

Semi-invariants of filtered quiver representations with at most two pathways

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ABSTRACT. A pathway from one vertex of a quiver to another is a reduced path. We modify the classical definition of quiver representations and we prove that semi-invariant polynomials for filtered quiver representations come from diagonal entries if and only if the quiver has at most two pathways between any two vertices. Such class of quivers includes finite ADE -Dynkin quivers, affine $\tilde{A}\tilde{D}\tilde{E}$ -Dynkin quivers, star-shaped and comet-shaped quivers. Next, we explicitly write all semi-invariant generators for filtered quiver representations for framed quivers with at most two pathways between any two vertices; this result may be used to study constructions analogous to Nakajima's affine quotient and quiver varieties, which are, in special cases, $\mathfrak{M}_0^{F^\bullet}(n, 1) := \mu_B^{-1}(0) // B$ and $\mathfrak{M}^{F^\bullet}(n, 1) := \mu_B^{-1}(0)^s / B$, respectively, where $\mu_B : T^*(\mathfrak{b} \times \mathbb{C}^n) \rightarrow \mathfrak{b}^* \cong \mathfrak{g}_n^* / \mathfrak{u}$, B is the set of invertible upper triangular $n \times n$ complex matrices, $\mathfrak{b} = \text{Lie}(B)$, and $\mathfrak{u} \subseteq \mathfrak{b}$ is the biggest unipotent subalgebra.

1. Introduction

As separating a G -space into invariant (or weight) spaces or constructing polynomials invariant under a group action is a fundamental and important procedure in mathematics, Schofield–van den Bergh ([26]), Derksen–Weyman ([5]), and Domokos–Zubkov ([6]) are a few mathematicians who have explicitly given strategies on producing (semi)-invariant polynomials for all quiver representations.

In recent years, quiver varieties have been emerging in the study of machine learning, cf. [17, 1, 12, 16]. Additionally, see [7] for the emergence of topological quantum field theories, topological quantum neural networks, and certain quiver representations in the context of information processing. In this paper, we modify the classical construction of quiver representations in such a way that our construction is related to the Grothendieck–Springer resolution ([2], [9], [13], [14], [15], [24]), Khovanov–Lauda and Rouquier (KLR) algebras ([18], [19], [25], [27]), universal quiver Grassmannians and universal quiver flag varieties ([3]), and Lusztig's upper half U^+ of the universal enveloping algebra of a Kac–Moody algebra ([20], [21], [22]), but such details will not be elaborated here. Instead, we describe the modification of quiver representations, which is as follows. Consider a sequence of vector spaces over each vertex of a quiver and restrict to the subspace of quiver representations that preserve this fixed sequence of vector spaces; we call such space *filtered quiver representations*. Furthermore, there is a unique largest unipotent subgroup of the set of complex invertible matrices over each vertex which preserves the filtration

of vector spaces and acts as a change-of-basis. Thus, considering the product of these unipotent groups acting on the filtered quiver representation space, we give an explicit description of the ring of invariant polynomials.

Let $Q = (Q_0, Q_1)$ be a finite, connected, nonframed quiver and $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector. Assume $F^\bullet$ is a filtration of vector spaces at each vertex and $F^\bullet \text{Rep}(Q, \beta)$ is a subspace of $\text{Rep}(Q, \beta)$ whose representations preserve $F^\bullet$; the product $\mathbb{U}_\beta$ of largest unipotent subgroups of a parabolic group acts on $F^\bullet \text{Rep}(Q, \beta)$ as a change-of-basis. A framed quiver $Q^\dagger = (Q_0^\dagger, Q_1^\dagger)$ is obtained from Q by adding a single new vertex i' to Q_0 together with an arrow from i' to i . Let $\beta^\dagger = (\beta_1, \dots, \beta_{Q_0}, m) \in \mathbb{Z}_{\geq 0}^{Q_0^\dagger}$ be the dimension vector for the framed quiver. We define

$$F^\bullet \text{Rep}(Q^\dagger, \beta^\dagger) := F^\bullet \text{Rep}(Q, \beta) \oplus M_{\beta_i \times m},$$

where $M_{\beta_i \times m}$ is the space of all $\beta_i \times m$ complex matrices.

Throughout this paper, assume $\beta = (n, \dots, n)$, a uniform dimension vector, and $F^\bullet$ is the complete standard filtration of vector spaces at each (nonframed) vertex of Q . A reason why we are using a uniform dimension vector is because this problem becomes significantly difficult if we are using a more general dimension vector. Let $\mathfrak{t}_n$ be the set of complex diagonal matrices in the set $\mathfrak{gl}_n$ of $n \times n$ complex matrices.

The definition of a pathway between two vertices is given in Definition 2.2.

THEOREM 1.1. *Q is a quiver with at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$.*

REMARK 1.2. *Quivers satisfying Theorem 1.1 include ADE-Dynkin quivers, affine $\tilde{ADE}$ -quivers, star-shaped and comet-shaped quivers ([13]).*

Theorem 1.1 has a number of important consequences, including if $\mathbb{U}_\beta$ -invariants for filtered quiver representations only come from diagonal blocks (i.e., the semisimple part), then Q has at most two pathways between any two vertices. Furthermore, this implies that Domokos-Zubkov's technique is applicable to the filtered quiver representation space if the quiver has at most two pathways between any two vertices. It is, in fact, shown in [13] that off-diagonal entries contribute as invariant polynomials if Q has more than two pathways between some of its vertices. We note that Theorem 1.1 could easily be generalized if we take the filtration of vector spaces at each (nonframed) vertex $i \in Q_0$ to be $G_i^\bullet : \mathbb{C}^0 \subseteq \mathbb{C}^{\gamma_1} \subseteq \mathbb{C}^{\gamma_2} \subseteq \dots \subseteq \mathbb{C}^{\gamma_k} = \mathbb{C}^n$; in such setting, only the Levi subalgebra components in the filtered representation space contribute $\mathbb{U}_\beta$ -invariant polynomials.

THEOREM 1.3. *Let $Q^\dagger$ be a framed quiver with at most two distinct pathways between any two vertices. Let $\beta = (n, \dots, n, m) \in \mathbb{Z}_{\geq 0}^{Q_0^\dagger}$ be a dimension vector, where m is associated to the framed vertex. Then $\mathbb{C}[F^\bullet \text{Rep}(Q^\dagger, \beta^\dagger)]^{\mathbb{U}_\beta} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}] \otimes_{\mathbb{C}} \mathbb{C}[\{f\}]$, where*

$$(1) \quad f = \sum_{\nu} g_{\nu}({}_{(\alpha)}a_{st})(J_{\nu}|I_{\nu})_{\Phi\Psi\dots\Gamma},$$

where $g_{\nu}({}_{(\alpha)}a_{st}) \in \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$ and $(J_{\nu}|I_{\nu})_{\Phi\Psi\dots\Gamma}$ is the block standard bideterminant given in Definition 4.10.

