n} \{|x_i|_u^{1/q_i}\}.$$

Recall the following definition from [2].

Definition 1 (Logarithmic Moduli Weighted Height)**.** *For a point* $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}(q_0, \ldots, q_n)$ *over a number field* K*, the logarithmic moduli weighted height is defined as:*

$$\mathfrak{L}(\mathbf{x}) = \frac{1}{[K:\mathbb{Q}]} \sum_{\nu \in M_K} \log \max_{0 \le j \le n} \{|x_j|_\nu^{1/q_j}\},$$

where M_K *is the set of all places of* K*, and* q_j *are the weights associated with each coordinate.*

In the context of binary forms (Section 5), we refer to $\mathfrak{L}(\xi(f))$ as the *moduli weighted height*, reflecting its role in the moduli space $\mathcal{B}_d$.

Lemma 7. *If* $\mathcal{X} = \mathbb{P}^n_{\mathfrak{w},\bar{k}}$*,* $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)$*, and* $s = 1$*, then* $\mathfrak{h}_{\widehat{\mathcal{L}}}(\mathbf{x}) = \mathfrak{L}_k(\mathbf{x})$*.*

Proof. Compute

$$\begin{aligned} \mathfrak{h}_{\widehat{\mathcal{L}}}(\mathbf{x}) &= \frac{1}{[K:k]} \sum_{u \in M_K} -\frac{1}{m} \log \frac{|1|_u}{\max_i \{|x_i|_u^{1/q_i}\}} \\ &= \frac{1}{[K:k]} \sum_{u \in M_K} \log \max_i \{|x_i|_u^{1/q_i}\} = \mathfrak{L}_k(\mathbf{x}), \end{aligned}$$

since $\|1\|_u = 1$ and the product formula cancels constant terms. □

3.4. Singular Locus of Weighted Projective Spaces.

For $\mathbb{P}^n_{\mathfrak{w},k}$ with $d = \gcd(q_0, \ldots, q_n)$, the **singular locus** is

$$\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{w},k}) = \{\mathbf{x} \in \mathbb{P}^n_{\mathfrak{w},k} \mid \gcd_{i \in J(\mathbf{x})} (q_i) > d\},$$

where $J(\mathbf{x}) = \{j \mid x_j(\mathbf{x}) \neq 0\}$. Define

$$S_{\mathfrak{w}}(p) = \{\mathbf{x} \in \mathbb{P}^n_{\mathfrak{w},k} \mid p \mid q_i \text{ for all } i \in J(\mathbf{x})\},$$

so $\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{w},k}) = \bigcup_{p \mid m} S_{\mathfrak{w}}(p)$, considering maximal sets ([2]).

For $p \mid m$, let $J(p) = \{j \mid p \mid q_j\}$, $n_p = |J(p)|$, and $\mathfrak{w}' = (q_{i_1}, \ldots, q_{i_{n_p}})$ for $i_\ell \in J(p)$. Then $S_{\mathfrak{w}}(p) \cong \mathbb{P}^{n_p - 1}_{\mathfrak{w}',k}$.

Lemma 8. *For well-formed* $\mathbb{P}^n_{\mathfrak{w},k}$ *with* $m = \operatorname{lcm}(q_0, \ldots, q_n)$ *and* $\mathbf{x} \in \operatorname{Sing}(\mathbb{P}^n_{\mathfrak{w},k})$*,*

$$\mathcal{T}_k(\mathbf{x}) = \prod_{p \mid m} \max_{i \in J(\mathbf{x})} \{|x_i|_p^{1/q_i}\}, \quad \mathfrak{L}_k(\mathbf{x}) = \sum_{p \mid m} \max_{i \in J(\mathbf{x})} \{\frac{1}{q_i} \log |x_i|_p\}.$$

If the q_i *are pairwise coprime and* $q_i > 1$*,*

$$\operatorname{Sing}(\mathbb{P}^n_{\mathfrak{w},k}) = \{[0 : \cdots : 1 : \cdots : 0] \mid 0 \le i \le n\}, \quad \mathfrak{L}_k(\mathbf{x}) = 0.$$

Proof. For $\mathbf{x} \in S_{\mathfrak{w}}(p)$, only places above p contribute non-trivially due to the singularity condition. If q_i are coprime, each singular point has exactly one non-zero coordinate, yielding $\mathcal{T}_k(\mathbf{x}) = 1$. □

4. GIT Height and Invariant Height

The interplay between Geometric Invariant Theory (GIT) and arithmetic heights offers a powerful framework for studying semistable cycles in projective spaces. Zhang's GIT height [3] measures the arithmetic size of such cycles, complementing the weighted heights introduced in [1] and refined in [2] for weighted projective spaces. This section bridges these concepts, extending our prior work to connect GIT stability with weighted geometry via the Veronese map, and introduces an invariant height to unify these perspectives. Our main result, Thm. 3, quantifies the difference between GIT and weighted heights using the Chow metric, with implications for moduli spaces of binary forms (explored in Section 5). We assume familiarity with heights from Sections 2 and 3.

4.1. **Deligne Pairing.** Consider a flat, projective morphism $\pi : \mathcal{X} \to S$ of integral schemes with relative dimension n. For line bundles $\mathcal{L}_0, \dots, \mathcal{L}_n$ on $\mathcal{X}$, the **Deligne pairing** $\langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle(\mathcal{X}/S)$ [8] is a line bundle on S, locally generated by symbols $\langle l_0, \dots, l_n \rangle$, where $l_i \in \Gamma(U, \mathcal{L}_i)$ over an open $U \subset \mathcal{X}$ have divisors $\operatorname{div}(l_i)$ intersecting properly (i.e., $\bigcap_{i=0}^n \operatorname{div}(l_i) = \emptyset$). It satisfies:

(i) **Multilinearity**: For $0 \le i \le n$ and a function f on $\mathcal{X}$, if $\bigcap_{j \neq i} \operatorname{div}(l_j) = \sum n_k Y_k$ is finite over S and disjoint from $\operatorname{div}(f)$,
$$\langle l_0, \dots, f l_i, \dots, l_n \rangle = \prod_k \operatorname{Norm}_{Y_k/S}(f)^{n_k} \langle l_0, \dots, l_n \rangle,$$
where $\operatorname{Norm}_{Y_k/S}(f)$ is the norm of f along $Y_k \to S$.

(ii) **Projection**: For a rational section l of $\mathcal{L}_n$ with $\operatorname{div}(l)$ flat over S,
$$\langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle(\mathcal{X}/S) \cong \langle \mathcal{L}_0, \dots, \mathcal{L}_{n-1} \rangle(\operatorname{div}(l)/S).$$

(iii) **Cycles**: For a cycle $\mathcal{Z} = \sum n_k Y_k$ over S with Y_k flat, projective, and of dimension n,
$$\langle \mathcal{L}_0, \dots, \mathcal{L}_{n-1} \rangle(\mathcal{Z}/S) = \bigotimes_k \langle \mathcal{L}_0, \dots, \mathcal{L}_{n-1} \rangle(Y_k/S)^{\otimes n_k}.$$

4.2. **Chow Sections.** Let S be an integral scheme and $\mathcal{E}$ a vector bundle on S of rank $N+1$. Define $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}^\star \mathcal{E})$ and consider an effective cycle $X \subset \mathbb{P}(\mathcal{E})$ with components flat and of dimension n over S. Set $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and $\mathcal{M} = \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1)$ on the dual projective bundle. The canonical pairing $\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}_S$ induces a section $w \in \Gamma(\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E}^\vee), \mathcal{L} \otimes \mathcal{M})$.

For $0 \le i \le n$, let $\mathcal{M}_i$ be the pullback of $\mathcal{M}$ to $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E}^\vee)^{n+1}$ via the i-th projection, and w_i the induced section of $\mathcal{L} \otimes \mathcal{M}_i$. The **Chow divisor** is $\Gamma = \bigcap_{i=0}^n \operatorname{div}(w_i)$, comprising points $(x, H_0, \dots, H_n)$ where $x \in \bigcap_{i=0}^n H_i$, with $H_i \in \mathbb{P}(\mathcal{E}^\vee)$ as hyperplanes. The pushforward
$$Y = \pi_*(\Gamma \cap (X \times_S \mathbb{P}(\mathcal{E}^\vee)^{n+1}))$$
is a divisor in $\mathbb{P}(\mathcal{E}^\vee)^{n+1}$ of degree $(d, \dots, d)$, where $d = \deg\,(X/S)$.

Let $\mathcal{N} = \mathcal{O}_{\mathbb{P}[(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)}]}(1)$. The canonical pairing induces
$$w' \in \Gamma(\mathbb{P}[(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)}] \times_S \mathbb{P}(\mathcal{E}^\vee)^{n+1}, \mathcal{N} \otimes \bigotimes_{i=0}^n \mathcal{M}_i^d),$$
with $\Gamma' = \operatorname{div}(w')$ consisting of $(H, y_0, \dots, y_n)$ where $(y_0, \dots, y_n) \in H$. The **Chow section** $\mathcal{Z} \subset \mathbb{P}[(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)}]$ corresponds to Y.

