Linear Algebra

Tanush Shaska

Linear Algebra for Machine Learning

Tony Shaska

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Preface

Linear algebra is one of the cornerstones of modern mathematics, with profound applications in computer science, engineering, physics, and the social sciences. It serves as the language of transformations, optimizations, and high-dimensional spaces, making it indispensable not only for pure mathematicians but also for practitioners in fields such as artificial intelligence, data science, and machine learning.

Traditionally, linear algebra is introduced at the sophomore level as the first rigorous encounter with vector spaces, linear transformations, and matrices. However, the way it is taught often varies. Some textbooks emphasize applications at the expense of mathematical depth, leaving students with a collection of computational techniques but little insight into the underlying structures. Others focus on the formal mathematical framework, avoiding computations and real-world connections. This book aims to strike a balance by presenting both the theoretical foundations and computational techniques of linear algebra while maintaining a strong geometric perspective.

One of the distinguishing features of this book is its integration of geometry throughout. The transformations of conic sections, for example, illustrate how diagonalizing a matrix corresponds to changing the basis of a vector space, revealing the deep connection between algebra and geometry. The notion of invariants—quantities that remain unchanged under transformations—appears repeatedly in discussions on eigenvalues, singular value decomposition, and other fundamental topics.

Beyond its traditional applications in physics and engineering, linear algebra has become an essential tool in artificial intelligence and data science. Machine learning algorithms, at their core, rely on linear algebraic structures. The representation of data as high-dimensional vectors, the optimization of loss functions, and the efficient computation of gradients in deep learning frameworks are all built on fundamental linear algebraic operations. Techniques such as Principal Component Analysis (PCA) for dimensionality reduction, singular value decomposition (SVD) for data compression, and gradient descent for optimization all rely on a solid understanding of linear algebra. Even neural networks, often perceived as highly nonlinear systems, can be analyzed as compositions of linear transformations with nonlinearity introduced via activation functions.

This book provides a comprehensive introduction to linear algebra while preparing students for more advanced applications in modern computational sciences. In later chapters, we explore optimization techniques, probability, statistics, and linear models—key topics for machine learning practitioners. The final chapters introduce neural networks from a linear algebraic perspective, offering insights into how matrices, vectors, and transformations underlie deep learning architectures.

A wide range of exercises is provided, from fundamental problems reinforcing key concepts to more challenging ones that connect to broader areas of mathematics. While this book assumes familiarity with calculus and basic discrete mathematics, no prior knowledge of linear algebra is required.

The material in this book has been shaped by years of teaching at the University of California, Irvine; the University of Idaho; the University of Vlora; and Oakland University. I am grateful to my students, whose engagement and curiosity have influenced this text. It is my hope that this book not only equips students with the mathematical tools necessary for their fields but also inspires an appreciation for the elegance and power of linear algebra.

Tony Shaska
Rochester, 2018

Chapter 1

Euclidean spaces, linear systems

We start this chapter with the familiar notion of Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ from previous lecture. Intuition from $\mathbb{R}^2$ and $\mathbb{R}^3$ will be used to generalize concepts for $\mathbb{R}^n$ including the norm, dot product of vectors, angles among vectors, and the geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$.

In Sec. 1.3, we introduce the matrices and their algebra. Using matrices to solve linear systems of equations involves computing the row-echelon form and the reduced row-echelon form of matrices. These are the so-called Gauss algorithm and Gauss - Jordan algorithm and are studied in Sec. 1.4. In Sec. 1.5 we study the inverses of matrices and algorithms of computing such matrices.

1.1 Vectors in Physics and Geometry

We will denote by $\mathbb{R}^2$ the xy-plane and by $\mathbb{R}^3$ the coordinate system in space. For any two given points P and Q, an **directed line segment** (P, Q) is the segment PQ. We call P the **initial point** and Q the **terminal point**. Two directed line segments (A, B) and (C, D) are called **equipollent** when the points A, B, D, C, in this order, form a parallelogram.

1.1.1 The plane $\mathbb{R}^2$

Every point in xy-plane is represented uniquely by an ordered pair (x, y). For any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ their distance is given by

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Let's denote the set of all directed line segments in $\mathbb{R}^2$ by S. In this set S we define the following relation: $(P_1, Q_1) \sim (P_2, Q_2)$ if the following hold

(i) lines P_1Q_1 and P_2Q_2 are parallel

(ii) $d(P_1, Q_1) = d(P_2, Q_2)$

(iii) directed line segments (P_1, Q_1) and (P_2, Q_2) have the same direction

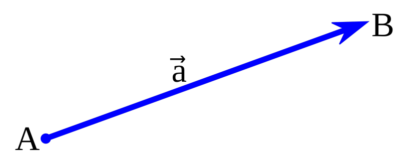


Figure 1.1: A Euclidean vector

Exercise 1. *Prove that $\sim$ is an equivalence relation.*

A **vector** is called an equivalence class from the above relation. Geometrically two directed line segments (A,B) and (C,D) are equivalent when they are equipollent. The equivalence class of the (A,B) will be denoted by $\overrightarrow{AB}$. The **magnitude** (or **length**) of the vector $\overrightarrow{AB}$ is simply the distance

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.1}$$

and from now on will be denoted by $\left\| \overrightarrow{AB} \right\|$.

Denote the set of all such equivalence classes by $V := S/\sim$. Hence, V the set of all vectors from the xy-plane. Moreover, the above three conditions are geometrically equivalent with moving the vector $\overrightarrow{P_1Q_1}$ in a parallel way over $\overrightarrow{OP}$, where O is the origin of the coordinate system. So we can assume that all vectors of V start at the origin O by picking for each equivalence class the representative that starts at the origin O. Elements of V will be denoted by bold letters throughout these lectures. Hence we have the following:

Lemma 1.1. *Thus, there is a one-to-one correspondence between the set of elements of V and points of the xy-plane, namely for any $P(x,y)$*

$$\mathbf{u} = \overrightarrow{OP} \longleftrightarrow P = (x,y)$$

Proof. Exercise $\square$

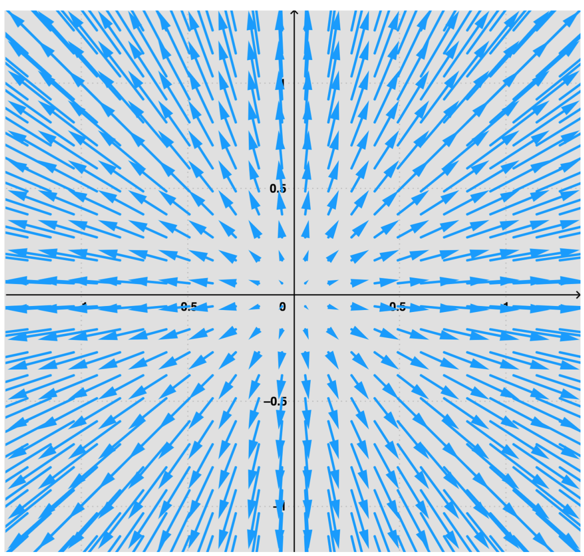


Figure 1.2: Vectors in $\mathbb{R}^2$

Hence, a vector $\mathbf{u} = \overrightarrow{OP}$ is identified with an ordered pair (x,y) and will be denoted by $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, in order to distinguish it from the point $P(x,y)$. Because of the above correspondence, from now on we will identify $V = \mathbb{R}^2$. We say that x and y are the **coordinates** of $\mathbf{u}$.

Example 1.1. *Let $P(1,2)$ and $Q(3,7)$ be given in $\mathbb{R}^2$. Find the coordinates of vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.*

Next we will see the addition and scalar multiplication of vectors. Most likely the reader is not new to such concepts since they are studied in a first course in elementary physics. We will focus on the algebraic and geometric point of view.

For any two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in V define the addition and scalar multiplication as

$$\mathbf{u} + \mathbf{v} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}, \qquad \textit{and} \qquad r \cdot \mathbf{u} := \begin{bmatrix} ru_1 \\ ru_2 \end{bmatrix}, \tag{1.2}$$

where $r \in \mathbb{R}$. Geometrically scalar multiplication $r\mathbf{u}$ is described as in Fig. 1.3, where $r\mathbf{u}$ is a new vector with the same direction as $\mathbf{u}$ and length r-times the length of $\mathbf{u}$.

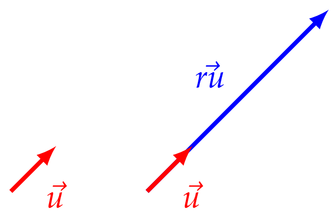


Figure 1.3: Multiplying by a scalar

Addition of two vectors $\mathbf{u}$ and $\mathbf{v}$ geometrically is described in Fig. 1.4.

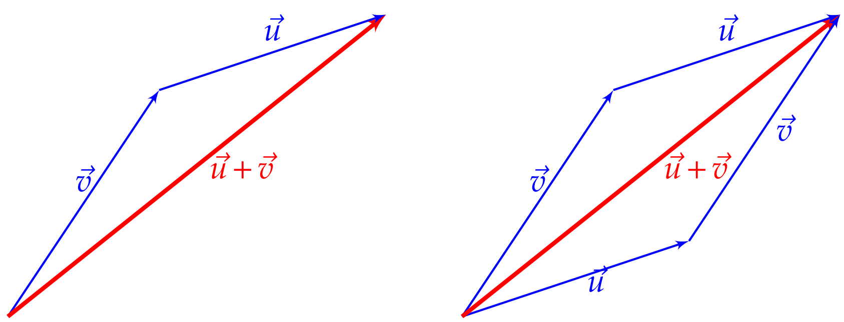


Figure 1.4: Addition of vectors

Exercise 2. *Prove that such definitions agree with addition and scalar multiplication defined in Eq. (1.2)*

The following exercise is elementary, but very interesting when we discuss determinants of matrices in coming lectures.

Exercise 3. *Given two vectors*

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

in $\mathbb{R}^2$ we can assume that both start at the origin.

(i) *Prove that the area of the parallelogram determined by these two vectors is* $A = |u_1 v_2 - u_2 v_1|$.
(ii) *Prove that the lines determined by* $\mathbf{u}$ *and* $\mathbf{v}$ *are perpendicular if and only if* $u_1 v_1 = u_2 v_2$.
(iii) *Determine the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

The following exercises explore further the geometry of the vectors in $\mathbb{R}^2$.

Exercises:

1. *If* A, B, C *are vertices of a triangle, find* $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

2. *Let* c *a positive real number and* O_1, O_2 *points on the* xy*-plane with coordinates* $(c, 0)$ *and* $(-c, 0)$ *respectively. Find the equation of all points* P *such that*

$$\left\| \overrightarrow{PO_1} \right\| + \left\| \overrightarrow{PO_2} \right\| = 2a,$$

for $a > c$.

3. *Let* $\triangle ABC$ *be a given triangle and* θ *the angle between* AB *and* AC. *Prove the Law of Cosines*

$$BC^2 = AB^2 + AC^2 - 2\, AB \cdot AC \cdot \cos\theta \qquad (1.3)$$

4. *Let* a *and* b *sides of a parallelogram and* d_1, d_2 *its diagonals. Prove that*

$$d_1^2 + d_2^2 = 2(a^2 + b^2).$$

5. *Prove that the diagonals of a parallelogram are perpendicular if and only if all sides are equal.*

6. *Prove that the distance* d *of a point* $P = (x_0, y_0)$ *from the line*

$$ax + by + c = 0$$

is given by

$$d = \frac{\left| a x_0 + b y_0 + c \right|}{\sqrt{a^2 + b^2}}.$$

1.1.2 The space $\mathbb{R}^3$

Next we review briefly the geometry of the space and vectors in $\mathbb{R}^3$. Definition of vectors in $\mathbb{R}^3$ goes exactly the same with their definition in $\mathbb{R}^2$, by adding a third coordinate.
Recall that $\mathbb{R}^3$ is the Cartesian product

$$\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$$

and a point P in $\mathbb{R}^3$ is represented by an ordered triple (x_0, y_0, z_0) as shown in Fig. 1.5.

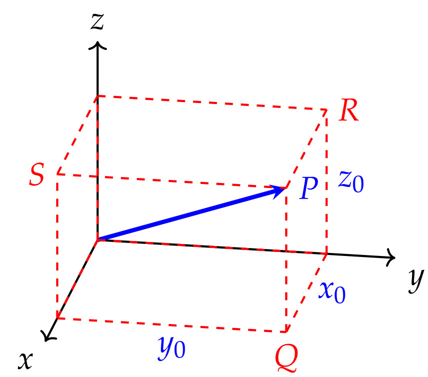


Figure 1.5: Coordinates of $P(x,y,z)$.

Let be given two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ in $\mathbb{R}^3$. We will show that the distance $|P_1P_2|$ between the two points is

$$||P_1P_2|| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

To verify this formula we construct a parallelepiped where the points P_1 and P_2 are vertices across from each other as in Fig. 1.6. If $A(x_2,y_1,z_1)$ and $B(x_2,y_2,z_1)$ are the other vertices as in Fig. 1.6, then

$$|P_1A| = |x_2-x_1|, \quad |AB| = |y_2-y_1|, \quad |BP_2| = |z_2-z_1|$$

Since the triangles $\triangle P_1BP_2$ and $\triangle P_1AB$ are right triangles, from the Pythagorean theorem we have

$$|P_1B|^2 = |P_1A|^2 + |AB|^2 \text{ and } |P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

Combining the two equations we have

$$\begin{aligned} ||P_1P_2||^2 &= ||P_1A||^2 + |||AB||^2 + ||BP_2||^2 \\ &= ||x_2-x_1||^2 + \left\|y_2-y_1\right\|^2 + ||z_2-z_1||^2 \\ &= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \end{aligned}$$

Thus,

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} \tag{1.4}$$

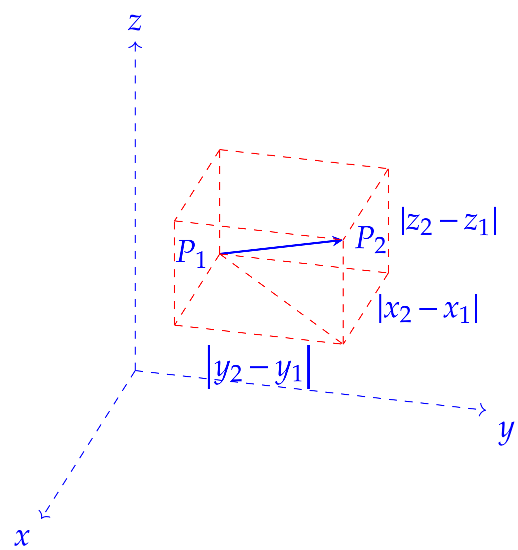
The distance between a point $P(x,y,z)$ and the origin is

$$\left\|\overrightarrow{OP}\right\| = \sqrt{x^2+y^2+z^2}.$$

Example 1.2. *Let $P(1,2,3)$ and $Q(4,2,1)$. Find the coordinates and the magnitude of the vector $\overrightarrow{PQ}$*

Solution: The coordinates of $\overrightarrow{PQ}$ are $\overrightarrow{PQ} = [3,0,-2]^t$ and its magnitude $\left\|\overrightarrow{PQ}\right\| = \sqrt{3^2+0^2+(-2)^2} = \sqrt{13}$. $\square$

Figure 1.6: Distance between two points



Every point in $3d$-space is represented uniquely by an ordered triple (x, y, z). For any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ a **Euclidean vector** (or simply a **vector**) is frequently represented by a ray (a line segment with a definite direction), or graphically as an arrow connecting an **initial point** P_1 with a **terminal point** P_2, and denoted by $\overrightarrow{P_1P_2}$.

The **magnitude** (or **length**) of $\overrightarrow{P_1P_2}$ is simple the distance

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Let's denote the set of all 'vectors' in $\mathbb{R}^3$ by S. In this set S we define the following relation: $\overrightarrow{P_1Q_1} \sim \overrightarrow{P_2Q_2}$ if the following hold

(i) lines P_1Q_1 and P_2Q_2 are parallel

(ii) $\left\| \overrightarrow{P_1Q_1} \right\| = \left\| \overrightarrow{P_2Q_2} \right\|$

(iii) $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$ have the same direction

Exercise 4. *Prove that $\sim$ is an equivalence relation.*

Denote the set of all such equivalence classes by $V := S/\sim$. Hence, V the set of all equivalence classes of vectors from the xy-plane. Moreover, the above three conditions are geometrically equivalent with moving the vector $\overrightarrow{P_1Q_1}$ in a parallel way over $\overrightarrow{OP}$, where O is the origin of the coordinate system.

Then, a **vector** is called an equivalence class from the above relation. So we can assume that all vectors of V start at the origin O by picking for each equivalence class the representative that starts at the origin O. Elements of V will be denoted by bold letters throughout these lectures.

Lemma 1.2. *Thus, there is a one-to-one correspondence between the set of elements of V and points of the 3d-space, namely for any $P(x,y,x)$*

$$\mathbf{u} = \overrightarrow{OP} \longleftrightarrow P = (x,y,z)$$

Proof. Exercise $\square$

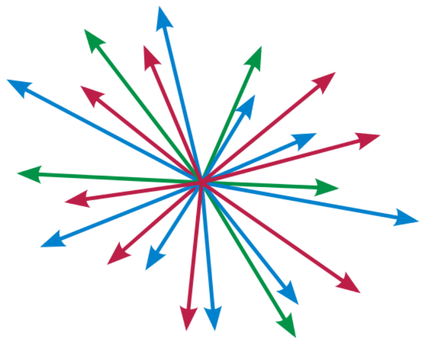


Figure 1.7: Representatives of the equivalence classes

Hence, a vector $u = \overrightarrow{OP}$ is identified with an ordered triple (x,y,z) and will be denoted by $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, in order to distinguish it from the point $P(x,y,z)$. Because of the above correspondence, from now on we will identify $V = \mathbb{R}^3$. We say that x, y and z are the **coordinates** of $\mathbf{u}$.

For any two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ we define the addition and scalar multiplication as in $\mathbb{R}^2$, namely

$$\mathbf{u} + \mathbf{v} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}, \quad r \cdot \mathbf{u} := \begin{bmatrix} ru_1 \\ ru_2 \\ ru_3 \end{bmatrix}.$$

where $r \in \mathbb{R}$. Since any two generic lines determine a plane, the geometric interpretation of addition and scalar multiplication of $\mathbb{R}^2$ is still valid in $\mathbb{R}^3$.

Sometimes it is more convenient to write vectors as **row vectors**. The **transpose** of the vector $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the row vector $u^t = [x,y,z]$ and the transpose of the row vector $[x,y,z]$ is the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. With these conventions the vector $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ will also be denoted by $\mathbf{u} = [x,y,z]^t$.

Exercise 5. *Find the coordinates of the vector $\overrightarrow{P_1P_2}$ when $P_1(1,1,2)$ and $P_2(2,4,6)$.*

Exercise 6. *Find* $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $\|\mathbf{v}\|$ *and* $\|\mathbf{v}-\mathbf{w}\|$, $\|\mathbf{v}+\mathbf{w}\|$, *and* $-2\mathbf{v}$, *if* $\mathbf{v}=[1,2,3]^t$ *and* $\mathbf{w}=[-1,2,-3]^t$.

Exercise 7. *Find* $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $\|\mathbf{v}\|$ *and* $\|\mathbf{v}-\mathbf{w}\|$, $\|\mathbf{v}+\mathbf{w}\|$, *and* $-2\mathbf{v}$, *if* $\mathbf{v}=[1,0,1]^t$ *and* $\mathbf{w}=[-1,-2,2]^t$

Properties of vector addition and multiplying by a scalar we can summarize below:

Theorem 1.1. *If* $\mathbf{u},\mathbf{v},\mathbf{w}$ *are three vectors in* $\mathbb{R}^3$ *and* $c,d\in\mathbb{R}$ *are scalars, then the following hold:*

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
(ii) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
(iii) $\mathbf{u}+\mathbf{0}=\mathbf{u}$
(iv) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
(vii) $(cd)\mathbf{u}=c(d\mathbf{u})$
(viii) $1\mathbf{u}=\mathbf{u}$

Proof. The proof is left as an exercise for the reader. □

Let's denote by V_3 the set of all vectors in the 3-dimensional space $\mathbb{R}^3$. Three vectors which play a special role in V_3 are

$$\mathbf{i}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{j}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

These vectors are called vectors of the **standard basis**. We will explain this terminology in more detail in the coming sections.

Exercise 8. *Prove that every vector in* $\mathbb{R}^3$ *is expressed in terms of vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. *In other words, if* $\mathbf{u}=\begin{bmatrix}a\\b\\c\end{bmatrix}$, *then we have*

$$\mathbf{u}==a\mathbf{i}+b\mathbf{j}+c\mathbf{k}.$$

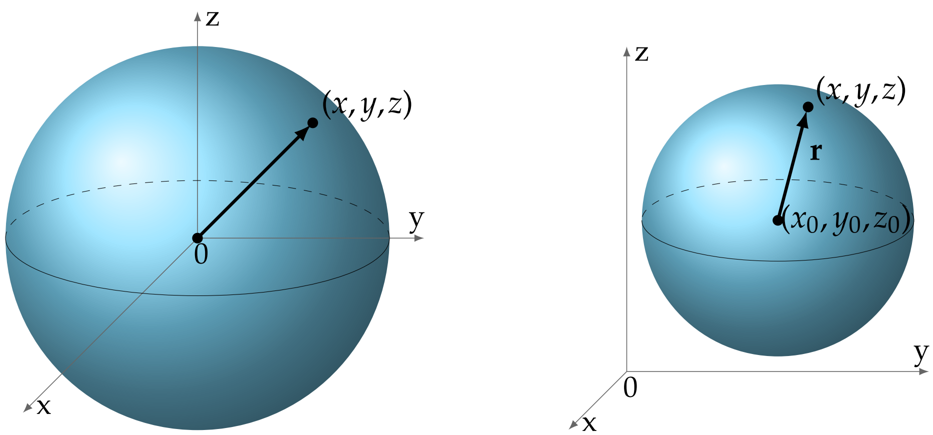
A vector u is called a **unit vector** if it has length 1. For example, vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors. A unit vector which has the same direction with a given vector $\mathbf{u}$ is a vector $\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{\mathbf{u}}{\|\mathbf{u}\|}$. In the next section we will formalize such definitions to the case of $\mathbb{R}^n$. The reader should make sure to fully understand the concepts from $\mathbb{R}^2$ and $\mathbb{R}^3$ before proceeding to $\mathbb{R}^n$.

Exercise 9. *Let* $\mathbf{v}=[x_0,y_0,z_0]^t$ *be a fixed vector in* $\mathbb{R}^3$. *Describe the set of all points* $P(x,y,z)$ *which satisfy* $\|\mathbf{u}-\mathbf{v}\|=1$, *where* $\mathbf{u}=[x,y,z]^t$.

Equation of the sphere

Using the distance formula above we can easily determine the equations of some simple geometric objects. The equation of the sphere with center at the point with coordinates (x_0,y_0,z_0) and radius r is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$



(a) Radius r, center $(0,0,0)$ (b) Radius r, center (x_0,y_0,z_0)

Figure 1.8: Spheres in $\mathbb{R}^3$

To prove this we just need the definition of the sphere, which is the set of all points $P(x,y,z)$ equidistant from the fixed point $Q(x_0,y_0,z_0)$ with a distance r from it. Thus, $\left\|\overrightarrow{QP}\right\| = r$. Squaring both sides we have $\left\|\overrightarrow{QP}\right\|^2 = r^2$ or

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2}$$

So the sphere with center at Q is the set of all terminal points of vectors with initial point at Q and magnitude r. When the center of the sphere is at the origin we have $x^2+y^2+z^2 = r^2$ and in this case the sphere is the set of all terminal points of vectors with magnitude r and initial point at the origin.

However, not every sphere has an equation as above. Consider the following example:

Example 1.3. *Prove that the following equation represent a sphere and find its radius and its center*

$$4x^2+4y^2+4z^2-8x+16y = 1.$$

Solution: Complete squares for $4x^2-8x$, $4y^2+16y$, and we have

$$(x-1)^2+(y+2)^2+z^2 = \frac{21}{4}$$

Thus the equation represents a sphere with center $(1,-2,0)$ and radius $\sqrt{\frac{21}{4}}$.

□

Remark 1.1. *Notice that the process of completing the square in each variable x,y,z gets complicated when the equation has cross terms xy, xz, and yz. We will learn how to handle such equations in later chapters.*

An equation in variables x and y represents a curve in $\mathbb{R}^2$ and a surface in $\mathbb{R}^3$. We illustrate with an easy example for which we construct the graph in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 1.4. *Construct the graph of* $x^2 + y^2 = 4$ *in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$

Solution: In $\mathbb{R}^2$ this equation represents a circle with radius 2 and center at the origin.

In $\mathbb{R}^3$, the graph is all points $P(x, y, z)$, where $x^2 + y^2 = 4$ and the z-coordinate takes any value $z \in \mathbb{R}$. Hence, it is a right cylinder with radius $r = 2$ and exists the z-axis as in Fig. 1.9 □

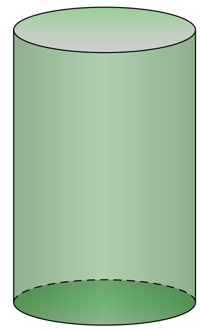


Figure 1.9: $x^2 + y^2 = 4$ in $\mathbb{R}^3$

1.1.3 Dot product

In $\mathbb{R}^2$, the **dot product** of two vectors $\mathbf{u} = [u_1, u_2]^t$ and $\mathbf{v} = [v_1, v_2]^t$ is defined as follows

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

For every two given vectors in $\mathbb{R}^3$, $\mathbf{u} = [u_1, u_2, u_3]^t$ and $\mathbf{v} = [v_1, v_2, v_3]^t$, **dot product** is called the real number $\mathbf{u} \cdot \mathbf{v}$ given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example 1.5. *Find the dot product in each case:*

(i) $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

(ii) $\mathbf{u} = [3, 0, -1]^t$, $\mathbf{v} = [2, 1, 7]^t$.

Solution: We have

i) $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 1 + 2 \cdot (-2) = 3 - 4 = -1$

ii) $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + 0 \cdot 1 + (-1) \cdot 7 = 6 + 0 - 7 = -1$.

□

The proof of the following is left as an exercise.

Theorem 1.2. *For every three vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *in* V_3 *and* $r \in \mathbb{R}$ *we have*

(i) $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$

(ii) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,

(iii) $(r\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (r\mathbf{v})$

(iv) $\mathbf{u} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{u}$

(v) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

Definition 1.1. ***The angle*** *between two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^3$ *is called the smallest angle between them measured counterclockwise.*

Theorem 1.3. *If we denote by* θ *the angle between* $\mathbf{u}$ *and* $\mathbf{v}$, *then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \cos\theta$$

Proof. Using the cosine formula for the triangle OAB we have

$$||AB||^2 = ||OA||^2 + ||OB||^2 - 2\,||OA|| \cdot ||OB|| \cdot \cos\theta \qquad (1.5)$$

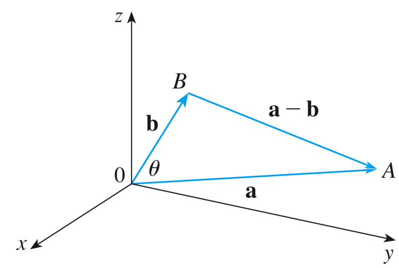


Figure 1.10

Since $||OA|| = ||\mathbf{u}||$, $||OB|| = ||\mathbf{v}||$, *and* $||BA|| = ||\mathbf{u}-\mathbf{v}||$, Eq. (1.5) becomes

$$||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\,||\mathbf{u}||\,||\mathbf{v}||\cos\theta \qquad (1.6)$$

The expression $||\mathbf{v}-\mathbf{w}||$, can be re-written as

$$\begin{aligned} ||\mathbf{u}-\mathbf{v}||^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = ||\mathbf{u}||^2 - 2\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2 \end{aligned}$$

Substituting in Eq. (1.6), we have

$$||\mathbf{v}||^2 - 2\mathbf{v}\cdot\mathbf{w} + ||\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 - 2\,||\mathbf{v}||\cdot||\mathbf{w}||\cos\theta$$

which implies $-2\mathbf{v}\cdot\mathbf{w} = -2\,||\mathbf{v}||\cdot||\mathbf{w}||\cos\theta$ and finally $\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||\cdot||\mathbf{w}||\cos\theta$. □

Corollary 1.1. *The angle θ between two vectors $\mathbf{v}$ and $\mathbf{w}$ is given by*

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||}.$$

Example 1.6. *Find the angle between the vectors $\mathbf{v} = [1,-2,2]^t$ and $\mathbf{w} = [2,-2,-1]^t$.*

Solution: First $||\mathbf{v}|| = \sqrt{1+4+4} = \sqrt{9} = 3$ and $||\mathbf{w}|| = \sqrt{4+4+1} = \sqrt{9} = 3$. Also, $\mathbf{v}\cdot\mathbf{w} = 1(2) + (-2)(-2) + 2(-1) = 4$. Then $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \frac{4}{3\cdot 3} = \frac{4}{9}$ and the angle between two vectors is $\theta = \cos^{-1}\left(\frac{4}{9}\right)$.