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2. Background

We refer to the author's doctoral thesis ([13]), Crawley-Boevey ([4]) and Ginzburg ([10]) for background on quivers and their representations.

DEFINITION 2.1. *Let Q be a quiver. Let $p = a_k \cdots a_2 a_1$ be a path where $a_i \in Q_1$ are arrows. If p is a cycle, then we define p^m to be the path composed with itself m times, i.e.,*

$$p^m := p \circ p \circ \cdots \circ p = \underbrace{(a_k \cdots a_2 a_1) \cdots (a_k \cdots a_2 a_1)}_m = (a_k \cdots a_2 a_1)^m.$$

A path p is reduced if $[p] \neq 0$ in $\mathbb{C}Q/\langle q^2 : q \in \mathbb{C}Q, l(q) \geq 1 \rangle$.

DEFINITION 2.2. *A pathway from vertex i to vertex j is a reduced path from i to j . We define pathways of a quiver Q to be the set of all pathways from vertex i to vertex j , where $i, j \in Q_0$.*

Note that pathways (of a quiver Q) include trivial paths and they form a finite set since Q is a finite quiver. We will now give an example of Definition 2.2.

EXAMPLE 2.3. *Consider the 2-Jordan quiver:*

$$a_2 \begin{array}{c} \curvearrowright \\ \bullet \\ \curvearrowleft \end{array} a_1.$$

Then $a_2^2 a_1$ is a path but not a pathway since it is not reduced. However, the path $a_2 a_1$ is a pathway.

Next, we make a distinction between quiver varieties and quiver representations. When one refers to quiver varieties, one usually means Nakajima quiver varieties, i.e., the Hamiltonian reduction of a (double) framed quiver representation space twisted by a nontrivial character ([23]). In this paper, we will only work with quiver representations.

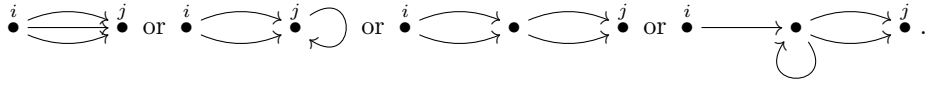
2.1. Filtered quiver representations. We give the construction of filtered quiver representations in the general setting. Let Q be a quiver and let $\beta = (\beta_1, \dots, \beta_{Q_0}) \in \mathbb{Z}_{\geq 0}^{Q_0}$, a dimension vector. Let $F^\bullet : 0 \subseteq \mathbb{C}^{\gamma^1} \subseteq \mathbb{C}^{\gamma^2} \subseteq \dots \subseteq \mathbb{C}^\beta$ be a filtration of vector spaces such that the filtration $F_i^\bullet : 0 \subseteq \mathbb{C}^{\gamma_i^1} \subseteq \mathbb{C}^{\gamma_i^2} \subseteq \dots \subseteq \mathbb{C}^{\beta_i}$ of vector spaces is fixed at vertex i for each $i \in Q_0$. Let $\text{Rep}(Q, \beta)$ be the representation space in the classical sense (without the filtration of vector spaces imposed). Then $F^\bullet \text{Rep}(Q, \beta)$ is a subspace of $\text{Rep}(Q, \beta)$ whose linear maps preserve the filtration of vector spaces at every level. Let $U_i \subseteq \text{GL}_{\beta_i}(\mathbb{C})$ be the largest unipotent group preserving the filtration of vector spaces at vertex i . Then the product $\mathbb{U}_\beta := \prod_{i \in Q_0} U_i$

of unipotent groups acts on $F^\bullet \text{Rep}(Q, \beta)$ as a change-of-basis.

We will say a representation (or a matrix) is *general* if it has indeterminates or zero in its entries.

3. Proof of Theorem 1.1

PROOF. First, we will prove from right to left. Assume $\mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$. We will prove that Q has at most two distinct pathways between any two vertices. For a contradiction, suppose Q has 3 or more pathways at some vertices. There are two cases to consider:

- [1] Q has 3 or more pathways from a vertex to itself, e.g., 
 [2] Q has 3 or more pathways from vertex i to vertex j , where $i \neq j$, e.g., 

Consider Case [1]. Without loss of generality, relabel the vertex with 3 or more pathways as 1 and label the distinct pathways from vertex 1 to itself as

$$a_1 = p_{i_1} \cdots p_{i_\alpha}, \quad a_2 = p_{j_1} \cdots p_{j_\beta}, \quad \dots, \quad a_m = p_{k_1} \cdots p_{k_\gamma},$$

where $m \geq 2$ and each a_l is not the trivial path. Write a general representation of $a_1, \dots, a_m$ as $A_1, \dots, A_m$, each of which is in $\mathfrak{b}$ with polynomial entries. Let $A_1 = ({}_{(1)}a_{ij})$ and $A_2 = ({}_{(2)}a_{ij})$, and consider the polynomial

$$f(A_1, \dots, A_m) = ({}_{(1)}a_{11} - {}_{(1)}a_{22})({}_{(2)}a_{12} - ({}_{(2)}a_{11} - {}_{(2)}a_{22})({}_{(1)}a_{12})) - ({}_{(2)}a_{11} - {}_{(2)}a_{22})({}_{(1)}a_{12} + u_{12}({}_{(1)}a_{22} - {}_{(1)}a_{11})).$$

For $u \in U \cong U \times I_n^{Q_0-1} \subseteq \mathbb{U}_\beta$ where u acts on the restricted tuple $(A_1, \dots, A_m)$ of the filtered representation space via $u.(A_1, \dots, A_m) = (uA_1u^{-1}, \dots, uA_mu^{-1})$ and I_n is the $n \times n$ identity matrix, the coordinates change as ${}_{(l)}a_{12} \mapsto {}_{(l)}a_{12} + u_{12}({}_{(l)}a_{22} - {}_{(l)}a_{11})$ under the group action. A restricted tuple means we ignore and suppress all other components of $F^\bullet \text{Rep}(Q, \beta)$ for simplicity as they are not currently relevant. So

$$\begin{aligned} u.f(A_1, \dots, A_m) &= ({}_{(1)}a_{11} - {}_{(1)}a_{22})({}_{(2)}a_{12} + u_{12}({}_{(2)}a_{22} - {}_{(2)}a_{11})) - ({}_{(2)}a_{11} - {}_{(2)}a_{22})({}_{(1)}a_{12} + u_{12}({}_{(1)}a_{22} - {}_{(1)}a_{11})) \\ &= ({}_{(1)}a_{11} - {}_{(1)}a_{22})({}_{(2)}a_{12} - ({}_{(2)}a_{11} - {}_{(2)}a_{22})({}_{(1)}a_{12})) \\ &\quad + u_{12}({}_{(1)}a_{11} - {}_{(1)}a_{22})({}_{(2)}a_{22} - {}_{(2)}a_{11}) - u_{12}({}_{(2)}a_{11} - {}_{(2)}a_{22})({}_{(1)}a_{22} - {}_{(1)}a_{11}) \\ &= f(A_1, \dots, A_m). \end{aligned}$$

This implies $\mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta} \supsetneq \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$, which is a contradiction.