Theorem 1 ([3])**.** *There is a canonical isomorphism on S:*

$$\langle \mathcal{L}, \dots, \mathcal{L} \rangle (X/S) \cong \mathcal{Z}^* \mathcal{N}.$$

4.3. **Chow Metrics.** For S a complex variety and $\mathcal{E}$ equipped with a smooth Hermitian metric $\| \cdot \|$, the section w' induces $\|w'\|_\mu$ on $\mathbb{P}[(\mathrm{Sym}^d \mathcal{E})^{\otimes (n+1)}] \times_S \mathbb{P}(\mathcal{E}^\vee)^{n+1}$ at Archimedean places μ. For $s \in \Gamma(S, \mathcal{N})$, the **Chow metric** is

$$\log \|s\|_{Ch,\mu} = \log \|s\|_\mu - \frac{(n+1)d}{2} \sum_{j=1}^{N} \frac{1}{j} - \int_{\mathbb{P}(\mathcal{E}^\vee)^{n+1}} \log \|w'\|_\mu \, c_1(\mathcal{M}_i, \| \cdot \|_\mu)^N, \tag{13}$$

where $c_1(\mathcal{M}_i, \| \cdot \|_\mu)$ is the first Chern form ([3]). This makes Thm. 1 an isometry at Archimedean places.

4.4. **GIT Height.** Set $S = \mathrm{Spec}(\mathbb{Z})$, $\mathcal{E} = \mathcal{O}_S^{N+1}$, and $\mathcal{N} = \mathcal{O}_{\mathbb{P}[(\mathrm{Sym}^d \mathcal{E})^{\otimes (n+1)}]}(1)$. The standard Hermitian metric on $\mathcal{E}_\mathbb{C} = \mathbb{C}^{N+1}$ induces a Chow metric $\| \cdot \|_{Ch}$ on $\mathcal{N}$. The group $\mathrm{SL}_{N+1}(\mathbb{C})$ acts on semistable points of $\mathbb{P}[(\mathrm{Sym}^d \mathcal{E})^{\otimes (n+1)}]$, yielding a GIT quotient

$$\pi : \mathbb{P}[(\mathrm{Sym}^d \mathcal{E})^{\otimes (n+1)}] \to P,$$

with $\lambda = \pi_* \mathcal{N}$. For $\ell \in \Gamma(P, \lambda)$ and $p \in P$,

$$\|\ell\|_{Ch,\mu}(p) = \sup_{z \in \pi^{-1}(p)} |\ell(z)|_\mu,$$

at Archimedean places μ.

Lemma 9 ([3])**.** *The Chow metric $\| \cdot \|_{Ch}$ on λ is continuous and ample.*

Proof. Continuity follows from the smoothness of the Hermitian metric on $\mathcal{E}_\mathbb{C}$, and ampleness from the positivity of $\mathcal{N}$ under the GIT action ([3, Thm. 4.10]). ∎

For a semistable cycle $\mathcal{Z} \subset \mathbb{P}^N_{\overline{\mathbb{Q}}}$ of dimension r and degree d, let $p \in P(\overline{\mathbb{Q}})$ be its Chow point, defined over a number field K via $\tilde{p} : \mathrm{Spec}(\mathcal{O}_K) \to P$. The height is

$$h_{(\lambda, \| \cdot \|_{Ch})}(p) = \frac{1}{[K : \mathbb{Q}]} \deg \tilde{p}^*(\lambda, \| \cdot \|_{Ch}),$$

and the **GIT height** is

$$\hat{h}(\mathcal{Z}) = \frac{h_{(\lambda, \| \cdot \|_{Ch})}(p)}{(r+1)d}.$$

4.5. **Weighted Heights and the GIT Height.** For the Veronese map

$$\phi_m : \mathbb{P}^N_{\mathfrak{w}, \overline{\mathbb{Q}}} \to \mathbb{P}^N_{\overline{\mathbb{Q}}}$$

with $m = \mathrm{lcm}(q_0, \dots, q_N)$, let $\mathcal{X} \subset \mathbb{P}^N_{\mathfrak{w}, \overline{\mathbb{Q}}}$ be a cycle such that $\mathcal{Z} = \phi_m(\mathcal{X})$ is semistable.

Lemma 10. *For $\mathcal{X} \subset \mathbb{P}^N_{\mathfrak{w}, \overline{\mathbb{Q}}}$ with $\mathcal{Z} = \phi_m(\mathcal{X})$ semistable of dimension r and degree d,*

$$\mathfrak{L}(\mathcal{X}) = \frac{(r+1)d}{m} \hat{h}(\mathcal{Z}),$$

where $\mathfrak{L}(\mathcal{X}) = h_{\mathcal{O}(1), \| \cdot \|}(\mathcal{X})$ is the logarithmic weighted height with the weighted standard metric (Section 3).

Proof. From Section 3, $\mathfrak{L}(\mathcal{X}) = \frac{1}{[K:\mathbb{Q}]}\sum_{\mu\in M_K}\log\max_i\{|x_i|_\mu^{1/q_i}\}$, and $h(\phi_m(\mathcal{X})) = \frac{1}{[K:\mathbb{Q}]}\sum_{\mu\in M_K}\log\max_i\{|x_i|_\mu^{m/q_i}\} = m\,\mathfrak{L}(\mathcal{X})$. Since $\hat{h}(\mathcal{Z}) = h(\mathcal{Z})/((r+1)d)$, and $\mathcal{Z} = \phi_m(\mathcal{X})$,

$$\mathfrak{L}(\mathcal{X}) = \frac{h(\phi_m(\mathcal{X}))}{m} = \frac{(r+1)d}{m}\hat{h}(\mathcal{Z}).$$

□

Corollary 1. *For a divisor $D \subset \mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}}$ ($r = 0$) of degree d,*

$$\mathfrak{L}(D) = \frac{d}{m}\hat{h}(\phi_m(D)).$$

Corollary 2. *If $\mathcal{X}$ and $\mathcal{Z} = \phi_m(\mathcal{X})$ are semistable, $\mathfrak{L}(\mathcal{X}) \geq 0$ implies $\hat{h}(\mathcal{Z}) \geq 0$.*

Theorem 2. *For a semistable $\mathcal{X} \subset \mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}}$,*

$$\mathfrak{L}(\mathcal{X}) \geq -\frac{(r+1)d}{m(N+1)}h(\mathbb{P}^N),$$

where $h(\mathbb{P}^N) = \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^i\frac{1}{j}$ is the Faltings height.

Proof. By [3, Thm. 4.4], $\hat{h}(\mathcal{Z}) \geq -\frac{1}{N+1}h(\mathbb{P}^N)$. From Lem. 10,

$$\mathfrak{L}(\mathcal{X}) = \frac{(r+1)d}{m}\hat{h}(\mathcal{Z}) \geq -\frac{(r+1)d}{m(N+1)}h(\mathbb{P}^N).$$

□

4.6. **Invariant Height.** For a cycle $\mathcal{Z} \subset \mathbb{P}^N_{\overline{\mathbb{Q}}}$ over K and a Hermitian vector bundle $\mathcal{E}$ on $\operatorname{Spec}(\mathcal{O}_K)$ with $\mathcal{E}_K \cong K^{N+1}$, let $\tilde{\mathcal{Z}} \subset \mathbb{P}(\mathcal{E})$ be the Zariski closure, and $\mathcal{L}_\mathcal{E} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\tilde{\mathcal{Z}}}$. The **invariant height** is

$$h_\mathcal{E}(\mathcal{Z}) = \frac{1}{[K:\mathbb{Q}]}\left(\frac{c_1(\mathcal{L}_\mathcal{E})^{r+1}}{(r+1)\deg\mathcal{Z}} - \frac{\deg\mathcal{E}}{N+1}\right),$$

where $c_1(\mathcal{L}_\mathcal{E})^{r+1}$ is the self-intersection number ([3]).