□

Dy vectors are called **orthogonal** if the angle between them is $\theta = \pi/2$. Thus, we have a corollary of Thm. 1.3, which gives an if and only if condition to determine if two vectors are orthogonal.

Corollary 1.2. *Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.*

For orthogonal vectors we use the notation $\mathbf{v} \perp \mathbf{w}$.

Example 1.7. *Determine if vectors $\mathbf{v} = [1,-5,2]^t$ and $\mathbf{w} = [3,1,1]^t$ are orthogonal.*

Solution: We have $\mathbf{v}\cdot\mathbf{w} = 1\cdot 3 + (-5)\cdot 1 + 2\cdot 1 = 0$, so vectors $\mathbf{w}, \mathbf{w}$ are orthogonal. □

Since $\cos\theta > 0$, for $0 \le \theta \le \pi/2$ and $\cos\theta < 0$ for $\pi/2 \le \theta \le \pi$, we have another corollary of Thm. 1.3

Corollary 1.3. *If θ is the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$, then*

$$\mathbf{v}\cdot\mathbf{w} = \begin{cases} > 0 & \text{for} \quad 0 \le \theta < \pi/2 \\ 0 & \text{for} \quad \theta = \pi/2 \\ < 0 & \text{for} \quad \pi/2 < \theta \le \pi \end{cases} \qquad (1.7)$$

Directional angles of a nonzero vector $\mathbf{u}$ are the angles α, β, γ of this vector with the major axes of the coordinate system as in Fig. 1.11. The cosine functions of these angles, $\cos\alpha, \cos\beta, \cos\gamma$, are called **directional cosines** of the vector $\mathbf{u}$.

Using Cor 1.1 we have

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{i}}{\|\mathbf{u}\|\cdot\|\mathbf{i}\|} = \frac{u_1}{\|\mathbf{u}\|} \tag{1.8}$$

and similarly for the other two angles

$$\cos\beta = \frac{\mathbf{u}\cdot\mathbf{j}}{\|\mathbf{u}\|\cdot\|\mathbf{j}\|} = \frac{u_2}{\|\mathbf{u}\|} \qquad \cos\gamma = \frac{\mathbf{u}\cdot\mathbf{k}}{\|\mathbf{u}\|\cdot\|\mathbf{k}\|} = \frac{u_3}{\|\mathbf{u}\|} \tag{1.9}$$

Using equations Eq. (1.8) and Eq. (1.9) , we square them and get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1.10}$$

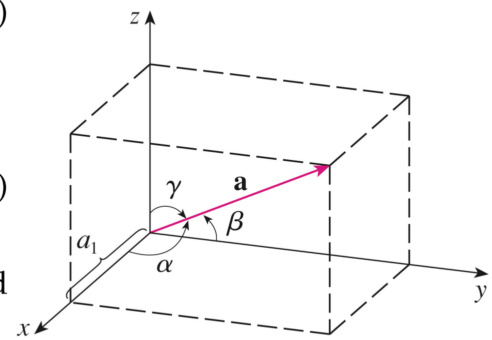


Figure 1.11: Directional angles

For $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ we have $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \|\mathbf{u}\|\cos\alpha \\ \|\mathbf{u}\|\cos\beta \\ \|\mathbf{u}\|\cos\gamma \end{bmatrix} = \|\mathbf{u}\| \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$. Then,

$$\boxed{\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}} \tag{1.11}$$

So, the directional cosines of the vector u are the components of a unit vector with the same direction as $\mathbf{u}$.

Example 1.8. *Determine directional cosines and directional angles for the vector* $\mathbf{u} = [2, 1, -4]^t$

Solution: First $\|\mathbf{u}\| = \sqrt{4+1+16} = \sqrt{21}$ then from Eq. (1.8) and Eq. (1.9), we have $\cos\alpha = \frac{2}{\sqrt{21}}$, $\cos\beta = \frac{1}{\sqrt{21}}$, $\cos\gamma = \frac{-4}{\sqrt{21}}$ and respectively $\alpha = 1.119$, $\beta = 1.351$, $\gamma = 2.632$. □

1.1.4 Cross product

Given vectors $\mathbf{u} = [u_1, u_2, u_3]^t$ and $\mathbf{v} = [v_1, v_2, v_3]^t$, then their **cross product** is defined as

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Another way to remember this formula is as the determinant of the 3 by 3 matrix

$$\begin{aligned} \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\,\mathbf{i} + (u_3v_1 - u_1v_3)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k} \end{aligned} \tag{1.12}$$

Let us see an example.

Example 1.9. *For vectors* $\mathbf{u} = [2,1,-1]^t$ *and* $\mathbf{v} = [-3,4,1]^t$, *find* $\mathbf{u}\times\mathbf{v}$ *and* $\mathbf{v}\times\mathbf{u}$.

Solution: From the definition we have

$$
\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ -3 & 4 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} \mathbf{k} = (1+4)\cdot\mathbf{i} - (2-3)\cdot\mathbf{j} + (8+3)\cdot\mathbf{k} = 5\cdot\mathbf{i} + \mathbf{j} + 11\cdot\mathbf{k}
\end{aligned}
$$

Also, $\mathbf{v}\times\mathbf{u}$

$$
\begin{aligned}
\mathbf{v}\times\mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 2 & 1 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 4 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 4 \\ 2 & 1 \end{vmatrix} \mathbf{k} = (-4-1)\,\mathbf{i} - (3-2)\,\mathbf{j} + (-3-8)\,\mathbf{k} = -5\,\mathbf{i} - \mathbf{j} - 11\,\mathbf{k}
\end{aligned}
$$

□

Theorem 1.4. *If* θ *is the angle between the vectors* $\mathbf{u}$ *and* $\mathbf{v}$, $(0 \le \theta \le \pi)$, *then*

$$
\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta. \tag{1.13}
$$

Proof. From the definition we have:

$$
\begin{aligned}
\|\mathbf{u}\times\mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 \quad + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\cos^2\theta = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\,(1-\cos^2\theta) = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\sin^2\theta
\end{aligned}
$$

taking square roots of both sides and keeping in mind that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \ge 0$ when $0 \le \theta \le \pi$, we have

$$
\boxed{\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta}
$$

□

Corollary 1.4. *Two nonzero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel if and only if* $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

The geometric interpretation of Thm. 1.4 is the area of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$, have the same initial point then they define a parallelogram with base $\|\mathbf{u}\|$ and height $\|\mathbf{v}\|\sin\theta$. Its area is

$$
\boxed{S = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta = \|\mathbf{u}\times\mathbf{v}\|} \tag{1.14}
$$

Thus, geometrically the magnitude of the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.

Example 1.10. *Find the area of the parallelogram determined by the points $P=(1,4,6)$, $Q=(-2,5,-1)$, and $R=(1,-1,1)$.*

Solution: From the discussion above in Thm. 1.4, we have $\overrightarrow{PQ}=\begin{bmatrix}-1-2\\5-4\\-1-6\end{bmatrix}=\begin{bmatrix}-3\\1\\-7\end{bmatrix}$ and $\overrightarrow{PR}=\begin{bmatrix}1-1\\-1-4\\1-6\end{bmatrix}=\begin{bmatrix}0\\-5\\-5\end{bmatrix}$. Their cross product is

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&1&-7\\0&-5&-5\end{vmatrix}=\begin{vmatrix}1&-7\\-5&-5\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3&-7\\0&-5\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3&1\\0&-5\end{vmatrix}\mathbf{k}=-40\,\mathbf{i}-15\,\mathbf{j}+15\,\mathbf{k}$$

and its magnitude $\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\sqrt{(-40)^2+(-15)^2+(15^2)}=5\sqrt{82}.$ □

Theorem 1.5. *The cross product of two nonzero vectors* **u** *and* **v** *is orthogonal with the vectors* **u** *and* **v**.

Proof. To show that $\mathbf{u}\times\mathbf{v}$ is orthogonal with $\mathbf{u}$, it is enough to show that their dot product is zero. So

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}=\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=u_2v_3u_1-u_3v_2u_1+u_3v_1u_2-u_1v_3u_2+u_1v_2u_3-u_2v_1u_3=0$$

Similarly $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=0$. Thus the cross product is orthogonal with vectors **u** and **v**.

□

In the picture it is illustrated the **right hand rule** of determining the direction of the cross product.

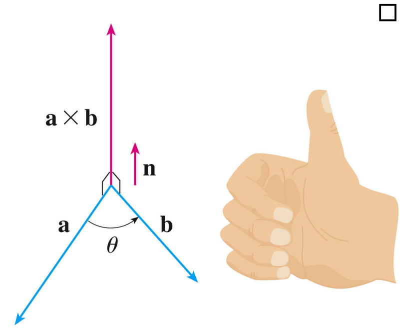


Figure 1.12: Cross product

Example 1.11. *If a plan is defined by the points $A(1,0,0)$, $B(2,-1,3)$ and $C=(1,1,1)$, find a vector orthogonal with it.*

Solution: We take

$$\overrightarrow{AB}=\begin{bmatrix}2-1\\-1-0\\3-0\end{bmatrix}=\begin{bmatrix}1\\-1\\3\end{bmatrix}\quad\text{and}\quad\overrightarrow{AC}=\begin{bmatrix}1-1\\1-0\\1-0\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}.$$

The cross product is

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&3\\0&1&1\end{vmatrix}=\begin{vmatrix}-1&3\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&3\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\0&1\end{vmatrix}\mathbf{k}=-4\,\mathbf{i}-\mathbf{j}+\mathbf{k}$$

Thus, the vector $-4\,\mathbf{i}-\mathbf{j}+2\,\mathbf{k}$ is orthogonal to the plane passing through A, B, and C □

Theorem 1.6. *For any vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ in V^3, and $r\in\mathbb{R}$, the following are true:*

(i) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$
(ii) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$
(iii) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$
(iv) $(r\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(r\mathbf{v}) = r(\mathbf{u}\times\mathbf{v})$
(v) $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$
(vi) $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
(vii) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$
(viii) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

Proof. We will only prove vii), since the rest are easy exxercises. If $\mathbf{u} = [u_1,u_2,u_3]^t$, $\mathbf{v} = [v_1,v_2,v_3]^t$ and $\mathbf{w} = [w_1,w_2,w_3]^t$, then

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= u_1(v_2w_3-v_3w_2)+u_2(v_3w_1-v_1w_3)+u_3(v_1w_2-v_2w_1)\\ &= u_1v_2w_3-u_1v_3w_2+u_2v_3w_1-u_2v_1w_3+u_3v_1w_2-u_3v_2w_1\\ &= (u_2v_3-u_3v_2)w_1+(u_3v_1-u_1v_3)w_2+(u_1v_2-u_2v_1)w_3 = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}\end{aligned} \tag{1.15}$$

This completes the proof. □

1.1.5 Mixed product

Given vectors $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3$ with coordinates $\mathbf{u} = [u_1,u_2,u_3]^t$, $\mathbf{v} = [v_1,v_2,v_3]^t$, and $\mathbf{w} = [w_1,w_2,w_3]^t$. The **mixed product** of $\mathbf{u},\mathbf{v},\mathbf{w}$ is called expression $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$. From Eq. (1.15) we notice that

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} \tag{1.16}$$

The geometric interpretation is that the absolute value of the mixed product is: the volume of the parallelepiped defined by vectors $\mathbf{u},\mathbf{v},\mathbf{w}$. Thus

$$\boxed{V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|} \tag{1.17}$$

Example 1.12. *Find the volume of the parallelepiped defined by vectors* $\mathbf{u} = [2,1,3]^t$, $\mathbf{v} = [-1,3,2]^t$ *and* $\mathbf{w} = [1,1,-2]^t$.

Solution: We have

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 2 & 1 & 3\\ -1 & 3 & 2\\ 1 & 1 & -2\end{vmatrix} = 2\begin{vmatrix} 3 & 2\\ 1 & -2\end{vmatrix} - 1\begin{vmatrix} -1 & 2\\ 1 & -2\end{vmatrix} + 3\begin{vmatrix} -1 & 3\\ 1 & 1\end{vmatrix} = 2(-8)-1(0)+3(-4) = -28$$

So $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |-28| = 28$. □

Example 1.13. *Prove that vectors* $\mathbf{u} = [1,4,-7]^t$, $\mathbf{v} = [2,-1,4]^t$ *and* $\mathbf{w} = [0,-9,18]^t$ *lie on the same plane.*

Solution: We have

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 0$$

Since the volume is zero, then the vectors lie on the same plane. □

Exercises:

7. *For given vectors find their cross product and verify if the vectors are orthogonal.*
(i) $\mathbf{u} = [5,2,-1]^t$, $\mathbf{v} = [7,2,-10]^t$
(ii) $\mathbf{u} = [4,4,-3]^t$, $\mathbf{v} = [2,6,4]^t$
(iii) $\mathbf{u} = [1,2,0]^t$, $\mathbf{v} = [1,0,3]^t$
(iv) $\mathbf{u} = [5,1,-1]^t$, $\mathbf{v} = [-1,0,2]^t$
(v) $\mathbf{u} = 3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$, $\mathbf{v} = \mathbf{i}-2\mathbf{j}-3\mathbf{k}$
(vi) $\mathbf{u} = -\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{v} = -3\mathbf{i}+6\mathbf{j}+3\mathbf{k}$

8. *Find* $\mathbf{u}\times\mathbf{v}$ *and* $\mathbf{v}\times\mathbf{u}$, *for vectors* $\mathbf{u} = [0,1,3]^t$ *and* $\mathbf{v} = [1,1,2]^t$

9. *For vectors* $\mathbf{u} = [3,1,2]^t$, $\mathbf{v} = [-1,1,0]^t$, *and* $\mathbf{w} = [0,0,-4]^t$, *prove that* $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

10. *Find the area of the triangle determined by*
(i) $P = (5,1,-2)$, $Q = (4,-4,3)$, $R = (2,4,0)$
(ii) $P = (4,0,2)$, $Q = (2,1,5)$, $R = (-1,0,-1)$.

11. *Find a unit vector which is orthogonal with vectors* $\mathbf{u} = [1,0,1]^t$ *and* $\mathbf{v} = [1,3,5]^t$.

12. *Prove that* $\mathbf{0}\times\mathbf{u} = \mathbf{u}\times\mathbf{0}$ *for every vector* $\mathbf{u}$ *in* V_3.

13. *Prove that* $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$ *for all vectors in* V_3.

14. *Find the area of the parallelogram with vertices:*
(i) $A(2,1,3)$, $B(1,4,5)$, $C(2,5,3)$, $D(3,2,1)$.
(ii) $A(-2,2)$, $B(1,4)$, $C(6,6)$, *and* $D(3,0)$.
(iii) $A(1,2,3)$, $B(1,3,6)$, $C(3,7,3)$, $D(3,8,6)$.

15. *Find* $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$ *and* $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})$.

16. *Prove that* $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.

17. *The angle between two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is* $\pi/6$ *and* $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$. *Find* $\|\mathbf{u}\times\mathbf{v}\|$.

18. *Find a vector which is orthogonal to the plane passing through* P, Q, R, *and find the area of the triangle* PQR.
(i) $P(3,0,6)$, $Q(2,1,5)$, $R(-1,3,4)$.
(ii) $P(1,2,3)$, $Q(1,0,1)$, $R(-1,3,1)$.
(iii) $P(2,0,-3)$, $Q(5,2,2)$, $R(3,1,0)$.

19. *Find the volume of the parallelepiped determined by the vectors*
(i) $\mathbf{u} = [1,1,3]^t$, $\mathbf{v} = [2,1,4]^t$, $\mathbf{w} = [5,1,-2]^t$
(ii) $\mathbf{u} = [1,3,2]^t$, $\mathbf{v} = [7,2,-10]^t$, $\mathbf{w} = [1,0,1]^t$.

20. *For the given vectors compute* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ *and* $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$.
(i) $\mathbf{u} = [1,1,1]^t$, $\mathbf{v} = [3,0,2]^t$, $\mathbf{w} = [2,2,2]^t$.
(ii) $\mathbf{u} = [1,0,2]^t$, $\mathbf{v} = [-1,0,3]^t$, $\mathbf{w} = [2,0,-2]^t$.

21. *Show that vectors* $\mathbf{u} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $\mathbf{v} = \mathbf{i}-\mathbf{j}$, *and* $\mathbf{w} = 7\mathbf{i} = 3\mathbf{j}+2\mathbf{k}$ *are coplanar.*

22. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are unit vectors in* V_3, *when is the vector* $\mathbf{v}\times\mathbf{w}$ *also a unit vector?*

23. *Prove that if* $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ *for all* $\mathbf{v}$ *in* V_3, *then* $\mathbf{v} = \mathbf{0}$.

24. *Prove that for all vectors* $\mathbf{v}, \mathbf{w}$ *in* V_3*:*

$$\|\mathbf{v}\times\mathbf{w}\| + |\mathbf{v}\cdot\mathbf{w}| = \|\mathbf{v}\|^2\cdot\|\mathbf{w}\|^2.$$

25. *Given* $\mathbf{u}, \mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ *such that* $\mathbf{u}\times\mathbf{x} = \mathbf{v}$, *where* $\mathbf{u} \neq \mathbf{0}$. *Prove that*
(i) $\mathbf{u}\cdot\mathbf{v} = 0$
(ii) $\mathbf{x} = \frac{\mathbf{v}\times\mathbf{u}}{\|\mathbf{u}\|^2} + \lambda\mathbf{u}$ *is a solution of the equation* $\mathbf{u}\times\mathbf{x} = \mathbf{v}$ *for every scalar* $\lambda \in \mathbb{R}$.

26. *Prove the Jacobi identity*

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}.$$

27. *For all vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *in* V_3, *prove that*

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b}))\mathbf{c} - (\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}))\mathbf{d}$$

1.1.6 Equation of lines

A line in $\mathbb{R}^3$ is uniquely determined when it passes through a point P and has a given direction. Let $P = (x_0, y_0, z_0)$ a point in $\mathbb{R}^3$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a nonzero vector. Denote by L the line passing through P and is parallel to a vector $\mathbf{v}$; Fig. 1.13. Denote by $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ the vector $\vec{OP}$. The vector

$$\mathbf{r}(t) := \mathbf{r}_0 + t\cdot\mathbf{v}, \quad \text{for} \quad t\in\mathbb{R} \tag{1.18}$$

determines every point of the line L. Hence,

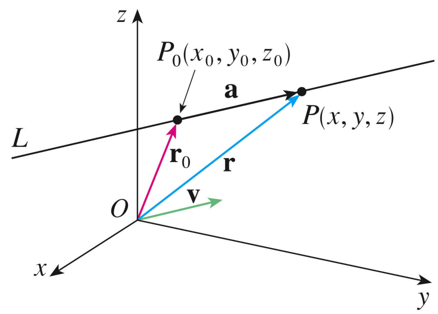


Figure 1.13: Equation of lines

Lemma 1.3. *For a given point* $P = (x_0, y_0, z_0)$ *and a nonzero vector* $\mathbf{v}\in\mathbb{R}^3$, *the line* L *which passes through* P *and is parallel with the vector* $\mathbf{v}$ *has equation*

$$\boxed{r(t) = r_0 + t\cdot\mathbf{v}, \qquad \textit{for} \quad r_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad \textit{and} \quad t\in(-\infty,\infty)} \tag{1.19}$$

Notice the correspondence between a the vector and its endpoint. Since $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then its endpoint $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ is the point

$$(x_0 + at, y_0 + bt, z_0 + ct).$$

Hence, we have a parametric representation of the line L in terms of the parameter t:

For a point $P(x_0, y_0, z_0)$ and a nonzero vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the line L, passing through P and parallel $\mathbf{v}$, consists in all points (x, y, z) such that

$$\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \textit{for} \quad -\infty < t < \infty} \tag{1.20}$$

Notice that in the above two interpretations, the point P is obtained when $t = 0$. Coordinates a, b, c are called **directional numbers** and the vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is called the **directional vector** of the line L.

If in Eq. (1.20), we have that $a \neq 0$, then solving for t, we have: $t = \frac{x-x_0}{a}$. We can also solve for y or z with the condition that b or c are nonzero. So we have $t = \frac{y-y_0}{b}$ or $t = \frac{z-z_0}{c}$. Hence,

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \tag{1.21}$$

If $a = 0$ then $x = x_0 + at$, hence $x = x_0 + 0 \cdot t = x_0$. Then, we have

$$x = x_0 \qquad \frac{y-y_0}{b} = \frac{z-z_0}{c} \tag{1.22}$$

Hence, the line L is on the plane $x = x_0$. Similarly for $b = 0$, or $c = 0$.

Example 1.14. *Find the equation of the line L passing through $P(2,3,5)$ and parallel to the vector $\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$, in all three forms. Find two points of L different from P.*

Solution: Denote by $\mathbf{r}_0 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and from Eq. (1.19), the line L has equation

$$\mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}, \quad \textit{for } -\infty < t < \infty.$$

For the parametric form , L consists of all points (x, y, z) such that

$$x = 2 + 4t, \quad y = 3 - t, \quad z = 5 + 6t, \quad \textit{for } -\infty < t < \infty$$

The symmetric equation of L is all points (x, y, z) such that

$$\frac{x-2}{4} = \frac{y-3}{-1} = \frac{z-5}{6}$$

Taking $t = 2$ and $t = 3$ we get $(10, 1, 17)$ and $(14, 0, 23)$ in L. □

The line going through two points

Given $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ two distinct points in $\mathbb{R}^3$ and L the line going through them. Denote by

$$\mathbf{r}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

two vectors with endpoints P_1 and P_2. Then, from Fig. 1.14, $\mathbf{r}_2 - \mathbf{r}_1$ is the vector from P_1 to P_2. Thus, $\overrightarrow{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1$. If we multiply $\mathbf{r}_2 - \mathbf{r}_1$ with a scalar t, and add that to the vector $\mathbf{r}_1$, we will have the line L for all values of t in $\mathbb{R}$. Thus points of the line are given by

$$\mathbf{r}(t) = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1),$$

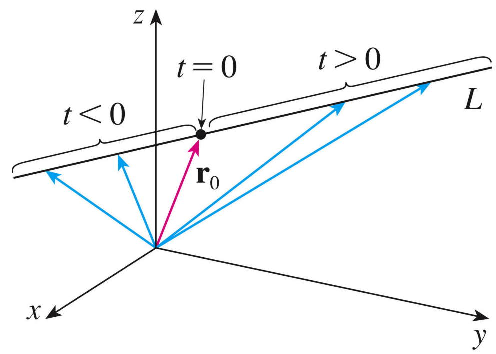


Figure 1.14: Equation of the line

for $t \in \mathbb{R}$. Then, the vector, parametric, or symmetric, equation of the line passing through P_1 and P_2 are.

Vector equation:

$$\mathbf{r}(t) = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1), \quad \textit{for} - \infty < t < \infty \tag{1.23}$$

Parametric equation:

$$x = x_1 + (x_2 - x_1) \cdot t, \quad y = y_1 + (y_2 - y_1) \cdot t, \quad z = z_1 + (z_2 - z_1) \cdot t, \quad \textit{for} - \infty < t < \infty \tag{1.24}$$

Symmetric equation:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \textit{for } x_1 \neq x_2,\ y_1 \neq y_2, \text{ and } z_1 \neq z_2 \tag{1.25}$$

1.1.7 Planes

Let $\mathbf{n} = [a, b, c]^t$ be a nonzero vector which is orthogonal to the plane P. Such vector is called **normal vector** of the plane. Let (x, y, z) be a point of P. Then, the vector $\mathbf{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$ is on the plane P; see Fig. 1.15.

Thus if $\mathbf{r} \neq \mathbf{0}$, then $\mathbf{r} \perp \mathbf{n}$, and so $\mathbf{n} \cdot \mathbf{r} = 0$. If $\mathbf{r} = \mathbf{0}$ then we have $\mathbf{r} \cdot \mathbf{n} = 0$. Conversely, if (x, y, z) is a point in $\mathbb{R}^3$ such that $\mathbf{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \neq \mathbf{0}$ and $\mathbf{n} \cdot \mathbf{r} = 0$, then $\mathbf{r} \perp \mathbf{n}$ and (x, y, z) is in P. Thus, we have:

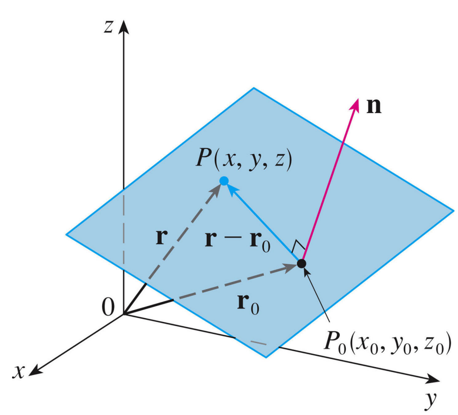


Figure 1.15: A normal vector with the plane

Lemma 1.4. *Let P be a plane and (x_0, y_0, z_0) a point in P. Let $\boldsymbol{n} = [a, b, c]^t$ be a nonzero vector orthogonal to the plane P. Then, the plane P consists of all points (x, y, z) such that*

$$\boldsymbol{n} \cdot \boldsymbol{r} = 0 \tag{1.26}$$

where $\boldsymbol{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$; *or*

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \tag{1.27}$$

Eq. (1.26) is called **vector equation** of the plane and Eq. (1.27) is called **scalar equation** of the plane. Expanding Eq. (1.27) we get

$$\boxed{ax + by + cz + d = 0} \tag{1.28}$$

where $d = -(ax_0 + by_0 + cz_0)$. Eq. (1.28) is called **linear equation of the plane**.

Example 1.15. *Find the equation of the plane passing through $Q(1,3,2)$, $R(3,-1,6)$ and $S(5,2,0)$.*

Solution: Vectors $\overrightarrow{QR}$ and $\overrightarrow{QS}$ are given by $\overrightarrow{QR} = [2, -4, 4]^t$ and $\overrightarrow{QS} = [4, -1, -2]^t$. Since these vectors are on the plane, their cross product is orthogonal to the plane and it is a normal vector of the plane. Thus

$$\mathbf{n} = \overrightarrow{QR} \times \overrightarrow{QS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{i} + 14\,\mathbf{k}$$

With the point Q and normal vector $\mathbf{n}$, the equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

Thus $6x + 10y + 7z = 50$. □

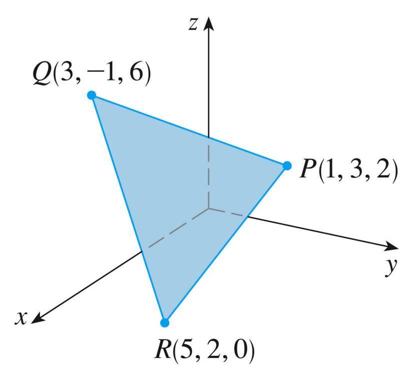


Figure 1.16

Two planes are **parallel** if their normal vectors are parallel . If planes are not parallel , then they intersect along a line. The **angle between two planes** is called the angle between their normal vectors.

Example 1.16. (i) *Find the angle between two planes $x + y + z = 1$ and $x - 2y + 3z = 1$.*

(ii) *Find the equation of the line of intersection between these two planes.*

Solution: i) Normal vectors are $\mathbf{n}_1 = \begin{bmatrix} 1 \ 1 \\ 1 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ Then the angle is

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|} = \frac{2}{\sqrt{42}}.$$

Thus, $\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$. Part ii) is left as an exercise. □

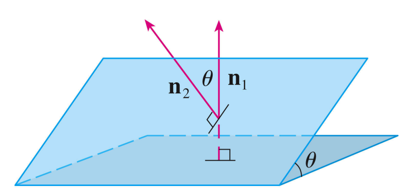


Figure 1.17

1.1.8 The distance between a point and a plane

The distance between a point and a plane is the length of the orthogonal line from the given point to the point of intersection with the plane.