Now consider Case [2]. Suppose

$$a_1 = p_{i_1} \cdots p_{i_\alpha}, \quad a_2 = p_{j_1} \cdots p_{j_\beta}, \quad \dots, \quad a_k = p_{l_1} \cdots p_{l_\gamma}$$

are distinct pathways from vertex i to vertex j , where $i \neq j$, $k \geq 3$, and each a_l is not the trivial path. Write a general representation of the pathways $a_1, \dots, a_k$ as $A_1 = ({}_{(1)}a_{ij}), \dots, A_k = ({}_{(k)}a_{ij}) \in \mathfrak{b}$. Consider the $\mathbb{U}_\beta$ -action on $F^\bullet \text{Rep}(Q, \beta)$. In particular, consider $I_n^{Q_0-2} \times U^2$ acting locally on $\mathfrak{b}^{\oplus k}$ via

$$(I_n, \dots, I_n, u, v).(A_1, \dots, A_k) = (uA_1v^{-1}, \dots, uA_kv^{-1}).$$

Consider the polynomial

$$\begin{aligned} g(A_1, \dots, A_k) &= ({}_{(1)}a_{11}({}_{(2)}a_{22} - {}_{(1)}a_{22}({}_{(3)}a_{12} + ({}_{(3)}a_{11}({}_{(1)}a_{22} - {}_{(3)}a_{22}({}_{(1)}a_{11}))({}_{(2)}a_{12} \\ &\quad + ({}_{(2)}a_{11}({}_{(3)}a_{22} - {}_{(2)}a_{22}({}_{(3)}a_{11}))({}_{(1)}a_{12})). \end{aligned}$$

Then

$$\begin{aligned}
(I_n, \dots, I_n, u, v).g(A_1, \dots, A_k) &= ({}_{(1)}a_{11}({}_2)a_{22} - ({}_1)a_{22}({}_2)a_{11})({}_3)a_{12} - ({}_3)a_{22}u_{12} + ({}_3)a_{11}v_{12}) \\
&\quad + ({}_3)a_{11}({}_1)a_{22} - ({}_3)a_{22}({}_1)a_{11})({}_2)a_{12} - ({}_2)a_{22}u_{12} + ({}_2)a_{11}v_{12}) \\
&\quad + ({}_2)a_{11}({}_3)a_{22} - ({}_2)a_{22}({}_3)a_{11})({}_1)a_{12} - ({}_1)a_{22}u_{12} + ({}_1)a_{11}v_{12}) \\
&= g(A_1, \dots, A_k) + ({}_1)a_{11}({}_2)a_{22} - ({}_1)a_{22}({}_2)a_{11})(-{}_3)a_{22}u_{12} + ({}_3)a_{11}v_{12}) \\
&\quad + ({}_3)a_{11}({}_1)a_{22} - ({}_3)a_{22}({}_1)a_{11})(-{}_2)a_{22}u_{12} + ({}_2)a_{11}v_{12}) \\
&\quad + ({}_2)a_{11}({}_3)a_{22} - ({}_2)a_{22}({}_3)a_{11})(-{}_1)a_{22}u_{12} + ({}_1)a_{11}v_{12}) = g(A_1, \dots, A_k).
\end{aligned}$$

This implies that g is an invariant polynomial, which contradicts that $\mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$. Thus, Q is a quiver with at most two pathways between any two vertices.

Now suppose Q is a quiver with at most two distinct pathways between any two vertices. We first define a notion of total ordering on pairs of integers and then choose the least pair. We then list all possible local models of arrows of Q at a fixed vertex. Writing an invariant polynomial which depends on this least pair, we carefully choose a subgroup of $\mathbb{U}_\beta$ and show that the invariant polynomial must only depend on diagonal coordinates of each general matrix in the filtered representation space. We will now give the full proof.

We label the arrows of Q as $a_1, a_2, \dots, a_{Q_1}$. Since it is clear that $\mathbb{C}[\mathfrak{t}^{\oplus Q_1}] \subseteq \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta}$, we will prove the other inclusion. Consider a general representation of $F^\bullet \text{Rep}(Q, \beta)$, which is a tuple of matrices. We define a total ordering $\leq$ on pairs (i, j) , where $1 \leq i \leq j \leq n$, by defining $(i, j) \leq (i', j')$ if either

- $i < i'$ or
- $i = i'$ and $j > j'$.

Let $f \in \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta}$. For each (i, j) , we can write

$$(2) \quad f = \sum_{|K| \leq d} a_{ij}^K f_{ij,K}, \text{ where } f_{ij,K} \in \mathbb{C}[\{(a)_{st} : (s, t) \neq (i, j)\}], a_{ij}^K := \prod_{\alpha \in Q_1} (a)_{ij}^{k_\alpha}, \text{ and } |K| = \sum_{\alpha=1}^{Q_1} k_\alpha.$$

Fix the least pair (under $\leq$) (i, j) with $i < j$ for which there exists $K \neq (0, \dots, 0)$ with $f_{ij,K} \neq 0$; we will continue to denote it by (i, j) . If no such (i, j) exists, then $f \in \mathbb{C}[(a)_{ii}]$ and we are done. Let $K = (k_1, \dots, k_{Q_1})$. Let $m \geq 1$ be the least integer satisfying the following: for all $p < m$, if some component k_p in K is strictly greater than 0, then $f_{ij,K} = 0$. Relabel the head of the arrow corresponding to a general representation A_m of arrow a_m as vertex m (this is the same m as in the previous sentence).