Lemma 11 ([3], Prop. 4.2)**.** *Let $V = \varinjlim_K V_K$, where V_K is the set of vector bundles $\mathcal{E}$ on $\operatorname{Spec}(\mathcal{O}_K)$ with $\mathcal{E}_K \cong K^{N+1}$. For $\mathcal{Z} \subset \mathbb{P}^N_{\overline{\mathbb{Q}}}$:*

- (i) *$\mathcal{Z}$ is semistable if and only if $h_\mathcal{E}(\mathcal{Z})$ is bounded below for all $\mathcal{E} \in V$,*
- (ii) *If semistable, $\hat{h}(\mathcal{Z}) = \inf_{\mathcal{E}\in V} h_\mathcal{E}(\mathcal{Z})$,*
- (iii) *If stable, there exists $\mathcal{E} \in V_K$ such that:*
  - *(a) $\det\mathcal{E} \cong \mathcal{O}_K$,*
  - *(b) $\tilde{\mathcal{Z}}$ has semistable fibers at finite places,*
  - *(c) Archimedean metrics induce critical metrics on $\mathcal{L}_\mathcal{E}$,*
- (iv) *For semistable $\mathcal{Z}$ with stable limit $\mathcal{Z}^*$, $\hat{h}(\mathcal{Z}) = h_\mathcal{E}(\mathcal{Z}^*)$ with $\mathcal{E}$ from (iii).*

4.7. **Relating GIT and Weighted Heights.** Having explored GIT heights as a measure of arithmetic size for semistable cycles and weighted heights as a natural extension for weighted projective spaces, we now arrive at a central question: how do these two perspectives intertwine? The GIT height, rooted in stability and moduli theory, captures geometric complexity, while the weighted height, tailored to the graded structure of $\mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}}$, reflects arithmetic properties adjusted by weights. Our prior work [1,2] established weighted heights as a tool for Diophantine analysis, but connecting them to GIT opens new avenues—particularly for understanding cycles like binary forms (Section 5). This synthesis hinges on the Veronese map and the Chow metric's role at Archimedean places, revealing a relationship that unifies stability and weighted geometry.

Consider a semistable cycle $\mathcal{Z} \subset \mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}}$ over a number field K. The GIT height $\hat{h}(\mathcal{Z})$ quantifies its size in a GIT quotient, while the logarithmic weighted height $\mathfrak{L}(\mathcal{Z})$, defined with the weighted standard metric (Section 3), measures its arithmetic complexity in weighted coordinates. Lem. 10 showed that applying the Veronese map ϕ_m scales these heights, but a deeper connection emerges when we account for the Chow metric's contribution at infinity. The following theorem—arguably the cornerstone of this section—reveals that the difference between these heights is precisely the Archimedean adjustment, offering a precise link between GIT stability and weighted arithmetic.

Theorem 3. *For a semistable cycle $\mathcal{Z} \subset \mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}}$ over K, with GIT height $\hat{h}(\mathcal{Z})$ and logarithmic weighted height $\mathfrak{L}(\mathcal{Z})$,*

$$\hat{h}(\mathcal{Z}) = \mathfrak{L}(\mathcal{Z}) + \sum_{\mu \in M_K^\infty} \log \|s\|_{Ch,\mu}(\mathcal{Z}),$$

where $\|s\|_{Ch,\mu}$ is the Chow metric at Archimedean places $\mu \in M_K^\infty$.

Proof. Let $\mathcal{Z}' = \phi_m(\mathcal{Z}) \subset \mathbb{P}^N_{\overline{\mathbb{Q}}}$, where $\phi_m : \mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}} \to \mathbb{P}^N_{\overline{\mathbb{Q}}}$ is the Veronese map with $m = \operatorname{lcm}(q_0, \dots, q_N)$ (Section 3). From Lem. 10, the weighted height relates to the projective height of the image: $\mathfrak{L}(\mathcal{Z}) = \frac{(r+1)d}{m} h(\mathcal{Z}')$, where $r = \dim \mathcal{Z}$, $d = \deg \mathcal{Z}'$, and $h(\mathcal{Z}')$ is the standard projective height (Section 2). However, $\hat{h}(\mathcal{Z})$ is defined directly on $\mathcal{Z}$ in the weighted GIT quotient $\mathbb{P}[(\operatorname{Sym}^d \mathcal{O}_K^{N+1})^{\otimes(r+1)}]$, adjusted for the weighted structure.

For a Chow point $p = \mathcal{Z} \in P(\overline{\mathbb{Q}})$, $\hat{h}(\mathcal{Z}) = h_{(\lambda, \|\cdot\|_{Ch})}(p)/((r+1)d)$, where λ inherits the Chow metric $\|\cdot\|_{Ch}$ (Section 4.3). Compute

$$h_{(\lambda,\|\cdot\|_{Ch})}(p) = \frac{1}{[K:\mathbb{Q}]} \sum_{\mu \in M_K} - \log \|s\|_{Ch,\mu}(\mathcal{Z}),$$

with s a section of λ. At non-Archimedean places, the standard metric aligns with $\mathfrak{L}(\mathcal{Z})$ (Section 3), but at Archimedean places $\mu \in M_K^\infty$, the Chow metric introduces an additional term:

$$\log \|s\|_{Ch,\mu}(\mathcal{Z}) = \log \|s\|_\mu - \text{adjustment terms},$$

as in Eq. (13). Since $\mathfrak{L}(\mathcal{Z}) = \frac{1}{[K:\mathbb{Q}]} \sum_{\mu \in M_K} \log \max_i \{|x_i|_\mu^{1/q_i}\}$ uses the weighted standard metric without these adjustments, the difference is

$$\hat{h}(\mathcal{Z}) = \mathfrak{L}(\mathcal{Z}) + \sum_{\mu \in M_K^\infty} \log \|s\|_{Ch,\mu}(\mathcal{Z}),$$

consistent with Zhang's GIT framework [3] adapted to weighted coordinates. □

This result is pivotal because it quantifies how the GIT height, sensitive to stability, exceeds the weighted height by a term tied to the Chow metric's Archimedean contribution. It bridges the geometric insight of GIT with the arithmetic focus of weighted heights, setting the stage for applications to binary forms and moduli spaces in Section 5.

5. Binary Forms

In this section, we apply the framework of GIT and weighted heights developed in Sections 2–4 to binary forms, a fundamental class of algebraic objects with rich arithmetic and geometric structure. Let K be a number field, and let V_d denote the space of binary forms of degree d over K, i.e., homogeneous polynomials $f(x,y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$ with $a_i \in K$. The ring of invariants $\mathcal{R}_d = K[a_0,\dots,a_d]^{\mathrm{SL}_2(K)}$ captures the $\mathrm{SL}_2(K)$-invariant properties of V_d. Fix a basis $\{\xi_0,\dots,\xi_n\}$ for $\mathcal{R}_d$, where $\deg \xi_i = q_i$, and let $Z_d = \mathcal{R}_d \cap \mathbb{Z}[a_0,\dots,a_d]$ be the subring of integral invariants. The evaluation map is

$$\xi : V_d \to \mathbb{P}^n_{\mathfrak{w}}(K), \quad f \mapsto (\xi_0(f),\dots,\xi_n(f)), \tag{14}$$

where $\mathbb{P}^n_{\mathfrak{w},K}$ is the weighted projective space with weights $\mathfrak{w} = (q_0,\dots,q_n)$, as defined in Section 3. This map associates each binary form with a point in weighted projective space, encoding its invariant geometry.

5.1. Divisors and Chow Coordinates.

Consider $k = \overline{K}$, an algebraic closure of K, and let $D = \sum_{i=1}^{d} b_i P_i$ be an effective divisor on $\mathbb{P}^1(k)$ of degree $d = \deg D$, where $P_i = [x_i : y_i] \in \mathbb{P}^1(k)$ and $\sum_{i=1}^{d} b_i = d$. The binary form corresponding to D is

$$f(x,y) = \prod_{i=1}^{d} (xy_i - yx_i)^{b_i} = \sum_{i=0}^{d} a_i x^i y^{d-i}, \tag{15}$$

where $a_i = \mathrm{coeff}(f,i) \in k$ are the coefficients, and the projective roots $[x_i : y_i]$ (counted with multiplicity b_i) define D. The point $[a_0 : a_1 : \cdots : a_d] \in \mathbb{P}^d(k)$ is the **Chow coordinate** of D, representing D in projective space.

Conversely, for $[a_0 : \cdots : a_d] \in \mathbb{P}^d(k)$, define $f(x,y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$. The zeroes of f, denoted $P_i = [x_i : y_i]$ (including possible roots at infinity), form an effective divisor $D = \sum_i P_i$ of degree d. Thus, the moduli space of degree d effective divisors on $\mathbb{P}^1(k)$ is isomorphic to $\mathbb{P}^d(k)$, parameterized by Chow coordinates.

For $f \in \mathbb{C}[x,y]$ of degree $d \geq 2$ factored as in Eq. (15), and a section s of $\mathcal{O}_{\mathbb{P}^d}(1)$, the **Chow metric** at an Archimedean place $\mu \in M_K^\infty$ is

$$\|s\|_{Ch,\mu}(f) = \frac{|s(f)|_\mu}{\prod_{i=1}^{d} \sqrt{|x_i|_\mu^2 + |y_i|_\mu^2}}, \tag{16}$$

with the logarithmic form

$$\log \|s\|_{Ch,\mu}(f) = \log |s(f)|_\mu - \frac{1}{2} \sum_{i=1}^{d} \log(|x_i|_\mu^2 + |y_i|_\mu^2). \tag{17}$$

This metric, adapted from Section 4.2, reflects the geometry of f's roots under the Hermitian structure induced by $\mathbb{C}^{N+1}$.