Lemma 1.5. *Let $P_1(x_1, y_1, z_1)$ be a point and P a plan with equation $l\ ax + by + cz + d = 0$, which does not contain P_1. Then, the distance of P_1 from P is:*

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{1.29}$$

Proof. Let $P_0(x_0, y_0, z_0)$ be a point of the plane P, and denote by $\mathbf{v}$ the corresponding vector $\overrightarrow{P_0P_1}$. Then,

$$\mathbf{v} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix}$$

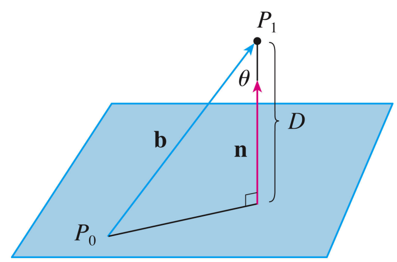


Figure 1.18: Distance of the point to the plane

From Fig. 1.18 we can see that the distance D from P_1 to the plane P, is the magnitude of the projection of $\mathbf{v}$ over the normal vector $\mathbf{n} = [a, b, c]^t$.

Thus

$$d = \left\| \mathrm{proj}_{\mathbf{v}}(\mathbf{n}) \right\| = \frac{\mathbf{n} \cdot \mathbf{v}}{\|\mathbf{n}\|} = \frac{\left| a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) \right|}{\sqrt{a^2 + b^2 + c^2}} = \frac{\left| (ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0) \right|}{\sqrt{a^2 + b^2 + c^2}}$$

Since P_0 is in the plane, then it satisfies the equation of the plane. Hence, $ax_0 + by_0 + cz_0 + d = 0$, from which we have $(ax_0 + by_0 + cz_0) = -d$. Therefore, the distance D is

$$\boxed{d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

□

Example 1.17. *Find the distance of the point $(2, 4, -5)$ to the plane $5x - 3y + z - 10 = 0$.*

Solution: Using the above formula we have

$$D = \frac{|5(2) - 3(4) + 1(-5) - 10|}{\sqrt{5^2 + (-3)^2 + 1^2}} = \frac{|-17|}{\sqrt{35}} = \frac{17}{\sqrt{35}} \approx 2.87$$

□

Example 1.18. *Find the distance between the two planes* $\quad 10x + 2y - 2z = 5 \quad$ *and* $\quad 5x + y - z = 1$

Solution: Normal vectors of these two planes are $\begin{bmatrix}10\\2\\-2\end{bmatrix}$ and $\begin{bmatrix}5\\1\\-1\end{bmatrix}$. They are parallel, and therefore planes are parallel . To find the distance, it is enough to take a point in one of the planes and find its distance to the other plane using the formula (1.29).

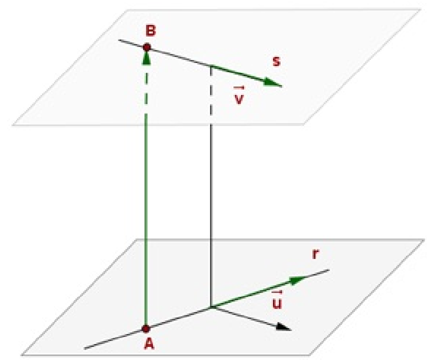
Take the point $\left(\frac{1}{2},0,0\right)$ in the first plane. Then we have

$$D=\frac{\left|5\cdot\frac{1}{2}+1\cdot 0-1\right|}{\sqrt{5^2+1^2+(-1)^2}}=\frac{\frac{3}{2}}{3\sqrt{3}}=\frac{\sqrt{3}}{6}$$

□

Two lines which are not in the same plane and do not intersect ate called **skew lines**

Example 1.19. *Given two lines with parametric equations as follows*



$$L_1: \quad x=1+t, \quad y=-2+3t, \quad z=4-t$$
$$L_2: \quad x=2s, \quad y=3=s, \quad z=-3+4s$$

Prove that these are skew lines. Find the distance between them.

Solution: The lines are not parallel because their directional vectors

$$\mathbf{u}=\begin{bmatrix}1\\3\\-1\end{bmatrix} \text{ and } \mathbf{v}=\begin{bmatrix}2\\1\\4\end{bmatrix},$$

are not parallel . They also do not intersect because the system

$$\begin{cases} 1+t=2s \\ -2+3t=3+s \\ 4-t=-3+4s \end{cases}$$

has no solutions. Thus, these are skew lines.

Since they do not intersect we can consider them in two parallel planes, say P_1 and P_2. The distance between L_1 and L_2 is is the same as the distance between P_1 and P_2, which can be found as follows.

A normal vector with these two planes must be orthogonal with vectors **u** and **v**. Thus a normal vector could be their cross product. Thus,

$$\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & -1\\ 2 & 1 & 4\end{vmatrix}=13\,\mathbf{i}-6\,\mathbf{j}-5\,\mathbf{k}$$

Now we can find the equation of each plane, say P_2.

Take a point in L_2 by choosing $s=0$. Then the point $(0,3,-3)$ is in L_2 and therefore in P_2. Thus, the equation for P_2 is

$$13(x-0)-6(y-3)-5(z+3)=0$$

or $13x-6y-5z+3=0$. Taking $t=0$ in the equation for L_1 we find the point $(1,-2,4)$ in P_1. Thus the distance between the lines L_1 and L_2 is the same as the distance from the point $(1,-2,4)$ to the plane $13-6y-5z+3=0$. From above we have

$$D = \frac{|13\cdot 1 - 6(-2) - 5\cdot 4 + 3|}{13^2+(-6)^2+(-5)^2} = \frac{8}{\sqrt{230}}.$$

□

Exercises:

28. *Determine if the lines L_1 and L_2 are parallel, intersect, or are skew lines.*

(i) $L_1 : \mathbf{u}(t) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $L_2 : \mathbf{v}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

(ii) $L_1 : \mathbf{u}(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, $L_2 : \mathbf{v}(t) = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$

29. *Is the line passing through points $P_1(-4,-6,1)$ and $P_2(-2,0,-3)$ parallel to the line passing through the points $Q_1(10,18,4)$ and $Q_2(5,3,14)$?*

30. *Find a and c such that the point $(a,1,c)$ is on the line passing through the points $P(0,2,3)$ and $Q(2,7,5)$.*

31. *Find the equation of the plane which contains the point $(-1,2,-3)$ and is orthogonal to the vector $[4,5,-1]^t$.*

32. *Find the equation of the plane which contains the point $(6,3,2)$ and is orthogonal to the vector $[-2,1,5]^t$.*

33. *Find the equation of the plane which contains the point $(4,0,-3)$ and has normal vector $\mathbf{j}+2\mathbf{k}$.*

34. *Find the equation of the plane which contains the point $(5,1,-2)$ and has normal vector $[4,-4,3]^t$.*

35. *Find the equation of the plane which passes through the point $(-2,8,10)$ and is orthogonal with the line $x=1=t, y=2t, z=4-3t$.*

36. *Find the equation of the plane which passes through the point $(4,-2,3)$ and is parallel with the plane $3x-7z=12$.*

37. *Find the equation of the plane which passes through points $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$.*

38. *Find the equation of the plane which passes through points $(1,0,3)$, $(2,01)$, and $(3,3,1)$.*

39. *Find the equation of the plane which contains the point $(1,0,0)$ and the line $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.*

40. *Find the equation of the plane which passes through the origin and is orthogonal to the plane $x+y-z=2$.*

41. *Find the equation of the plane which passes through the point $-1,2,1$ and contains the intersection line of the two planes $x=y-z=2$ and $2x-y+3z=1$.*

42. *Find the intersection line of the two planes:*

(i) $x+3y-3z-6=0$ *and* $2x-y+z+2=0$.

(ii) $3x+y-5z=0$ *and* $x+2y+z+4=0$.

43. *Find point of intersection of the line $\frac{x-6}{4} = y+3 = z$ with the plane $x+3y+2z-6=0$.*

44. *Find point of intersection of the line $x=y-1=2z$ with the plane $4x-y+3z=8$.*

45. *Find point of intersection of the line* $x = 1 + 2t, y = 4t, z = 2 - 3t$ *with the plane* $x + 2y - z + 1 = 0$.

46. *How can we find the the angle between two planes? Find the angle between the two planes* $x + y + z = 2$ *and* $x + 2y + 3z = 8$.

47. *Find cosine of the angle between two planes* $x + y + z = 0$ *and* $x + 2y + 3z = 1$.

48. *Find the lengths of the sides of the triangle with vertices* $A(3,-2,1)$, $B(1,2,-3)$, $C(3,4,-2)$. *Determine if this triangle is regular.*

49. *Finds the distance of the point* $(-5,3,4)$ *from each coordinate plane.*

50. *Find the magnitude of the force which has its projections on the coordinate axis as* $x = -6$, $y = -2$, *and* $z = 9$.

51. *Prove that the triangle with vertices* $A(1,-2,1)$ $B(3,-3,1)$ *and* $C(4,0,3)$ *is a right triangle.*

52. *Find the equation of the sphere with center at the point* $(4,-2,3)$ *and radius* $r = \sqrt{3}$.

53. *Find the equation of the sphere with center at the point* $(-1,3,2)$ *and radius* $r = \sqrt{3}$.

54. *Find the equation of the sphere with center at the point* $(2,3,4)$ *and radius 5. Where does the sphere intersect the coordinate planes?*

55. *Find the equation of the sphere which passes through the point* $(4,3,-1)$ *and has the center at* $(3,8,1)$.

56. *Prove that the following equations represent a sphere, find its center and its radius.*

(i) $x^2 + y^2 + z^2 - 6x + 4y + 2z = -17$
(ii) $x^2 + y^2 + z^2 = 4x - 2y$
(iii) $x^2 + y^2 + z^2 = x + y + z$
(iv) $x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$
(v) $16x^2 + 16y^2 + 16z^2 - 96x + 32y = 5$

57. *(a) Prove that the middle of the segment which is determined by the points* $A(a_1,b_1,c_1)$ *and* $B(a_2,b_2,c_2)$ *is the point with coordinates*

$$\left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}, \frac{c_1 + c_2}{2}\right)$$

(b) Find the lengths of the three medians of the triangle with vertices $A(4,1,5)$, $B(1,2,3)$, $C(-2,0,5)$.

Determine the inequalities which determine the following regions.

58. *The region between the plane* xy *and* $z = 5$.

59. *The region which consists of all points between spheres of radii* r *and* R *with center at the origin, where* $r < R$.

60. *Find the equation of the sphere with has the same center with* $x^2 + y^2 + z^2 - 6x + 4z - 36 = 0$ *and passes through the point* $(2,5,-7)$.

61. *Prove that the set of all points whose distance from* $A(-1,5,3)$ *is twice the distance from* $B(6,2,-2)$, *is a sphere.*

62. *Determine an equation for the set of points equidistant from* $A(-1,5,3)$ *and* $B(6,2,-2)$.

63. *Draw the vector* $\overrightarrow{AB}$, *when A and B are given as below and find its equivalent with the initial point at the origin.*

(i) $A = (0,3,1), \quad B = (2,3,-1)$
(ii) $A = (4,0,-2), \quad B = (4,2,1)$
(iii) $A = (2,0,3), \quad B = (3,4,5)$
(iv) $A = (0,3,-2), \quad B = (2,4,-1)$

64. *Find* $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} - 3\mathbf{b}$, $\|\mathbf{a}\|$ *and* $\|\mathbf{a} - \mathbf{b}\|$, *if* $\mathbf{a} = [5,-12]^t$ *and* $\mathbf{b} = [3,6]^t$.

65. *Find* $\mathbf{a} - \mathbf{b}$, $\mathbf{a} + 2\mathbf{b}$, $\|\mathbf{a}\|$ *and* $\|\mathbf{a} - \mathbf{b}\|$, *if* $\mathbf{a} = [1,2,-3]^t$ *and* $\mathbf{b} =]-2,-1,5]^t$.

66. *Find* $\mathbf{a} + \mathbf{b}$, $3\mathbf{a} - 2\mathbf{b}$, $\|\mathbf{a}\|$ *and* $\|\mathbf{a} - \mathbf{b}\|$, *if* $\mathbf{a} = [2,-4,4]^t$ *and* $\mathbf{b} = [0,2,-1]^t$.

67. *Find* $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$, $\|\mathbf{v}\|$ *and* $\|\mathbf{v} - \mathbf{w}\|$, $\|\mathbf{v} + \mathbf{w}\|$, *and* $-2\mathbf{v}$, *if* $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$.

68. *Find the unit vector which has the same direction with the vector* $-3\mathbf{i}+7\mathbf{j}$.

69. *Find the unit vector which has the same direction with the vector* $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

70. *Find the unit vector which has the same direction with the vector* $\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$.

71. *Find a vector which has the same direction with the vector* $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, *but has length 3*

72. *Find a vector which has the same direction with the vector* $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$, *but has length 6.*

73. *Let be given the vectors* $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(i) *Find the vector* $\mathbf{u}$ *such that* $\mathbf{u}+\mathbf{v}+\mathbf{w} = \mathbf{i}$.

(ii) *Find the vector* $\mathbf{u}$ *such that* $\mathbf{u}+\mathbf{v}+\mathbf{w} = 2\mathbf{j}+\mathbf{k}$.

74. *If* A, B, C *are vertices of a triangle, find* $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$.

75. *Draw the vectors* $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. *Determine graphically if there exist the scalars* s *and* t *such that* $\mathbf{w} = s\mathbf{u}+t\mathbf{v}$. *Find the values for* s *and* t.

76. *Let be given* $\mathbf{u}$ *and* $\mathbf{v}$ *two nonzero vectors not parallel in* $\mathbb{R}^2$. *Prove that if* $\mathbf{w}$ *is any vector in* $\mathbb{R}^2$, *then there exist two scalars* s *and* t *such that* $\mathbf{w} = s\mathbf{u}+t\mathbf{v}$.

77. *Is the property from the previous problem true for* $\mathbb{R}^3$? *Explain.*

78. *Let* $\mathbf{a} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{a}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, *and* $\mathbf{a}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. *Describe the set of all points* (x, y, z) *which satisfy*

$$\|\mathbf{a}-\mathbf{a}_1\| + \|\mathbf{a}-\mathbf{a}_2\| = \lambda,$$

where $\lambda > \|\mathbf{a}_1 - \mathbf{a}_2\|$.

1.2 Euclidean n- space $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the following Cartesian product

$$\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

A vector $\mathbf{u}$ in $\mathbb{R}^n$ will be defined as an ordered tuple $(u_1, \dots, u_n)$ for $u_i \in \mathbb{R}$, $i = 1, \dots, n$ and denoted by $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$.

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such as $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ we define the **vector addition** and **scalar multiplication** as follows:

$$\mathbf{u}+\mathbf{v} := \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}, \qquad r\,\mathbf{v} := \begin{bmatrix} rv_1 \\ \vdots \\ rv_n \end{bmatrix}. \tag{1.30}$$

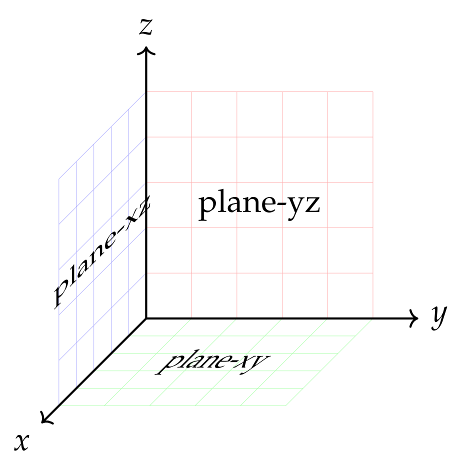


Figure 1.19: Euclidean space $\mathbb{R}^3$.

A **Euclidean** n**-space** is the set of vectors together with **vector addition** and **scalar multiplication** defined as above. Elements of $\mathbb{R}^n$ are called **vectors** and all $r \in \mathbb{R}$ are called **scalars**. The vector $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is called the **zero vector**. By a vector $\mathbf{u}$ we usually mean a **column vector** unless otherwise stated. The row vector $[u_1, \dots, u_n]$ is called the **transpose** of $\mathbf{u}$ and denoted by

$$\mathbf{u}^t = [u_1, \dots, u_n]$$

For the addition and scalar multiplication we have the following properties.

Theorem 1.7. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$ and r, s scalars in $\mathbb{R}$. The following are satisfied:*

(i) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$,
(ii) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$,
(iii) $\mathbf{0}+\mathbf{u} = \mathbf{u}+\mathbf{0} = \mathbf{u}$,
(iv) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$,
(v) $r(\mathbf{u}+\mathbf{v}) = r\mathbf{u}+r\mathbf{v}$,
(vi) $(r+s)\mathbf{u} = r\mathbf{u}+s\mathbf{u}$,
(vii) $(rs)\mathbf{u} = r(s\mathbf{u})$,
(viii) $1\mathbf{u} = \mathbf{u}$.

Proof. Exercise. □

Two vectors $\mathbf{v}$ and $\mathbf{u}$ are called **parallel** if there exists an $r \in \mathbb{R}$ such that $\mathbf{v} = r\mathbf{u}$.

Definition 1.2. *Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_s \in \mathbb{R}^n$ and $r_1, \dots, r_s \in \mathbb{R}$, the vector*

$$r_1\mathbf{v}_1 + \cdots + r_s\mathbf{v}_s$$

is called a ***linear combination of vectors*** *$\mathbf{v}_1, \dots, \mathbf{v}_s$.*

Definition 1.3. *Let $\mathbf{v}_1, \dots, \mathbf{v}_s$ be vectors in $\mathbb{R}^n$. The* ***span*** *of these vectors, denoted by Span $(\mathbf{v}_1, \dots, \mathbf{v}_s)$, is the set in $\mathbb{R}^n$ of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_s$.*

$$\textit{Span}\ (\mathbf{v}_1, \dots, \mathbf{v}_s) = \left\{ r_1\mathbf{v}_1 + \cdots + r_s\mathbf{v}_s \mid r_i \in \mathbb{R}, i = 1, \dots, s \right\}$$

Exercise 10. *Let $V = \mathbb{R}^3$ be the 3-dimensional Euclidean space and*

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

vectors in V. Determine Span $(\mathbf{i}, \mathbf{j})$. What about Span $(\mathbf{i}, \mathbf{j}, \mathbf{k})$?

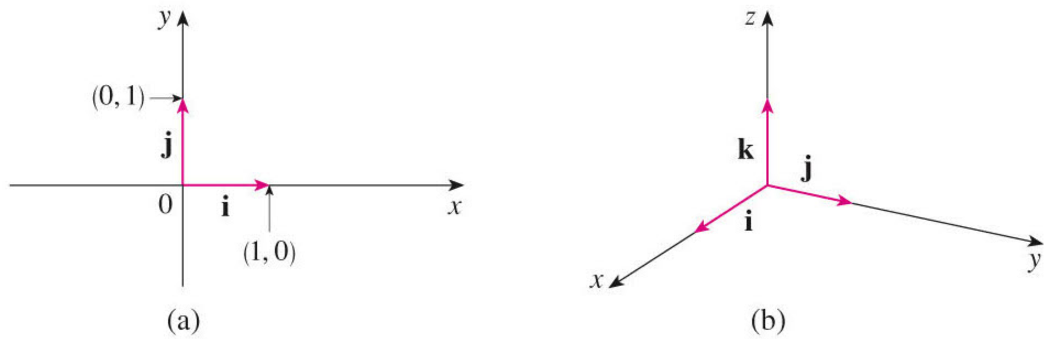


Figure 1.20: Standard basis for $\mathbb{R}^2$ and $\mathbb{R}^3$

Proof. If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then

$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$$

Hence, every vector in $\mathbb{R}^3$ can be expressed as a linear combination of vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Therefore, Span $(\mathbf{i}, \mathbf{j}, \mathbf{k}) = \mathbb{R}^3$. □

Definition 1.4. *Vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ are called **linearly independent** if*

$$r_1 \mathbf{u}_1 + \cdots + r_n \mathbf{u}_n = 0$$

implies that

$$r_1 = \cdots = r_n = 0,$$

*otherwise, we say that $\mathbf{u}_1, \dots, \mathbf{u}_n$ are **linearly dependent**.*

Exercise 11. *Prove that $\mathbf{i}, \mathbf{j}, \mathbf{k}$, given above, are linearly independent.*

In the coming sections we will see that the concept of linear independence is one of the most important concepts of linear algebra. Our strategy will be to try to generalize all concepts of $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$. Of course the geometric interpretation in $\mathbb{R}^n$ doesn't make sense, but this will not deter us to assign the same names to abstract concepts in $\mathbb{R}^n$ as we had for $\mathbb{R}^2$ and $\mathbb{R}^3$.

1.2.1 Subspaces of $\mathbb{R}^n$

A subset $U \subset \mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if the following hold:

(i) $\mathbf{0} \in U$

(ii) $\forall \mathbf{u}, \mathbf{v} \in U$, $\mathbf{u} + \mathbf{v} \in U$

(iii) $\forall \lambda \in \mathbb{R}$, $\forall \mathbf{u} \in U$, we have that $\lambda \mathbf{u} \in U$.

Property ii) is usually referred to as U is **closed under addition** and property iii) as U is **closed under scalar multiplication**. A subspace U of $\mathbb{R}^n$ is called **proper** if $U \neq \{0\}$ and $U \neq \mathbb{R}^n$. The concept of a subspace is very important and we will study it in detail in the next chapter.

Exercise 12. *Prove that every line and every plane in $\mathbb{R}^3$ which passes through the point $O(0,0,0)$ is a subspace.*

Lemma 1.6. *Let $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{R}^n$. Prove that Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is a subspace of $\mathbb{R}^n$.*

Proof. The zero vector $\mathbf{0}$ is in Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ since it can be written as

$$\mathbf{0} = 0\,\mathbf{u}_1 + \cdots + 0, u_n.$$

Let $\mathbf{v}_1, \mathbf{v}_2 \in$ Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$. Then exist scalar $r_1, \dots r_n$ and $s_1 \dots s_n$ such that

$$\mathbf{v}_1 = r_1 \mathbf{u}_1 + \cdots r_n \mathbf{u}_n \quad \text{and} \quad \mathbf{v}_2 = s_1 \mathbf{u}_1 + \cdots s_n \mathbf{u}_n$$

Thus

$$\mathbf{v}_1 + \mathbf{v}_2 = (r_1 + s_1)\mathbf{u}_1 + \cdots + (r_n + s_n)\mathbf{u}_n$$

is also a vector in Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$. Hence, Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is closed under addition. Similarly we show that it is also closed under scalar multiplication.

□

Exercise 13. *Let P be a plane in $\mathbb{R}^3$ with equation*

$$ax + by + cz = d.$$

Determine the values of a, b, c, d such that the set of points of P forms a subspace of $\mathbb{R}^3$.

Solution: For P to be a subspace the vector $\mathbf{0}$ must be in P. Hence, point $O(0,0,0)$ must be in P. This implies that $d = 0$. The plane P is closed under addition and scalar multiplication since the sum of any two vectors is on the same plane determined by the two vectors (similarly for the multiplication by a scalar).

□

Exercise 14. *From Exe. 10 we know that every vector $\mathbf{u} \in \mathbb{R}^2$, such that $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, can be written as $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$. Using this fact, can you determine all subspaces of $\mathbb{R}^2$?*

1.2.2 Norm and dot product

In this section we study two very important concepts of Euclidean spaces; that of the dot product and the norm. The concept of the dot product will be generalized later to that of inner product for any vector space.

Definition 1.5. *Let* $\mathbf{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$. *The **norm** of $\mathbf{u}$, denoted by $\|\mathbf{u}\|$, is defined as*

$$\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

The norm has the following properties:

Theorem 1.8. *For any vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and any scalar* $r \in \mathbb{R}$ *the following are true:*

(i) $||\mathbf{u}|| \geq 0$ *and* $||\mathbf{u}|| = 0$ *if and only if* $\mathbf{u} = 0$

(ii) $||r\mathbf{u}|| = |r| \, ||\mathbf{u}||$

(iii) $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$

Proof. The proof of i) and ii) are easy and left as exercises. The proof of iii) is completed in Lem. 1.9

□

A **unit vector** is a vector with norm 1. Notice that for any nonzero vector $\mathbf{u}$ the vector $\frac{\mathbf{u}}{||\mathbf{u}||}$ is a unit vector. Let

$$\mathbf{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} := \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

be vectors in $\mathbb{R}^n$. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ (sometimes called the **inner product**) is defined as follows:

$$\boxed{\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n,} \tag{1.31}$$

or sometimes denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$. Notice the identity $||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$, which is very useful.

Lemma 1.7. *The dot product has the following properties:*

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(ii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(iii) $r(\mathbf{u} \cdot \mathbf{v}) = (r\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (r\mathbf{v})$

(iv) $\mathbf{u} \cdot \mathbf{u} \geq 0$, *and* $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = 0$

Proof. Use the definition of the dot product to check all i) through iv).

□

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are called **perpendicular** if $\mathbf{u} \cdot \mathbf{v} = 0$.

Lemma 1.8 (Cauchy-Schwartz inequality)**.** *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any vectors in* $\mathbb{R}^n$. *Then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$$

Proof. If one of the vectors is the zero vector, then the inequality is obvious. So we assume that $\mathbf{u}, \mathbf{v}$ are nonzero. For any $r, s \in \mathbb{R}^n$ we have $||r\mathbf{v} + s\mathbf{u}|| \geq 0$. Then,

$$||r\mathbf{v} + s\mathbf{u}||^2 = (r\mathbf{v} + s\mathbf{u}) \cdot (r\mathbf{v} + s\mathbf{u}) = r^2 (\mathbf{v} \cdot \mathbf{v}) + 2rs (\mathbf{v} \cdot \mathbf{u}) + s^2 (\mathbf{u} \cdot \mathbf{u}) \geq 0$$

Take $r = \mathbf{u} \cdot \mathbf{u}$ and $s = -\mathbf{v} \cdot \mathbf{u}$. Substituting in the above we have:

$$\begin{aligned} ||r\mathbf{v} + s\mathbf{u}||^2 &= (\mathbf{u} \cdot \mathbf{u})^2 (\mathbf{v} \cdot \mathbf{v}) - 2(\mathbf{u} \cdot \mathbf{u}) (\mathbf{v} \cdot \mathbf{u})^2 + (\mathbf{v} \cdot \mathbf{u})^2 (\mathbf{u} \cdot \mathbf{u}) \\ &= (\mathbf{u} \cdot \mathbf{u}) \left[(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{u})^2 \right] \geq 0 \end{aligned}$$

Since $(\mathbf{u} \cdot \mathbf{u}) = ||\mathbf{u}||^2 > 0$ then $\left[(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{u})^2 \right] \geq 0$. Hence,

$$(\mathbf{v} \cdot \mathbf{u})^2 \leq (\mathbf{u} \cdot \mathbf{u}) (\mathbf{v} \cdot \mathbf{v}) = ||\mathbf{u}||^2 \cdot ||\mathbf{v}||^2$$

and $|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\cdot\|\mathbf{v}\|$.

□

Lemma 1.9 (Triangle inequality). *For any two vectors* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^n$ *the following hold*

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\|+\|\mathbf{v}\|$$

Proof. We have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ &= (\mathbf{u}\cdot\mathbf{u})+2(\mathbf{u}\cdot\mathbf{v})+(\mathbf{v}\cdot\mathbf{v}) = \|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2+2\,|\mathbf{u}\cdot\mathbf{v}|+\|\mathbf{v}\|^2\\ &\leq \|\mathbf{u}\|^2+2\cdot\|\mathbf{u}\|\cdot\|\mathbf{v}\|+\|\mathbf{v}\|^2 = (\|\mathbf{u}\|+\|\mathbf{v}\|)^2\end{aligned}$$

Hence, $\|\mathbf{v}+\mathbf{u}\| \leq \|\mathbf{v}\|+\|\mathbf{u}\|$.

□

Example 1.20. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two given vectors and* θ *the angle between them. Prove that*

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cos\theta$$

Hence, we have the following definition. The **angle** between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined to be

$$\theta := cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|}\right)$$

From Lem. 1.8 we have that

$$-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|} \leq 1$$

Hence, the angle between two vectors is well defined.

Example 1.21. *Find the angle between* $\mathbf{u} = \begin{bmatrix}2\\-1\\2\end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$.

Solution: *Using the above formula we have* $\theta = \cos^{-1}\left(\frac{(2,-1,2)\cdot(-1,-1,1)}{\sqrt{9}\cdot\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{9}\right)$. *Then* $\theta \approx 1.377$ *radians or* $\theta \approx 78.90^\circ$. □

Let $P(x_1,\dots,x_n)$ and $Q(y_1,\dots,y_n)$ be points in $\mathbb{R}^n$. The **Euclidean distance** between P and Q is defined as

$$d(P,Q) := \left\|\overrightarrow{PQ}\right\| = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

The **distance between two vectors** $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is defined as the distance between P and Q.

Exercise 15. *Prove that the distance* $d(\mathbf{u},\mathbf{v})$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *is* $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$.