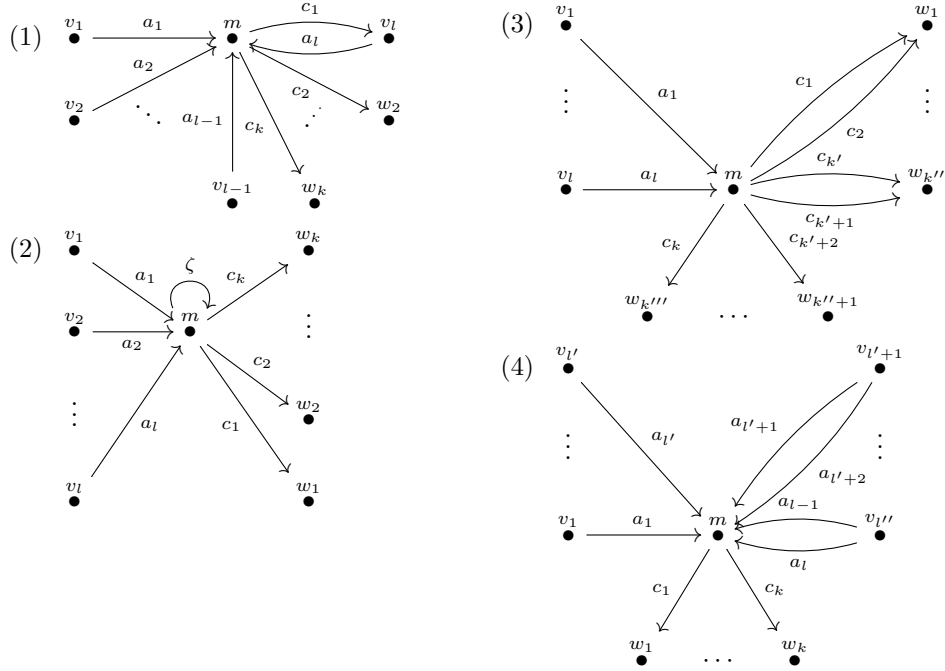
Let U_{ij} be the subgroup of matrices of the form $u_{ij} := (I_n, \dots, I_n, \hat{u}_m, I_n, \dots, I_n)$, where $\hat{u}_m$ is the matrix with 1 along the diagonal, the variable u in the (i, j) -entry, and 0 elsewhere. Let $u_{ij}^{-1} := (I_n, \dots, I_n, \hat{u}_m^{-1}, I_n, \dots, I_n)$. Then since u_{ij}^{-1} acts on f via

$$\begin{aligned}
u_{ij}^{-1}.f(A_1, \dots, A_{Q_1}) &= f(u_{ij}.(A_1, \dots, A_{Q_1})) \\
&= \begin{cases} f(A_1, \dots, \hat{u}_m A_{m'}, \dots, A_{Q_1}) & \text{whenever vertex } m \text{ is a sink of arrow } a_{m'}, \\ f(A_1, \dots, A_{m'} \hat{u}_m^{-1}, \dots, A_{Q_1}) & \text{whenever vertex } m \text{ is a source of arrow } a_{m'}, \\ f(A_1, \dots, \hat{u}_m A_{m'} \hat{u}_m^{-1}, \dots, A_{Q_1}) & \text{whenever arrow } a_{m'} \text{ is a loop at vertex } m, \end{cases}
\end{aligned}$$

(3)

$$u_{ij}^{-1} \cdot {}_{(\alpha)}a_{st} = \begin{cases} (m')a_{ij} + (m')a_{jj}u & \text{if } \alpha = m', (s, t) = (i, j), \text{ and } m \text{ is a sink to } a_{m'}, \\ (m')a_{ij} - (m')a_{ii}u & \text{if } \alpha = m', (s, t) = (i, j), \text{ and } m \text{ is a source to } a_{m'}, \\ (m')a_{ij} + ((m')a_{jj} - (m')a_{ii})u & \text{if } \alpha = m', (s, t) = (i, j), \text{ and } a_{m'} \text{ is a loop at } m, \\ {}_{(\alpha)}a_{st} & \text{if } s > i \text{ or } s = i \text{ and } t < j. \end{cases}$$

Locally at vertex m , Q has one of the following local models:



The following argument holds for all four cases. Relabel the arrows in the following way: write $a_1, \dots, a_l$ if $ha_i = m$, $c_1, \dots, c_k$ if $tc_j = m$, and ζ if ζ is the loop at m . Let $\{q_\phi\}_{0 \leq \phi \leq Q_1 - l - k - 1}$ (or $\{q_\phi\}_{0 \leq \phi \leq Q_1 - l - k}$ if there is no loop at vertex m) be all the other arrows of Q , where $q_0 := \emptyset$. Write general representations of a_α as $A_\alpha = ({}_{(\alpha)}a_{st})$, c_γ as $C_\gamma = ({}_{(\gamma)}c_{st})$, and ζ as $\Xi = (\zeta_{st})$, and let $({}_{(\phi)}q_{st})$ be a general representation of q_ϕ . Write

$$f = \sum_{\rho + |K'| + |\Gamma| \leq d} \zeta_{ij}^\rho \prod_{\alpha=1}^l {}_{(\alpha)}a_{ij}^{k_\alpha} \prod_{\gamma=1}^k {}_{(\gamma)}c_{ij}^{\mu_\gamma} f_{ij, \rho, K', \Gamma},$$

where $K' = (k_1, \dots, k_l)$, $\Gamma = (\mu_1, \dots, \mu_k)$, $|K'| = \sum_{\alpha=1}^l k_\alpha$, $|\Gamma| = \sum_{\gamma=1}^k \mu_\gamma$, and $f_{ij, K', \Gamma} \in \mathbb{C}[\{({}_{(\alpha)}a_{st}, {}_{(\gamma)}c_{st}, \zeta_{st}, {}_{(\phi)}q_{st} : (s, t) \neq (i, j) \text{ and } \alpha \notin \{1, \dots, l\} \text{ and } \gamma \notin \{1, \dots, k\}\}]$

$\gamma \notin \{1, \dots, k\}\} =: R_1$. Then

$$\begin{aligned}
0 &= u_{ij}^{-1} \cdot f - f = \sum_{\rho+|K'|+|\Gamma|\leq d} (\zeta_{ij} + (\zeta_{jj} - \zeta_{ii})u)^\rho \prod_{\alpha=1}^l ((\alpha)a_{ij} + (\alpha)a_{jj}u)^{k_\alpha} \prod_{\gamma=1}^k ((\gamma)c_{ij} - (\gamma)c_{ii}u)^{\mu_\gamma} f_{ij,\rho,K',\Gamma} \\
&\quad - \sum_{\rho+|K'|+|\Gamma|\leq d} \zeta_{ij}^\rho \prod_{\alpha=1}^l (\alpha)a_{ij}^{k_\alpha} \prod_{\gamma=1}^k (\gamma)c_{ij}^{\mu_\gamma} f_{ij,\rho,K',\Gamma} \\
&= \sum_{1\leq\rho+|K'|+|\Gamma|\leq d} \sum_{\substack{\tau\leq\rho \\ r_\alpha\leq k_\alpha, s_\gamma\leq\mu_\gamma}} \binom{\rho}{\tau} \binom{k_1}{r_1} \binom{k_2}{r_2} \dots \binom{k_l}{r_l} \binom{\mu_1}{s_1} \binom{\mu_2}{s_2} \dots \binom{\mu_k}{s_k} \\
&\quad \cdot \zeta_{ij}^{\rho-\tau} \prod_{\alpha=1}^l (\alpha)a_{ij}^{k_\alpha-r_\alpha} \prod_{\gamma=1}^k (\gamma)c_{ij}^{\mu_\gamma-s_\gamma} \cdot u^{\tau+|R|+|S|} (\zeta_{jj} - \zeta_{ii})^\tau \prod_{\alpha=1}^l ((\alpha)a_{jj})^{r_\alpha} \prod_{\gamma=1}^k ((-\gamma)c_{ii})^{s_\gamma} f_{ij,\rho,K',\Gamma},
\end{aligned}$$