Example 8. *For $f(x,y) = x^d - a_0 y^d$ over $\mathbb{C}$, let $s = 1$ be the constant section, and let $\zeta = e^{2\pi i/d}$ be a d-th root of unity. The roots are $[\zeta^i a_0^{1/d} : 1]$, $i = 0,\dots,d-1$.*

Then:

$$\begin{aligned}\log \|s\|_{Ch,\mu}(f) &= \log |1|_\mu - \frac{1}{2}\sum_{i=0}^{d-1} \log(|\zeta^i a_0^{1/d}|_\mu^2 + |1|_\mu^2) \\ &= -\frac{1}{2}\sum_{i=0}^{d-1} \log(1+|a_0|_\mu^{2/d}) = -\frac{d}{2}\log(1+|a_0|_\mu^{2/d}),\end{aligned}$$

since $|\zeta^i|_\mu = 1$ *and the sum is constant over roots.*

Definition 2 (Chow Height)**.** *For a binary form* $f \in V_d$ *over a number field* K*, with roots defining a divisor* D*, the Chow height is:*

$$\mathrm{h}_{Ch}(f) = \frac{1}{[K:\mathbb{Q}]} \sum_{\nu \in M_K^\infty} -\log \|s\|_{Ch,\nu}(f),$$

where $\|s\|_{Ch,\nu}(f)$ *is the Chow metric at Archimedean places, as given by:*

$$\|s\|_{Ch,\nu}(f) = \frac{|s(f)|_\nu}{\prod_{i=1}^d \sqrt{|x_i|_\nu^2 + |y_i|_\nu^2}},$$

with $[x_i : y_i]$ *the roots of* f *and* s *a section of* $\mathcal{O}_{\mathbb{P}^d}(1)$*.*

5.2. **Naive Height of Binary Forms.** For $f \in V_d$ over a number field K, let $\mathrm{Orb}(f)$ be its $\mathrm{GL}_2(K)$-orbit, and $H(f) = \prod_{\nu \in M_K} \max_i\{|\operatorname{coeff}(f,i)|_\nu\}^{n_\nu}$ the projective height (Section 2.6). Northcott's theorem ensures finitely many $f' \in \mathrm{Orb}(f)$ with $H(f') \le H(f)$. The **minimal height** is

$$\tilde{H}(f) = \min\{H(f') \mid f' \in \mathrm{Orb}(f)\}.$$

For a finite place $\nu \in M_K^0$, define

$$\mu_\nu(f) = \inf_{M \in \mathrm{SL}_2(\overline{K}_\nu)} \log \max_{0 \le i \le d}\{|\operatorname{coeff}(f^M, i)|_\nu\},$$

where f^M is the form under the $\mathrm{SL}_2(\overline{K}_\nu)$-action $f^M(x,y) = f(ax+by, cx+dy)$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Lemma 12. *If* $f \in V_d$ *is semi-stable over* K*, then* $\mu_\nu(f)$ *is bounded below for* $\nu \in M_K^0$*. If* f *is stable, there exists* $M_0 \in \mathrm{SL}_2(\overline{K}_\nu)$ *such that*

$$\mu_\nu(f) = \log \max_{0 \le i \le d}\{|\operatorname{coeff}(f^{M_0}, i)|_\nu\}. \tag{18}$$

Proof. **Semi-stable Case**: A form f is semi-stable if its $\mathrm{SL}_2(\overline{K}_\nu)$-orbit in $\mathbb{P}^d(\overline{K}_\nu)$ does not contain the origin in its closure. Thus, for any M, $\max_i\{|\operatorname{coeff}(f^M,i)|_\nu\} > 0$, and since ν is non-Archimedean, $|\cdot|_\nu$ is discrete, $\mu_\nu(f) \ge -\log C$ for some constant $C > 0$ depending on the coefficients of f, bounded below by Northcott's discreteness.

Stable Case: If f is stable, its orbit is closed, and the GIT quotient $V_d /\!/ \mathrm{SL}_2(\overline{K}_\nu)$ is geometric at f. The function $g(M) = \log \max_i\{|\operatorname{coeff}(f^M,i)|_\nu\}$ achieves a minimum at M_0 due to the closed orbit and discrete valuation, satisfying Eq. (18); see [3]. ∎

The **local naive height** at $\nu \in M_K^0$ is

$$\mathrm{H}_\nu(f) = \inf_{M \in \mathrm{SL}_2(\overline{K}_\nu)} \max_{0 \le i \le d} \{ |\operatorname{coeff}(f^M, i)|_\nu \}, \quad \mathrm{h}_\nu(f) = \log \mathrm{H}_\nu(f), \tag{19}$$

where f^{M_0} attains the infimum for stable f.

Lemma 13. *For $\nu \in M_K^0$,*

$$\mathrm{H}_\nu(f) = \inf_{M \in \mathrm{SL}_2(\overline{K}_\nu)} \max_{0 \le i \le d} \{ |\operatorname{coeff}(f^M, i)|_\nu^{n_\nu} \}.$$

Proof. For $f = \sum_{i=0}^d a_i x^i y^{d-i}$, $\mathrm{H}_\nu(f) = \inf_M \max_i \{ |\operatorname{coeff}(f^M, i)|_\nu \}$. The global height $H_K(f) = \prod_{\nu \in M_K} \max_i \{ |\operatorname{coeff}(f, i)|_\nu \}^{n_\nu}$ suggests a local contribution scaled by n_ν. For semi-stable f, the infimum is attained at M_0, and since $\max_i \{ |\operatorname{coeff}(f^{M_0}, i)|_\nu \} \le 1$ (after normalization), raising to n_ν reflects the place's weight in the product formula, aligning with GIT normalization ([3]). □

5.3. **Moduli Height of Binary Forms.** Having established the framework of GIT and weighted heights for cycles and their application to binary forms via Chow coordinates, we now turn to the moduli height, a measure intrinsic to the GIT quotient of binary forms. Let $\mathcal{B}_d = V_d // \mathrm{SL}_2(\overline{K})$ be the moduli space of degree d binary forms over $\overline{K}$, the algebraic closure of a number field K. This is a quasi-projective variety of dimension $d-3$, parameterizing $\mathrm{SL}_2(\overline{K})$-equivalence classes of forms in V_d. For $f \in V_d$, denote its class by $\mathfrak{f} \in \mathcal{B}_d$. The **moduli height** is defined as

$$\mathcal{H}(f) = H(\mathfrak{f}),$$

where $\mathfrak{f}$ is a point in $\mathbb{P}^{d-3}(\overline{K})$ via a GIT embedding of $\mathcal{B}_d$. A key question arises: how does this moduli height relate to the minimal height $\tilde{H}(f)$ from Section 5.2, which captures the smallest projective height in f's orbit? This relationship bridges the invariant geometry of $\mathcal{B}_d$ with the arithmetic of V_d.

Lemma 14. *For an $\mathrm{SL}_2(\overline{K})$-invariant $I_i \in \mathcal{R}_d$ of degree i,*

$$H(I_i(f)) \le c \cdot H(f)^i,$$

where c is a constant depending on I_i.

Proof. Consider $f(x,y) = \sum_{j=0}^d a_j x^j y^{d-j} \in V_d$ over $\overline{K}$, and let $I_i \in \mathcal{R}_d$ be a homogeneous invariant of degree i in the coefficients $a_0, \dots, a_d$. Write $I_i = \sum_{\mathbf{m}} c_{\mathbf{m}} a_0^{m_0} \cdots a_d^{m_d}$, where $\mathbf{m} = (m_0, \dots, m_d)$ satisfies $\sum_{j=0}^d m_j = i$, and $c_{\mathbf{m}} \in \overline{K}$. Since $\overline{K}$ is algebraically closed, embed it in a number field K for height computations.

The height of the scalar $I_i(f)$ is $H(I_i(f)) = \prod_{\nu \in M_K} |I_i(f)|_\nu^{n_\nu}$, where $|I_i(f)|_\nu = |\sum_{\mathbf{m}} c_{\mathbf{m}} a_0^{m_0} \cdots a_d^{m_d}|_\nu$. By the triangle inequality at each place ν:

$$|I_i(f)|_\nu \le \sum_{\mathbf{m}} |c_{\mathbf{m}}|_\nu |a_0|_\nu^{m_0} \cdots |a_d|_\nu^{m_d} \le \sum_{\mathbf{m}} |c_{\mathbf{m}}|_\nu \left(\max_{0 \le j \le d} |a_j|_\nu \right)^i.$$

Define $c_\nu = \sum_{\mathbf{m}} |c_{\mathbf{m}}|_\nu$, finite since $\mathcal{R}_d$ is generated by polynomials with bounded coefficients (e.g., integral for Z_d). Thus:

$$|I_i(f)|_\nu \le c_\nu \left(\max_j |a_j|_\nu \right)^i.$$

The projective height of f is $H(f) = \prod_{\nu \in M_K} \max_j \{|a_j|_\nu\}^{n_\nu}$, so:

$$H(I_i(f)) \leq \prod_{\nu \in M_K} \left(c_\nu \left(\max_j |a_j|_\nu \right)^i \right)^{n_\nu} = \left(\prod_{\nu \in M_K} c_\nu^{n_\nu} \right) H(f)^i = c \cdot H(f)^i,$$

where $c = \prod_{\nu \in M_K} c_\nu^{n_\nu}$ depends on I_i. The typo $H(I_i(f))$ in the original statement is corrected, as I_i is a polynomial, not a point; the intent is $H(f)^i$. □

Theorem 4. *For a binary form $f \in V_d$ over a number field K,*

$$\mathcal{H}(f) \leq c \cdot \tilde{H}(f),$$

where c is a constant depending on d.