Consider now a subspace V in $\mathbb{R}^n$. The distance between P and V is defined as

$$d(P,V) := \ min \ \left\{ d(\overrightarrow{OP}, \mathbf{v}) \mid \mathbf{v} \in V \right\}$$

The concept of the distance on Euclidean spaces is widely used in communication theory and more specifically coding theory. A **linear code** C is a subspace of a vector space $\mathbb{F}^n$, where $\mathbb{F}$ is a finite field. Its **minimum distance** is

$$d(C) := \ min_{\ \mathbf{v} \neq \mathbf{0}} \left\{ d(\mathbf{0}, \mathbf{v}) \mid \mathbf{v} \in C \right\}$$

Then we say that this is an $[n,d]$ code. One of the classical results of coding theory is that we can detect up to $(d-1)$ errors and can correct up to $\left[\frac{d-1}{2}\right]$ of them.

1.2.3 Projections

Consider vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ having the same initial point. The **projection vector** of $\mathbf{v}$ onto $\mathbf{u}$, denoted by $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is the vector with initial point the same as that of $\mathbf{v}$ and terminal point obtained by dropping a perpendicular from the terminal point of $\mathbf{v}$ on the line determined by $\mathbf{u}$. Thus,

$$\left\| \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) \right\| := \left\| \overrightarrow{AO} \right\| = \|\mathbf{v}\| \cdot \cos(C\hat{A}B) = \|\mathbf{v}\| \cdot \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}.$$

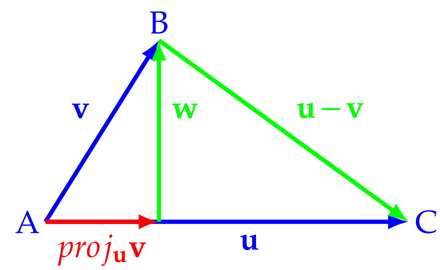


Figure 1.21: The projection of $\mathbf{v}$ onto $\mathbf{u}$

We can multiply by the unit vector $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ to get

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) \ = \ \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u}^2} \, \mathbf{u} \tag{1.32}$$

If we want a vector perpendicular to $\mathbf{u}$ we have

$$\boxed{\mathbf{w} = \mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u}^2} \, \mathbf{u}.} \tag{1.33}$$

We will see later in the course how this idea is generalized in $\mathbb{R}^n$ to the process of orthogonalization.

Exercise 16. *The above discussion provides a method that for any two given vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *we can determine a vector* $\mathbf{w}$ *which is perpendicular to* $\mathbf{u}$*. Can you devise a similar argument for three vectors* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$*? In other words, determine* $\mathbf{v}$ *and* $\mathbf{w}$ *from* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *such that the set of vectors* $\{\mathbf{u}_1, \mathbf{v}, \mathbf{w}\}$ *are pairwise perpendicular.*

Exercise 17. *Show that the distance from a point* $P = (x_0, y_0)$ *to a line* $L : ax + by + c = 0$ *is given by* $d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$.

Solution: The line L intersect the y-axis at $A\left(0,-\frac{c}{b}\right)$ and the x-axis at $B\left(-\frac{c}{a},0\right)$. Let $\mathbf{u} = \overrightarrow{AB} = c\begin{bmatrix} -\frac{1}{a} \\ \frac{1}{b} \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AP} = \begin{bmatrix} x_0 \\ y_0 + \frac{c}{b} \end{bmatrix}$. Then the distance d from the point P to the line L is $d = \left\|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\right\|$. Use the formula from Eq. (1.33) to prove the result. $\square$

Exercise 18. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Prove that the formulas Eq. (1.32) and Eq. (1.33) still hold.*

Next we consider the problem of finding the projection of a vector $\mathbf{w}$ on the plane P determined by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Denote by $\mathbf{n} = \frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}$ the unit normal vector to the plane P, say $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then the plane has equation $ax + by + cz = 0$. The projection of $\mathbf{w}$ onto the plane P is

$$\text{proj}_P(\mathbf{w}) = \mathbf{w} - \text{proj}_{\mathbf{n}}(\mathbf{w}) = \mathbf{w} - \frac{\mathbf{n}\cdot\mathbf{w}}{\mathbf{n}^2}\mathbf{n} = \mathbf{w} - (\mathbf{n}\cdot\mathbf{w})\mathbf{n}, \quad (1.34)$$

since $\mathbf{n}$ is a unit vector. Summarizing, we have:

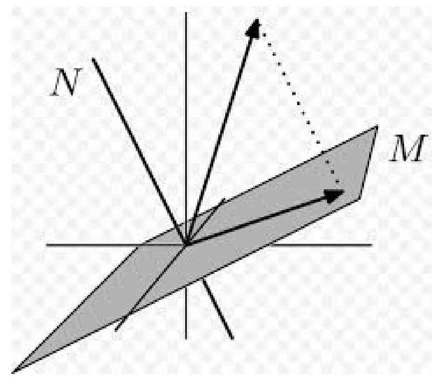


Figure 1.22: Projection on a plane

Lemma 1.10. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $U =$ Span $(\mathbf{u}, \mathbf{v})$. If $\mathbf{u}$ and $\mathbf{v}$ are not co-linear then the projection of any vector $\mathbf{w} \in \mathbb{R}^3$ onto the space U is given by the formula*

$$\boxed{proj_U(\mathbf{w}) = \mathbf{w} - (\boldsymbol{n}\cdot\mathbf{w})\boldsymbol{n},} \quad (1.35)$$

where $\boldsymbol{n}$ is a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Before we generalize this result to $\mathbb{R}^n$ let us see a computational example.

Example 1.22. *Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Find the projection of $\mathbf{u}$ onto the $\mathbf{vw}$-plane.*

Solution: The normal vector for the $\mathbf{vw}$-plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -3 \\ -1 & -1 & -1 \end{vmatrix} = -5\mathbf{i} + 5\mathbf{j} + 0\mathbf{k} = -5\mathbf{i} + 5\mathbf{j}$$

We normalize this vector as $\mathbf{n} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. From the above formula we have

$$\text{proj}_P(\mathbf{u}) = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = \mathbf{u} - \frac{1}{\sqrt{2}}\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

□

In Lem. 1.11 we give another formula for $\text{proj}_{\mathbb{P}}(\mathbf{x})$, which does not include the normal vector $\mathbf{n}$. While the Lemma has a very simple proof, its generalization to $\mathbb{R}^n$ is quite important as we will see in Lem. 1.10 and Lem. 5.1.

Lemma 1.11. *Let V be a subspace of $\mathbb{R}^3$ such that $V = \text{ Span } (\mathbf{v}_1, \mathbf{v}_2)$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors and perpendicular to each other. Prove that*

$$proj_V(\mathbf{x}) = (\mathbf{v}_1 \cdot \mathbf{x}) \cdot \mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{x}) \cdot \mathbf{v}_2. \tag{1.36}$$

Solution: This is a simple geometry problem. Let P denote the endpoint of the vector $\mathbf{x}$ and Q the endpoint of the vector $\text{proj}_V(\mathbf{x})$. Denote by $\mathbf{a}$ and $\mathbf{b}$ the projections of Q on $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Then

$$\text{proj}_V(\mathbf{x}) = \|\mathbf{a}\| \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \|\mathbf{b}\| \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \|a\| \, \mathbf{v}_1 + \|b\| \, \mathbf{v}_2,$$

since $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$ However, since $\mathbf{a} = \text{proj}_{\mathbf{v}_1}(\mathbf{x})$ and $\mathbf{b} = \text{proj}_{\mathbf{v}_2}(\mathbf{x})$. we have

$$\|\mathbf{a}\| = \frac{\mathbf{v}_1 \cdot \mathbf{x}}{\|\mathbf{v}_1\| \cdot \|\mathbf{x}\|} \cdot \mathbf{x} = \frac{\mathbf{v}_1 \cdot \mathbf{x}}{\|\mathbf{v}_1\|} = \mathbf{v}_1 \cdot \mathbf{x}.$$

Similarly $\|\mathbf{b}\| = \mathbf{v}_2 \cdot \mathbf{x}$. This completes the proof. □

Exercise 19. *Let V be a subspace in $\mathbb{R}^3$ and P a point in $\mathbb{R}^3$. The* ***distance*** *$d(P,V)$ between P and the subspace V is called the shortest distance between P and all points of V. In other words,*

$$d(P,V) = \ min \ \{d(P,Q) \mid \overrightarrow{OP} \in V\}$$

Prove that

$$d(P,V) = \left\| \overrightarrow{OP} - proj_V(\overrightarrow{OP}) \right\|$$

We will generalize the concept of the projection to a subspace of $\mathbb{R}^n$ in coming lectures when we study projections; see Lem. 5.1. Projection formulas will be used in the so called Gram-Schmidt algorithm and in the QR-factorization of matrices and will be generalized to any positive definite inner product; see Chap. 5.

Exercises:

79. *Show that the formal definitions of the addition and scalar multiplication in $\mathbb{R}^2$ agree with the geometric interpretations.*

80. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ given as $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. Compute $2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$.*

81. *Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ -6 \end{bmatrix}$. Compute $2\mathbf{u} + 3\mathbf{v}$.*

82. *Let $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Find scalars r, s such that*

$$r\,\mathbf{v} + s\,\mathbf{u} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

83. *What does it mean for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ to* *be linearly dependent?*

84. *What is the span of* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *in* $\mathbb{R}^2$*?*

85. *Let* $\mathbf{u}, \mathbf{v}$*, and* $\mathbf{w}$ *be given vectors as below*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Can $\mathbf{w}$ *be a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$*? What is geometrically the span of* $\mathbf{u}$ *and* $\mathbf{v}$*?*

86. *Find the area of the triangle determined by the vectors*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}.$$

87. *Use vectors to decide whether the triangle with vertices* $A = (1, -3, -2)$*,* $B = (2, 0, -4)$*, and* $C = (6, -2, -5)$ *is right angled.*

88. *Prove that the triangle with vertices* $A(-2, 4, 0)$*,* $B(1, 2, -1)$ *and* $C(-1, 1, 2)$ *is regular.*

89. *In the third octant find the point P the distances of which from the three coordinate axis are* $d_x = \sqrt{10}$*,* $d_y = \sqrt{5}$*,* $d_z = \sqrt{13}$*.*

90. *Show that for any two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *the following is true*

$$(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = 0 \iff ||\mathbf{v}|| = ||\mathbf{w}||$$

91. *Find the angle between the vectors* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$ *and the area of the triangle determined by them.*

92. *Let* $\mathbf{u}$ *be the unit vector tangent to the graph of* $y = x^2 + 1$ *at the point* $(2, 5)$*. Find a vector* $\mathbf{v}$ *perpendicular to* $\mathbf{u}$*.*

93. *For what values of t are the vectors* $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} t \\ -t \\ t^2 \end{bmatrix}$ *perpendicular?*

94. *Let the vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and coordinates*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

Compute the volume of the parallelepiped determined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$*.*

95. *Let the vectors* $\mathbf{u}, \mathbf{v} \in\in \mathbb{R}^3$ *be given as* $[\mathbf{u} = [1, 2, 2]^t$ *and* $\mathbf{v} = [1, 2, -3]^t$*. Find the projection of* $\mathbf{u}$ *on* $\mathbf{v}$*.*

96. *Let* $\mathbf{u} = [1, 2, 2]^t$*,* $\mathbf{v} = [2, 2, -3]^t$*, and* $\mathbf{w} = [-1, -1, -1]^t$*. Find the projection of* $\mathbf{u}$ *onto the* $\mathbf{vw}$*-plane.*

1.3 Matrices and their algebra

A matrix is a list of vectors. Consider for example vectors $\mathbf{u}_i \in \mathbb{R}^m$, for $i = 1, \ldots, n$. An ordered list of such vectors, say

$$A = [\mathbf{u}_1 | \ldots | \mathbf{u}_n]$$

is called a **matrix**. If each $\mathbf{u}_i$ is given by $\mathbf{u}_i = \begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,m} \end{bmatrix}$, then A is a m by n table of scalars from $\mathbb{R}$.

In general an $m \times n$ **matrix** A is an array of numbers which consists of m rows and n columns and is represented as follows:

$$A = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \ldots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \ldots & a_{3,n} \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ a_{m,1} & a_{m,2} & a_{m,3} & \ldots & a_{m,n} \end{bmatrix} \tag{1.37}$$

The ***i*-th row** of A is the vector $R_i := [a_{i,1}, \ldots, a_{i,n}]$ and the ***j*-th column** is the vector $\mathbf{u}_j := \begin{bmatrix} a_{1,j} \\ \vdots \\ a_{n,j} \end{bmatrix}$.

Let $A = [a_{i,j}]$ be an $m \times n$ matrix and $B = [b_{i,j}]$ be a $n \times s$ matrix. The **matrix product** AB is the $m \times s$ matrix $C = [c_{i,j}]$ such that

$$\boxed{c_{i,j} := R_i(A) \cdot C_j(B)}$$

is the dot product of the i-th row vector of A and the j-th column vector of B.

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & \boxed{c_{ij}} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

For example, in the case that A and B are 3×3 matrices we will have

$$AB = \begin{bmatrix} \vec{r_1} \cdot \vec{c_1} & \vec{r_1} \cdot \vec{c_2} & \vec{r_1} \cdot \vec{c_3} \\ \vec{r_2} \cdot \vec{c_1} & \vec{r_2} \cdot \vec{c_2} & \vec{r_2} \cdot \vec{c_3} \\ \vec{r_3} \cdot \vec{c_1} & \vec{r_3} \cdot \vec{c_2} & \vec{r_3} \cdot \vec{c_3} \end{bmatrix}$$

Example 1.23. *Find AB and BA, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$.*

$$A \bullet B = \begin{array}{c} \\ \\ \vec{r_1} \to \\ \vec{r_2} \to \\ \vec{r_3} \to \end{array} \begin{array}{c} \\ \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{array} \bullet \begin{array}{ccc} \vec{c_1} & \vec{c_2} & \vec{c_3} \\ \downarrow & \downarrow & \downarrow \\ \multicolumn{3}{c}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 7 & 2 & 5 \end{bmatrix}} \end{array} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Figure 1.23: Matrix multiplication

The **matrix addition** is defined as

$$A + B = [a_{i,j} + b_{i,j}],$$

and the **multiplication by a scalar** $r \in \mathbb{R}$ is defined to be the matrix the matrix $rA := [ra_{i,j}]$. The $m \times n$ **zero matrix**, denoted by 0, is the $m \times n$ matrix which has zeroes in all its entries. An m by n matrix A is called a **square matrix** if $m = n$. If $A = [a_{i,j}]$ is a square matrix then all entries $a_{i,i}$ form the **main diagonal** of A. A **diagonal matrix** is a matrix that has nonzero entries only in its mail diagonal.

Example 1.24. *The matrix*

$$M = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

is a diagonal matrix.

The n by n **identity matrix**, denoted by I_n, is the matrix which has 1's in the main diagonal and zeroes elsewhere. A matrix that can be written as rI is called a **scalar matrix**. Two matrices are called **equal** if their corresponding entries are equal.

Notice that the arithmetic of matrices is not the same as the arithmetic of numbers. For example, in general $AB \neq BA$, or $AB = 0$ does not imply that $A = 0$ or $B = 0$. We will study some of these properties in detail in the next few sections. Next we state the main properties of the algebra of matrices.

Theorem 1.9. *Let A, B, C be matrices of sizes such that the operations below are defined. Let r, s be scalars. Then the following hold:*

(i) $A + B = B + A$
(ii) $(A + B) + C = A + (B + C)$
(iii) $A + \mathbf{0} = \mathbf{0} + A = A$
(iv) $r(A + B) = rA + rB$
(v) $(r + s)A = rA + sA$
(vi) $(rs)A = r(sA)$
(vii) $(rA)B = A(rB) = r(AB)$
(viii) $A(BC) = (AB)C$

(ix) $IA = A = AI$
(x) $A(B+C) = AB + AC$
(xi) $(A+B)C = AC + BC$

Proof. Most of the proofs are elementary and we will leave them as exercises for the reader. ◻

The **trace** of a square matrix $A = [a_{i,j}]$ is the sum of its diagonal entries:

$$\mathrm{tr}(A) := a_{11} + \cdots + a_{nn}.$$

The trace will be used in the coming lectures. Some of its properties are quite useful and easy to prove.

Lemma 1.12. *The following hold:*
(i) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$,
(ii) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Proof. The first part is obvious. We prove only part ii). Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be $n \times n$ matrices. Denote $AB = C = [c_{i,j}]$ and $BA = D = [d_{i,j}]$. Then

$$c_{i,i} = R_i(A) \cdot C_i(B) = C_i(B) \cdot R_i(A) = d_{i,i},$$

where $R_i(A)$ is the i-th row of A and $C_i(B)$ is the i-th column of B. This completes the proof. ◻

Example 1.25. *For matrices A and B given below*

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 21 & 10 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 61 \\ 3 & -3 & 1 \\ 31 & 2 & 1 \end{bmatrix}$$

compute the following $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(A+B)$, $\mathrm{tr}(AB)$, *and* $\mathrm{tr}(BA)$.

Solution: *It is clear that* $tr(A) = 5$, $tr(B) = -1$. *Then,* $tr(A+B) = 4$. *We have*

$$AB = \begin{bmatrix} 74 & 6 & 248 \\ 41 & -7 & 4 \\ -13 & 8 & 1289 \end{bmatrix}.$$

Hence, $\mathrm{tr}(AB) = \mathrm{tr}(BA) = 1356$. ◻

Given the matrix $A = [a_{i,j}]$ its **transpose** is defined to be the matrix

$$A^t := [a_{j,i}].$$

A is called **symmetric** if $A = A^t$. Note that for a square matrix A its transpose is obtained by simply rotating the matrix along its main diagonal.

Lemma 1.13. *For any matrix A the following hold*
(i) $(A^t)^t = A$,
(ii) $(A+B)^t = A^t + B^t$,

(iii) $(AB)^t = B^t A^t$.

Proof. Parts i) and ii) are easy. We prove only part iii). Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$. Denote $AB = [c_{i,j}]$. Then, $(AB)^t = [c_{j,i}]$ where

$$c_{j,i} = R_j(A) \cdot C_i(B) = C_j(A^t) \cdot R_i(B^t) = R_i(B^t) \cdot C_j(A^t).$$

This completes the proof. □

Remark 1.2. *Notice that for a vector* $\mathbf{u}$*, its transpose* $\mathbf{u}^t$ *is a row vector. Moreover, its norm is given by*

$$\|\mathbf{u}\|^2 = \mathbf{u}^t \star \mathbf{u},$$

where $\star$ *is the matrix multiplication.*

Example 1.26. *For matrices* A *and* B *given below*

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 21 & 10 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 61 \\ 3 & -3 & 1 \\ 31 & 2 & 1 \end{bmatrix}$$

compute the following A^t, B^t, $(A+B)^t$, $(AB)^t$, *and* $(BA)^t$.

Solution: *We have*

$$A^t = \begin{bmatrix} 4 & 0 & 21 \\ 2 & 3 & 10 \\ 2 & 1 & -2 \end{bmatrix}, \qquad B^t = \begin{bmatrix} 1 & 3 & 31 \\ 2 & -3 & 2 \\ 61 & 1 & 1 \end{bmatrix}.$$

Computing $(A+B)^t$, $(AB)^t$, *and* $(BA)^t$ *is left as an exercise for the reader.* □

Let A be a square matrix. If there is an integer n such that $A^n = I$ then we say that A has **finite order**, otherwise A has **infinite order**. The smallest integer n such that $A^n = I$ is called the **order** of A.

A **submatrix** of a matrix A is called any matrix that is obtained by deleting a number of rows or columns of A. A **principal submatrix** is a square submatrix obtained by removing the last few rows and columns.

A square matrix A that is equal to its transpose, that is, $A = A^t$, is a symmetric matrix. If instead, A is equal to the negative of its transpose, that is, $A = -A^t$, then A is a skew-symmetric matrix.

In complex matrices, symmetry is often replaced by the concept of Hermitian matrices, which satisfy $A^* = A$, where the star or asterisk denotes the conjugate transpose of the matrix, that is, the transpose of the complex conjugate of A.

A square matrix is called **upper triangular** if all entries below the mail diagonal are zero.

Exercise 20. *What is the sum and product of upper triangular matrices? Justify your answer.*

A square matrix is called **lower triangular** if all entries above the mail diagonal are zero.

Exercise 21. *What is the sum and product of lower triangular matrices?*

Let $V := \mathrm{Mat}_{n\times n}(\mathbb{R})$ be the set of all $n \times n$ matrices with entries in $\mathbb{R}$, W_1 the set of all upper triangular matrices of V, and W_2 the set of all lower triangular matrices of V.

Exercise 22. *What is the intersection* $W_1 \cap W_2$*?*

1.3.1 Groups of $\mathrm{Mat}_n(\mathbb{F})$

Understanding the structure of $\mathrm{Mat}_n(\mathbb{F})$ is a very important part of linear algebra and other parts of mathematics. In the coming lectures we will learn about some very important subsets of $\mathrm{Mat}_n(\mathbb{F})$, namely invertible matrices, special linear group, orthogonal matrices, etc.

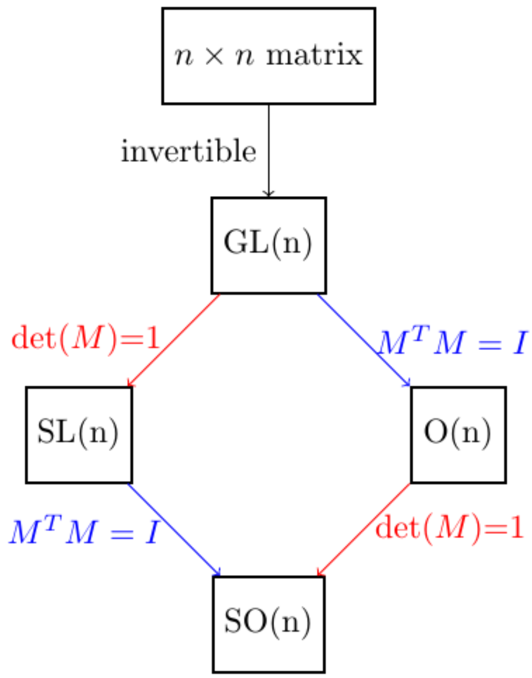


Figure 1.24: Linear groups

Exercises:

97. *Find the trace of the matrices A, B, A + B, and A − B, where A and B are*

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 21 & 10 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 6 \\ 3 & -3 & 1 \\ 31 & 0 & 13 \end{bmatrix}$$

98. *We call a matrix A **idempotent** if $A^2 = A$. Find a 2 by 2 idempotent matrix A not equal to the identity matrix I_2. Using A, give an example of two matrices B, C such that $BC = 0$, but $B \neq 0$ and $C \neq 0$.*

99. *Let*

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Find A^2. What about A^n?

100. *A square matrix A is said to be **nilpotent** if there is an integer $r \geq 1$ such that $A^r = 0$. Let A, B be matrices such that $AB = BA$, $A^2 = 0$ and $B^2 = 0$. Show that AB and $A + B$ are nilpotent.*

101. *Let*

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

If possible, find a matrix B such that $AB = 2A$.

102. *Prove that:*

(i) *For any matrix A, the matrix AA^t is symmetric*
(ii) *If A is a square matrix then $A+A^t$ is symmetric.*

103. *Let A be a square matrix. Show that $(A^n)^t = (A^t)^n$.*

104. *Prove or disprove the identity*

$$(A+B)^2 = A^2 + 2AB + B^2,$$

for any two $m \times n$ matrices A and B.

105. *Let A and B be two matrices such that $AB = BA$. Prove that*

$$(A-B)(A+B) = A^2 - B^2.$$

106. *Let A and B be two matrices such that $AB = BA$. Prove that*

$$(A-B)(A^2+AB+B^2) = A^3 - B^3.$$

107. *Let Q be the following set of complex matrices:*

$$\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

such that $i^2 = -1$. Further, let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Prove the following statements $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -I$ and

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ji} = -\mathbf{k}, \quad \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ik} = -\mathbf{j}.$$

These matrices are sometimes called ***quaternions****. Show that $\pm\mathbf{i}$, $\pm\mathbf{j}$, $\pm\mathbf{k}$ have order 4.*

108. *Let $A \in \mathrm{Mat}_n(\mathbb{R})$ such that $\mathrm{tr}(AB) = 0$ for all $B \in \mathrm{Mat}_n(\mathbb{R})$. Can we conclude that A is the zero matrix?*

1.4 Linear systems of equations, Gauss method

In this section we will study the classical problem of solving linear systems of equations. In high school we have learned how to solve small systems of equations by using substitutions, adding equations side by side and eliminating variables, etc. Now we want to streamline this process so we can solve any systems of linear equations no matter how large. Think of it as we want to write a computer program that should work for any linear system.

Recall that by a vector $\mathbf{x} \in \mathbb{R}^n$ we denote a column vector. Let a linear system of m equations with n unknowns be given as follows:

$$\begin{cases} a_{1,1}x_1 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + \cdots + a_{2,n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

We write this system in the matrix form as

$$A \cdot \mathbf{x} = \mathbf{b}$$

where

$$A = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n} \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & a_{m,n} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ x_m \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \\ \\ \\ b_m \end{bmatrix}.$$

We want to use matrices and design an algorithm which can determine if such a system has a solution and in the case it does, find that solution. Since all the data for the system is contained in A and $\mathbf{b}$, the new matrix $[A \mid \mathbf{b}]$ denotes the following matrix

$$[A \mid \mathbf{b}] := \left[\begin{array}{ccccc|c} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n} & b_3 \\ & & \cdot & & \cdot & \cdot \\ & & \cdot & & \cdot & \cdot \\ & & \cdot & & \cdot & \cdot \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & a_{m,n} & b_m \end{array}\right]$$

and is called the **augmented matrix** of the corresponding system $\mathbb{A}\mathbf{x} = \mathbf{b}$.

1.4.1 Elementary row operations

We would like to manipulate the augmented matrix $[A \mid \mathbf{b}]$ such that the solution set of the linear system does not change. As learned in elementary mathematics the solution to any system does not change if we changed the order of the equations, we multiply any equation by any nonzero constant, or replacing any equation by a sum of any other two. These rules motivate the following definition.

We define as **elementary row operations** performed on a matrix the following operations:

- *Interchange the i-th row with the j-th row* (denoted by $R_i \longrightarrow R_j$)
- *Multiply the i-th row by a nonzero scalar r* (denoted by $R_i \to rR_i$)
- *Add the i-th row to r times the j-th row* (denoted by $R_i \to R_i + rR_j$)

It is obvious that such operations on the augmented matrix do not change the solution set of the system. If the matrix B is obtained by performing row operations on A then matrices A and B are called **row equivalent**

1.4.2 Row-echelon form of a matrix

Definition 1.6. *A matrix is in **row echelon form** if:*

- *All rows containing all zeroes are below rows with nonzero entries.*
- *The first nonzero entry in a row appears in a column to the right of the first nonzero entry in any preceding row.*

*For a matrix in row-echelon form, the first nonzero entry in a row is the **pivot** for that row.*

A row echelon form of a matrix A will be denoted by **ref** A.

Example 1.27. *Using row operations find the row echelon form of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}$.

Solution: *We perform the following row operations:*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix} \overset{R2 \to \frac{1}{2}R_2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & \frac{1}{2} \\ 3 & 2 & 2 \end{bmatrix} \overset{R_2 \to R_1 - R_2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \frac{5}{2} \\ 3 & 2 & 2 \end{bmatrix}$$

$$\overset{R_3 \to \frac{1}{3}R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \frac{5}{2} \\ 1 & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \overset{R_3 \to R_1 - R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \frac{5}{2} \\ 0 & \frac{4}{3} & \frac{7}{3} \end{bmatrix} \overset{R_3 \to R_2 - \frac{3}{2}R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \frac{5}{2} \\ 0 & 0 & -1 \end{bmatrix}$$

□

Row operations are fast and inexpensive operations. Below we summarize the algorithm.

Algorithm 1. ***Computing the row-echelon form***

Input: A matrix A.
*Output: The row-echelon form **ref** (A).*

(i) *Start with the first column which has nonzero entries.*
(ii) *By row interchange get a pivot p in the first row of this column. Make entries in this column below the pivot all zeroes.*
(iii) *Continue this way with the next column.*

The row-echelon form of matrices is used to solve linear systems of equations. Let $A\mathbf{x} = \mathbf{b}$, be a linear equation. We create the augmented matrix $[A \mid b]$ and find its row-echelon form, say $[H \mid \mathbf{c}]$. Using **back substitution** we solve the system

$$H\mathbf{x} = \mathbf{c}.$$

We illustrate with an example.

