COMPUTING THE WEIERSTRASS NORMAL FORM OF SUPERELLIPTIC CURVES

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Abstract. We develop algorithms to determine whether an irreducible algebraic curve $C : f(x, y) = 0$ over $\mathbb{C}$ is superelliptic of level n and, if so, to compute its Weierstrass normal form $v^n = h(u)$, where $h(u)$ is a polynomial with distinct roots. Our approach identifies the minimal n via a Wronskian-based method and constructs a birational transformation to the normal form.

1. Introduction

Superelliptic curves represent a natural generalization of hyperelliptic curves, extending the rich algebraic geometry framework from degree 2 projections to higher-order cyclic covers of $\mathbb{P}^1$. While hyperelliptic curves, defined by equations of the form $y^2 = h(x)$, have been extensively studied for their connections to elliptic integrals, theta functions, and Jacobian varieties, superelliptic curves—characterized by a degree $n \geq 2$ cyclic Galois cover—offer a broader landscape for exploration; see [1] for a detailed account of this subject. In [2] is highlighted this progression, demonstrating how tools such as Weierstrass points, automorphism groups, and differential techniques can be adapted from the hyperelliptic case to superelliptic contexts. A more detailed account of the subject can be found in [1]. Motivated by this vision, we tackle a fundamental computational challenge: given an irreducible algebraic curve $C : f(x, y) = 0$ over $\mathbb{C}$, can we algorithmically detect its superellipticity, determine the minimal level n, and transform it into its Weierstrass normal form?

Our work builds on the theoretical insights of [2] and [3], translating them into actionable algorithms. We propose a two-stage computational framework: first, an algorithm to identify whether C is superelliptic by examining the nongap sequence of holomorphic differentials at ramification points, inspired by Wronskian method in [4] and [5, 6]; second, a procedure to compute the birational transformation to $v^n = h(u)$, where $h(u)$ is a polynomial with distinct roots. This approach avoids exhaustive automorphism group computations, relying instead on differential properties to achieve both efficiency and generality. The resulting algorithms are implemented in pseudo-code, with practical considerations for tools like Maple, SageMath, or Singular, making them accessible for computational algebraic geometry.

The paper is structured as follows: Section 2 defines superelliptic curves and their differential properties, drawing from foundational works. Section 3 presents the detection algorithm for superellipticity, including pseudo-code and mathematical validation. Section 4 develops the transformation to Weierstrass normal form,

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integrating the detection results with a computational method to standardize the curve equation.

2. Preliminaries

Following [3], a *superelliptic automorphism of level* n is defined as a conformal automorphism τ of order $n \geq 2$ of a closed Riemann surface $\mathcal{X}$ of genus $g \geq 2$, which is central in the full automorphism group $G = \mathrm{Aut}(\mathcal{X})$, and such that the quotient $\mathcal{X}/\langle\tau\rangle$ has genus zero (i.e., is isomorphic to $\mathbb{P}^1$). The cyclic group $H = \langle\tau\rangle$ is called a *superelliptic group of level* n, and $\mathcal{X}$ is called a *superelliptic curve of level* n. This definition imposes a stronger condition than that of cyclic n-gonal curves, where τ need only satisfy $\mathcal{X}/\langle\tau\rangle$ having genus zero, by requiring centrality in G, distinguishing it from generalized superelliptic automorphisms where τ is central only in its normalizer $N \subset G$.

A superelliptic curve $\mathcal{X}$ of level n can be represented by an affine algebraic curve:

$$y^n = \prod_{j=1}^{s} (x - p_j)^{l_j},$$

where $p_1, \ldots, p_s \in \mathbb{C}$ are distinct branch points, and the exponents $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$ satisfy:

- $\sum_{j=1}^{s} l_j \equiv 0 \pmod{n}$, ensuring the covering $\pi : \mathcal{X} \to \mathbb{P}^1$ defined by $\pi(x, y) = x$ is well-defined up to normalization,
- $\gcd(n, l_1, \ldots, l_s) = 1$, guaranteeing that $H = \langle\tau\rangle$ acts transitively, generating the full cyclic group of order n.

In this model, $\tau(x, y) = (x, \omega_n y)$, with $\omega_n = e^{2\pi i/n}$, is the superelliptic automorphism, and π is a degree n map. Define $h(x) = \prod_{j=1}^{s}(x - p_j)^{l_j}$ with degree $m = \sum_{j=1}^{s} l_j$. If a branch point is at infinity (e.g., $p_s = \infty$), the factor $(x - p_s)^{l_s}$ is omitted from the product, and the equation is adjusted to reflect the ramification at $x = \infty$.