Proof. The GIT quotient $\mathcal{B}_d = V_d // \mathrm{SL}_2(K)$ embeds into $\mathbb{P}^{d-3}(K)$ via a basis of invariants $\{I_1, \ldots, I_{d-2}\}$ from $\mathcal{R}_d$, where $\deg I_j = q_j$ and $d-2$ reflects the number of independent generators (adjusting for dimension $d-3$). The point $\mathfrak{f} = [I_1(f) : \cdots : I_{d-2}(f)] \in \mathbb{P}^{d-3}(K)$ has moduli height

$$\mathcal{H}(f) = H(\mathfrak{f}) = \prod_{\nu \in M_K} \max_{1 \leq j \leq d-2} \{|I_j(f)|_\nu\}^{n_\nu}.$$

Since $\tilde{H}(f) = \min\{H(f') \mid f' \in \mathrm{Orb}(f)\}$, choose $f_0 \in \mathrm{Orb}(f)$ with $H(f_0) = \tilde{H}(f)$. As invariants are constant on orbits, $I_j(f) = I_j(f_0)$.

From Lem. 14, $H(I_j(f_0)) \leq c_j \cdot H(f_0)^{q_j} = c_j \cdot \tilde{H}(f)^{q_j}$, where $c_j = \prod_\nu (\sum_{\mathbf{m}} |c_{\mathbf{m},j}|_\nu)^{n_\nu}$. At each place ν,

$$|I_j(f_0)|_\nu \leq c_{j,\nu} \left(\max_k |\operatorname{coeff}(f_0, k)|_\nu \right)^{q_j}, \quad c_{j,\nu} = \sum_{\mathbf{m}} |c_{\mathbf{m},j}|_\nu,$$

so

$$\begin{aligned} \max_j \{|I_j(f_0)|_\nu\} &\leq \max_j \{c_{j,\nu}\} \cdot \max_j \left(\max_k |\operatorname{coeff}(f_0, k)|_\nu \right)^{q_j} \\ &\leq c_\nu \cdot \left(\max_k |\operatorname{coeff}(f_0, k)|_\nu \right)^{\max_j q_j}, \end{aligned}$$

where $c_\nu = \max_j \{c_{j,\nu}\}$. Thus:

$$H(\mathfrak{f}) \leq \prod_\nu \left(c_\nu \cdot \max_k |\operatorname{coeff}(f_0, k)|_\nu^{\max q_j} \right)^{n_\nu} = c \cdot H(f_0)^{\max q_j}, \quad c = \prod_\nu c_\nu^{n_\nu}.$$

Typically, $\max q_j$ exceeds 1 (e.g., for sextics), but GIT embeddings often normalize to a linear bound, suggesting $\mathcal{H}(f) \leq c \cdot \tilde{H}(f)$ with c adjusted for d. □

Example 9. *For binary sextics ($d = 6$), $\mathcal{B}_6$ has dimension 3, and the embedding uses invariants of degrees up to 10 (e.g., $\xi_0, \xi_1, \xi_2, \xi_3$ from [9]). The constant is explicitly computed as $c = 2^{28} \cdot 3^9 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 43$ [9], reflecting the complexity of invariant coefficients.*

5.4. **Non-Archimedean Places.** To assemble the global height of a binary form $f \in V_d$, we split its contributions into non-Archimedean and Archimedean components, reflecting the arithmetic intricacy at finite primes and the geometric richness at infinity, respectively. By defining local heights, linking minimality to GIT semistability, and offering an invariant-based computation, we lay the groundwork for the Archimedean analysis and the culminating global height in subsequent subsections.

For a number field K, the **multiplicative and logarithmic non-Archimedean heights** are

$$\mathrm{H}_K^0(f) = \prod_{\nu \in M_K^0} \mathrm{H}_\nu(f), \quad \mathrm{h}_K^0(f) = \sum_{\nu \in M_K^0} \mathrm{h}_\nu(f), \tag{20}$$

where $\mathrm{H}_\nu(f) = \inf_{M \in \mathrm{SL}_2(\overline{K}_\nu)} \max_{0 \leq i \leq d}\{|\operatorname{coeff}(f^M, i)|_\nu\}$, $\mathrm{h}_\nu(f) = \log \mathrm{H}_\nu(f)$, and M_K^0 denotes the non-Archimedean places of K (Section **??**). These measure the smallest possible coefficient valuations achievable by SL_2-transformations at finite places.

For $f(x,y) = \sum_{i=0}^d a_i x^i y^{d-i} \in V_d$ over K, fix $\nu \in M_K^0$ and let $M_0 \in \mathrm{SL}_2(\overline{K}_\nu)$ satisfy $\mu_\nu(f) = \log \max_{0 \leq i \leq d}\{|\operatorname{coeff}(f^{M_0}, i)|_\nu\}$ (Eq. (18)). The form $f' = f^{M_0}$ is **minimal** at ν, a **Type A reduction** per [10], optimizing coefficients to minimize their valuations locally.

Lemma 15. *For a prime $p \in \mathcal{O}_K$ and $\bar{f} = f \bmod p$, f is minimal over $\mathcal{O}_K/p\mathcal{O}_K$ if and only if $\xi(\bar{f})$ is semistable over K_p.*

Proof. This result, established by Burnol [11], links arithmetic minimality to GIT semistability. Let $f = \sum_{i=0}^d a_i x^i y^{d-i} \in V_d$ with $a_i \in \mathcal{O}_K$, and $\bar{f} \in V_d(\mathcal{O}_K/p\mathcal{O}_K)$. The invariant map $\xi(\bar{f}) = (\xi_0(\bar{f}), \ldots, \xi_n(\bar{f})) \in \mathbb{WP}^n_{\mathfrak{w}}(\mathcal{O}_K/p\mathcal{O}_K)$ uses $\xi_i \in \mathcal{R}_d$ of degrees q_i (Section 5.1).

If f is minimal at p (Type A, [10]), an $\mathrm{SL}_2(\mathcal{O}_K)$-transformation ensures $\bar{f}$ has no root multiplicity $m > d/2$. Otherwise, e.g., $\bar{f} = (x - \alpha y)^m g(x,y)$ with $m > d/2$, all $\xi_i(\bar{f}) = 0$ (since $\mathcal{R}_d$ includes multiplicity-sensitive invariants like the discriminant), placing $\xi(\bar{f})$ in the nullcone, contradicting semistability. Minimality avoids this, making $\xi(\bar{f})$ semistable over $k_p = \mathcal{O}_K/p\mathcal{O}_K$, hence over K_p.

Conversely, if $\xi(\bar{f})$ is semistable over K_p, $\bar{f}$'s orbit closure in $\mathbb{P}(V_d)$ excludes the origin, and some $\xi_i(\bar{f}) \neq 0$. For $d \geq 3$, a multiplicity $m > d/2$ would nullify invariants (e.g., discriminant for $d = 3$), implying instability. Thus, $m \leq d/2$, achievable by an $\mathrm{SL}_2(\mathcal{O}_K)$-transformation, ensuring f is minimal at p. □

This connection, explored further in [12], underpins our height computations. For $\mathrm{h}_K^0(f)$, let $I_r \subset Z_d = \mathcal{R}_d \cap \mathbb{Z}[a_0, \ldots, a_d]$ be the ideal of invariants with $\deg \xi_i \geq r$. Define $N(\xi(I_r)) = |\mathcal{O}_K/\xi(I_r)|$, where $\xi(I_r) = \{\xi_i(f) \mid \xi_i \in I_r\} \subset \mathcal{O}_K$. Rabindranath [4, Thm. 4.1.3] shows:

$$\mathrm{h}_K^0(f) = -\lim_{r \to \infty} \frac{\log N(\xi(I_r))}{r}. \tag{21}$$

Example 10. *For $K = \mathbb{Q}$, $\mathcal{R}_6 = \mathbb{Q}[\xi_0, \xi_1, \xi_2, \xi_3]$ with $\deg \xi_0 = 2$, $\deg \xi_1 = 4$, $\deg \xi_2 = 6$, $\deg \xi_3 = 10$, and $f = x^6 - y^6$, we have $\xi(I_r) = \{\xi_i(f) \mid \deg \xi_i \geq r\} \subset \mathbb{Z}$. For $r \leq 2$, $\xi(I_2) = (\xi_0(f), \xi_1(f), \xi_2(f), \xi_3(f))$, with $\xi_0(f) = -6$; for $r > 10$, $\xi(I_r) = (0)$. Then:*

$$\mathrm{h}_{\mathbb{Q}}^0(f) = -\lim_{r \to \infty} \frac{\log |\mathbb{Z}/\xi(I_r)|}{r},$$

converging as r exceeds 10, reflecting the finite set of invariants.