where $|R| = \sum_{\alpha=1}^l r_\alpha$ and $|S| = \sum_{\gamma=1}^k s_\gamma$, and we see that

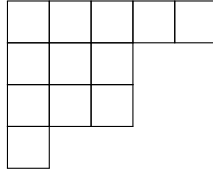
$$\left\{ \zeta_{ij}^{\rho-\tau} \prod_{\alpha=1}^l (\alpha)a_{ij}^{k_\alpha-r_\alpha} \prod_{\gamma=1}^k (\gamma)c_{ij}^{\mu_\gamma-s_\gamma} u^{\tau+|R|+|S|} : \tau \leq \rho, \ r_\alpha \leq k_\alpha, \ s_\gamma \leq \mu_\gamma \ \forall 1 \leq \alpha \leq l, \ 1 \leq \gamma \leq k \right\}$$

is linearly independent over R_1 . This implies that $f_{ij,\rho,K',\Gamma} = 0$ whenever $|K'| \geq 1$, $|\Gamma| \geq 1$, or $\rho \geq 1$, which contradict our choices of (i, j) and m . We conclude that $f \in \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$. $\square$

4. Proof of Theorem 1.3

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0)$ be a *partition* of size $|\lambda| = \sum_{i=1}^l \lambda_i$. One identifies to λ a left-justified shape of l rows of boxes of length $\lambda_1, \lambda_2, \dots, \lambda_l$, which is called the *Young diagram* associated to λ . A (*Young*) *filling* or a *Young tableau* of λ assigns a positive integer to each box of Young diagram.

EXAMPLE 4.1. Associated to $\lambda = (5, 3, 3, 1)$ is the Young diagram



and a Young tableau

4	1	2	3	3
3	4	5		
2	5	6		
1				

DEFINITION 4.2. A Young tableau is normal if the entries in each row are strictly increasing from left to right. It is called standard if it is normal and the entries in each column are nondecreasing from top to bottom. A bitableau $J|I$ is a pair of Young tableaux J and I having the same shape, and the bitableau is called standard if both J and I are standard Young tableaux.

REMARK 4.3. In Fulton and Harris ([8]), a Young tableau is called standard if the entries in each row and column are strictly increasing from left to right and from top to bottom, and a Young tableau is called semistandard if the entries in each row are nondecreasing from left to right while the entries in each column are strictly increasing from top to bottom. We will not use their definition in this paper.

The bideterminant $(J|I)$ of a bitableau is defined in the following way: the positive integer entries in J in a fixed row correspond to the rows of a matrix while the positive integer entries in I in the same row correspond to the columns of a matrix. Take the determinant of these minors of a matrix and repeat for each row in $J|I$ to obtain the bideterminant $(J|I)$.

Note that the bideterminant $(J|I)$ associated to a bitableau $J|I$ is a product of minors of a matrix, where J are the row indices and I are the column indices. Furthermore, a matrix (or a product of matrices) must be specified when calculating the bideterminant associated to a bitableau; such specification will be denoted on the lower-right corner of each row of the bitableau (and the bideterminant), i.e., see (5) and (6).

EXAMPLE 4.4. Consider the two Young tableaux:

$$J = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 4 & \\ \hline 1 & 4 & \\ \hline \end{array} \quad \text{and} \quad I = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 4 & \\ \hline 2 & 5 & \\ \hline \end{array}.$$

The tableau J is normal but not standard (since the entries in the first column are not nondecreasing when reading from top to bottom), while I is standard.

From this point forward, we will not draw a box around each entry of a Young (bi)tableau or the bideterminant of a bitableau.

Let $Q^\dagger$ be a quiver with one framed vertex labelled as $1'$ and all other nonframed vertices labelled as $1, 2, \dots, |Q_0|$, and the arrows labelled as $a_0, a_1, \dots, a_{|Q_1|}$, where $ta_0 = 1'$ and $ha_0 = 1$. Let A_{ϕ_u} be a general representation of the arrow a_{ϕ_u} . The product $A_{\phi_1}A_{\phi_2}\cdots A_0$ of general matrices is associated to the quiver path $a_{\phi_1}a_{\phi_2}\cdots a_0$, which begins at the framed vertex. Moreover, $A_{\phi_1}A_{\phi_2}\cdots A_0$ is uniquely associated to the sequence

$$(4) \quad \Phi := [\phi_1, \phi_2, \dots, 0]$$

of integers obtained by reading the indices of $A_{\phi_1}A_{\phi_2}\cdots A_0$. Consider all Φ whose quiver paths begin at the framed vertex; we will fix a partial ordering $\leq$ on these finite sequences of nonnegative integers. We say

$$\Phi := [\phi_1, \phi_2, \dots, 0] \leq [\psi_1, \psi_2, \dots, 0] =: \Psi$$

if the number of components in Φ is less than the number of components in Ψ , or if the number of components in Φ equals the number of components in Ψ and the right-most nonzero entry in $\Psi - \Phi$ is positive when subtracting component-wise.