Example 1.28. *Solve the linear system*

$$\begin{cases} x_2 - 3x_3 = -5 \\ 2x_1 + 3x_2 - x_3 = 7 \\ 4x_1 + 5x_2 - 2x_3 = 10 \end{cases}$$

Solution: *Then*

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{array}\right] \rightsquigarrow [H \mid \mathbf{c}] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

by performing the operations $R_1 \longrightarrow R_2$, $R_3 \to R_3 - 2R_1$, $R_3 \to R_3 + R_2$. Thus the linear system is equivalent with the following system

$$\begin{cases} 2x_1 + 3x_2 - x_3 = \ 7 \\ x_2 - 3x_3 = -5 \\ -3x_3 = -9 \end{cases}$$

By back substitution we have $\mathbf{x} = [-1, 4, 3]^t$. □

This method is known as the **Gauss method**.

Theorem 1.10. *Let $A\mathbf{x} = \mathbf{b}$ be a linear system, and denote by*

$$[H \mid \mathbf{c}] := \textbf{\textit{ref}}\ ([A \mid \mathbf{b}]).$$

Then one of the following hold:
(i) *$A\mathbf{x} = \mathbf{b}$ has **no solution** if and only if H has a row of all zeroes and in the same row $\mathbf{c}$ has a nonzero entry.*
(ii) *If $A\mathbf{x} = \mathbf{b}$ has solutions then one of the following holds:*
   *i) it has **a unique solution** if every column of H contains a pivot*
   *ii) it has **infinitely many solutions** if some column of H contains no pivot*

Proof. We recall from elementary algebra that a linear equation $ax = b$ has no solution if and only if $a = 0$ and $b \neq 0$. It has a unique solution if and only if $a \neq 0$ and $b \neq 0$ and infinitely many solutions if and only if $a = b = 0$.

If H has a row of all zeroes and in the same row $\mathbf{c}$ has a nonzero entry $c_n \neq 0$ then the equation $0 \cdot x_n = c_n$ has no solution and therefore the linear system $A\mathbf{x} = \mathbf{b}$ has no solution. The converse also hold from the definition of the row-echelon form. Parts 2, i) and 2, ii) follow similarly. ☐

Example 1.29. *Find how many solutions the following system has:*

$$\begin{cases} 2x + 5y = 3 \\ 6x + 15y = 9 \end{cases}$$

Solution: *The augmented matrix is*

$$[A \mid \mathbf{b}] = \left[\begin{array}{cc|c} 2 & 5 & 3 \\ 6 & 15 & 9 \end{array}\right] \rightsquigarrow [H \mid \mathbf{c}] = \left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

From the above theorem the system has infinitely solutions. Of course, this is easy to see since the second equation of the system is obtained by multiplying the first equation by 3. ☐

The above theorem can be interpreted geometrically in the case of a 2 by 2 or a 3 by 3 coefficient matrix. For example in the case of a linear system of 2 equations and 2 variables we have the well known situation of two lines on the plane. It is known from geometry that two lines intersect in one point, no points, or infinitely many points.

1.4.3 Reduced row-echelon form, Gauss-Jordan method

Let $[A \mid \mathbf{b}]$ be a matrix in row-echelon form. Can we manipulate $[A \mid \mathbf{b}]$ even further so that the solution of the corresponding system is read directly from the matrix equation? This leads to the following definition

Definition 1.7. *A matrix is in* ***reduced row-echelon form*** *if it is in row-echelon form, all pivots are 1, and all terms above the pivots are 0.*

As we will see, once the coefficient matrix is in the reduced row-echelon form then the solution of the corresponding linear system is read directly in the last column of the augmented matrix. We illustrate with an example.

Example 1.30. *Let* $[H \mid \mathbf{c}]$ *be the matrix in row-echelon form as in the Exa. 1.29:*

$$[H \mid \mathbf{c}] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right].$$

Find its reduced row-echelon form.

Solution: *To find the reduced row-echelon form we perform the following row-operations*

$$[H \mid \mathbf{c}] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1,\, R_3 \to -\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1 - \frac{3}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_2 \to 3R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1 - 4R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence, we can directly conclude that the solution to the corresponding system is $\mathbf{x} = [-1, 4, 3]^t$*, as concluded previously.* □

Remark 1.3. *Notice that the reduced row-echelon form of a matrix A, on contrary to the row-echelon form, is unique.*

The method that transforms the augmented matrix to the reduced row-echelon form is called the **Gauss-Jordan method**.

Remark 1.4. *Even though the Gauss-Jordan method gives the solution in a "nicer" form, it is not necessarily better than the Gauss method. For large linear systems the number of operations performed becomes significant. Using the Gauss-Jordan method, it takes roughly 50% more arithmetic operations than using the Gauss method.*

Example 1.31. *Find the reduced row-echelon form of the matrix.*

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 1 \\ -2 & 1 & 1 & 2 \\ -2 & -1 & 2 & 2 \end{array}\right]$$

Show all the row operations. What are the solutions of the corresponding system $A\mathbf{x} = \mathbf{b}$*?*

Solution: *The reduced row-echelon form is*

$$[H \mid \mathbf{c}] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, the system has no solutions.

Example 1.32. *Determine values of b such that the following system has one solution, infinitely many solutions, or no solutions*

$$\begin{cases} x_1 + 2x_2 - x_3 = b \\ x_1 + x_2 + 2x_3 = 1 \\ 2x_1 - x_2 + x_3 = 2 \end{cases}$$

Solution: *The augmented matrix is*

$$[A \mid b] = \begin{bmatrix} 1 & 2 & -1 & b \\ 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{bmatrix}$$

and its reduced row-echelon form is:

$$[H \mid \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 & \frac{b+3}{4} \\ 0 & 1 & 0 & \frac{b-1}{4} \\ 0 & 0 & 1 & \frac{b-1}{4} \end{bmatrix}$$

The system has a solution for any b.

1.4.4 Homogenous systems

A linear system is called a **homogenous system** if it is in the form

$$A\mathbf{x} = \mathbf{0}. \tag{1.38}$$

Clearly $\mathbf{x} = \mathbf{0}$ is a solution of such systems and is called the **trivial solution**. The augmented matrix for such systems is $[A \mid \mathbf{0}]$ and its row-echelon form will be $[H \mid \mathbf{0}]$. The system has nontrivial solutions if there is a row of H with no pivots.

Lemma 1.14. *A homogenous system $A\mathbf{x} = \mathbf{0}$ which has a non-trivial solution has infinitely many solutions. This occurs if and only if the row-echelon form of A has at least a row with no pivots.*

The set of solutions of Eq. (1.38) is called the **nullspace** of A and denoted by $\mathbf{Null}(A)$.

Example 1.33. *If the matrix A has an inverse, then $\mathbf{Null}(A) = \{\mathbf{0}\}$.*

Solution: Indeed, if A^{-1} exists then

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0} \implies \mathbf{x} = \mathbf{0}.$$

□

Example 1.34. *Let $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ and assume that $\mathbf{Null}(A)$ is not trivial. Denote its columns by $\mathbf{u}_1$ and $\mathbf{u}_2$. Give a geometric interpretation of the $\mathbf{Null}(A)$.*

In the coming lectures we will determine exactly which matrices have trivial nullspace.

Example 1.35. *Determine the set of solutions of the homogenous linear system $A\mathbf{x} = \mathbf{0}$ when*

$$A = \begin{bmatrix} 3 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: *In this case, the matrix is already in the row-echelon form. We can think of the system as*

$$\begin{cases} 3x_1 - x_2 + x_3 + x_4 = 0 \\ \qquad x_2 + x_3 + x_4 = 0 \\ \qquad 0 \cdot x_3 + 0 \cdot x_4 = 0 \\ \qquad\qquad 0 \cdot x_4 = 0 \end{cases}$$

Starting from the last equation $0x_4 = 0$, this is satisfied for any value of x_4. Let say $x_4 = t$ for any $t \in \mathbb{R}$. Then the third equation becomes $0 \cdot x_3 + 0 \cdot t = 0$, which is satisfied for any value of x_3, say $x_3 = s$, for any $s \in \mathbb{R}$. So now we have

$$\begin{cases} 3x_1 - x_2 + s + t = 0 \\ \qquad x_2 + s + t = 0 \end{cases}$$

which implies $x_2 = -s - t$ and $x_1 = \frac{1}{3}(x_2 - s - t) = \frac{1}{3}(-2s - 2t) = -\frac{2}{3}s - \frac{2}{3}t$. Finally the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}s - \frac{2}{3}t \\ -s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{2}{3} \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{2}{3} \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the set of solutions is Span $(\mathbf{v}_1, \mathbf{v}_2)$, where $\mathbf{v}_1 = \left[-\frac{2}{3}, -1, 1, 0\right]^t$ and $\mathbf{v}_2 = \left[-\frac{2}{3}, -1, 0, 1\right]^t$. □

We will use such method in Sec. 3.2 to determine the nullspace and in Chap. 4 to determine eigenspaces of matrices.

Exercises:

Solve the linear systems using the Gauss method with back substitution.

109.
$$\begin{cases} x + 5y = 2 \\ 3x + 2y = 9 \end{cases}$$

110.
$$\begin{cases} 2x + y - 3z = 0 \\ 6x + y - 8z = 0 \\ 2x - y + 5z = -4 \end{cases}$$

111.
$$\begin{cases} y - 2z = 3 \\ x + 2y - 3z = 2 \\ 5x - 3y + z = -1 \end{cases}$$

Find the row-echelon form of the following matrices.

112. $\begin{bmatrix} 0 & 1 & -3 & -5 \\ 0 & 3 & 0 & 1 \\ 4 & 5 & -2 & 10 \end{bmatrix}$

113. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & -3 \\ 1 & 3 & 0 & 0 \\ 2 & 5 & -2 & 1 \end{bmatrix}$

114. *Determine all values of b_1, b_2 such that the following system has solutions*

$$\begin{cases} x_1 + 11x_2 = b_1 \\ 3x_1 + 33x_2 = b_2 \end{cases}$$

115. *Determine all values of b_1, b_2 such that the following system has no solutions*

$$\begin{cases} x_1 + 2x_2 = b_1 \\ -2x_1 - 4x_2 = b_2 \end{cases}$$

116. *Find a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(1, -4)$, $(-1, 0)$, and $(2, 3)$.*

117. *Find a, b, c and d such that the quartic $y = ax^4 + bx^3 + cx^2 + d$ passes through the points $(3, 2)$, $(-1, 6)$, and $(-2, 38)$, and $(2, 6)$.*

118. *Find a polynomial function going through the points $(3, 1, -2)$, $(1, 4, 5)$, and $(2, 1, -4)$.*

119. *Find **ref** A and solve the linear system $A\mathbf{x} = \mathbf{0}$, for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}$.*

120. *Find* ***ref*** *A and solve the linear system* $A\mathbf{x} = \mathbf{0}$, *for* $A = \begin{bmatrix} 0 & 1 & -3 & -5 \\ 0 & 3 & 0 & 1 \\ 4 & 5 & -2 & 10 \end{bmatrix}$.

121. *Find* ***ref*** *A and solve the linear system* $A\mathbf{x} = \mathbf{0}$, *for* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & -3 \\ 1 & 3 & 0 & 0 \\ 2 & 5 & -2 & 1 \end{bmatrix}$.

122. *Solve the following system using the Gauss-Jordan method*

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_1 + x_2 + 2x_3 = 3 \\ 2x_1 - x_2 + x_3 = -2 \end{cases}$$

123. *Solve the following system using the Gauss method*

$$\begin{cases} 5x_1 + 3x_2 - x_3 = -2 \\ 2x_1 + 2x_2 + 2x_3 = 3 \\ -x_1 - x_2 + x_3 = 6 \end{cases}$$

124. *Solve the following system using the Gauss-Jordan method*

$$\begin{cases} 11x_1 + 12x_2 - 3x_3 = 2 \\ -x_1 + 3x_2 + 2x_3 = 3 \\ 2x_1 + 3x_2 + x_3 = -2 \end{cases}$$

125. *Prove that the reduced row-echelon form of a matrix is unique.*

126. *Let* $A\mathbf{x} = \mathbf{0}$ *be a homogenous system which has no nontrivial solutions. What is the reduced row-echelon form of A?*

127. *Find* a, b, *and* c *such that the parabola*

$$y = ax^2 + bx + c$$

passes through the points (1,2), (-1,1), and (2,3).

128. *Find* a, b, c *and* d *such that the quartic*

$$y = ax^4 + bx^3 + cx^2 + d$$

passes through the points (3,2), (-1,6), and (-2,1), and (0,0).

Johann Carl Friedrich Gauss (1777 - 1855)

Carolus Fridericus Gauss; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields in mathematics and science. Sometimes referred to as the Princeps mathematicorum (Latin for ′"the foremost of mathematicians"′) and "the greatest mathematician since antiquity", Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.

Johann Carl Friedrich Gauss was born on 30 April 1777 in Brunswick (Braunschweig), in the Duchy of Brunswick-Wolfenbüttel (now part of Lower Saxony, Germany), to poor, working-class parents. His mother was illiterate and never recorded the date of his birth, remembering only that he had been born on a Wednesday, eight days before the Feast of the Ascension (which occurs 39 days after Easter). Gauss later solved this puzzle about his birthdate in the context of finding the date of Easter, deriving methods to compute the date in both past and future years. He was christened and confirmed in a church near the school he attended as a child.

Gauss was a child prodigy. In his memorial on Gauss, Wolfgang Sartorius von Waltershausen says that when Gauss was barely three years old he corrected a math error his father made; and that when he was seven, he confidently solved an arithmetic series problem (commonly said to be 1 + 2 + 3 + ... + 98 + 99 + 100) faster than anyone else in his class of 100 students. Many versions of this story have been retold since that time with various details regarding what the series was the most frequent being the classical problem of adding all the integers from 1 to 100. There are many other anecdotes about his precocity while a toddler, and he made his first groundbreaking mathematical discoveries while still a teenager. He completed his magnum opus, Disquisitiones Arithmeticae, in 1798, at the age of 21 though it was not published until 1801. This work was fundamental in consolidating number theory as a discipline and has shaped the field to the present day.

Gauss's intellectual abilities attracted the attention of the Duke of Brunswick, who sent him to the Collegium Carolinum (now Braunschweig University of Technology), which he attended from 1792 to 1795, and to the University of Göttingen from 1795 to 1798. While at university, Gauss independently rediscovered several important theorems. His breakthrough occurred in 1796 when he showed that a regular polygon can be constructed by compass and straightedge if the number of its sides is the product of distinct Fermat primes and a power of 2. This was a major discovery in an important field of mathematics; construction problems had occupied mathematicians since the days of the Ancient Greeks, and the discovery ultimately led Gauss to choose mathematics instead of philology as a career. Gauss was so pleased with this result that he requested that a regular heptadecagon be inscribed on his tombstone. The stonemason declined, stating that the difficult construction would essentially look like a circle.

The year 1796 was productive for both Gauss and number theory. He discovered a construction of the heptadecagon on 30 March. He further advanced modular arithmetic, greatly simplifying manipulations in number theory. On 8 April he became the first to prove the quadratic reciprocity law. This remarkably general law allows mathematicians to determine the solvability of any quadratic equation in modular arithmetic. The prime number theorem, conjectured on 31 May, gives a good understanding of how the prime numbers are distributed among the integers.

Gauss also discovered that every positive integer is representable as a sum of at most three triangular numbers on 10 July and then jotted down in his diary the note:

$$\text{EYPHKA! num } = \Delta + \Delta' + \Delta''.$$

On 1 October he published a result on the number of solutions of polynomials with coefficients in finite fields, which 150 years later led to the Weil conjectures.

1.5 Inverses of matrices

In this section we study the important concept of the inverse of a matrix. Let $A = [a_{i,j}]$ be a $n \times n$ square matrix. A is called **invertible** if there exists an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n.$$

A^{-1} is called the **inverse** of A. If A is not invertible then it is called **singular**. Consider if A is invertible and we want to solve the linear system $A\mathbf{x} = \mathbf{b}$. Then $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I_n\mathbf{x} = \mathbf{x}$. Hence the solution to the system would be $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 1.11 (Uniqueness of the inverse)**.** *Let A be an invertible matrix. Then, its inverse is unique.*

Proof. Suppose that A has two inverses C and D. Then, $AC = I = AD$ and $CA = I = DA$. Then we have $D(AC) = DI = D$ and $D(AC) = (DA)C = IC = C$. Hence, $C = D$. ◻

We also have the following useful result.

Exercise 23. *Let A, B be invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Any matrix that can be obtained from the identity matrix I_n by one row operation is called an **elementary matrix**.

Theorem 1.12. *Let A be an $m \times n$ matrix and E an $m \times m$ elementary matrix. Then EA affects the same row operation on A as the one performed in I_n to obtain E.*

Proof. Let E be the elementary matrix obtained as

$$I_m \xrightarrow{R_i \longleftrightarrow R_j} E.$$

Then the new $R_i(E) = (0, \dots, 0, 1, 0, \dots 0)$ where 1 is in the j-th position. Hence, the entries of $R_i(EA)$ are

$$R_i(E) \cdot C_r(A), \quad \textit{for} \quad r = 1, \dots n$$

and $R_i(EA) = R_j(A)$. Similarly, $R_j(EA) = R_i(A)$. The cases in which E is obtained by a row-scaling and row-addition go similarly and are left as exercises to the reader. ◻

Exercise 24. *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

1.5.1 Computing the inverses using the row-echelon form

Let A be a given matrix. We want to find its inverse A^{-1} if it exists. Consider first the elementary matrices.

Let E be an elementary row matrix obtained by one row-interchange of I. Then performing the same row-interchange to E would give us back I. Hence, $EE = I$ and the inverse of E is E itself. If E is obtained by multiplying a row by a scalar then we divide that row with the same scalar to get back I. If E is obtained by $R_i \to R_i + rR_j$ then performing $R_i \to R_i - rR_j$ would result in I. Hence, we have the following:

Lemma 1.15. *Elementary matrices are invertible*

Proof. Let E_1 be an elementary matrix. Then E_1 is obtained by some row operation on the identity I. Since every row-operation can be undone then we can perform a new row-operation on E_1 to obtain I. The second row operation corresponds to another elementary matrix E_2 such that $E_2E_1 = I$; see the previous theorem. Thus, E_1 has an inverse. □

Example 1.36. *Let E be given as below*

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Find its inverse.

Solution: *E is obtained by interchanging rows $R_2 \longleftrightarrow R_4$ of the identity matrix. So E is an elementary matrix and therefore invertible. Its inverse is E since $E^2 = I$.*

□

Lemma 1.16. *Let A, B be $n \times n$ matrices. Then, $AB = I_n$ if and only if $BA = I_n$.*

Proof. It is enough to show that if $AB = I_n$ then $BA = I_n$, the other direction goes by symmetry of A and B. Hence, we assume that $AB = I_n$. Let $\mathbf{b}$ be any vector in $\mathbb{R}^n$. Then $AB\mathbf{b} = \mathbf{b}$. Thus the system $A\mathbf{x} = \mathbf{b}$ has always a solution (namely $\mathbf{x} = B\mathbf{b}$). By Thm. 1.10 the reduced row-echelon form of A is I_n. Hence, there are $E_1, \dots, E_k$ such that

$$E_k \cdots E_1 A = I_n \tag{1.39}$$

Multiplying both sides on the right by B we have

$$E_k \cdots E_1 (AB) = B.$$

But $AB = I_n$, hence $E_k \cdots E_1 = B$. Thus, Eq. (1.39) we have $BA = I_n$.

□

Now, we go back to the main question of this section, that of computing inverses. In general we proceed as follows. Let $A = [a_{i,j}]$ be given. To find A^{-1} we have the following algorithm:

Algorithm 2. ***Computing the inverse of a matrix***

Input: A square matrix A.
Output: Determines if A^{-1} exists and finds it in that case.

(i) *Form the augmented matrix $[A \mid I]$*
(ii) *Apply the Gauss-Jordan method to reduce $[A \mid I]$ to $[I \mid C]$. If this is possible then $C = A^{-1}$, otherwise A^{-1} does not exist.*

Example 1.37. *Find the inverse of the following matrix*

$$A = \begin{bmatrix} -1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

Solution: *Create the matrix $[A \mid I]$. Then its reduced row-echelon form is:*

$$[I \mid C] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -5 & 2 & -9 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -3 & 1 & -5 \end{array}\right]$$

Hence,

$$A^{-1} = C = \begin{bmatrix} -1 & -5 & 2 & -9 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & -3 & 1 & -5 \end{bmatrix}$$

□

Example 1.38. *Find the inverse of*

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Solution: Create $[A \mid I]$ and find its reduced row-echelon form

$$[A \mid I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow [I \mid A^{-1}] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

□

Remark 1.5. *We have illustrated above how to find the inverse of a matrix. However, such an inverse does not always exist. In the next chapter we will study some necessary and sufficient conditions such that the inverse of a matrix exists.*

Example 1.39. (i) *Let A be a square matrix such that $A^2 = 0$. Find the inverse of $I - A$.*

(ii) *Let A be a square matrix such that $A^2 + 2A + I = 0$. Find the inverse of A.*

(iii) *Let A be a square matrix such that $A^3 - A + I = 0$. Find the inverse of A.*

(iv) *Let A be a square matrix such that $A^n = 0$. Find the inverse of $I - A$.*

Solution: i) Notice that since $A^2 = 0$ then $I^2 - A^2 = I$. However, $I^2 - A^2 = (I - A)(I + A)$. Hence, the inverse of $(I - A)$ is $(I + A)$.

ii) Since $A^2 + 2A + I = 0$ then $I = -(A^2 + 2A) = -(A + 2I)A$. Hence, $A^{-1} = -(A + 2I)$.

iii) Since $A^3 - A + I = 0$ then $I = A - A^3 = A(I - A^2)$. Hence, $A^{-1} = I - A^2$.

iv) Since $A^n = 0$ then $I^n - A^n = 0$. From Calculus you should remember the formula for the geometric sum

$$1 - x^n = (1 - x)(x^{n-1} + x^{n-2} + \cdots x + 1).$$

This suggests that

$$I - A^n = (I - A)\left(A^{n-1} + A^{n-2} + \cdots A^2 + A + I\right)$$

This can be easily verified by multiplying the right hand side to get

$$\begin{aligned}(I - A)\left(A^{n-1} + A^{n-2} + \cdots A^2 + A + I\right) =& (A^{n-1} + A^{n-2} + \cdots A^2 + A + I) - A^n - A^{n-1} - \cdots - A^2 - A \\ =& I - A^n\end{aligned}$$

Hence, $(I - A)^{-1} = A^{n-1} + A^{n-2} + \cdots A^2 + A + I$. ◻

Example 1.40. *For any matrices A and B such that the product BAB^{-1} is defined, prove that*

$$\operatorname{tr}(A) = \operatorname{tr}\left(BAB^{-1}\right).$$

Solution: Recall from Lem. 1.12 that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any two matrices A and B. Hence

$$\operatorname{tr}\left(BAB^{-1}\right) = \operatorname{tr}\left(AB^{-1}B\right) = \operatorname{tr} A$$

◻

Example 1.41. *Let*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

If possible, find a matrix B such that $AB = 2I$.

Solution: Then we have

$$\frac{1}{2}AB = I$$

which implies that B is invertible. Hence, $B = \left(\frac{1}{2}A\right)^{-1} = 2A^{-1}$. ◻

Exercises:

129. *Find the inverse of*

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

Does A have an inverse for any value of a?

130. *For what values of a, b, c, d does the inverse of*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

exist? Find the inverse for such values of a, b, c, d.

131. *Solve the linear system*

$$A\mathbf{x} = \mathbf{b}$$

when A is invertible.

132. *Find the inverse of the following*

$$A = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 2 & 4 & -1 & 2 \end{bmatrix}.$$

133. *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix},$$

be given. Find the following: $\operatorname{tr}(A)$, $\operatorname{tr}(B)$, A^t, AB, B^tA^t, $\operatorname{tr}(BAB^{-1})$.

134. *Show that if A is invertible then so is A^t.*

135. *Let r be a positive integer and A an invertible matrix. Is A^r necessarily invertible? Justify your answer.*

136. *Find the reduced row-echelon form of the matrix. Show all the row operations.*

$$\begin{bmatrix} 4 & 2 & 3 & 3 \\ -2 & 1 & 1 & 2 \\ 3 & -1 & 2 & 1 \end{bmatrix}$$

137. *Find the angle between the vectors* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \\ 8 \end{bmatrix}$.

138. *Determine all values of b_1 and b_2 such that the following system has no solutions*

$$\begin{cases} x_1 + 2x_2 - x_3 = b_1 \\ -2x_1 - 4x_2 + 2x_3 = b_2 \\ x_1 - x_2 + x_3 = 2 \end{cases}$$

139. *Find the area of the triangle between the three points* (1,2), (3,4), (5,6).

140. *Let the matrices*

$$A = \begin{bmatrix} 3 & 2 & 3 \\ -2 & 1 & 2 \\ 0 & 1 & \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix},$$

be given. Find the following: $\operatorname{tr}(A)$, $\operatorname{tr}(B)$, A^t, AB, B^tA^t, $\operatorname{tr}(BAB^{-1})$.

141. *Show that if AB is invertible then so are A and B.*

142. *Let A be a 3 by 2 matrix. Show that there is a vector* $\mathbf{b}$ *such that the linear system*

$$A\mathbf{x} = \mathbf{b}$$

is unsolvable.

143. *Let A be an $m \times n$ matrix with $m > n$. Show that there exists a* $\mathbf{b}$ *such that the linear system* $A\mathbf{x} = \mathbf{b}$ *is unsolvable.*

144. *Let A be a $m \times n$ matrix and B an $n \times m$ matrix, where $m > n$. Use the above result to show that the row-echelon form of the matrix AB has at least one row of all zeroes.*

145. *Find all matrices B such that*

i) $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

ii) $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

146. *Find all matrices which commute with*

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

147. *Show that if A commutes with B then A^t commutes with B^t.*

148. *Let V be the set of all m by n matrices with entries in $\mathbb{R}$. Show that scalar matrices commute with all matrices from V. Are there any other matrices which commute with all matrices of V?*

149. *Let a, b, c, d be real number not all zero. Show that the following system has exactly one solution*

$$\begin{cases} ax_1 + bx_2 + cx_3 + dx_4 = 0 \\ bx_1 - ax_2 + dx_3 - cx_4 = 0 \\ cx_1 - dx_2 - ax_3 + bx_4 = 0 \\ dx_1 + cx_2 - bx_3 - ax_4 = 0 \end{cases}$$

150. *For what value of λ does the following system has a solution:*

$$\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 7x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$$

151. *The following system has a unique solution:*

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a. \end{cases}$$

Show that $abc \neq 0$. Find the solution of the system.

152. *Find the following:*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^n, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^n$$

153. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

such that $A^2 = I$. Show that the following relation is satisfied when x is substituted by A:

$$x^2 - (a+d)x + (ad - bc) = 0.$$

154. *Let A be a 3 by 3 matrix. Can you generalize the above problem in that case? What about if A is an n by n matrix?*

155. *Find the order of the following matrices*

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

Paul Gordan: The King of Invariant Theory

Paul Albert Gordan (April 27, 1837 – December 21, 1912) was a towering figure in 19th-century German mathematics, renowned especially for his mastery of invariant theory. His epithet, "the king of invariant theory," speaks to his profound contributions and his dominant influence in this field.


Figure 1.25: Paul Albert Gordan

Born in Breslau, Germany (now Wroclaw, Poland), Gordan's academic journey began at the University of Königsberg, where he studied under the tutelage of the eminent Carl Gustav Jacobi. He earned his Ph.D. from the University of Breslau in 1862 and subsequently embarked on a distinguished professorial career at the University of Erlangen-Nuremberg. He remained there for the rest of his life, contributing significantly to establishing Erlangen as a global center for mathematical research.

Gordan's most celebrated achievement was proving the finite generation of the ring of invariants of binary forms of a fixed degree. This was a monumental task, achieved through intricate and laborious calculations, a hallmark of Gordan's approach to mathematics. He was a master of computational techniques, often undertaking complex calculations by hand. A testament to this dedication is his computation of all 70 invariants of binary sextics – a feat of remarkable perseverance and skill.

His collaboration with Alfred Clebsch led to the development of the now-famous Clebsch-Gordan coefficients, which are crucial in representation theory and quantum mechanics. These coefficients arise in the decomposition of tensor products of representations and have far-reaching applications in physics and other areas.

Gordan's influence extended beyond his own research. He played a key role in making Erlangen a leading mathematical center, working alongside Felix Klein and Max Noether. This trio fostered a vibrant intellectual environment that attracted mathematicians from around the world.

One of Gordan's most significant legacies is his role as the doctoral advisor to Emmy Noether, one of the most important mathematicians of the 20th century. He recognized her exceptional talent and guided her early research, even though her later work in abstract algebra eventually diverged significantly from his own computational focus.