The genus g of $\mathcal{X}$ is determined by the Riemann-Hurwitz formula:

$$2g - 2 = n(2 \cdot 0 - 2) + \sum_{p \in \mathcal{X}} (e_p - 1),$$

where e_p is the ramification index at point p. For a finite branch point $p = (p_j, 0)$ where $y = 0$, $e_p = n/\gcd(n, l_j)$. At infinity, the ramification index e_∞ is computed as follows: under the map $\pi(x, y) = x$, the degree of the polynomial $y^n - h(x) = 0$ in y is n, and in x at infinity, the leading term of $h(x)$ has degree m. Thus, $e_\infty = n/\gcd(n, m)$ when $m \not\equiv 0 \pmod{n}$, with the total number of ramified points typically $s + 1$ if infinity is ramified. This form, $y^n = h(x)$, is the Weierstrass normal form we seek, generalizing the hyperelliptic case where $n = 2$ and τ is the hyperelliptic involution.

2.1. Description of the Space of Holomorphic Differentials.

The space of holomorphic 1-forms on $\mathcal{X}$, denoted $V = H^0(\mathcal{X}, \Omega^1_{\mathcal{X}})$, has dimension equal to the genus g. For a superelliptic curve $\mathcal{X} : y^n = h(x)$ with $h(x)$ of degree m, a basis for V consists of differentials:

$$\omega_{i,j} = \frac{x^i \, dx}{y^j},$$

where the indices (i,j) range over $i \geq 0$ and $1 \leq j \leq n-1$, subject to holomorphicity constraints. A differential $\omega_{i,j}$ is holomorphic if it has no poles at any point of $\mathcal{X}$. At a finite branch point $x = p_j$, the order of $\omega_{i,j}$ is $i - j \cdot l_j / \gcd(n, l_j)$, requiring $i \geq j \cdot l_j / \gcd(n, l_j)$ for non-negativity, though typically i is small relative to m. At infinity, using a local parameter $t = 1/x$, the differential becomes $\omega_{i,j} = -t^{-i-2} t^{jm/n}\, dt$ (assuming $m \equiv 0 \pmod{n}$ for simplicity), with pole order $-(i + 2 - jm/n)$, which must be non-positive. Thus, the basis is:

$$\{\omega_{i,j} \mid 0 \leq i \leq \lfloor (n-1)(m-1)/2 \rfloor / n,\ 1 \leq j \leq n-1,\ in + jm \leq (n-1)(m-1)\},$$

adjusted to yield exactly g elements, where $g = (n-1)(m-1)/2$ if m and n are coprime and s is sufficiently large, with corrections for specific cases.

For a general plane curve $C : f(x,y) = 0$ of degree d, holomorphic differentials are:

$$\omega = \frac{p(x,y)\, dx}{\frac{\partial f}{\partial y}},$$

where $p(x,y)$ is a polynomial of degree at most $d-3$, chosen to be regular at all points, including singularities. At a singularity p, the differential must be checked for poles, often requiring resolution of singularities to compute V accurately. The action of an automorphism τ on V is crucial: for $\tau(x,y) = (x, \zeta_n y)$, $\tau^* \omega_{i,j} = \zeta_n^{-j} \omega_{i,j}$, and the eigenspace $V_0 = \{\omega \in V \mid \tau^* \omega = \omega\}$ must be trivial if $C/\langle \tau \rangle \cong \mathbb{P}^1$, reflecting the absence of holomorphic differentials on $\mathbb{P}^1$.

3. Detecting Superellipticity

With the theoretical groundwork for superelliptic curves established in the preceding sections, we now address the task of determining whether an irreducible algebraic curve C, defined by $f(x,y) = 0$ over $\mathbb{C}$, is superelliptic. A curve C is superelliptic if it admits a degree $n \geq 2$ map $\pi : C \to \mathbb{P}^1$ such that the field extension $\mathbb{C}(C)/\mathbb{C}(\mathbb{P}^1)$ is Galois with cyclic group $\mathbb{Z}/n\mathbb{Z}$, and the automorphism generating this group is central in $\mathrm{Aut}(C)$, with the quotient $C/\langle \tau \rangle$ isomorphic to $\mathbb{P}^1$. We adapt the generalized Wronskian framework of Towse [4] to detect this property by analyzing the orders of holomorphic differentials at ramification points, identifying the minimal n for which C is superelliptic. This approach is enriched by insights from Shor and Shaska [6], who detail differential bases and properties of q-Weierstrass points for superelliptic curves, enhancing our computational strategy.