DEFINITION 4.5. *The s^{th} row of the bitableau*

$$(5) \quad \begin{array}{cccc|cccc} j_1^{(\phi)} & j_2^{(\phi)} & \dots & j_{v_1}^{(\phi)} & i_1^{(\phi)} & i_2^{(\phi)} & \dots & i_{v_1}^{(\phi)} \\ j_1^{(\psi)} & j_2^{(\psi)} & \dots & j_{v_2}^{(\psi)} & i_1^{(\psi)} & i_2^{(\psi)} & \dots & i_{v_2}^{(\psi)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ j_1^{(\mu)} & j_2^{(\mu)} & \dots & j_{v_l}^{(\mu)} & i_1^{(\mu)} & i_2^{(\mu)} & \dots & i_{v_l}^{(\mu)} \end{array} \begin{array}{l} A_{\phi_1} \dots A_0 \\ A_{\psi_1} \dots A_0 \\ \\ A_{\mu_1} \dots A_0 \end{array}$$

is defined to be the bitableau associated to rows $j_1^{(\psi)}, j_2^{(\psi)}, \dots, j_{v_s}^{(\psi)}$ and columns $i_1^{(\psi)}, i_2^{(\psi)}, \dots, i_{v_s}^{(\psi)}$ in the product $A_{\psi_1} A_{\psi_2} \dots A_0$ of general matrices. The bideterminant

$$(6) \quad \begin{array}{cccc|cccc} j_1^{(\phi)} & j_2^{(\phi)} & \dots & j_{v_1}^{(\phi)} & i_1^{(\phi)} & i_2^{(\phi)} & \dots & i_{v_1}^{(\phi)} \\ j_1^{(\psi)} & j_2^{(\psi)} & \dots & j_{v_2}^{(\psi)} & i_1^{(\psi)} & i_2^{(\psi)} & \dots & i_{v_2}^{(\psi)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ j_1^{(\mu)} & j_2^{(\mu)} & \dots & j_{v_l}^{(\mu)} & i_1^{(\mu)} & i_2^{(\mu)} & \dots & i_{v_l}^{(\mu)} \end{array} \begin{array}{l} A_{\phi_1} \dots A_0 \\ A_{\psi_1} \dots A_0 \\ \\ A_{\mu_1} \dots A_0 \end{array}$$

is the product of bideterminants of the form

$$(j_1^{(\psi)} \ j_2^{(\psi)} \ \dots \ j_{v_s}^{(\psi)} \ | \ i_1^{(\psi)} \ i_2^{(\psi)} \ \dots \ i_{v_s}^{(\psi)})_{A_{\psi_1} A_{\psi_2} \dots A_0}$$

obtained by taking the determinant of minors of rows $j_1^{(\psi)}, j_2^{(\psi)}, \dots, j_{v_s}^{(\psi)}$ and columns $i_1^{(\psi)}, i_2^{(\psi)}, \dots, i_{v_s}^{(\psi)}$ in the product $A_{\psi_1} A_{\psi_2} \dots A_0$ of general matrices.

Note that the s^{th} row of (5) is associated to the sequence Ψ of integers. We say the bitableau (5) is in *block standard form* if the sequence Φ of integers associated to each row of the bitableau is in nondecreasing order (with respect to the partial ordering $\leq$ defined earlier in this section) when ascending down the rows, and in the case the sequence Φ for multiple rows are identical, then these rows are in standard form as defined in Definition 4.2. In the case that the bitableau (5) is in block standard form, we will interchangeably say the bideterminant (associated to the bitableau) is in block standard form.

EXAMPLE 4.6. *Let*

$$A_0 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

Then

$$\begin{array}{ccc|ccc} (2 & | & 1 & & &)_{A_0} \\ (2 & | & 2 & & &)_{A_0} \\ (1 & 2| & 1 & 2 & &)_{A_1 A_0} \\ (2 & | & 1 & & &)_{A_1 A_0} \end{array}$$

is a bideterminant in block standard form, and the polynomial associated to the bideterminant is

$$\begin{aligned} x_{21} x_{22} \det(A_1 A_0) \cdot (A_1 A_0)_{2,1} &= x_{21} x_{22} a_{11} a_{22} (x_{11} x_{22} - x_{12} x_{21}) a_{22} x_{21} \\ &= x_{21}^2 x_{22} a_{11} a_{22}^2 (x_{11} x_{22} - x_{12} x_{21}). \end{aligned}$$

We recall [11, Theorem 13.1].

THEOREM 4.7. *Let $R = \mathbb{C}[\{x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}]$. Then bideterminants of standard bitableaux form a basis of R over $\mathbb{C}$.*

LEMMA 4.8. *Let the set $B \subseteq \mathrm{GL}_n(\mathbb{C})$ of invertible upper triangular matrices act on $M_{n \times m}$ via left translation. Consider the character $\chi_p(b) = \prod_{i=p}^n b_{ii}$ of B and let $f = (p \ p+1 \ \cdots \ n | i_1 \ \cdots \ i_{n-p+1}) \in \mathbb{C}[M_{n \times m}]$ where $1 \leq i_1 < i_2 < \cdots < i_{n-p+1} \leq m$. Then $b.f = \chi_p(b)f$.*

PROOF. Write $b = tu$, where $t = (t_{ii}) \in T$ and $u \in U$, T is the maximal torus in B and U is the maximal unipotent subgroup of B . Then $t.f = \prod_{i=p}^n t_{ii}f = \chi_p(t)f$ and for the subgroup $U_{i,j}$ which has 1 along the main diagonal, the variable u in the (i,j) -entry and 0 elsewhere, we will show that $U_{i,j}$ fixes f . So let $\hat{u} \in U_{i,j}$. Then since $\hat{u}.f(x) = f(\hat{u}^{-1}.x) = f(\hat{u}^{-1}x)$,

$$\hat{u}.x_{st} = \begin{cases} x_{it} - x_{jt}u & \text{if } s = i, \\ x_{st} & \text{otherwise.} \end{cases}$$

So $\hat{u}.f = f - u(p \ p+1 \ \cdots \ j \ \cdots \ j \ \cdots \ n | i_1 \ \cdots \ i_{n-p+1}) = f$ if $p \leq i$ and $\hat{u}.f = f$ if $p > i$. This concludes the proof. $\square$

The following lemma generalizes Lemma 4.8.

LEMMA 4.9. *Consider the equioriented (all arrows are pointing in the same direction) quiver*

$$\begin{array}{ccccccc} 1' & \xrightarrow{a_0} & 1 & \xrightarrow{a_1} & 2 & \cdots & r & \xrightarrow{a_r} & r+1 \\ \circ & & \bullet & & \bullet & & \bullet & & \bullet \end{array},$$

where $\beta_{1'} = m$ and $\beta_i = n$ for $1 \leq i \leq r+1$. Let $\mathbb{U}_\beta \subseteq B^{r+1}$ be the product of largest unipotent subgroups and let A_m be a general matrix associated to arrow a_m for each $0 \leq m \leq r$. Then $F^\bullet \mathrm{Rep}(Q, \beta) = M_{n \times m} \oplus \mathfrak{b}^{\oplus r}$ and standard bideterminants of the form

$$(7) \quad (p \ p+1 \ \cdots \ n | i_1 \ i_2 \ \cdots \ i_{n-p+1})_{A_m \cdots A_0}, \quad \text{where } 1 \leq p \leq n \text{ and } 0 \leq m \leq r,$$

are $\mathbb{U}_\beta$ -invariant polynomials.