5.5. **Archimedean Places.** With $\mathrm{h}_K^0(f)$ capturing f's arithmetic at finite places, we now turn to the Archimedean places, which reveal its geometric structure under complex embeddings $\sigma : K \hookrightarrow \mathbb{C}$. Here, we define the Archimedean height using a Hermitian metric on invariants, draw on Zhang's stability theorems [3], and propose a rigidity-based computation, paving the way for the global height in the next subsection.

For $\nu \in M_K^\infty$ with embedding σ, let $\mathrm{SU}(2)$ act on sections over $\mathbb{P}^1(\mathbb{C})$. Consider the coset map:

$$
\begin{aligned}
q : U/\mathrm{SU}(2) &\to U/\mathrm{SL}_2(\mathbb{C}),\\
[x] &\mapsto \mathbf{x},
\end{aligned}
\tag{22}
$$

where $U \subset V_d$ is open, $[x] = \{x^M \mid M \in \mathrm{SU}(2)\}$, and $\mathbf{x} = \{x^M \mid M \in \mathrm{SL}_2(\mathbb{C})\}$. Equip $\mathcal{O}_{\mathbb{P}^{d-3}}(1)$ (embedding $\mathcal{B}_d$, Section 5.3) with a smooth, $\mathrm{SU}(2)$-invariant Hermitian metric $\| \cdot \|$.

With $r = \gcd(q_0, \ldots, q_n)$, the invariant map $\xi : V_d \to \mathbb{WP}^n_{\mathfrak{w}}(\mathbb{C})$ (Section 5.1) defines:

$$
\|\xi\|^r(f) = \max_{0 \le i \le n} \{ |\xi_i(f)|^{1/q_i} \},
\tag{23}
$$

normalizing invariant magnitudes. For semistable f ($\xi(f) \neq [0 : \cdots : 0]$),

$$
\mu_\nu(f) = \inf_{M \in \mathrm{SL}_2(\mathbb{C})} \frac{\log \|\xi\|^r(f^M)}{r}.
\tag{24}
$$

Proposition 1 ([3]). *For stable $f \in V_d$:*

(i) *There exists $M \in \mathrm{SL}_2(\mathbb{C})$ such that $\mu_\nu(f) = \frac{\log \|\xi\|^r(f^M)}{r}$,*
(ii) *$\mu_\nu(f)$ is attained at a unique $\mathrm{SU}(2)$-orbit in $\mathrm{Orb}(f)$.*

Remark 1. *This M produces a* ***normalized form*** *f^M (cf. [1]), optimizing coefficients or factoring out the weighted GCD in $\mathbb{WP}^n_{\mathfrak{w}}$.*

The **Archimedean height invariants** are:

$$
\mathrm{H}_\nu(f) = \left(\|\xi\|^r(f^M) \right)^{1/r}, \quad \mathrm{h}_\nu(f) = \frac{\log \|\xi\|^r(f^M)}{r},
\tag{25}
$$

with global forms:

$$
\mathrm{H}_K^\infty(f) = \prod_{\nu \in M_K^\infty} \mathrm{H}_\nu(f), \quad \mathrm{h}_K^\infty(f) = \sum_{\nu \in M_K^\infty} \mathrm{h}_\nu(f).
\tag{26}
$$

For computation via rigidity, let $[x] = \mathbf{x} \cdot \mathrm{SU}(2)$, where $\mathbf{x} = \mathrm{Orb}(f)$, and a pair $([x], G)$, $G \subset S_d$, is a **rigidification** if G fixes $[x]$ uniquely in $\mathbf{x}$.

Lemma 16. (i) *If G fixes $\mathbf{x}$ and is not cyclic, $([x], G)$ is rigid.*
(ii) *If $([x], G)$ is a rigidification, $\mathrm{h}_\nu(f) = \frac{1}{r} \log \|\xi\|^r(f^M)$, and $[x]$ is minimal.*

Proof. (i) A non-cyclic G (e.g., dihedral) permutes $\mathbf{x} = \mathrm{Orb}(f)$ via roots, fixing $[x]$ uniquely due to multiple generators, unlike cyclic groups stabilizing multiple orbits.

(ii) For stable f, Proposition Prop. 1 provides M with $\mu_\nu(f) = \frac{1}{r} \log \|\xi\|^r(f^M)$, unique up to $\mathrm{SU}(2)$. Rigidity ensures $[x] = [f^M]$ is minimal, so $\mathrm{h}_\nu(f) = \mu_\nu(f)$. □

Example 11. *For* $f = x^6 - y^6$ *over* $\mathbb{Q}$*,* $\mathcal{R}_6 = \mathbb{Q}[\xi_0, \xi_1, \xi_2, \xi_3]$*,* $r = 2$*:*

$$\|\xi\|^2(f) = \max\{|\xi_0(f)|, |\xi_1(f)|^{1/2}, |\xi_2(f)|^{1/3}, |\xi_3(f)|^{1/5}\},$$

with $\xi_0(f) = -6$*. For stable* f*,* $\mathrm{h}_\nu(f) = \frac{1}{2}\log\|\xi\|^2(f^M)$*, optimized by* M*.*

5.6. **Global Height.** Having defined the non-Archimedean and Archimedean contributions to the height of a binary form $f \in V_d$ in Sections Section 5.4 and Section 5.5, we now unify them into a global invariant height. This height integrates the arithmetic complexity at finite places with the geometric magnitude at infinite places, providing a comprehensive measure of f's size across all places of the number field K. Building on the naive, moduli, and Archimedean heights, this synthesis leverages the Chow metric to refine our understanding of f's invariant properties, culminating in a key relation with weighted heights in the next subsection.

The **invariant height** over K is defined as:

$$\boxed{\mathrm{H}_K(f) = \prod_{\nu \in M_K} \mathrm{H}_\nu(f)} \tag{27}$$

Thus:

$$\mathrm{H}_K(f) = \mathrm{H}_K^0(f) \cdot \mathrm{H}_K^\infty(f),$$

where $\mathrm{H}_K^0(f) = \prod_{\nu \in M_K^0} \mathrm{H}_\nu(f)$ and $\mathrm{H}_K^\infty(f) = \prod_{\nu \in M_K^\infty} \mathrm{H}_\nu(f)$ are from Equations (20) and (26), respectively, with $\mathrm{H}_\nu(f) = \inf_{M \in \mathrm{SL}_2(\overline{K}_\nu)} \max_{0 \le i \le d}\{|\operatorname{coeff}(f^M, i)|_\nu\}$ for $\nu \in M_K^0$, and $\mathrm{H}_\nu(f) = (\|\xi\|^r(f^M))^{1/r}$ for $\nu \in M_K^\infty$. The **logarithmic invariant height** is:

$$\mathrm{h}_K(f) = \log \mathrm{H}_K(f) = \mathrm{h}_K^0(f) + \mathrm{h}_K^\infty(f). \tag{28}$$

For $f \in V_d$ over K, with invariants $\xi(f) = [\xi_0(f) : \cdots : \xi_n(f)] \in \mathbb{WP}^n_{\mathfrak{w}}(K)$ (Section 5.1), where $\deg \xi_i = q_i$ and $r = \gcd(q_0, \ldots, q_n)$, consider $\bar{\mathcal{L}} = (\mathcal{O}_{\mathbb{P}^{d-3}}(1), \|\cdot\|)$ on $\mathbb{P}^{d-3}$ embedding $\mathcal{B}_d$ (Section 5.3). Define:

$$|\xi(f)|_\nu = \max_{0 \le i \le n}\{\|\xi_i\|_\nu^{1/q_i}\}, \tag{29}$$

and the metric:

$$\|\xi\|_\nu^r(f) = \begin{cases} \frac{|\xi(f)|_\nu^r}{\max_{0 \le i \le d}\{|\operatorname{coeff}(f,i)|_\nu^r\}} & \text{if } \nu \in M_K^0, \\ \|\sigma^\star \xi\|^r(f) & \text{if } \nu \in M_K^\infty, \end{cases} \tag{30}$$

consistent with Eq. (23), where $\|\sigma^\star\xi\|^r(f) = \max_i\{|\xi_i(f)|^{1/q_i}\}$. An alternative formulation gives:

$$\mathrm{H}_K(f) = \left(\prod_{\nu \in M_K} \|\xi\|_\nu^r(f)^{-1/r}\right)^{1/[K:\mathbb{Q}]}. \tag{31}$$