3.1. Initial Computations.

The analysis commences with the computation of the genus g of C, a key invariant governing superellipticity. The genus is the dimension of the space of holomorphic 1-forms, $V = H^0(C, \Omega^1_C)$, with basis $\{b_1, b_2, \ldots, b_g\}$, where:

$$b_i = f_i(x,y)\, dx, \quad f_i(x,y) = \frac{p_i(x,y)}{\frac{\partial f}{\partial y}},$$

and $p_i(x,y)$ is a polynomial of degree at most $\deg(f) - 3$, adjusted for regularity at all points, including singularities. For a smooth plane curve of degree d, the genus is:

$$g = \frac{(d-1)(d-2)}{2},$$

reduced by the sum of delta invariants δ_p at singularities. If $g < 1$, $C \cong \mathbb{P}^1$ (genus 0) and cannot support a non-trivial cyclic cover to itself with $g \geq 2$, as required for superellipticity beyond $n = 1$. Thus, we proceed only if $g \geq 1$, rejecting such curves otherwise.

3.2. **Defining the Search Scope.** For C with $g \geq 1$, we seek the smallest $n \geq 2$ admitting a superelliptic map. The gonality of C, the minimal degree of a map to $\mathbb{P}^1$, is bounded above by $\lfloor (g+3)/2 \rfloor$ for non-hyperelliptic curves, though superelliptic curves may exhibit higher degrees. We set:

$$n_{\max} = \lfloor (g+3)/2 \rfloor + 1,$$

exceeding the gonality to encompass potential cyclic maps, justified by the fact that superelliptic n may exceed minimal gonality when ramification aligns with a central automorphism. We test n from 2 to $n_{\max}$, constrained by:

$$g \geq n - 1,$$

derived from the Riemann-Hurwitz formula for a degree n map to a genus-0 quotient, ensuring sufficient ramification. If $g < n - 1$, we skip to the next n.

3.3. **Generalized Wronskian Framework.** We employ Towse's generalized Wronskian method [4] to detect superellipticity by identifying the nongap sequence of differential orders at ramification points. Let $\mathcal{F} = \{f_1, f_2, \ldots, f_g\}$ where $b_i = f_i dx$ forms a basis for V. Select a set of candidate points $P_{\text{list}} \subset C$ including:

- Points $P = (0, y_i)$ where $f(0, y_i) = 0$,
- Singularities of C, solutions to $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$,
- Points where $\frac{\partial f}{\partial y} = 0$, potential ramification loci under a map $\pi(x, y) = x$.

For each $P \in P_{\text{list}}$, choose a local uniformizing parameter (e.g., $x - P[1]$ if $\frac{\partial f}{\partial y}(P) \neq 0$, or adjust via resolution if singular). Define the Wronskian matrix for a sequence $s = (m_0, m_1, \ldots, m_{g-1})$, $0 \leq m_0 < m_1 < \cdots < m_{g-1}$:

$$M_{\mathcal{F}}[s] = \begin{bmatrix} f_1^{(m_0)} & f_2^{(m_0)} & \cdots & f_g^{(m_0)} \\ f_1^{(m_1)} & f_2^{(m_1)} & \cdots & f_g^{(m_1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m_{g-1})} & f_2^{(m_{g-1})} & \cdots & f_g^{(m_{g-1})} \end{bmatrix},$$

where $f_i^{(m)} = \frac{d^m f_i}{dx^m}$. The determinant $\det M_{\mathcal{F}}[s](P) \neq 0$ indicates s is the nongap sequence at P.

3.4. **Constructing Superelliptic Sequences.** For C to be superelliptic of level n, the nongap sequence at a ramification point must reflect a cyclic n-gonal structure. For $y^n = h(x)$ with $h(x)$ of degree m and distinct roots, a basis $\{x^i dx / y^{n-1} \mid 0 \leq i < k\}$ yields orders i up to $m - 2$, then n-spaced gaps, where k adjusts for infinity. Initially, estimate:

$$m_0 = \left\lceil \frac{2g - 2 + 2n}{n - 1} \right\rceil,$$

assuming full ramification ($e_p = n$). Define:

$$q = \max\left(0, g - \left\lfloor \frac{(n-1)m_0}{2} \right\rfloor\right),$$

and the candidate sequence:

$$s_n = (0, 1, \ldots, q - 1, q, q + n, q + 2n, \ldots, q + (g - q - 1)n).$$

Refine m iteratively: if $\det M_{\mathcal{F}}[s_n](P) \neq 0$, compute the number of n-gaps $r = g - q$, and adjust $m = r + 1$ if infinity ramifies, verifying:

$$2g - 2 = -2n + m(n-1).$$

3.5. **Superelliptic Test.** For each $P \in P_{\text{list}}$, compute $\det M_{\mathcal{F}}[s_n](P)$. If non-zero, s_n is the nongap sequence at P (Towse [4]), and we check the Riemann-Hurwitz condition with refined m. If consistent, C is superelliptic with level n, and we take the smallest such n; otherwise, test the next n up to $n_{\max}$.