A well-known anecdote, often repeated, involves Gordan and David Hilbert's groundbreaking proof of Hilbert's basis theorem. This theorem, which drastically generalized Gordan's work on invariants, demonstrated the existence of a finite basis for invariants in a much broader context using non-constructive methods. The quote attributed to Gordan, "This is not mathematics; this is theology," reflects the initial shock and perhaps skepticism that some mathematicians felt towards Hilbert's abstract, existence-based approach, which contrasted sharply with the constructive, computational methods prevalent at the time.

However, the historical accuracy and intended meaning of this quote are debated. The earliest known reference to it appears long after the events and Gordan's death. Furthermore, the narrative of Gordan as being opposed to Hilbert's work is largely a myth. In reality, Gordan recognized the power of Hilbert's methods, used them in his own research, and even supported Hilbert's publications. It's likely that the quote, if indeed Gordan uttered it, was meant as a humorous or nuanced observation, not as a categorical rejection of Hilbert's approach. Gordan himself acknowledged the significance of Hilbert's work, and the two mathematicians maintained a professional respect for each other.

Paul Gordan's legacy is multifaceted. He was a master of classical invariant theory, a key figure in the development of Erlangen's mathematical school, and a mentor to one of the most influential mathematicians of the 20th century. While the anecdote about Hilbert persists, it's crucial to understand it in the context of the evolving mathematical landscape of the late 19th century and to recognize Gordan's own contributions to and acceptance of the new, more abstract mathematics that was emerging.

Chapter 2

Vector Spaces

In this chapter we formally define vector spaces. After discussing Euclidean spaces in the previous chapter, the concept of the vector space here will be more intuitive. Throughout this chapter $\mathbb{F}$ denotes a field. For our purposes $\mathbb{F}$ is always one of the following: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{F}_q$.

2.1 Definition of vector spaces

In this section we generalize the concept of Euclidean spaces studied in the previous chapter to a more abstract object, that of a vector space. Let V be a given set and $''+''$ a binary operation defined as

$$\begin{aligned} ''+'': \ & V \times V \to V \\ & (\mathbf{u}, \mathbf{v}) \to \mathbf{u} + \mathbf{v} \end{aligned} \tag{2.1}$$

Let $\mathbb{F}$ be a field $''*''$ be another binary operation

$$\begin{aligned} ''*'': \ & \mathbb{F} \times V \to V \\ & (r, \mathbf{u}) \to r * \mathbf{u} \end{aligned} \tag{2.2}$$

Definition 2.1. *The set V together with the binary operations above, denoted by $(V, +, *)$, is a **vector space** over the field of scalars $\mathbb{F}$ if the following are satisfied:*

(i) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}), \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$
(ii) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}, \forall \mathbf{u}, \mathbf{v} \in V$
(iii) $\exists \mathbf{0} \in V, \ s.t. \ \mathbf{0}+\mathbf{u} = \mathbf{u}+\mathbf{0} = \mathbf{u}, \quad \forall u \in V$
(iv) $\forall \mathbf{u} \in V$, *there is* $-\mathbf{u} \in V$ *such that* $\mathbf{u}-\mathbf{u} = \mathbf{0}$
(v) $\forall r \in \mathbb{F}, \forall \mathbf{u}, \mathbf{v} \in V, \quad r*(\mathbf{u}+\mathbf{v}) = r*\mathbf{u} + r*\mathbf{v}$
(vi) $\forall r, s \in \mathbb{F}, \mathbf{u} \in V, \quad (r+s)*\mathbf{u} = r*\mathbf{u} + s*\mathbf{u}$
(vii) $\forall r, s \in \mathbb{F}, \mathbf{u} \in V, \quad (rs)*\mathbf{u} = r*(s*\mathbf{u})$
(viii) $\exists 1 \in \mathbb{F}, s.t. \forall \mathbf{u} \in V, \quad 1*\mathbf{u} = \mathbf{u}$

Property 1) and 2) say that addition is associative and commutative. By property 3) we have an **additive identity** and by property 8) a **multiplicative identity**. Property 4) assures there is an **additive inverse** normally called the **opposite**. Elements $r, s \in \mathbb{F}$ are called **scalars**. From now on we will suppress $'*'$.

Exercise 25. *Every field is a vector space over itself.*

Elements of a vector space are called **vectors**. From now on $V/\mathbb{F}$ will denote a vector space over some field $\mathbb{F}$. When there is no confusion we will simply use V. Next we give some examples of some classical vector spaces.

Exercise 26. *Prove that $\mathbb{R}$ is a vector space over $\mathbb{Q}$. If $\mathbb{F}$ and K are fields such that $\mathbb{F} \subset K$, prove that K is a vector space over $\mathbb{F}$.*

Example 2.1 (Euclidean spaces $\mathbb{R}^n$). *Show that $\mathbb{R}^n$ is a vector space over the field of scalar $\mathbb{R}$, with the usual vector addition and scalar multiplication. What is the additive and multiplicative identity?*

Example 2.2 (The space of polynomials with coefficients in $\mathbb{F}$). *Let $\mathbb{F}[x]$ denote the set of polynomials*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

*where $a_0, \ldots, a_n \in \mathbb{F}$. We define the **sum** and the **scalar product** of two polynomials to be*

$$\begin{aligned}(f+g)(x) &:= f(x) + g(x)\\ (rf)(x) &:= r\,f(x)\end{aligned} \tag{2.3}$$

*for any $r \in \mathbb{F}$. Then, $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$. $\mathbb{F}[x]$ is also called the **polynomial ring** of univariate polynomials; see Chapter 4 for more details.*

Example 2.3 (The space of $n \times n$ matrices). *The set of $n \times n$ matrices with entries in a field $\mathbb{F}$, together with matrix addition and scalar multiplication forms a vector space. We denote this space by $\mathrm{Mat}_{n\times n}(\mathbb{F})$.*

Example 2.4 (The space of functions from $\mathbb{R}$ to $\mathbb{R}$). *Let $\mathcal{L}(\mathbb{R})$ denote the set of all functions*

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

*We define the **sum** and the **scalar product** of two functions to be*

$$\begin{aligned}(f+g)(x) &:= f(x) + g(x)\\ (rf)(x) &:= r\,f(x)\end{aligned} \tag{2.4}$$

for any $r \in \mathbb{R}$. Show that $\mathcal{L}(\mathbb{R})$ is a vector space over $\mathbb{R}$.

We can generalize the above example as follows:

Example 2.5 (Function Spaces). *Let S denote a set and $\mathbb{F}$ a field. A function is called $\mathbb{F}$-**valued** if*

$$f : S \longrightarrow \mathbb{F}$$

*Let V denote the set of all $\mathbb{F}$-valued functions. We define the **sum** and the **scalar product** of two functions in V to be*

$$\begin{aligned}(f+g)(x) &:= f(x) + g(x)\\ (rf)(x) &:= r\,f(x)\end{aligned} \tag{2.5}$$

for any $r \in \mathbb{F}$. Then V is a vector space over $\mathbb{F}$.

Exercise 27. *Prove that the set of **integrable functions** over $\mathbb{R}$ is a vector space. It is denoted by $L^1(\mathbb{R})$.*

Example 2.6 (Square-integrable functions)**.** *A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is called a square-integrable function if*

$$\int_{-\infty}^{\infty} \left| f(x) \right|^2 \mathrm{dx} < \infty$$

Let $L^2(\mathbb{R}$ denote the space of square-integrable functions. Show that $L^2(\mathbb{R})$ is a vector space. Moreover, $L^2(\mathbb{R})$ is a Hilbert space and comes with an inner product which we will see in coming chapters.

Exercise 28 (Complex numbers as a vector space). *Prove that $\mathbb{C}$ together with the addition and scalar multiplication defined* **??** *form a vector space over $\mathbb{R}$.*

Exercise 29 (Binary quadratics). *A degree two homogenous polynomial in two variables*

$$f(x,y) = ax^2 + bxy + cy^2,$$

*with coefficients $a,b,c, \in \mathbb{R}$ is called a **binary quadratic** over $\mathbb{R}$. Let $V_2(\mathbb{R})$ be the space of all binary quadratics with coefficients in $\mathbb{R}$. Prove that $V_2(\mathbb{R})$ together with addition of polynomials and multiplication by a constant forms a vectors space over $\mathbb{R}$.*

Example 2.7. *A **binary form** of degree $d \geq 2$ with coefficients from a field $\mathbb{F}$ is a homogenous polynomial*

$$f(x,y) = a_d x^d + a_{d-1} x^{d-1} y + \cdots a_1 x y^{d-1} + a_0 y^d,$$

where $a_i \in \mathbb{F}$ for all $i = 0, \dots, d$. Let V_d be the set of binary forms of degree $d \geq 2$ with coefficients from a field $\mathbb{F}$. Prove that V_d together with the addition and multiplying by a scalar for polynomials form a vectors space over $\mathbb{F}$.

2.1.1 Subspaces

There are some subsets of a vector space V which are of special importance.

Definition 2.2. *A subset $W \subset V$ is called a **subspace** (or a **linear subspace**) of V if it is a vector space by itself. Next we see some examples of subspaces of a vector space.*

Example 2.8. *Let $V = \mathbb{R}^3$. Then every $\mathbf{v} \in V$ is a triple (x,y,z), which we have denoted by $\mathbf{v} = [x,y,z]^t$. Let W be the set of vectors $\mathbf{v} \in V$ such that the last coordinate is 0,*

$$W = \left\{ \mathbf{v} = [x,y,0]^t \mid \mathbf{v} \in V \right\}.$$

Then W is $\mathbb{R}^2$ which is also a vector space. Hence, W is a subspace of V. The reader can provide a formal proof of this based on the definition above.

A set S of V is called **closed under addition** if for every $\mathbf{u}, \mathbf{v} \in S$ we have $\mathbf{u} + \mathbf{v} \in S$. It is called **closed under scalar multiplication** if for every $\mathbf{u} \in S$ and $r \in \mathbb{F}$ we have $r\mathbf{u} \in S$.

Lemma 2.1. *Any subset $W \subset V$ is a vector space if and only if it is closed under addition, scalar multiplication, and contains $\mathbf{0}$.*

Proof. Exercise

Example 2.9. *Let $V = \mathbb{R}^3$ and P be the plane determined by the vectors $\mathbf{u}$ and $\mathbf{v}$ going through the origin. This plane is a vector space because: it contains the zero vector, every sum of two vectors in P is again in P, and every vector in P multiplied by a scalar is again in P.*

Exercise 30. *Let $V = \mathrm{Mat}_{n\times n}(\mathbb{F})$ matrices with entries from a field of scalar $\mathbb{F}$. Above we showed that V is a vector space over $\mathbb{F}$. Let V_1 be the set of all upper triangular matrices from V and V_2 the set of all lower triangular matrices from V. Prove that:*
(i) *V_1 is a subspace of V*
(ii) *V_2 is a subspace of V*
(iii) *$V_1 \cap V_2$ is a subspace of V*

Example 2.10 (The nullspace of a matrix:). *Let A be a given matrix. Consider the set of all vectors in $\mathbb{R}^n$ which satisfy the equation $A\mathbf{x} = \mathbf{0}$. We call this set the* ***nullspace*** *of A and denoted by $\boldsymbol{Null}(A)$. Hence,*

$$\boldsymbol{Null}(A) := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

Prove that $\boldsymbol{Null}(A)$ is a subspace of $\mathbb{R}^n$.

Definition 2.3. *Let V be a vector space over $\mathbb{F}$ and $v_1, \dots v_n \in V$. Then, v is a* ***linear combination*** *of $v_1, \dots v_n$ if it can be written as*

$$v = r_1 v_1 + \dots + r_n v_n$$

where $r_1, \dots, r_n \in \mathbb{F}$.

As in Def. 1.3 we define Span $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ as the set of all linear combinations of $v_1, \dots v_n \in V$.

Lemma 2.2. *Let V be a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$. The set $W =$ Span $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a subspace of V.*

Proof. Indeed, let $W =$ Span $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be the set of all linear combinations. Obviously $\mathbf{0} \in W$ since $\mathbf{0}$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$ by taking all scalars 0. Thus we have to show that W is closed under addition and multiplication by a scalar. Both are easily checked. □

For an arbitrary set $S \subset V$ (not necessary finite), the **linear closure** of S is defined as the set of all linear combinations of elements from S and denoted by Span (S).

Exercise 31. *Prove that Span (S) is a subspace of V and the intersection of all subspaces of V containing S.*

2.1.2 Linear independence

Let V be a vector space and $\mathbf{u}_1, \dots, \mathbf{u}_n$ vectors in V.

Definition 2.4. *Vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ are called* ***linearly independent*** *if*

$$r_1 \mathbf{u}_1 + \dots + r_n \mathbf{u}_n = 0$$

implies that $r_1 = \dots = r_n = 0$, otherwise, we say that $\mathbf{u}_1, \dots, \mathbf{u}_n$ are ***linearly dependent****.*

Hence, a set of vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ are linearly dependent if one of them is expressed as a linear combination of the other ones. First we see what linear independence means in $\mathbb{R}^n$,

Example 2.11. *Show that* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, *and* $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *are linearly independent in* $\mathbb{R}^3$.

Solution: *We want to find if there exist* r_1, r_2, r_3, *not all zero such that* $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + r_3\mathbf{u}_3 = 0$. *We have*

$$\begin{bmatrix} 2r_1 + r_2 + r_3 \\ 3r_1 + 2r_2 + r_3 \\ r_1 + r_2 + r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and its reduced row-echelon form are

$$[A \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow [H \mid] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since every row has a pivot then the system has a unique solution $(r_1, r_2, r_3) = (0, 0, 0)$. *This concludes that* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly independent.* □

The approach of the above example can be generalized for any set of vectors in $\mathbb{R}^n$.

Lemma 2.3. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s \in \mathbb{R}^n$ *are linearly independent if and only if the raw-echelon form of the matrix*

$$A = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \dots \mid \mathbf{v}_s]$$

has a pivot in every column.

Proof. Assume that there are scalars $r_1, \dots, r_n \in \mathbb{R}$ such that

$$r_1\mathbf{v}_1 + \dots r_n\mathbf{v}_n = \mathbf{0}.$$

We want to determine $r_1, \dots, r_n$ by solving the system $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = [r_1, \dots, r_n]^t$ and $A = [\mathbf{v}_1 | \dots | \mathbf{v}_n]$. From Lem. 1.14, we know that it has a unique solution if and only if every column of the row-echelon form ref(A) has a pivot. In that case, the solution is $\mathbf{0}$ and therefore $r_1 = \dots = r_n = 0$.

□

The following example should be familiar to students who have had a course in differential equations:

Example 2.12. *Let* $\mathcal{L}(\mathbb{R})$ *be the vector space of all real-valued functions in* t. *Show that the following pair of functions* $\sin t, \cos t$ *are linearly independent.*

Solution: *Let* $r_1, r_2 \in \mathbb{R}$ *such that*

$$r_1 \sin t + r_2 \cos t = 0$$

for every $t \in \mathbb{R}$. *Take* $t = 0$, *then* $r_2 = 0$. *If we take* $t = \frac{\pi}{2}$ *then* $r_1 = 0$. *Hence* $\sin t$ *and* $\cos t$ *are linearly independent.*

Exercise 32. *Interpret geometrically what it means for two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *to be linearly independent in* $\mathbb{R}^2$*. What about* $\mathbb{R}^3$*? Prove that:*

(i) *Any two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ *are linearly independent if and only if they are not parallel to each other.*

(ii) *If* $\mathbf{u}, \mathbf{v}$ *are linearly independent then* $\mathit{Span}\,(\mathbf{u}, \mathbf{v})$ *is the plane passing through the origin and determined by* $\mathbf{u}$ *and* $\mathbf{v}$*.*

(iii) *Any three vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *are linearly independent if and only if they are not coplanar.*

(iv) *If* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are linearly independent then* $\mathit{Span}\,(\mathbf{u}, \mathbf{v}, \mathbf{w})$ *is all of* $\mathbb{R}^3$*.*

Exercises:

156. *Let* U, W *be subspaces of* V*. Define the* ***sum of subspaces*** *of* U *and* W *by*

$$U + W := \{u + w \,|\, u \in U, w \in W\}.$$

Show that $U \cap W$ *and* $U + W$ *are subspaces of* V*.*

157. *Let* $\mathbf{u} \in V = \mathbb{R}^n$ *and* $W_{\mathbf{u}} := \{\mathbf{v} \in V \,|\, \mathbf{u} \cdot \mathbf{v} = 0\}$*. Show that* W_u *is a subspace of* V*.*

158. *Let* S *be a set and* V *a vector space over the field* $\mathbb{F}$*. Show that the set of functions*

$$f : S \to \mathbb{F},$$

under function addition and multiplication by a constant is a vector space over $\mathbb{F}$*.*

159. *Let* $\mathcal{L}(\mathbb{R})$ *be the vector space of all real-valued functions in* t*. Show that the following pairs are linearly independent.*

i) t, e^t

ii) $\sin t, \cos 2t$

iii) te^t, e^{2t}

iv) $t, \sin t$.

160. *An* ***upper triangular*** *matrix is a matrix* $A = [a_{i,j}]$ *such that* $a_{i,j} = 0$ *for all* $i < j$*. Show that the space of upper triangular matrices is a subspace of* $\mathrm{Mat}_{n\times n}(\mathbb{R})$*.*

161. *Prove that* $\mathbb{F}[x]$ *is a vector space over the field* $\mathbb{F}$*.*

162. *Let* $\mathbb{F}$ *be a field and* $A := \mathbb{F}[x]$ *the polynomial ring. Denote by* A_n *the set of polynomials in* A *of degree* n*. Is* A_n *a subspace of* A*? Justify your answer.*

163. *Let* $\mathbb{F}$ *be a field and* $A := \mathbb{F}[x]$ *the polynomial ring. Denote by* P_n *the set of polynomials in* A *of degree* $\leq n$*. Is* P_n *a subspace of* A*? Justify your answer.*

164. *Let* $\mathbb{Q}$ *be the set of rational numbers and*

$$\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} \,|\, a, b \in \mathbb{Q}\}.$$

Prove that $\mathbb{Q}(\sqrt{2})$ *is a vector space over* $\mathbb{Q}$ *with the usual addition and scalar multiplication.*

165. *The following are the simplest and most classical examples of vector spaces.*

(i) *Is* $\mathbb{R}$ *a vector space over* $\mathbb{Q}$ *with the usual addition and scalar multiplication? Prove your answer.*

(ii) *The set of complex numbers* $\mathbb{C}$ *is given by*

$$\mathbb{C} := \{a + bi \,|\, a, b \in \mathbb{R}\}$$

where $i = \sqrt{-1}$*. Is* $\mathbb{C}$ *a vector space over* $\mathbb{R}$ *with the usual addition and scalar multiplication?*

166. *Let* V *be the set of 2 by 2 matrices of the form* $\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$ *where* $x, y \in \mathbb{R}$*. Is* V *a vector space over* $\mathbb{R}$*?*

2.2 Bases and dimension

In this section we will study two very fundamental concepts in the theory of vector spaces that of basis and dimension. Let V be a vector space over $\mathbb{F}$ and $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$. Denote by $W =$ Span $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$ in V. We say that W is **generated** by $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Definition 2.5. *Let V be a vector space over $\mathbb{F}$ and $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$. Then $\mathfrak{B}$ is a **basis** of V if the following hold:*

(i) *$V =$ Span $(\mathbf{v}_1, \dots, \mathbf{v}_n)$*

(ii) *$\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.*

Example 2.13. *Let $V = \mathbb{R}^2$. A basis of this vector space is $\mathfrak{B} = \{\mathbf{i}, \mathbf{j}\}$, where $\mathbf{i} = [1,0]^t$ and $\mathbf{j} = [0,1]^t$. Indeed, we know from Chap. 1 that every vector $\mathbf{v} \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$ as $\mathbf{v} = r_1 i + r_2 j$, for some real numbers r_1, r_2. This is called the **standard basis of** $\mathbb{R}^2$. The standard basis for $\mathbb{R}^3$ is $\mathfrak{B} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.* □

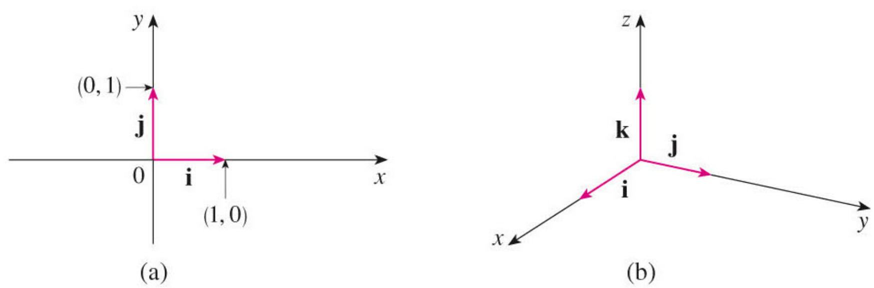


Figure 2.1: Standard basis for $\mathbb{R}^2$ and $\mathbb{R}^3$

Theorem 2.1. *Let V be a vector space over $\mathbb{F}$ and $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V. If*

$$x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n = y_1 \mathbf{v}_1 + \cdots + y_n \mathbf{v}_n, \tag{2.6}$$

then $x_i = y_i$, for $i = 1, \dots, n$.

Proof. From Eq. (2.6) we get that

$$(x_1 - y_1)\mathbf{v}_1 + \cdots + (x_n - y_n)\mathbf{v}_n = 0.$$

Since $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V then $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent. Hence $x_i = y_i$, for $i = 1, \dots, n$.

□

Hence, once a basis $\mathfrak{B}$ is fixed for a vector space V, the above theorem says that the expression of any vector $\mathbf{v} \in V$ as a linear combination of elements of $\mathfrak{B}$ is unique up to the reordering of elements of $\mathfrak{B}$.

2.2.1 Ordered bases and coordinates

The theorem motivates the following definition:

Definition 2.6. *Let V be a vector space, $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ a basis of V with a fixed ordering $\mathbf{v}_1 < \mathbf{v}_2 < \cdots < \mathbf{v}_n$, and $\mathbf{u} \in V$ given by*

$$\mathbf{u} := x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n.$$

Then $x_1, \dots, x_n$ are called the ***coordinates*** *of $\mathbf{u}$ with respect to $\mathfrak{B}$ and the above ordering.*

2.2.2 Cardinality of the basis, dimension of V

Theorem 2.2. *Let V be a vector space over the field $\mathbb{F}$ and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ bases for V such that $|\mathfrak{B}_1| = m$ and $|\mathfrak{B}_2| = n$. Then, $m = n$.*

Proof. Let the bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be

$$\mathfrak{B}_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_m\} \quad \textit{and} \quad \mathfrak{B}_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$$

and assume that $m < n$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis then there exist $x_1, \dots, x_n \in \mathbb{F}$ such that

$$\mathbf{w}_1 = x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m.$$

We know that $\mathbf{w}_1 \neq 0$ since $\mathfrak{B}_2$ is a basis, thus at least one of $x_1, \dots, x_m$ is $\neq 0$. Without loss of generality we may assume that $x_1 \neq 0$. Then we have

$$x_1\mathbf{v}_1 = \mathbf{w}_1 - x_2\mathbf{v}_2 - \cdots - x_m\mathbf{v}_m$$

Hence,

$$\mathbf{v}_1 = \frac{1}{x_1}\mathbf{w}_1 - \frac{x_2}{x_1}\mathbf{v}_2 - \cdots - \frac{x_m}{x_1}\mathbf{v}_m.$$

The subspace $W =$ Span $(\mathbf{w}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ contains $\mathbf{v}_1$. Hence, $W = V$. We continue this procedure until we replace all $\mathbf{v}_2, \dots, \mathbf{v}_m$ by $\mathbf{w}_2, \dots \mathbf{w}_m$. Thus, we have that the set. $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ generates V. Then for each $i > m$ we have $\mathbf{w}_i$ as a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_m$. This is a contradiction because $\mathbf{w}_1, \dots, \mathbf{w}_n$ are linearly independent since $\mathfrak{B}_2$ is a basis. Hence, $m \geq n$. Interchanging the roles of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ we get $m = n$.

$\square$

Hence, we have the following definition.

Definition 2.7. *Let V be a vector space and $\mathfrak{B}$ a basis of V. The cardinality $|\mathfrak{B}|$ is called is called the* ***dimension of*** *V and denoted by $\dim(V) := |\mathfrak{B}|$.*

Vector spaces with a finite dimension are called **finite dimensional**. In this book we will primarily study finite dimensional vector spaces.

Let V be a vector space. For analogy with the case $V = \mathbb{R}^3$, a subspace W of V of dimension $\dim(W) = 1$ is called a **line** and a subspace of dimension 2 is called a **plane**.

Theorem 2.3. *If $\dim(V) = n$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of linearly independent elements in V, then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V.*

Proof. Exercise. ◻

Corollary 2.1. *Let V be a vector space and W a subspace of V. If $\dim(W) = \dim(V)$ then $W = V$.*

Proof. Take a basis $\mathfrak{B} = \{\mathbf{w}_1, \dots \mathbf{w}_n\}$ of W. Hence, $\mathbf{w}_1, \dots, \mathbf{w}_n$ are linearly independent. Then from the above theorem they generate V. ◻

Corollary 2.2. *Let V be a vector space and W a subspace of V. Then $\dim(W) \leq \dim(V)$.*

Proof. Exercise. ◻

Example 2.14. *Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ be vectors in $V = \mathbb{R}^2$. What is the space $W =$ Span $(\mathbf{u}, \mathbf{v})$?*

Solution: *From the previous examples we know that $\dim(V) = 2$. Then from the previous corollary.* $\dim(W) \leq 2$. *Since $\mathbf{u}$ and $\mathbf{v}$ are not multiples of each other then they are independent. Hence,* $\dim(W) = 2$. *From Cor 2.1 we have that $W = \mathbb{R}^2$.* ◻

2.2.3 Finding a basis of a subspace in $\mathbb{F}^n$

Let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be vectors in $\mathbb{R}^n$ and $W =$ Span $(\mathbf{w}_1, \dots, \mathbf{w}_m)$. By Lem. 2.2, W is a subspace of $\mathbb{R}^n$. We want to find a basis for W. So first we need to check if $\mathbf{w}_1, \dots, \mathbf{w}_m$ are independent. Hence, we would like to find scalars $r_1, \dots, r_m \in \mathbb{R}$ such that

$$r_1 \mathbf{w}_1 + \cdots + r_m \mathbf{w}_m = 0.$$

Let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be as below

$$\mathbf{w}_1 = \begin{bmatrix} w_{1,1} \\ \vdots \\ w_{1,n} \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} w_{2,1} \\ \vdots \\ w_{2,n} \end{bmatrix}, \dots\dots \mathbf{w}_m = \begin{bmatrix} w_{m,1} \\ \vdots \\ w_{m,n} \end{bmatrix} \tag{2.7}$$

Then

$$r_1 \mathbf{w}_1 + \cdots + r_m \mathbf{w}_m = 0$$

implies

$$\begin{cases} w_{1,1} r_1 + w_{2,1} r_2 + \cdots + w_{m,1} r_m = 0 \\ w_{1,2} r_1 + w_{2,2} r_2 + \cdots + w_{m,2} r_m = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ w_{1,n} r_1 + w_{2,n} r_2 + \cdots + w_{m,n} r_m = 0 \end{cases}$$

Hence we have the system

$$\begin{bmatrix} w_{1,1} & w_{2,1} \dots & w_{m,1} \\ w_{1,2} & w_{2,2} \dots & w_{m,2} \\ \dots & \dots\dots & \dots \\ w_{1,n} & w_{2,n} \dots & w_{m,n} \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \\ r_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}.$$

which can be written as

$$[\mathbf{w}_1 \,|\, \mathbf{w}_2 \,|\, \cdots \,|\, \mathbf{w}_m] \cdot [r_1, r_2, \ldots, r_m]^t = \mathbf{0}.$$

To solve this system we find the row-echelon form of the matrix

$$A = [\mathbf{w}_1 \mid \mathbf{w}_2 \mid \ldots \mid \mathbf{w}_m].$$

If the row-echelon form has a pivot in every column then $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are linearly independent, otherwise they are linearly dependent. The vectors which form a basis in this case are the ones corresponding to columns with pivots. So we have the following algorithm:

Algorithm 3. ***Finding a basis of Span*** $(\mathbf{w}_1, \ldots, \mathbf{w}_m)$ ***in*** $\mathbb{F}^n$

Input: A subspace $W =$ *Span* $\mathbf{w}_1, \ldots, \mathbf{w}_m$ *in* $\mathbb{F}^n$.
Output: A basis of W
(i) *Form the matrix* $A = [\mathbf{w}_1 \mid \mathbf{w}_2 \mid \ldots \mid \mathbf{w}_m]$
(ii) *Find the row-echelon form of* A
(iii) *The columns with pivots come from* $\mathbf{w}_i$*'s which form a basis for* W.