PROOF. For $u = (u_1, \dots, u_{r+1}) \in \mathbb{U}_\beta$ and $(A_0, A_1, \dots, A_r) \in M_{n \times m} \oplus \mathfrak{b}^{\oplus r}$,

$$u.(A_0, A_1, \dots, A_r) = (u_1 A_0, u_2 A_1 u_1^{-1}, \dots, u_{\alpha+1} A_\alpha u_\alpha^{-1}, \dots, u_{r+1} A_r u_r^{-1}).$$

We will write the entries of the product $A_\alpha \cdots A_0$ of matrices as $((\alpha)y_{st})$, where $((0)y_{st}) = (x_{st}) \in M_{n \times m}$. Then for the subgroup U_{ij} of $\mathbb{U}_\beta$ which is

$$U_{ij} = \{u_{ij} = (I_n, \dots, \hat{u}_m, \dots, I_n) : \hat{u}_m \text{ is the matrix with the variable } u \text{ in the } (i,j)\text{-entry, where } i < j, \\ 1 \text{ along the diagonal entries, and } 0 \text{ elsewhere}\},$$

$u_{ij} \in U_{ij}$ acts on $((\alpha)y_{st}) \in A_\alpha \cdots A_1 A_0$ as follows: U_{ij} changes the coordinate polynomial $((\alpha)y_{st})$ via

$$u_{ij}.((\alpha)y_{st}) = \begin{cases} ({}_{(m-1)}y_{it} - {}_{(m-1)}y_{jt}u & \text{if } \alpha = m-1 \text{ and } s = i, \\ {}_{(\alpha)}y_{st} & \text{otherwise.} \end{cases}$$

So for $f = (p \ p+1 \ \cdots \ n | i_1 \ i_2 \ \cdots \ i_{n-p+1})_{A_\alpha \cdots A_0}$,

$$u_{ij}.f = \begin{cases} f - u(p \ p+1 \ \cdots \ j \ \cdots \ j \ \cdots \ n | i_1 \ i_2 \ \cdots \ i_{n-p+1})_{A_{m-1} \cdots A_0} = f & \text{if } \alpha = m-1 \text{ and } p \leq i, \\ f & \text{otherwise.} \end{cases}$$

Thus standard bideterminants of the form $(p \ p+1 \ \cdots \ n \mid i_1 \ i_2 \ \cdots \ i_{n-p+1})_{A_m \cdots A_0}$ are $\mathbb{U}_\beta$ -invariant polynomials, where $1 \leq p \leq n$ and $0 \leq m \leq r$. $\square$

DEFINITION 4.10. Assume $\beta = (n, \dots, n, m) \in \mathbb{Z}_{\geq 0}^{Q_0^\dagger}$, where m is associated to the framed vertex. Define

$$(8) \quad (J|I)_{\Phi\Psi\cdots\Gamma} := \begin{pmatrix} j_1 & j_1+1 & \cdots & n & | i_1^{(\phi)} & i_2^{(\phi)} & \cdots & i_{n-j_1+1}^{(\phi)} \\ j_2 & j_2+1 & \cdots & n & | i_1^{(\psi)} & i_2^{(\psi)} & \cdots & i_{n-j_2+1}^{(\psi)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ j_l & j_l+1 & \cdots & n & | i_1^{(\mu)} & i_2^{(\mu)} & \cdots & i_{n-j_l+1}^{(\mu)} \end{pmatrix}_{A_{\phi_1} \cdots A_0}$$

as the bideterminant in block standard form, where $A_{\psi_1} \cdots A_0$ is a general representation of the quiver path $a_{\psi_1} \cdots a_0$ which begins at the framed vertex.

Note that each sequence Φ of integers associated to each row of (8) corresponds to a general representation of a quiver path that begins at the framed vertex (this is important as this will imply the uniqueness of (8): if $(J|I)_{\Phi\Psi\cdots\Gamma} = (J'|I')_{\Phi'\Psi'\cdots\Gamma'}$ where $(J|I)_{\Phi\Psi\cdots\Gamma}$ and $(J'|I')_{\Phi'\Psi'\cdots\Gamma'}$ are in block standard form, then $J = J'$, $I = I'$, and $\Phi = \Phi'$, $\Psi = \Psi'$, $\dots$, $\Gamma = \Gamma'$). The following proof is a generalization of the proof of Theorem 13.3 in [11].

PROOF. By Theorem 1.1, it suffices to find all invariants for paths starting at a framed vertex. Let $_{(\alpha)}a_{st}$ be the entries of a general matrix A_α and let x_{st} be the entries of a general matrix A_0 . Suppose f is a $\mathbb{U}_\beta$ -invariant polynomial. Without loss of generality, if $f(x_{st}, _{(\alpha)}a_{st}) = g(x_{st}, _{(\alpha)}a_{st}) + h(_{(\alpha)}a_{ii})$, where all the monomials of g are divisible by some x_{st} for some s and t and $h \in \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$, then we only consider g by subtracting off h since we have already proved that $h = h(_{(\alpha)}a_{ii})$ is an invariant polynomial. By applying Lemma 4.9 to each row of (8), we see that (1) is a $\mathbb{U}_\beta$ -invariant polynomial. Now suppose there exists a polynomial not in $\mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$ or not of the form (1) which is in $\mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta}$. That is, suppose there exists $f \in \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]$ fixed by $\mathbb{U}_\beta$ with its monomials divisible by x_{st} for some s and t which cannot be written as (1). Let $j \leq n-1$ be the biggest integer which satisfies the following:

there exists a $\mathbb{U}_\beta$ -invariant $F \in \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]$ such that when F is written in terms of the block standard basis, i.e., $F = \sum_\nu g_\nu(_{(\alpha)}a_{st})(J_\nu|I_\nu)_{\Phi\Psi\cdots\Gamma}$ with each $(J_\nu|I_\nu)_{\Phi\Psi\cdots\Gamma}$ a standard Young bideterminant in block standard form and each $g_\nu \neq 0$, then there exists a ν and a row in J_ν where j is not followed by $j+1$. Let us label this choice of j as $(\dagger)$.