3.6. **Mathematical Validation.** The method's validity rests on Towse's theorem: $\det M_{\mathcal{F}}[s_n](P) \neq 0$ identifies the nongap sequence. For superelliptic $y^n = h(x)$, $\tau : (x, y) \mapsto (x, \zeta_n y)$ ensures $C/\langle \tau \rangle \cong \mathbb{P}^1$, with n-spaced gaps at ramification points, matching $g = (n-1)(m-1)/2$ when adjusted. Shor and Shaska [6] affirm this structure, supporting the detection process.

3.7. **Pseudocode Implementation.**

Require: Polynomial $f(x, y)$, variables x, y
Ensure: Boolean, minimal n if superelliptic

$b \leftarrow$ ComputeHolomorphicDifferentials(f, x, y)
$g \leftarrow$ Length(b)
if $g < 1$ **then**
    return false
end if
$n_{\max} \leftarrow \lfloor (g+3)/2 \rfloor + 1$
$P_{\text{list}} \leftarrow$ FindRamificationPoints(f) {Includes $f(0, y) = 0$, singularities, $\frac{\partial f}{\partial y} = 0$}
if $P_{\text{list}} = \emptyset$ **then**
    Error "No ramification points found"
    return false
end if
$F \leftarrow \{b_i/dx \mid b_i \in b\}$
for $n = 2$ to $n_{\max}$ **do**
    if $g \geq n - 1$ **then**
        $m_0 \leftarrow \left\lceil \frac{2g-2+2n}{n-1} \right\rceil$
        $q \leftarrow \max\left(0, g - \left\lfloor \frac{(n-1)m_0}{2} \right\rfloor\right)$
        $s_n \leftarrow [0, 1, \ldots, q-1, q, q+n, \ldots, q + (g-q-1)n]$
        for $P \in P_{\text{list}}$ **do**
            $M \leftarrow$ Matrix$(g, g, (i, j) \to \frac{d^{s_n[i]} F[j]}{dx^{s_n[i]}})$
            $\det_M \leftarrow$ Determinant$(M)(x = P[1], y = P[2])$
            if $\det_M \neq 0$ **then**
                $r \leftarrow g - q$, $m \leftarrow r + 1$
                if $2g - 2 = -2n + m(n-1)$ **then**
                    Output "Superelliptic level: n at point P"
                    return true
                end if
            end if
        end for
    end if