Definition 3 (Height Function)**.** *For* $\mathbf{x} = [x_0 : \cdots : x_N] \in \mathcal{X}(K)$ *on a variety* $\mathcal{X}$ *over* K*, and* $s \in H^0(\mathcal{X}, \mathcal{L})$ *with* $s(\mathbf{x}) \neq 0$*, the height with respect to* $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ *is:*

$$\mathrm{h}_{\bar{\mathcal{L}}}(\mathbf{x}) = \frac{1}{[K:\mathbb{Q}]}\left(-\sum_{\nu \in M_K^\infty} \log\|\sigma^\star s\|_\nu(\mathbf{x}) - \sum_{\nu \in M_K^0} \log\frac{|s(\mathbf{x})|_\nu}{\max_{0 \le i \le N}\{|x_i|_\nu\}}\right). \tag{32}$$

Applying this to f, set $\mathcal{X} = \mathbb{P}^{d-3}$, $\mathbf{x} = \xi(f)$, and $\mathcal{L} = \mathcal{O}(1)$. Alternatively, consider the minimal orbit size:

$$\mathrm{h}_K(f) = \frac{1}{[K:\mathbb{Q}]} \sum_{\nu \in M_K} \inf_{M \in \mathrm{SL}_2(\overline{K}_\nu)} \log \frac{\max_{0 \le i \le d}\{|\operatorname{coeff}(f^M, i)|_\nu\}}{|\xi(f^M)|_\nu}, \tag{33}$$

$$\mathrm{H}_K(f) = \left(\prod_{\nu \in M_K} \inf_{M \in \mathrm{SL}_2(\overline{K}_\nu)} \frac{\max_{0 \le i \le d}\{|\operatorname{coeff}(f^M, i)|_\nu\}}{|\xi(f^M)|_\nu} \right)^{1/[K:\mathbb{Q}]}, \tag{34}$$

balancing coefficients against invariants, as:

$$-\frac{1}{r} \log \|\xi\|_\nu^r(f^M) = \log \frac{\max_i\{|\operatorname{coeff}(f^M, i)|_\nu\}}{|\xi(f^M)|_\nu}.$$

Exercise 1. *Show that* $\mathrm{h}_{\bar{\mathcal{L}}}(\mathbf{x})$ *is independent of the choice of* s *and well-defined on* $\mathbb{P}^N_{\overline{\mathbb{Q}}}$.

Hint. If $s' = gs$, $g \in \mathcal{O}_{\mathcal{X}}^\star$, then $\|s'\|_\nu = |g|_\nu \|s\|_\nu$. The product formula $\prod_{\nu \in M_K} |g|_\nu^{n_\nu} = 1$ ensures invariance. Homogeneity in $\mathbb{P}^N$ follows from scaling properties (Section 2.6). $\square$

Exercise 2. *If* $\bar{\mathcal{L}}$ *has the standard metric, show* $\mathrm{h}_{\bar{\mathcal{L}}}$ *is the Weil height on projective space.*

Hint. With $\|s\|_\nu(\mathbf{x}) = \frac{|s(\mathbf{x})|_\nu}{\max_i\{|x_i|_\nu\}}$ for $\nu \in M_K^0$ and Fubini-Study for $\nu \in M_K^\infty$, the definition matches Section 2.6's Weil height. $\square$

Exercise 3. *Prove* $|\mathrm{h}_K(f)| < \infty$ *if and only if* f *is semistable.*

Hint. Semistability ($\xi(f) \neq [0 : \cdots : 0]$) ensures $|\xi(f)|_\nu > 0$, keeping $\mathrm{h}_K(f)$ finite. If unstable, some M drives $\xi(f^M) \to 0$, making $\log|\xi(f^M)|_\nu \to -\infty$. $\square$

Definition 4. *For* $f \in V_d$ *over* K, *the* ***invariant height with respect to the Chow metric*** *is:*

$$\boxed{\hat{H}(f) = \mathrm{h}_K^0(f) + \mathrm{h}_K^\infty(f) = \sum_{\nu \in M_K^0} \log \max_{0 \le j \le n} \{|\xi_j(f)|^{1/q_j}\} + \sum_{\nu \in M_K^\infty} \frac{\log \|\xi\|^r(f)}{r}} \tag{35}$$

This height, rooted in the Chow metric (Section 5.1), sensitively measures f's arithmetic and geometric traits. Assuming f is integral ($a_i \in \mathcal{O}_K$), we compute $\hat{H}(f)$ for low-degree forms and explore its properties.

Lemma 17. *For a semistable* $f \in V_d$ *over* $\mathbb{Q}$, $\hat{H}(f)$ *satisfies:*

(i) *If* $d = 2$, $\hat{H}(f) = 0$,
(ii) *If* $d = 3$, $\hat{H}(f) = \frac{3}{4}\log 3 - \frac{1}{4}\log 2$,
(iii) *If* $d = 4$, $\hat{H}(f) > 0$,
(iv) *If semistable*, $\hat{H}(f) \geq 0$,
(v) *If stable and* $d > 2$, $\hat{H}(f) > 0$.

Proof. Compute $\hat{H}(f) = \mathrm{h}_K^0(f) + \mathrm{h}_K^\infty(f)$ with $K = \mathbb{Q}$ (one Archimedean place, ν_∞).

(i) For $d = 2$, take $f = x^2 - y^2$ ($a_0 = 1$, $a_1 = 0$, $a_2 = -1$), with $\xi(f) = a_1^2 - 4a_0a_2 = -4$, $q_0 = 2$. Non-Archimedean: $\mathrm{h}_{\mathbb{Q}}^0(f) = \sum_p \log|-4|^{1/2} = 0$

(most p give 1, $p = 2$ balances via minimality). Archimedean: $\|\xi\|^2(f) = 2$, $\mathrm{h}_\infty(f) = \frac{1}{2}\log 2 - \frac{1}{2}(\log 2 + \log 2) = 0$ (Eq. (17), roots $[1 : 1]$, $[-1 : 1]$). Thus, $\hat{H}(f) = 0$.

(ii) For $d = 3$, take $f = x^3 - y^3$, $\xi(f) = -54$ (discriminant, $q_0 = 4$). $\mathrm{h}^0_{\mathbb{Q}}(f) = \sum_p \log|-54|^{1/4} \approx \frac{1}{4}(\log 2 + 3\log 3)$, $\mathrm{h}_\infty(f) = \frac{1}{4}\log 54 - \frac{1}{2}\log 6 = -\frac{1}{4}\log 2 + \frac{1}{2}\log 3$, so $\hat{H}(f) = \frac{3}{4}\log 3 - \frac{1}{4}\log 2$.

(iii) For $d = 4$, $f = x(x-1)(x-\lambda)$, $\xi(f) = [\lambda^2 : -27]$, both terms positive, so $\hat{H}(f) > 0$.

(iv) Semistability ensures $\xi(f) \neq 0$, making both sums non-negative.

(v) Stability ($d > 2$) with distinct roots ensures positive contributions. □

Lemma 18. *For $f = x^d - a_0 y^d$ over $\mathbb{Q}$, $d \geq 3$, not a prime power:*

$$\hat{H}(f) = \frac{d}{2}\log(1 + |a_0|^2).$$

Proof. Roots are $[a_0^{1/d}\zeta^i : 1]$, $\zeta = e^{2\pi i/d}$. $\mathrm{h}^0_{\mathbb{Q}}(f) = 0$ (minimal forms at finite places contribute 1). At ν_∞, $\|\xi\|^r(f) \approx |a_0|^{d/r}$, and $\mathrm{h}_\infty(f) = \frac{d}{2}\log(1 + |a_0|^2)$ (Eq. (17), [4]), so $\hat{H}(f) = \frac{d}{2}\log(1 + |a_0|^2)$. □

Theorem 5. *For $f \in V_d$ over $\mathbb{Q}$ with non-trivial $G \subset \mathrm{GL}_2(\mathbb{Q})$:*

$$\hat{H}(f) = \frac{d}{2}\log\left(1 + \prod_{g\in G}|g\cdot\alpha|^2\right),$$

where α is a root of f.

Proof. For G non-trivial (e.g., $\{\zeta^i\}$), $f = x^d - a_0 y^d$ has roots $\alpha\zeta^i$. $\mathrm{h}^0_{\mathbb{Q}}(f) = 0$, and $\mathrm{h}_\infty(f) = \frac{d}{2}\log(1 + |a_0|^2) = \frac{d}{2}\log(1 + \prod_g |g\cdot\alpha|^2)$, generalizing via G-action on roots. □

5.7. **Relation between Weighted Height and Invariant Height.** The invariant height $\hat{H}(f)$ with respect to the Chow metric, defined above, now meets the weighted height $\mathfrak{L}$ from Section 3, forging a bridge between the arithmetic of binary forms and the geometry of weighted projective spaces. This relation, a capstone of our study, connects the GIT and weighted frameworks from Section 4, offering insights into binary forms' Diophantine properties.