Let a_ϕ be the arrow associated to a general representation A_ϕ . Writing ha_ϕ to be the head of the arrow a_ϕ , let $U_{j,j+1}$ be the subgroup consisting of matrices of the form $u_{j,j+1} = (I_n, \dots, u_{ha_\phi}, \dots, I_n)$ where ha_ϕ has diagonal entries 1, the variable $-u$ in $(j, j+1)$ -entry, and 0 elsewhere. Let's write the entries of the product $A_\alpha \cdots A_0$ of matrices as y_{st} . Then for $u_{j,j+1} \in U_{j,j+1}$,

$$u_{j,j+1} \cdot y_{st} = \begin{cases} y_{jt} + u y_{j+1,t} & \text{if } \alpha = \phi \text{ and } s = j, \\ y_{st} & \text{otherwise} \end{cases}$$

since $A_\alpha \cdots A_0$ is a general representation of the quiver path $a_\alpha \cdots a_0$. To explain further, if the path $a_\alpha \cdots a_0$ includes a_ϕ somewhere strictly in the middle of the path,

i.e., $a_\alpha \cdots a_0 = a_\alpha \cdots a_\phi \cdots a_0$, then although u_{ha_ϕ} acts by left multiplication on A_ϕ , u_{ha_ϕ} acts by right (inverse) multiplication on the general representation of the arrow in the path immediately following a_ϕ (this is the arrow which is immediately to the left of a_ϕ in the concatenation of the arrows $a_\alpha \cdots a_0$). Thus, the action by u_{ha_ϕ} is canceled. So for any α , u fixes every minor of the form

$$(\cdots j \ j+1 \cdots | \cdots)_{\Phi \Psi \dots \Gamma}.$$

Now let us write

$$F = \sum_{\nu} g_{\nu}((\alpha) a_{st})(J_{\nu}|I_{\nu})_{\Phi \Psi \dots \Gamma} + \sum_{\gamma} g_{\gamma}((\alpha) a_{st})(J_{\gamma}|I_{\gamma})_{\Phi \Psi \dots \Gamma},$$

with the following properties:

- the g_{ν} are nonzero,
- there exists at least one row in each J_{ν} which contains j but not $j+1$,
- if j appears in any row of J_{γ} , then so does $j+1$,
- the $(J_{\nu}|I_{\nu})_{\Phi \Psi \dots \Gamma}$ and $(J_{\gamma}|I_{\gamma})_{\Phi \Psi \dots \Gamma}$ are unique.

By Lemma 4.9, $\mathbb{U}_{\beta}$ fixes each row of $(J_{\gamma}|I_{\gamma})_{\Phi \Psi \dots \Gamma}$. Since $(J_{\gamma}|I_{\gamma})_{\Phi \Psi \dots \Gamma}$ is the product of the rows in the bideterminant, $\mathbb{U}_{\beta}$ fixes $(J_{\gamma}|I_{\gamma})_{\Phi \Psi \dots \Gamma}$, which in turn fixes $\sum_{\gamma} g_{\gamma}((\alpha) a_{st})(J_{\gamma}|I_{\gamma})_{\Phi \Psi \dots \Gamma}$. Among those rows with identical sequence Φ in each J_{ν} in block standard form, the only possible occurrences of j and $j+1$ in its rows are as follows:

- (1) j is followed by $j+1$,
- (2) j is followed by an integer larger than $j+1$,
- (3) j ends in a row,
- (4) $j+1$ is preceded by an integer smaller than j ,
- (5) $j+1$ starts a row.

Since J_{ν} is in block standard form, all rows of type (i) must occur above all rows of type (i+1) within each block.

After re-numbering the indices ν , let J_1 have the greatest number of rows, say M , of types (2) and (3). There may be other Young tableaux, say $J_2, \dots, J_W$ in $(J_{\nu}|I_{\nu})_{\Phi \Psi \dots \Gamma}$, with M rows of types (2) and (3), but we may ignore them because of the uniqueness of $(J_{\nu}|I_{\nu})_{\Phi \Psi \dots \Gamma}$. We label the sequence Φ of integers associated to the rows of $(J_{\nu}|I_{\nu})_{\Phi \Psi \dots \Gamma}$ as $\Phi(\nu) \leq \Psi(\nu) \leq \dots \leq \Gamma(\nu)$. Let

$$U_{j,j+1} := \{u_{j,j+1} = (I_n, \dots, u_{ha_{\phi(1)}}, \dots, u_{ha_{\mu(1)}}, \dots, I_n) : u_{ha_{\phi(1)}} = \dots = u_{ha_{\mu(1)}} \text{ is the matrix with } 1 \text{ along the diagonal, the same variable } u \text{ in } (j, j+1)\text{-entry, and } 0 \text{ elsewhere}\}.$$

Applying $u_{j,j+1} \in U_{j,j+1}$ to F , we see that $g_1((\alpha) a_{st})(J_1|I_1)_{\Phi \Psi \dots \Gamma}$ gives a term

$$u^M g_1((\alpha) a_{st})(J'_1|I_1)_{\Phi \Psi \dots \Gamma},$$

where J'_1 is obtained from J_1 by replacing each j in rows of type (2) and (3) by $j+1$; furthermore, $(J'_1|I_1)_{\Phi \Psi \dots \Gamma}$ is block standard. Thus, the tableau J'_1 uniquely determines J_1 for the following reasons: first, all rows of type (3) have been changed to rows ending with $j+1$; such rows must end with $j+1$ in J_1 by our choice of j . Otherwise, to obtain J_1 , we change $j+1$ to j in M rows of J'_1 reading from top to bottom (while ignoring those rows which contain both j and $j+1$).

Now, in $u_{j,j+1}.F$, any other occurrence of $(J'_1|I_1)_{\Phi \Psi \dots \Gamma}$ is with a coefficient $u^k g'((\alpha) a_{st})$ where $k < M$ and g' is a polynomial in $(\alpha) a_{st}$ since j was carefully chosen such that j is the biggest integer satisfying $(\dagger)$. Since F is a polynomial over

a field of characteristic 0, the coefficient of $(J'_1|I_1)_{\Phi\Psi\cdots\Gamma}$ depends on u . Thus, F is not fixed by $\mathbb{U}_\beta$. This shows that j cannot appear in a row of J_ν without $j+1$, which shows that $\mathbb{U}_\beta$ -invariant polynomials must be of the form (1). Thus, if some of the terms of a $\mathbb{U}_\beta$ -invariant polynomial f are divisible by x_{st} for some s and t , then we write f as

$$f(x_{st}, (a)_{st}) = g(x_{st}, (a)_{st}) + h((a)_{st}),$$

where all the terms in g are divisible by x_{st} for some s and t and h is a polynomial in $(a)_{st}$. By the first part of this proof, $h \in \mathbb{C}[t^{\oplus Q_1}]$ while g must be of the form (1). It is immediate by the proof of Theorem 1.1 that if none of the terms in a $\mathbb{U}_\beta$ -invariant polynomial are divisible by x_{st} for all $1 \leq s \leq n$ and $1 \leq t \leq m$, then the polynomial is in $\mathbb{C}[t^{\oplus Q_1}]$. This concludes our proof. $\square$

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