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end for
return false
```

4. Weierstrass Normal Form for Superelliptic Curves

With the groundwork laid in Section 3 for detecting superellipticity, we now focus on transforming a superelliptic curve $C : f(x,y) = 0$ over $\mathbb{C}$ into its Weierstrass normal form. Assuming C is identified as superelliptic of level n via the algorithm in Section 3, we establish the existence of a birational transformation to the form $v^n = h(u)$, where $h(u)$ is a polynomial with distinct roots, and provide a computational method to construct this transformation explicitly.

4.1. **Existence of the Weierstrass Normal Form.** Let C be an irreducible algebraic curve defined by $f(x,y) = 0$, where $f(x,y) \in \mathbb{C}[x,y]$, and assume C is superelliptic of level n, as defined in Section 2 and confirmed by Section 3. We have the following theorem.

Theorem 1. *If C is a superelliptic curve of level n, then there exists a birational transformation $\phi : C \dashrightarrow C'$ mapping C to a curve C' in Weierstrass normal form:*

$$v^n = h(u),$$

where $h(u) \in \mathbb{C}[u]$ is a polynomial with distinct roots. Furthermore, this transformation ϕ can be constructed explicitly using the holomorphic differentials of C and the ramification data encoded in the nongap sequence at a ramification point of π.

Proof. Since C is superelliptic of level n, there exists a morphism $\pi : C \to \mathbb{P}^1$ of degree n such that $\mathbb{C}(C)/\mathbb{C}(\mathbb{P}^1)$ is Galois with $G \cong \mathbb{Z}/n\mathbb{Z}$. Let $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$, where $t \in \mathbb{C}(C)$ is a rational function, and $\mathbb{C}(C) = \mathbb{C}(t,s)$ with:

$$s^n = h(t) \in \mathbb{C}(t),$$

and $\tau(s) = \zeta s$ ($\zeta^n = 1$) fixes $\mathbb{C}(t)$.

From Section 3, we have a basis $\{b_1, \dots, b_g\}$ for $H^0(C, \Omega_C^1)$, where $b_i = f_i(x,y)\, dx$, and a ramification point P with nongap sequence $s_n = (0, 1, \dots, q-1, q, q+n, \dots, q+(g-q-1)n)$, where $\det M_{\mathcal{F}}[s_n](P) \neq 0$, $\mathcal{F} = \{f_1, \dots, f_g\}$. Define t such that $\operatorname{ord}_P(t - t(P)) = n$, ensuring $\pi(x,y) = t(x,y)$ has degree n, consistent with s_n's ramification structure.

To find s, use the resultant method on:

$$f(x,y) = 0, \quad t(x,y) - t = 0, \quad s(x,y) - S = 0.$$

Compute:

$$R_1(x,S) = \operatorname{Res}_y(f(x,y), s(x,y) - S),$$
$$p(S,t) = \operatorname{Res}_x(R_1(x,S), t(x,y) - t).$$

If $p(S,t) = S^n - h(t)$ is irreducible of degree n, then $s^n = h(t)$, and $\mathbb{C}(C) = \mathbb{C}(t,s)$.

If $h(t) = \frac{p(t)}{q(t)}$, define:

$$u = \frac{1}{t-a}, \quad a \text{ a root of } q(t),$$
$$v = s \cdot (t-a)^{\deg q},$$

so:

$$v^n = v^{n \deg q} \cdot \frac{p\left(a + \frac{1}{u}\right)}{q\left(a + \frac{1}{u}\right)} = h(u) \in \mathbb{C}[u],$$

with distinct roots by adjusting a.

The transformation $\phi : (x,y) \mapsto (u,v)$ is birational, as $t, s \in \mathbb{C}(x,y)$, and inversely, $t = a + \frac{1}{u}$, $s = \frac{v}{u^{\deg q}}$, mapping C to $C' : v^n = h(u)$. $\square$

4.2. **Computing the Transformation.** Building on Section 3, we construct t and s using the differential basis and s_n. If $t = x$ aligns with s_n, solve $f(x,y) = 0$ for $y = s$, setting $h(x) = s^n$. Otherwise, compute $t(x,y)$ via ramification orders and use resultants to find s.

4.3. **Pseudo-Code Implementation.**

1. **Input:** $f(x,y) \in \mathbb{C}[x,y]$, x, y, level n
2. **Output:** $u, v, h_v(u)$ such that $v^n = h_v(u)$
3. Set $t \leftarrow x$
4. Set $s \leftarrow y$
5. Compute poly $\leftarrow$ subs$(y = s, f)$
6. Compute $h_t \leftarrow -\text{subs}(y = s, f - y^n)$
7. Simplify $h_t \leftarrow \text{simplify}(\text{expand}(h_t))$
8. If h_t depends on y
    1. Compute $h_t \leftarrow \text{solve}(f, y^n)$
    2. If $h_t = \text{NULL}$ or list, error: "Adjust t"
    3. Simplify $h_t \leftarrow \text{simplify}(h_t)$
9. Compute $\text{num}_h \leftarrow \text{numer}(h_t)$, $\text{den}_h \leftarrow \text{denom}(h_t)$
10. If $\text{den}_h = 1$
    1. Set $u \leftarrow t$, $v \leftarrow s$, $h_v \leftarrow h_t$
11. Else
    1. Compute roots $\leftarrow \text{solve}(\text{den}_h, t)$
    2. If no roots, error: "Normalization failed"
    3. Set $a \leftarrow \text{roots}[1]$
    4. Set $u \leftarrow \frac{1}{t-a}$
    5. Compute $\deg_q \leftarrow \text{degree}(\text{den}_h, t)$
    6. Set $v \leftarrow s \cdot (t-a)^{\text{quo}(\text{degree}(\text{num}_h, t) - n \cdot \deg_q, n)}$
    7. Compute $h_v \leftarrow \text{simplify}(\text{subs}(t = \frac{1}{u} + a, h_t) \cdot (1/u)^{\text{degree}(\text{num}_h, t)} / (1/u)^{\text{degree}(\text{den}_h, t)})$
    8. Set $h_v \leftarrow \text{collect}(h_v, u)$
12. Return $[u, v, h_v]$

4.4. **Practical Considerations.** For singular curves, resolve singularities using Gröbner bases. The complexity depends on g and n, suggesting optimization for efficiency. This method extends differential techniques from [7], aligning with Section 3's framework.

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