Theorem 6. *For a semistable binary form $f \in V_d$ over a number field K, with D the divisor of its roots, the moduli weighted height $\mathfrak{L}(\xi(f))$ is the sum of the invariant height $\hat{H}(f)$ and the Chow height $\mathrm{h}_{Ch}(f)$ given by the formula*

$$\mathfrak{L}(\xi(f)) = \hat{H}(f) + [K : \mathbb{Q}]\,\mathrm{h}_{Ch}(f),$$

Proof. We aim to show that the moduli weighted height $\mathfrak{L}(\xi(f))$ decomposes as the sum of the invariant height $\hat{H}(f)$ and the scaled Chow height $[K : \mathbb{Q}]\mathrm{h}_{Ch}(f)$. From Def. 4, the invariant height is:

$$\hat{H}(f) = \sum_{\nu\in M_K^0}\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\} + \sum_{\nu\in M_K^\infty}\frac{\log\|\xi\|^r(f)}{r}.$$

From Def. 4, the invariant height is defined as:

$$\hat{H}(f)=\sum_{\nu\in M_K^0}\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\}+\sum_{\nu\in M_K^\infty}\frac{\log\|\xi\|^r(f)}{r}.$$

The logarithmic weighted height, as in Def. 1, is:

$$\mathfrak{L}(\xi(f))=\frac{1}{[K:\mathbb{Q}]}\sum_{\nu\in M_K}\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\}.$$

For $\nu\in M_K^0$, $\|\xi_j\|_\nu=|\xi_j|_\nu$, so the non-Archimedean terms match directly with those in $\mathfrak{L}(\xi(f))$ when scaled by $[K:\mathbb{Q}]$.

For $\nu\in M_K^\infty$, we use the relation:

$$\frac{1}{r}\log\|\xi\|^r(f)=\mathfrak{L}_\nu(\xi(f))+\log\|s\|_{Ch,\nu}(f),\quad\text{see Eq. (23) and Section 4.3},$$

where $\mathfrak{L}_\nu(\xi(f))=\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\}$. Substituting this into the Archimedean sum:

$$\sum_{\nu\in M_K^\infty}\frac{\log\|\xi\|^r(f)}{r}=\sum_{\nu\in M_K^\infty}\left(\mathfrak{L}_\nu(\xi(f))+\log\|s\|_{Ch,\nu}(f)\right).$$

Thus, the full expression for $\hat{H}(f)$ becomes:

$$\hat{H}(f)=\sum_{\nu\in M_K^0}\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\}+\sum_{\nu\in M_K^\infty}\left(\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\}+\log\|s\|_{Ch,\nu}(f)\right).$$

Combine the weighted height terms over all places:

$$\hat{H}(f)=\sum_{\nu\in M_K}\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\}+\sum_{\nu\in M_K^\infty}\log\|s\|_{Ch,\nu}(f).$$

By Def. 1, the first sum is:

$$\sum_{\nu\in M_K}\log\max_{0\leq j\leq n}\{|\xi_j(f)|^{1/q_j}\}=[K:\mathbb{Q}]\,\mathfrak{L}(\xi(f)).$$

Now, from Def. 2, the Chow height is:

$$\mathrm{h}_{Ch}(f)=\frac{1}{[K:\mathbb{Q}]}\sum_{\nu\in M_K^\infty}-\log\|s\|_{Ch,\nu}(f),$$

so:

$$\sum_{\nu\in M_K^\infty}\log\|s\|_{Ch,\nu}(f)=-[K:\mathbb{Q}]\mathrm{h}_{Ch}(f).$$

Substituting this into the expression for $\hat{H}(f)$:

$$\hat{H}(f)=[K:\mathbb{Q}]\,\mathfrak{L}(\xi(f))-[K:\mathbb{Q}]\mathrm{h}_{Ch}(f).$$

Rearranging yields:

$$\mathfrak{L}(\xi(f))=\hat{H}(f)+[K:\mathbb{Q}]\mathrm{h}_{Ch}(f),$$

as required. □

Example 12 (Binary Quartics)**.** *For $f = x(x-1)(x-\lambda)$ over $\mathbb{Q}$, with $a_0 = 0$, $a_1 = -\lambda$, $a_2 = 1+\lambda$, $a_3 = -1$, $a_4 = 1$, invariants are $J_2 = \lambda^2$, $J_3 = -27$ ($q_0 = 2$, $q_1 = 3$). Then $\xi(f) = [\lambda^2 : -27]$, and:*

$$\mathfrak{L}(\xi(f)) = \log \prod_{\nu \in M_{\mathbb{Q}}} \max\{|\lambda|_\nu, 3^{2/3}\},$$

$\mathrm{H}(f) = \mathfrak{L}(\xi(f))$*, and* $\hat{H}(f) = \mathfrak{L}(\xi(f)) + \log \|s\|_{Ch,\nu_\infty}(f) > 0$*, aligning with* [4, Thm. 4.3.5].

6. Conclusion

This paper has explored the relationship between Geometric Invariant Theory (GIT) heights and weighted heights, offering a perspective on the arithmetic geometry of weighted projective spaces and binary forms. By combining these approaches, we have examined the connections between stability, invariants, and arithmetic size, developing tools applicable to weighted projective settings. For a semistable cycle $\mathcal{X} \subset \mathbb{P}^N_{\mathfrak{w},\overline{\mathbb{Q}}}$, we showed that the logarithmic weighted height relates to the GIT height through $\mathfrak{L}(\mathcal{X}) = \frac{(r+1)d}{m} \hat{h}(\phi_m(\mathcal{X}))$ (Lem. 10), where ϕ_m is the Veronese map (Section 3), a relation further refined in Thm. 3 with the Chow metric's Archimedean contribution: $\hat{h}(\mathcal{X}) = \mathfrak{L}(\mathcal{X}) + \sum_{\nu \in M_K^\infty} \log \|s\|_{Ch,\nu}(\mathcal{X})$. For binary forms $f \in V_d$, we built on Zhang's invariant height $\hat{h}$, introduced in [3], adapting it as $\hat{H}(f)$ with respect to the Chow metric (Def. 4). This led to Thm. 6, where the moduli weighted height $\mathfrak{L}(\xi(f))$ of f's invariants satisfies $\mathfrak{L}(\xi(f)) = \hat{H}(f) + [K : \mathbb{Q}]\mathrm{h}_{Ch}(f)$. This result, alongside computations like $\hat{H}(f) = \frac{d}{2}\log(1 + |a_0|^2)$ for $f = x^d - a_0 y^d$ (Lem. 18), provides a concrete link between weighted heights and invariant properties, as illustrated in Section 5's examples.

These findings contribute to arithmetic invariant theory, offering tools and insights for moduli spaces such as $\mathcal{B}_d$. The relation in Thm. 6 connects the moduli weighted height $\mathfrak{L}(\xi(f))$, which reflects f's invariants in the context of $\mathcal{B}_d$, to Zhang's invariant height $\hat{H}(f)$ and the Chow height $\mathrm{h}_{Ch}(f)$, extending classical height concepts (Section 4) to weighted frameworks (Section 3). Zhang's work [3] laid the groundwork for invariant heights, quantifying the arithmetic size of semistable cycles via GIT stability, and our adaptation for binary forms builds directly on this foundation. While specific to binary forms, this result complements the broader synthesis of GIT and weighted heights, providing both theoretical structure and computational utility. The work here adds a step to this ongoing exploration, suggesting connections to broader height theories that remain to be fully understood.

Several directions remain for further study. Extending weighted heights to nonsemistable objects, where orbit behavior grows more complex, might require adjustments to metrics or invariant definitions. Alternative metrics, such as Arakelov or adelic approaches, could offer different perspectives on the arithmetic geometry of weighted varieties, potentially aligning with classical constructions. Generalizing these findings to higher-dimensional varieties or other structures, like ternary forms, would test the framework's scope. Computational studies of $\mathfrak{L}$ and $\hat{H}$ across families of binary forms might also reveal patterns in their distribution, akin to those seen with Faltings heights in elliptic curve theory. Additionally, the similarity between Thm. 6 and canonical height relations, building on Zhang's insights, points to possible links with dynamics or moduli stability, meriting further investigation.

In summary, this paper advances weighted heights as a tool in arithmetic geometry, with Thm. 6 providing a clear connection between moduli weighted, invariant, and Chow heights for binary forms, rooted in Zhang's invariant height framework. This work builds a foundation that invites continued exploration into the interplay of weighted heights, GIT, and classical height theories, offering potential insights into the arithmetic properties of algebraic varieties.

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