Hence, we have the following:

Theorem 2.4. *Let* $W =$ *Span* $(\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_s)$ *be a subspace in* $\mathbb{R}^n$. *Let* $A = [\mathbf{v}_1 | \ldots | \mathbf{v}_s]$. *Then a basis for* ***Col*** (A) *is also a basis for* W.

Proof. Exercise ◻

Example 2.15. *Let* $W =$ *Span* $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) \subset \mathbb{R}^4$ *such that*

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{w}_4 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 5 \end{bmatrix} \tag{2.8}$$

Find a basis for W.
Solution: *We form the matrix* $A = [\mathbf{w}_1, \ldots, \mathbf{w}_n]$ *and then find* ***ref*** (A) *which gives*

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 3 & 4 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 5 & 6 & 5 \end{bmatrix} \qquad \textbf{\textit{ref}}\,(A) = \begin{bmatrix} 5 & 0 & 0 & -2 \\ 0 & 5 & 0 & -3 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the basis of W *is* $\mathfrak{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. ◻

Theorem 2.5. $\dim(\mathbb{R}^n) = n$

Proof. Take the set

$$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

of elementary vectors. Obviously this set generates $\mathbb{R}^n$ since every vector in $\mathbb{R}^n$ can be written as a linear combination of elements in $\mathfrak{B}$.

Create the matrix $A = [\mathbf{w}_1, \dots, \mathbf{w}_n]$. Then $A = I$ so it is already in reduced row-echelon form. Since every column has a pivot, then elements of $\mathfrak{B}$ are linearly independent. $\square$

The basis $\mathfrak{B}$ is called the **standard basis** of $\mathbb{R}^n$.

Example 2.16. *Let P_4 be the vector space of polynomials with real coefficients and degree ≤ 4. Determine whether $\{f_1, f_2, f_3, f_4, f_5\}$ given as below*

$$f_1 = 2x^4 - x^3 + 2x^2 - 1, \; f_2 = x^4 - x, \; f_3 = x^4 + x^3 + x^2 + x + 1, \; f_4 = x^2 - 1, \; f_5 = x - 1$$

form a basis for P_4.

Solution: *We take the basis $\mathfrak{B} = \{x^4, x^3, x^2, x, 1\}$ for P_4. The reader should verify that this is a basis for P_4. Then the coordinates of f_1, f_2, f_3, f_4, f_5 with respect to the basis $\mathfrak{B}$ are*

$$f_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad f_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad f_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

We can determine whether the polynomials are independent by determining whether the corresponding coordinate vectors in $\mathbb{R}^5$ are independent. The corresponding matrix is

$$A = [f_1 \,|\, f_2 \,|\, f_3 \,|\, f_4 \,|\, f_5] = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 \end{bmatrix}$$

and its reduced row-echelon form is the identity matrix I_5. Since every column has a pivot then the vectors are independent in $\mathbb{R}^5$ and therefore $f_1, \dots f_5$ are independent in P_4. The dimension of P_4 is $\dim P_4 = 5$. *Hence, the set $\{f_1, f_2, f_3, f_4, f_5\}$ forms a basis for P_4.* $\square$

Exercise 33. *We have seen that for any field $\mathbb{F}$, the set of polynomials $\mathbb{F}[x]$ with coefficients from $\mathbb{F}$ forms a vector space over $\mathbb{F}$. Find a basis for $\mathbb{F}[x]$.*

2.2.4 A basis for $\mathrm{Mat}_{n\times n}(\mathbb{R})$

So far our examples of bases are from the spaces $\mathbb{R}^n$. However, the above results hold for any vector space. So what is a basis and the dimension of $\mathrm{Mat}_{n\times n}(\mathbb{R})$?

Example 2.17. *Let $V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$. Find a basis for V and its dimension.*

Solution: *First we notice that every matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ can be written as*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, the set $\mathfrak{B} = \{M_1, M_2, M_3, M_4\}$ *where*

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

generates all of V. *Are* M_1, M_2, M_3, M_4 *linearly independent? If*

$$r_1 M_1 + r_2 M_2 + r_3 M_3 + r_4 M_4 = 0,$$

then

$$\begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which gives $r_1 = r_2 = r_3 = r_4 = 0$. *Hence,* $\mathfrak{B}$ *is a basis of* V *and* $\dim(V) = 4$. □

Remark 2.1. *In general, one can find a basis of* $\mathrm{Mat}_{n\times n}(\mathbb{F})$ *as above and show that the dimension is* n^2.

2.2.5 Quaternions

A quaternion is an expression of the form

$$a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}\,,$$

where $a, b, c, d \in \mathbb{R}$ are real numbers, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are symbols that can be interpreted as unit-vectors pointing along the three spatial axes. In practice, if one of $a, b, c, d$is 0, the corresponding term is omitted; if a, b, c, d are all zero, the quaternion is the zero quaternion, denoted 0; if one of b, c, d equals 1, the corresponding term is written simply $\mathbf{i}, \mathbf{j}$, or $\mathbf{k}$.

Hamilton describes a quaternion $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, as consisting of a scalar part and a **vector** part. The quaternion

$$b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

is called the vector part (sometimes imaginary part) of q, and a is the scalar part (sometimes real part) of q. A quaternion that equals its real part (that is, its vector part is zero) is called a scalar or real quaternion, and is identified with the corresponding real number. That is, the real numbers are embedded in the quaternions. (More properly, the field of real numbers is isomorphic to a subset of the quaternions. The field of complex numbers is also isomorphic to three subsets of quaternions.) A quaternion that equals its vector part is called a vector quaternion.

The set of quaternions is made a 4 dimensional vector space over the real numbers, with basis $\mathfrak{B} = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and by the component wise addition

$$(a_1 + b_1\,\mathbf{i} + c_1\,\mathbf{j} + d_1\,\mathbf{k}) + (a_2 + b_2\,\mathbf{i} + c_2\,\mathbf{j} + d_2\,\mathbf{k}) = (a_1 + a_2) + (b_1 + b_2)\,\mathbf{i} + (c_1 + c_2)\,\mathbf{j} + (d_1 + d_2)\,\mathbf{k}\,,$$

and the component wise scalar multiplication

$$\lambda(a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}) = \lambda a + (\lambda b)\,\mathbf{i} + (\lambda c)\,\mathbf{j} + (\lambda d)\,\mathbf{k}.$$

A multiplicative group structure, called the Hamilton product, denoted by juxtaposition, can be defined on the quaternions in the following way:

The real quaternion 1 is the identity element. The real quaternions commute with all other quaternions, that is $aq = qa$ for every quaternion q and every real quaternion a. In algebraic terminology this is to say that the field of real quaternions are the center of this quaternion algebra. The product is first given for the basis elements (see next subsection), and then extended to all quaternions by using the distributive property and the center property of the real quaternions. The Hamilton product is not commutative, but is associative, thus the quaternions form an associative algebra over the reals. Additionally, every nonzero quaternion has an inverse with respect to the Hamilton product:

$$(a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k})^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}\,(a - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k}).$$

Exercises:

167. *Let V be a vector space over $\mathbb{F}$. If a set of vectors is linearly independent in V, prove that the set does not contain the zero vector.*

168. *Consider the set of all vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, such that $\sum_{i=1}^n x_i = 0$. Show that this set is a vector space and find a basis for it.*

169. *Let $W =$ Span $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \subset \mathbb{R}^4$ such that*

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 6 \end{bmatrix}$$

Find a basis for W.

170. *Let $W =$ Span $(\mathbf{w}_1, \mathbf{w}_2) \subset \mathbb{R}^6$ such that*

$$\begin{aligned} \mathbf{w}_1 &= [1,2,3,1,9,5]^t, \\ \mathbf{w}_2 &= [2,4,6,2,18,10]^t \\ \mathbf{w}_3 &= [4,8,12,4,36,20]^t \end{aligned}$$

Find a basis for W.

171. *Prove that any set $\mathfrak{B} \subset \mathbb{R}^n$ of n non-zero vectors which are mutually perpendicular form a basis for $\mathbb{R}^n$.*

172. *Let $V = \mathrm{Mat}_{3\times 3}(\mathbb{R})$. Find a basis for V and its dimension.*

173. *Let $V = \mathbb{F}[x]$. Show that $f_1 = x^6 + x^4$ and $f_2 = x^6 + 3x^4 - x$ are linearly independent.*

174. *Let $\mathbb{F}$ be a field and $V := \mathbb{F}[x]$ the vector space of polynomials in x. Denote by P_n the space of polynomials in V of degree $\leq n$. Find a basis for P_n.*

175. *Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Let W be the subspace of V such that*

$$W := \text{Span}\,(\sin^2 x, \cos^2 x).$$

Show that W contains all constant functions.

176. *Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Show that the set*

$$\{1, \sin x, \sin 2x, \ldots, \sin nx\}$$

is an independent set of vectors in V.

177. *Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Find a basis of the subspace $W =$ Span $(3 - \sin x, 2\sin 2x - \sin 3x, 3\sin 2x - \sin 4x, \sin 5x - \sin 2x\}$.*

178. *Let V be the set of all matrices $A \in \mathrm{Mat}_n(\mathbb{R})$ such that $\mathrm{tr}(A) = 0$. Prove that V is a vector space and find its dimension.*

2.3 Linear maps between vector spaces

In this section we will study maps between vector spaces. We are interested in maps which will preserve the operations on the vector space. Let V and W be vector spaces over the same field $\mathbb{F}$.

Definition 2.8. *A map $T : U \to V$ is called a **linear map** over $\mathbb{F}$ if the following hold:*
(i) $\forall \mathbf{u}_1, \mathbf{u}_2 \in U$, $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$,
(ii) $\forall r \in \mathbb{F}$, $\forall \mathbf{u} \in U$, $T(r \cdot \mathbf{u}) = r \cdot T(\mathbf{u})$

We see some examples:

Example 2.18. *Let $V = \mathbb{R}^n$ and $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ be an $n \times n$ matrix. We define the following map $T_A : V \longrightarrow V$ such that*

$$T(\mathbf{x}) = A \cdot \mathbf{x}$$

From the properties of matrices it is easily checked that this is a linear map. □

Exercise 34. *Let U and V be vector spaces over $\mathbb{F}$. Denote the set of all linear maps $L : U \to V$ by*

$$\mathcal{L}(U, V) := \{L : U \to V \mid L \text{ is linear }\}$$

We define an addition in $\mathcal{L}(U, V)$ as the usual addition of functions and the scalar multiplication will be the multiplication by a constant from $\mathbb{F}$. In other words,

$$(L + T)(\mathbf{u}) = L(\mathbf{u}) + T(\mathbf{u}) \quad \text{and} \quad r * L(\mathbf{u}) = r \cdot L(\mathbf{u})$$

Prove that $\mathcal{L}(U, V)$ is an $\mathbb{F}$-vector space.

From the above discussion, $T : U \to V$ is a linear map between the vector spaces U and V is equivalent as saying that $T \in \mathcal{L}(U, V)$. From now on we will use both notations.

Lemma 2.4. *Let $T : U \to V$ be a linear map between the $\mathbb{F}$-vector spaces U and V. Then the following hold:*
(i) $T(\mathbf{0}_U) = \mathbf{0}_V$.
(ii) $\forall \mathbf{u} \in U$, $T(-\mathbf{u}) = -T(\mathbf{u})$.

Proof. The proof is straight forward since

$$T(\mathbf{0}_U) = T(\mathbf{u} - \mathbf{u}) = T(\mathbf{u}) + T(-\mathbf{u}) = T(\mathbf{u}) - T(\mathbf{u}) = \mathbf{0}_W$$

Part ii) is obvious. □

Definition 2.9. *Let $T : U \to V$ be a linear map between the vector spaces U and V. The **kernel** of T, denoted by $ker(T)$, is defined to be the following subset of U;*

$$ker(T) := \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0}_W\}$$

*The **image** of T is defined to be following subset of V;*

$$\mathrm{Img}(T) := \{\mathbf{v} \in V \mid \exists \mathbf{u} \in U, T(\mathbf{u}) = \mathbf{v}\}$$

A graphical interpretation of the kernel and the image of T is given in Fig. 2.2.

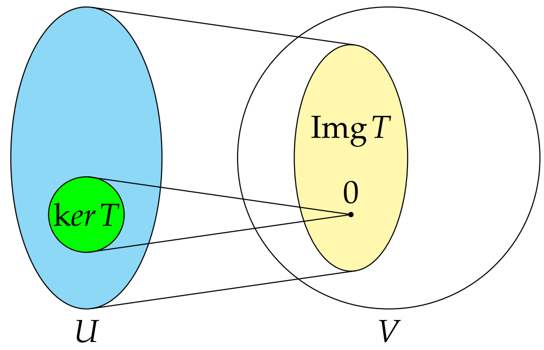


Figure 2.2: Kernel and image of a linear map

Exercise 35. *Let $T : U \to V$ be a linear map. Then,*
(i) *$ker\,(T)$ is a subspace of U*
(ii) $\mathrm{Img}(T)$ *is a subspace of V.*

The following lemma is helpful in checking whether a liner map is injective or not.

Lemma 2.5. *Let $T \in \mathcal{L}(U,V)$. Then $ker\,(T) = \{\mathbf{0}_V\}$ if and only if T is injective.*

Proof. Assume that $\mathrm{k}er\,(T) = \{\mathbf{0}_V\}$. Then, for every $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ we have

$$T(\mathbf{v}_1) - T(\mathbf{v}_2) = 0 \Longrightarrow T(\mathbf{v}_1 - \mathbf{v}_2) = 0 \Longrightarrow (\mathbf{v}_1 - \mathbf{v}_2) \in \mathrm{k}er\,(T)$$

which means that $\mathbf{v}_1 - \mathbf{v}_2 = 0$ or $\mathbf{v}_1 = \mathbf{v}_2$.

Assume that T is injective and let $\mathbf{v} \in \mathrm{k}er\,(T)$. Then $T(\mathbf{v}) = T(\mathbf{0}_V) = \mathbf{0}_W$ implies that $\mathbf{v} = \mathbf{0}_V$. □

Exercise 36. *Let $A = [a_{i,j}] \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ and $\mathrm{tr}(A)$ denote its trace. Show that the following map*

$$\begin{aligned} \mathrm{tr} : \mathrm{Mat}_{n\times n}(\mathbb{F}) &\longrightarrow \mathbb{F} \\ A &\longrightarrow \mathrm{tr}(A) \end{aligned}$$

is a linear map.

Example 2.19. *Let $C(\mathbb{R})$ denote the vector space all differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Consider the map*

$$\begin{aligned} D :\ C(\mathbb{R}) &\to C(\mathbb{R}) \\ f(x) &\to D(f(x)) = f'(x) \end{aligned}$$

where $f'(x)$ is the derivative of $f(x)$. Show that D is a linear map. □

Exercise 37. *Let $L :\ \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $L(\mathbf{x}) = A\mathbf{x}$, for some matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. Prove that $ker\,(L) = \boldsymbol{Null}(A)$.*

Theorem 2.6. *Let $T : V \to W$ be an injective linear map. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent elements in V, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent elements in W.*

Proof. Let
$$y_1T(\mathbf{v}_1)+\cdots+y_nT(\mathbf{v}_n)=\mathbf{0}_W$$
for $y_1,\dots,y_n$ scalars. Then
$$T(y_1\mathbf{v}_1)+\cdots+T(y_n\mathbf{v}_n)=\mathbf{0}_W$$
which implies that
$$T(y_1\mathbf{v}_1+\cdots+y_n\mathbf{v}_n)=\mathbf{0}_W$$
Since T is injective then $\ker(T)=\{\mathbf{0}\}$ and
$$y_1\mathbf{v}_1+\cdots+y_n\mathbf{v}_n=\mathbf{0}_V$$
This implies that $y_1=\ldots=y_n=0$, since $\mathbf{v}_1,\dots,\mathbf{v}_n$ are linearly independent. Thus $T(\mathbf{v}_1),\dots,T(\mathbf{v}_n)$ are linearly independent elements in W. □

Theorem 2.7. *If V is a finite dimensional vector space over $\mathbb{F}$ and $T\in\mathcal{L}(V,W)$, then*
$$\dim V=\dim ker(T)+\dim \operatorname{Img}(T)$$

Proof. Let $\mathfrak{B}=\{\mathbf{u}_1,\dots,u_m\}$ be a basis for $\ker(T)$. Then we can extend this to a basis for V, say $\mathfrak{B}'=\{\mathbf{u}_1,\dots,u_m,\mathbf{v}_1,\dots,\mathbf{v}_n\}$. Hence, $\dim V=m+n$. Now it is enough to prove that $\dim\operatorname{Img}(T)=n$.

Let $\mathbf{u}\in U$. Then
$$\mathbf{u}=a_1\mathbf{u}_1+\cdots+a_m\mathbf{u}_m+b_1\mathbf{v}_1+\cdots+b_n\mathbf{v}_n$$
which implies that
$$T(\mathbf{u})=b_1T(\mathbf{v}_1)+\cdots+b_nT(\mathbf{v}_n),$$
since $\mathbf{u}_1,\dots\mathbf{u}_m\in\ker(T)$. Hence, $T(\mathbf{v}_1),\dots,T(\mathbf{v}_n)$ span $\operatorname{Img}(T)$. From the above theorem $T(\mathbf{v}_1),\dots,T(\mathbf{v}_n)$ are linearly independent. Therefore, $\{T(\mathbf{v}_1),\dots,T(\mathbf{v}_n)\}$ is a basis for $\operatorname{Img}(T)$ and $\dim\operatorname{Img}(T)=n$. □

Theorem 2.8. *Let $T:V\to W$ be a linear map and $\dim V=\dim W$. If $ker(T)=\{0\}$ or $\operatorname{Img}(T)=W$, then T is bijective.*

Proof. If $\ker(T)=\{\mathbf{0}\}$ then T is injective and $\dim\operatorname{Img}(T)\geq\dim V=\dim W$. Thus $\operatorname{Img}T=W$ and T is surjective. If $\operatorname{Img}(T)=W$ then T is surjective and $\dim\ker(T)=0$. Thus $\ker(T)=\{\mathbf{0}_V\}$ and T is also injective. □

Exercise 38. *If V and W are finite dimensional vector spaces such that $\dim V>\dim W$, then no linear map from V to W is injective.*

Exercise 39. *If V and W are finite dimensional vector spaces such that $\dim V<\dim W$, then no linear map from V to W is surjective.*

Example 2.20. *Let $A=\begin{bmatrix}-1&2&3\\4&5&6\\7&8&9\end{bmatrix}$ and L_A the linear map $L_A:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ such that $L_A(\mathbf{x})=A\cdot\mathbf{x}$. Determine whether the map L_A is bijective.*

Solution: *We determine first* $\ker(L_A)$. *More precisely we want to find all* $\mathbf{x} \in \mathbb{R}^3$ *such that* $L_A(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$. *Hence* $\ker(L_A) = \mathbf{Null}(A)$. *To find the nullspace we proceed as before The reduced row-echelon form is* $\mathbf{ref}\,(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Hence* $\mathbf{rank}\,(A) = 3$, $\mathbf{null}\,(A) = 0$ *and nullspace of* A *is* $\mathbf{Null}(A) = \{\mathbf{0}\}$. *Thus,* $\ker(L_A) = \{\mathbf{0}\}$ *and* L_A *is injective. From the previous theorem we conclude that* L_A *is bijective.* □

2.3.1 Composition of linear maps, inverse maps, isomorphisms

It is natural to ask whether the composition of two linear maps is linear or if the inverse of a linear map is linear.

Theorem 2.9. *Let* U, V, W *be vector spaces over some field* $\mathbb{F}$ *and*

$$U \xrightarrow{f} V \xrightarrow{g} W$$

be linear maps. Then the composition map

$$g \circ f : U \longrightarrow W$$

is also linear.

Proof. Let $\mathbf{u}_1, \mathbf{u}_2 \in U$. Then

$$\begin{aligned} (g \circ f)(\mathbf{u}_1 + \mathbf{u}_2) &= g\,(f(\mathbf{u}_1 + \mathbf{u}_2)) \\ &= g\,(f(\mathbf{u}_1) + f(\mathbf{u}_2)) \\ &= (g \circ f)(\mathbf{u}_1) + (g \circ f)(\mathbf{u}_2) \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(r \cdot \mathbf{u}) &= g\,(f(r \cdot \mathbf{u}) \\ &= g(r \cdot f(\mathbf{u})) = r \cdot (g \circ f)(\mathbf{u}) \end{aligned}$$

Hence, $g \circ f$ is linear. □

Example 2.21. *Let* A *and* B *be matrices of dimension* $m \times n$ *and* $n \times s$ *respectively and* L_A, L_B *be the linear maps*

$$\mathbb{R}^m \xrightarrow{L_A} \mathbb{R}^n \xrightarrow{L_B} \mathbb{R}^s$$

such that $L_A(\mathbf{x}) = A\mathbf{x}$ *and* $L_B(\mathbf{x}) = B\mathbf{x}$. *The composition map* $L_B \circ L_A : \mathbb{R}^m \longrightarrow \mathbb{R}^s$ *is given by*

$$(L_B \circ L_A)(\mathbf{x}) = (BA)\,\mathbf{x}$$

and it is easily verified to be linear. □

Exercise 40. *Let* U, V *be vector spaces over a field* $\mathbb{F}$ *and* $f : U \longrightarrow V$ *be a linear map which has an inverse* $f^{-1} : V \longrightarrow U$. *Then,* f^{-1} *is linear.*

Definition 2.10. *Let U, V be vector spaces over the field $\mathbb{F}$ and*

$$L : U \longrightarrow V$$

*a linear map which has an inverse. Then, L is called an **isomorphism** and U and V are called **isomorphic spaces**.*

Let us see an example of an isomorphism.

Example 2.22. *Let $U = \mathbb{R}^2$ which is a vector space over $\mathbb{R}$ and take its standard basis $\mathfrak{B} = \{\mathbf{i}, \mathbf{j}\}$. Let $V = P_1$ the space of degree one polynomials with real coefficients. We have seen that P_1 is a vector space of dimension two over $\mathbb{R}$ and a basis of P_1 can be taken as $\mathfrak{B}' = \{1, x\}$. Define a map $L : U \to V$ such that*

$$L(a\mathbf{i} + b\mathbf{j}) = a + bx$$

Check that this is an isomorphism.

Exercise 41. *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be any invertible linear transformation. Show that the image of the unit circle under L is an ellipse E. Hence, isomorphisms do not necessarily preserve geometric shapes. In the coming chapters we will discuss in detail those linear maps which do preserve geometry.*

We assume that $\mathbb{F}$ is a field of characteristic 0 and V, W are vector spaces over $\mathbb{F}$.

Theorem 2.10. *Any two finite dimensional vector spaces V and W over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.*

Proof. If V and W are isomorphic, then there exists a bijective linear map $\phi : V \to W$. Since ϕ is an isomorphism, it preserves linear independence and spanning sets. That is:
- If $\{v_1, \ldots, v_n\}$ is a basis of V, then $\{\phi(v_1), \ldots, \phi(v_n)\}$ is a linearly independent set in W.
- Since ϕ is surjective, $\{\phi(v_1), \ldots, \phi(v_n)\}$ spans W.

Thus, $\{\phi(v_1), \ldots, \phi(v_n)\}$ is a basis for W, meaning W has dimension n, so $\dim W = \dim V$.

Assume that V and W are vector spaces over $\mathbb{F}$ with the same finite dimension, say $\dim V = \dim W = n$. By the definition of dimension, there exist bases

$$B_V = \{v_1, v_2, \ldots, v_n\} \text{ for } V, \quad B_W = \{w_1, w_2, \ldots, w_n\} \text{ for } W.$$

We define a map $\phi : V \to W$ by specifying how it acts on the basis elements:

$$\phi(v_i) = w_i, \quad \text{for } i = 1, 2, \ldots, n.$$

Since any vector in V can be uniquely expressed as a linear combination of basis elements, say

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,$$

we define ϕ on general vectors by

$$\phi(v) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n.$$

This map is linear because for any $v, v' \in V$ and scalars $\alpha, \beta \in \mathbb{F}$, we have

$$\phi(\alpha v + \beta v') = \alpha\phi(v) + \beta\phi(v').$$

Moreover, ϕ is bijective because

- **Injectivity**: Suppose $\phi(v) = 0$, meaning

$$a_1 w_1 + a_2 w_2 + \cdots + a_n w_n = 0.$$

Since $\{w_1, \dots, w_n\}$ is a basis, the only solution is $a_1 = a_2 = \cdots = a_n = 0$, which implies $v = 0$. Hence, $ker(\phi) = \{0\}$, so ϕ is injective.

- **Surjectivity**: Any element $w \in W$ can be written as

$$w = b_1 w_1 + b_2 w_2 + \cdots + b_n w_n$$

for some scalars $b_1, \dots, b_n$. But then

$$w = \phi(b_1 v_1 + b_2 v_2 + \cdots + b_n v_n),$$

which shows that w is in the image of ϕ, meaning ϕ is surjective.

Since ϕ is a bijective linear map, it is an **isomorphism**, so $V \cong W$. This completes the proof. □

Corollary 2.3. *Any n-dimensional vector space over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.*

Exercises:

179. *Let $T : \mathbb{R} \to \mathbb{R}$ such that $T(x) = \sin x$. Is T an isomorphism? Explain.*

180. *Let $\mathcal{L}([0,1], \mathbb{R})$ denote the set of integrable functions on the interval $[0,1]$. Check whether the map*

$$\phi : \mathcal{L}([0,1], \mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$$

such that

$$f(x) = \int_0^1 f(x)\,\mathrm{dx}$$

is a linear map.

181. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map given by $T(\mathbf{x}) = A\mathbf{x}$ for some $n \times n$ invertible matrix A. Show that T is a bijection.*

182. *Let P_4 be the vector space of degree ≤ 4 polynomials with real coefficients. Show that P_4 is isomorphic to $\mathbb{R}^5$. Generalize this result. In other words, prove that P_n is isomorphic to $\mathbb{R}^{n+1}$.*

183. *Can you find two vector spaces over the same field of finite dimension which are not isomorphic? Explain.*

184. *We know that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Define the map $T : \mathbb{C} \to \mathbb{C}$, such that $T(z) = \bar{z}$, where $\bar{z}$ is the complex conjugate of z; see* **??**. *Is T a linear map?*

185. *Let $T : \mathbb{C} \to \mathbb{C}$, such that $T(z) = z + z_0$, where z_0 is a fixed complex number. Is T an isomorphism?*

186. *Let $T : \mathbb{C} \to \mathbb{C}$, such that*

$$T(z) = \begin{cases} \dfrac{1}{z} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

Is T an isomorphism? Explain.

187. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $L(\mathbf{x}) = A\mathbf{x}$. Then L is an isomorphisms if and only if A is invertible.*

2.4 Direct sums and direct products

Let is start by recalling some basic facts about subspaces.

Exercise 42. *Let U and W be subspaces of a vector space V.*
 (i) *Show that* $U \cap W \subset U \cup W \subset U + W$.
 (ii) *When is $U \cup W$ a subspace of V?*
 (iii) *What is the smallest subspace of V containing $U \cup W$?*

Let V be a finite dimensional vector space and U, W its subspaces. We define the **sum** $U+W$ of subspaces U and W as follows

$$U + W := \{u + w \mid u \in U, w \in W\}$$

This set $U+W$ is a subspace of V; see Problem 156 at the end of this section.

Lemma 2.6. *Let V be a finite dimensional vector space and U and W subspaces of V. Then $U+W$ is a subspace of V of dimension*

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Proof. Let us show first that $U+W$ is a subspace. Clearly $\mathbf{0} \in U+W$ since $\mathbf{0} \in U$ and $\mathbf{0} \in W$. Let $\mathbf{v}_1, \mathbf{v}_2 \in U+W$. Then exists $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$ such that

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{w}_1 \qquad \text{and} \qquad \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{w}_2.$$

Hence,

$$\mathbf{v}_1 + \mathbf{v}_2 = (\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_1 + \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{w}_1 + \mathbf{w}_2) \in U + W.$$

Hence $U+W$ is closed under addition. Similarly we show that $U+W$ is closed under multiplying with a scalar. Thus, $U+W$ is a subspace.

Let $\dim U = m$ and $\dim W = n$. Since $U \cap W$ is a subspace of V, then $U \cap W$ is finite dimensional, say $\dim(U \cap W) = s$.

□

Example 2.23. *Consider $V = \mathbb{R}^3$, $U =$ Span $(\mathbf{u})$, and $W =$ Span $(\mathbf{v})$. Hence, U (resp. W) contains all vectors parallel to $\mathbf{u}$ (resp. parallel to $\mathbf{v}$). Therefore a vector $\mathbf{x} \in U+W$ will be written as*

$$\mathbf{x} = \lambda_1 \mathbf{u} + \lambda_2 \mathbf{v}, \quad \textit{for} \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Therefore, the sum of $U+W$ is the **uv***-plane.*

2.4.1 Direct sums

We say that V is a **direct sum** of U and W, denoted by $V = U \oplus W$, if every element $\mathbf{v}$ in V is expressed **uniquely** as a sum $\mathbf{v} = \mathbf{u} + \mathbf{w}$, for some $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

Theorem 2.11. *Let U, W be subspaces of the vector space V. If $V = U+W$ and $U \cap W = \{0\}$, then*

$$V = U \oplus W.$$

Proof. Let $\mathbf{v}$ in V and $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u} \in U$ and $\mathbf{w} \in W$. To prove that V is a direct sum we must show that $\mathbf{u}$ and $\mathbf{w}$ are uniquely determined. Assume that exist $\mathbf{u}'$ and $\mathbf{w}'$ such that $\mathbf{v} = \mathbf{u}' + \mathbf{w}'$. Then,

$$\mathbf{v} - \mathbf{v} = (\mathbf{u} - \mathbf{u}') + (\mathbf{w} - \mathbf{w}') = \mathbf{0}$$

Hence, $\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w}$. Since $\mathbf{u} - \mathbf{u}' \in U$ and $\mathbf{w} - \mathbf{w}' \in W$, then

$$(\mathbf{u} - \mathbf{u}') = (\mathbf{w}' - \mathbf{w}) \in U \cap W = \{\mathbf{0}\}$$

Therefore, $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$. This completes the proof. □

Theorem 2.12. *Let V be a finite dimensional vector space over $\mathbb{F}$ and W a subspace of V. Then, there is a subspace $U \subset V$ such that*

$$V = U \oplus W$$

Proof. Let $\dim V = n$ and $\dim W = r$ where $r < n$. Let $\mathfrak{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V. Then we can pick r elements of $\mathfrak{B}$ which form a basis for W, say $\mathbf{b}_1, \dots, \mathbf{b}_r$. Let $U := \{\mathbf{b}_{r+1}, \dots, \mathbf{b}_n\}$ Obviously $V = U + W$. Also $U \cap W = \{\mathbf{0}\}$, otherwise $\mathbf{b}_1, \dots, \mathbf{b}_n$ would not be linearly independent.

□

The subspace U is called the **complement** of W in V.

Example 2.24. *Let $V = \mathbb{R}^3$ and $\mathfrak{B} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ its standard basis. Let $U :=$ Span $(\mathbf{i}, \mathbf{j})$. Then, from the above theorem,*

$$V := U \oplus W$$

where $W =$ Span $(\mathbf{k})$. Thus $\mathbb{R}^3 =$ Span $(\mathbf{i}, \mathbf{j}) \oplus$ Span $(\mathbf{k})$. □

The next result is an immediate consequence of Lem. 2.6. We also provide a direct proof.

Theorem 2.13. *Let V be a finite dimensional vector space over $\mathbb{F}$ such that $V = U \oplus W$. Then,*

$$\dim(V) = \dim(U) + \dim(W)$$

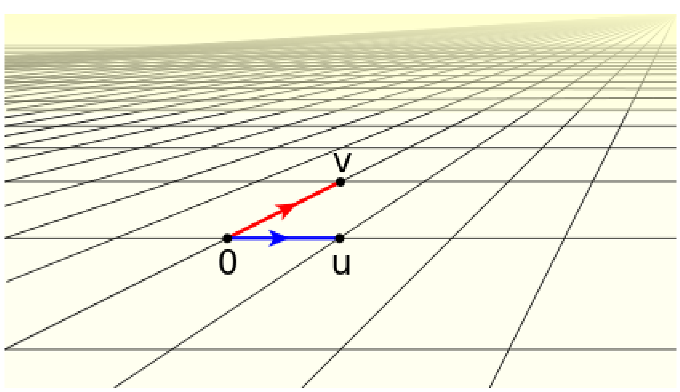


Figure 2.3: A subspace in $\mathbb{R}^3$

Proof. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be bases for U and W respectively. Say

$$\mathfrak{B}_1 = \{\mathbf{u}_1, \dots, \mathbf{u}_r\} \quad \textit{and} \quad \mathfrak{B}_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_s\}$$

Then every element of U can be written as a unique linear combination

$$\mathbf{u} = x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r$$

and every element of W can be written as a unique linear combination

$$\mathbf{w} = y_1\mathbf{w}_1 + \cdots + y_s\mathbf{w}_r$$

Hence, every element of V can be written as a unique linear combination

$$\mathbf{v} = x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r + y_1\mathbf{w}_1 + \cdots + y_s\mathbf{w}_r$$

Thus the set $\{\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{w}_1, \dots, \mathbf{w}_s\}$ forms a basis for V. □

The definition of the direct sum can be generalized to several summands. We say that

$$V = \bigoplus_{i=1}^{n} V_i = V_1 \oplus \cdots \oplus V_n$$

if every element in V can be written uniquely as a sum

$$\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_n, \quad \textit{with} \quad \mathbf{v}_i \in V_i.$$

2.4.2 Direct products

The notion of direct products is based on the Cartesian products. We review some of the standard terminology. Let U and W be vector spaces over some field $\mathbb{F}$. We let $U \times W$ be the set of all ordered pairs $(\mathbf{u}, \mathbf{w})$ such that $\mathbf{u} \in U$ and $\mathbf{w} \in W$, i.e.,

$$U \times W := \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u} \in U, \mathbf{w} \in W\}$$

We define the addition of any two pairs $(\mathbf{u}_1, \mathbf{w}_1)$ and $(\mathbf{u}_2, \mathbf{w}_2)$ as follows

$$(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2)$$

The scalar multiplication is defined as follows: for every $r \in \mathbb{F}$,

$$r(\mathbf{u}, \mathbf{w}) = (r\mathbf{u}, r\mathbf{w})$$

Exercise 43. *Show that $U \times W$ with this addition and scalar multiplication is a vector space over $\mathbb{F}$.*

Definition 2.11. *The vector space $U \times W$ is called the* ***direct product*** *of U and W.*

Lemma 2.7. *Let U and W be finite dimensional vector spaces over some field $\mathbb{F}$. Then,*

$$\dim(U \times W) = \dim U + \dim W \tag{2.9}$$

Proof. Assume that $\mathfrak{B}_1 = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis for U and $\mathfrak{B}_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis for W. Consider the set

$$\mathfrak{B} = \{(\mathbf{u}_1, \mathbf{0}_W), (\mathbf{u}_2, \mathbf{0}_W), \dots, (\mathbf{u}_n, \mathbf{0}_W), (\mathbf{0}_U, \mathbf{w}_1), (\mathbf{0}_U, \mathbf{w}_2), \dots, (\mathbf{0}_U, \mathbf{w}_m).\},$$

It is easy to show that $\mathfrak{B}$ is a linearly independent set. Let $(\mathbf{u}, \mathbf{w}) \in U \times W$. Then

$$\mathbf{u} = x_1\mathbf{u}_1 + \ldots + x_n\mathbf{u}_n \quad \text{and} \quad \mathbf{w} = y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m,$$

for some scalars $x_1, \dots, x_n, y_1, \dots, y_m \in \mathbb{F}$. Hence $(\mathbf{u}, \mathbf{w})$ can be expressed as a linear combination of elements in $\mathfrak{B}$ via

$$\begin{aligned}
(\mathbf{u}, \mathbf{w}) &= (x_1\mathbf{u}_1 + \ldots + x_n\mathbf{u}_n, y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m) \\
&= (x_1\mathbf{u}_1 + \ldots + x_n\mathbf{u}_n, \mathbf{0}_W) + (\mathbf{0}_U, y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m) \\
&= (x_1\mathbf{u}_1, \mathbf{0}_W) + \ldots + (x_n\mathbf{u}_n, \mathbf{0}_W) + (\mathbf{0}_U, y_1\mathbf{w}_1) + \cdots + (y_m\mathbf{w}_m) \\
&= x_1(\mathbf{u}_1, \mathbf{0}_W) + \cdots + x_n(\mathbf{u}_n, \mathbf{0}_W) + y_1(\mathbf{0}_U, \mathbf{w}_1) + \cdots + y_m(\mathbf{0}_U, \mathbf{w}_m)
\end{aligned}$$

This completes the proof. □

Example 2.25. *The simplest illustration of the idea of the proof above is when we take $U = \mathbb{R}$ and $W = \mathbb{R}$. Take $\mathfrak{B}_1 = \mathfrak{B}_2 = \{1\}$. then,*

$$\mathfrak{B} = \{(1,0), (0,1)\},$$

which is our standard bases $\{\mathbf{i}, \mathbf{j}\}$.

The definition of the direct product can be generalized to several factors. For example

$$V := \prod_{i=1}^{n} V_i = V_1 \times \cdots \times V_n$$

is the set of n-tuples where addition and scalar multiplication are defined coordinate-wise. Using the lemma above we can prove that

$$\dim\left(\prod_{i=1}^{n} V_i\right) = \dim(V_1) + \cdots + \dim(V_n)$$

Exercise 44. *Prove that:*

(i) $\mathbb{R}^n$ *is a direct product* $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$

(ii) $\mathbb{R}^n \times \mathbb{R}^m$ *is a vector space of dimension* $m + n$*.*

Exercises:

188. *Let $V = \mathbb{R}^2$ and W be the subspace generated by $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Let U be the subspace generated by $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Show that V is the direct sum of W and U. Can you generalize this to any two vectors $\mathbf{u}$ and $\mathbf{w}$?*

189. *Let $V = \mathbb{R}^3$. Let W be the space generated by $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and let U be the subspace generated by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Show that V is the direct sum of W and U.*

190. *Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^2$. If there is no $c \in \mathbb{R}$ such that $\mathbf{u} = c\mathbf{v}$, show that $\{\mathbf{u}, \mathbf{v}\}$ is a basis of $\mathbb{R}^2$ and that $\mathbb{R}^2$ is a direct sum of the subspaces generated by $U =$ Span $(\mathbf{u})$ and $V =$ Span $(\mathbf{v})$ respectively.*

191. *Let U and W be subspaces of V. What are $U+U$, $U+V$? Is $U+W = W+U$?*

192. *Let U, W be subspaces of a vector space V. Show that*

$$\dim U + \dim W = \dim(U+W) + \dim(U \cap W)$$

193. *Let $\mathbb{F}$ be a field, $V = \mathrm{Mat}_{2\times 2}(\mathbb{F})$,*

$$U := \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{F} \right\}$$

and

$$W := \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \mid a, b \in \mathbb{F} \right\}.$$

Show that:

(i) *U and W are subspaces of V.*

(ii) *$V = U \oplus W$*

194. *Let V be a vector space over $\mathbb{F}$ and $\mathcal{S}$ the set of all subspaces of V.*

(i) *Consider the operation of subspace addition in $\mathcal{S}$. Show that there is a zero in $\mathcal{S}$ for this operation and that the operation is associative.*

(ii) *Consider the operation of intersection in $\mathcal{S}$. Show that this operation is associative. Is there an identity for this operation (i.e., there is an $E \in \mathcal{S}$ such that $A \cap E = A$ for all E in $\mathcal{S}$)?*

195. *Let A be an invertible 3 by 3 matrix. Prove that $\mathfrak{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ if and only if $\mathfrak{B}' = \{A\mathbf{u}, A\mathbf{v}, A\mathbf{w}\}$ is a basis for $\mathbb{R}^3$.*

196. *Let $A\mathbf{x} = \mathbf{b}$ be a linear system of n equations and n unknowns. How many solutions has this system if **rank** $(A) = n$? What if **rank** $(A) < n$? Explain.*

197. *Find a basis for Span $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$ in $\mathbb{R}^4$ where $\mathbf{w}_1, \dots, \mathbf{w}_4$ are given as below:*

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{w}_4 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

198. *Let $V = \mathbb{R}^2$, $W =$ Span $\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$, and $U =$ Span $\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$. Show that V is the direct sum of W and U.*

199. *Let $\mathfrak{B} := \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ such that*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Is $\mathfrak{B}$ a basis for $\mathbb{R}^3$? Justify your answer.

200. *Let $V = \mathrm{Mat}_n(\mathbb{R})$. Find the matrices that commute with every element of V.*

201. *Let $GL_2(\mathbb{F})$ denote the set of matrices in $\mathrm{Mat}_2(\mathbb{F})$ which have an inverse. Is $GL_2(\mathbb{F})$ a subspace of $\mathrm{Mat}_2(\mathbb{F})$? Justify your answer.*

2.5 Quotient spaces

In many mathematical contexts, we often encounter situations where we want to study a space by identifying elements that differ by some well-defined structure. Quotient spaces provide a natural way to do this in linear algebra by "collapsing" a subspace to a single point and considering the resulting structure. This concept plays a fundamental role in simplifying problems, defining new spaces, and understanding the relationship between vector spaces and their subspaces.

Let V be a vector space over a field $\mathbb{F}$ and W be a subspace of V. We define an equivalence relation $\sim$ on V by stating that

$$\mathbf{x} \sim \mathbf{y} \text{ if } \mathbf{x} - \mathbf{y} \in W$$

Exercise 45. *Prove that this is an equivalence relation as defined in* **??**

Denote the set of all equivalence classes of this relation by V/W. The equivalence class (or, in this case, the coset) of $\mathbf{x}$ is often denoted by

$$[x] = x + W$$

since it is given by

$$[\mathbf{x}] = \{\mathbf{x} + \mathbf{w} : \mathbf{w} \in W\}.$$

Definition 2.12. *The* ***quotient space*** *V/W is then defined as $V/\sim$, the set of all equivalence classes over V by $\sim$. Scalar multiplication and addition are defined on the equivalence classes by*

$$\begin{aligned} &[x] + [y] = [x + y] \\ &\alpha[x] = [\alpha x], \end{aligned}$$

for all $\alpha \in \mathbb{F}$.

Lemma 2.8. *These operations are well-defined (i.e. do not depend on the choice of representative of the equivalence class).*

Proof. To show that the operations are well-defined, we need to prove that the choice of representatives does not affect the results of addition and scalar multiplication.

Let $x, y \in V$ and suppose $x' \sim x$ and $y' \sim y$, meaning that there exist $w_1, w_2 \in W$ such that $x' = x + w_1$ and $y' = y + w_2$. We show that addition is well-defined:

$$[x'] + [y'] = [x + w_1] + [y + w_2] = [(x + w_1) + (y + w_2)] = [x + y + (w_1 + w_2)].$$

Since $w_1 + w_2 \in W$ (because W is a subspace), it follows that $[x + y] = [x' + y']$, proving that addition is well-defined.

For scalar multiplication, let $\alpha \in \mathbb{F}$. Then,

$$\alpha[x'] = \alpha[x + w_1] = [\alpha x + \alpha w_1].$$

Since $\alpha w_1 \in W$ (again because W is a subspace), it follows that $[\alpha x] = [\alpha x']$, proving that scalar multiplication is well-defined. $\square$

Lemma 2.9. *The V/W together with above operations is a vector space over $\mathbb{F}$.*

Proof. To verify that V/W is a vector space, we need to check that it satisfies the vector space axioms.

1. *Closure under addition and scalar multiplication*: This follows directly from the definition of the operations and the proof that they are well-defined.

2. *Associativity of addition*: Given $[x],[y],[z] \in V/W$, we have

$$([x]+[y])+[z] = [x+y]+[z] = [(x+y)+z] = [x+(y+z)] = [x]+[y+z] = [x]+([y]+[z]).$$

3. *Commutativity of addition*: Since addition in V is commutative, we have $[x]+[y] = [x+y] = [y+x] = [y]+[x]$.

4. *Existence of an additive identity*: The set $[0]$ (i.e., the equivalence class of the zero vector) serves as the additive identity since $[x]+[0] = [x+0] = [x]$.

5.*Existence of additive inverses*: For each $[x]$, its inverse is given by $[-x]$, since $[x]+[-x] = [x-x] = [0]$.

6. *Distributive properties*: The distributive laws follow from those in V:

$$\alpha([x]+[y]) = \alpha[x+y] = [\alpha(x+y)] = [\alpha x+\alpha y] = [\alpha x]+[\alpha y],$$

$$(\alpha+\beta)[x] = [(\alpha+\beta)x] = [\alpha x+\beta x] = [\alpha x]+[\beta x].$$

7. *Associativity of scalar multiplication*: For $\alpha,\beta \in \mathbb{F}$,

$$(\alpha\beta)[x] = [(\alpha\beta)x] = [\alpha(\beta x)] = \alpha[\beta x].$$

8. *Identity in scalar multiplication*: Since $1x = x$ in V, we have $1[x] = [1x] = [x]$.

Since all vector space axioms are satisfied, V/W is a vector space over $\mathbb{F}$. □

□

The mapping that associates to $\mathbf{v} \in V$ the equivalence class $[\mathbf{v}]$ is known as the **quotient map** or the **canonical projection** of V onto W. The concept of a quotient space is fundamental in mathematics.

Theorem 2.14 (Isomorphism Theorem)**.** *Let U, V be vector spaces over a scalar field $\mathbb{F}$ and $\phi : U \to V$ a linear map. Then the quotient space $U/ker\,\phi$ is isomorphic to* $\mathrm{Img}\,\phi$*,*

$$U/ker\,\phi \cong \mathrm{Img}\,\phi$$

Proof. Define a map $\Psi : U/\mathrm{ker}\,\phi \to \mathrm{Im}\,\phi$ by

$$\Psi([u]) = \phi(u),$$

where $[u]$ denotes the equivalence class of u in $U/\mathrm{ker}\,\phi$.

Suppose $u' \sim u$, meaning $u' - u \in \mathrm{ker}\,\phi$. Then

$$\phi(u') - \phi(u) = \phi(u'-u) = 0,$$

so $\phi(u') = \phi(u)$, ensuring that $\Psi([u']) = \Psi([u])$. Thus, Ψ is well-defined.

For any $[u_1],[u_2] \in U/\ker\phi$ and $\alpha \in \mathbb{F}$,

$$\Psi([u_1]+[u_2]) = \Psi([u_1+u_2]) = \phi(u_1+u_2) = \phi(u_1)+\phi(u_2) = \Psi([u_1])+\Psi([u_2]).$$

Similarly,

$$\Psi(\alpha[u]) = \Psi([\alpha u]) = \phi(\alpha u) = \alpha\phi(u) = \alpha\Psi([u]).$$

Thus, Ψ is linear.

Suppose $\Psi([u]) = 0$, i.e., $\phi(u) = 0$. This means $u \in \ker\phi$, so its equivalence class is $[0]$. Thus, $[u] = [0]$, proving injectivity.

By definition, the image of Ψ is $\mathrm{Im}\,\phi$, making Ψ surjective.

Since Ψ is a bijective linear map, it is an isomorphism. □

Lemma 2.10. *Let U be a finite-dimensional vector space over a field $\mathbb{F}$, and let W be a subspace of U. If $\{w_1, w_2, \dots, w_m\}$ is a basis for W, then there exist vectors $\{u_{m+1}, u_{m+2}, \dots, u_n\}$ in U such that*

$$\{w_1, w_2, \dots, w_m, u_{m+1}, u_{m+2}, \dots, u_n\}$$

forms a basis of U. The set

$$\{[u_{m+1}], [u_{m+2}], \dots, [u_n]\}$$

forms a basis of the quotient space U/W, and

$$\dim U/W = \dim U - \dim W.$$

Proof. Since W is a subspace of U, we can extend the basis $\{w_1, \dots, w_m\}$ of W to a basis of U by adding vectors $\{u_{m+1}, \dots, u_n\}$ so that

$$\{w_1, \dots, w_m, u_{m+1}, \dots, u_n\}$$

forms a basis for U. This implies that every vector in U can be uniquely expressed as

$$\sum_{i=1}^{m} \alpha_i w_i + \sum_{j=m+1}^{n} \beta_j u_j,$$

where $\alpha_i, \beta_j \in \mathbb{F}$.

Now, consider the quotient space U/W. The elements of U/W are cosets of the form

$$[x] = x + W.$$

We claim that the set of cosets

$$\{[u_{m+1}], [u_{m+2}], \dots, [u_n]\}$$

forms a basis of U/W.

Any coset in U/W has a representative of the form

$$x = \sum_{i=1}^{m} \alpha_i w_i + \sum_{j=m+1}^{n} \beta_j u_j.$$

Since elements of W are identified with zero in U/W, we have

$$[x] = \left[\sum_{j=m+1}^{n} \beta_j u_j\right] = \sum_{j=m+1}^{n} \beta_j [u_j].$$

Thus, every coset is a linear combination of $[u_{m+1}], \ldots, [u_n]$.

Suppose there exist scalars $\gamma_{m+1}, \ldots, \gamma_n$ such that $\sum_{j=m+1}^{n} \gamma_j [u_j] = 0$. This means that $\sum_{j=m+1}^{n} \gamma_j u_j \in W$. Since the vectors $\{u_{m+1}, \ldots, u_n\}$ are linearly independent from W in U, it follows that all $\gamma_j = 0$. Thus, the cosets $[u_{m+1}], \ldots, [u_n]$ are linearly independent.

Since these cosets are linearly independent and span U/W, they form a basis of U/W. The number of these basis elements is

$$n - m = \dim U - \dim W.$$

Thus, $\dim U/W = \dim U - \dim W$. □

Theorem 2.15. *Let V be an n-dimensional vector space over $\mathbb{F}$ and W a subspace of V. Then*

$$\dim V = \dim W + \dim V/W$$

Proof. Let $\mathfrak{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V. If all $\mathbf{v}_1, \ldots, \mathbf{v}_n \in W$ then $V = W$ and the theorem is trivial. Without loss of generality assume $\mathbf{v}_1, \ldots, \mathbf{v}_r \notin W$ and $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n \in W$. Then $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ form a basis for W. Hence, $\dim W = n - r$.

Let $\pi : V \to V/W$ be the natural projection. From Lem. 2.10 the elements $\pi(\mathbf{v}_1), \ldots, \pi(v_r)$ form a basis of V/W. Thus, $\dim V/W = r$. This completes the proof. □

A subspace W of V is said to have **co-dimension** r if V/W has dimension r.

Exercises:

202. *Let $V = \mathbb{R}^3$ and $W =$ Span $\left\{ \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \right\}$.*

1. *Describe the cosets of W in V geometrically.*
2. *Give three different column vectors that belong to the same coset in V/W.*
3. *Is the coset $\left[\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right]$ the same as the coset $\left[\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} \right]$? Explain.*

203. *Let $V = \mathbb{R}^n$ and W be a subspace of V. What does the quotient space V/W "look like" geometrically? (Hint: Think about the case where W is a line or a plane in $\mathbb{R}^3$ represented by column vectors.)*

204. *Consider $V = \mathbb{Z}$ (integers) as a vector space over $\mathbb{Q}$ (rationals) and $W = 5\mathbb{Z}$ (multiples of 5). The equivalence relation is $x \sim y$ if $x - y \in 5\mathbb{Z}$. Work through the well-definedness proof for addition and scalar multiplication in this specific example. What goes wrong if we try to define scalar multiplication with real numbers instead of rationals?*

205. *Let $T : V \to W$ be a linear transformation. Explain how the Isomorphism Theorem relates the quotient space $V/\ker T$ to the image of T. Give a concrete example with specific vector spaces (using* *column vector notation) and a linear transformation (represented by a matrix).*

206. *Let* $V = \mathbb{R}^2$ *and* $W = \text{Span}\left\{\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right\}$. *Describe the quotient map* $\pi : V \to V/W$. *What is* $\pi\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$*?*

207. *Let* $V = \mathbb{R}^4$ *and* $W = \text{Span}\left\{\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}\right\}$. *Find a basis for* V/W. *Express your basis vectors as cosets.*

208. *Let* $V = \mathbb{R}^3$ *and* $W = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in V : x+y+z=0\right\}$. *Construct an explicit isomorphism between* V/W *and* $\mathbb{R}$.

209. *Let* V *be a finite-dimensional vector space and* W_1, W_2 *be subspaces of* V. *Prove that*

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

(Hint: Consider the quotient space $(W_1+W_2)/W_2$.*)*

210. *Let* V *be a finite-dimensional vector space and* W *be a subspace of* V. *The annihilator of* W*, denoted* W^0*, is the set of all linear functionals* $f : V \to \mathbb{F}$ *such that* $f(w) = 0$ *for all* $w \in W$. *Prove that* $(V/W)^* \cong W^0$ *where* $*$ *denotes the dual space.*

211. *Let* V *be a vector space and* W *be a subspace. Prove the following universal property: If* U *is any vector space and* $T : V \to U$ *is a linear transformation such that* $T(w) = 0$ *for all* $w \in W$*, then there exists a unique linear transformation* $\bar{T} : V/W \to U$ *such that* $\bar{T} \circ \pi = T$*, where* $\pi : V \to V/W$ *is the canonical projection.*

212. *Let* V *be a vector space, and* W *and* U *be subspaces of* V *such that* $U \subseteq W$. *Prove that* $V/U/W/U \cong V/W$.

213. *Explore the relationship between exact sequences of vector spaces and quotient spaces. For example, if* $0 \to U \to V \to W \to 0$ *is an exact sequence, how do the dimensions of* U*,* V*, and* W *relate?*

2.6 Bilinear maps and the dual space

This section introduces bilinear maps and the dual space, two interconnected concepts that are fundamental in advanced linear algebra and have significant applications in various fields, including machine learning, physics, and engineering. Understanding these concepts provides a deeper understanding of vector spaces and lays the groundwork for more sophisticated mathematical tools.

Bilinear maps generalize the familiar dot product and provide a powerful way to study relationships between vector spaces. They are essential for understanding quadratic forms, tensor products, and other important mathematical objects. The dual space, closely related to bilinear forms, provides a new perspective on vector spaces by considering the set of all linear functionals defined on them. This perspective is crucial in functional analysis and optimization, and it plays a subtle but important role in machine learning, particularly in areas like kernel methods and regularization. This section will not only define these concepts but also motivate their importance and illustrate their connections to previously studied topics, preparing you for their use in more advanced contexts.

2.6.1 Bilinear maps

Let U, V and W be three vector spaces over the same base field $\mathbb{F}$. A **bilinear map** is a function $\phi : U \times V \to W$ such that it satisfies the following properties.

(i) For any $\lambda \in \mathbb{F}$, $\phi(\lambda u, v) = \phi(v, \lambda u) = \lambda(\phi(u,v)$

(ii) For any $u_1, u_2 \in U$ and $v_1, v_2 \in V$ $\phi(u_1 + u_2, v) = \phi(u_1, v) + \phi(u_2, v)$ and

If $U = V$ and we have $\phi(u,v) = \phi(v,u)$ for all $u, v \in U$, then we say that ϕ is symmetric. If $W = \mathbb{F}$, then the map is called a **bilinear form**. We will study bilinear form again in the coming lectures. First let us see an example which captures the heart of this topic.

Example 2.26. *Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{F})$, say $A = [a_{i,j}]$. Define a map*

$$\begin{aligned} \phi_A : \mathbb{F}^m \times \mathbb{F}^n &\to \mathbb{F} \\ (\mathbf{u}, \mathbf{v}) &\to \mathbf{u}^t A \mathbf{v} \end{aligned}$$

*Then ϕ is bilinear. We will call ϕ_A the **associated map** of A. Thus,*

$$\phi_A(\mathbf{u}, \mathbf{v}) = [u_1, \dots, u_n] \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Remark 2.2. *Notice that the properties of the bilinear maps are very similar to those of the dot product. We will explore this similarity further in the coming lectures.*

Exercise 46. *Let A be an $n \times n$ symmetric matrix. Show that the map*

$$\begin{aligned} \phi_A :\ \mathbb{F}^n \times \mathbb{F}^n &\to \mathbb{F} \\ (\mathbf{u}, \mathbf{v}) &\to \mathbf{u}^t A \mathbf{v} \end{aligned}$$

satisfies

$$\phi_A(\mathbf{u}, v) = \phi_A(\mathbf{v}, \mathbf{u})$$

and therefore has the properties of the dot product.

Exercise 47. *Prove the converse of the above statement.*

Lemma 2.11. *Let U, V, and W be vector spaces over $\mathbb{F}$. The set of bilinear maps $\mathrm{Bil}(U \times V, W)$ is a vector space.*

Proof. To prove that $\mathrm{Bil}(U \times V, W)$ is a vector space, we need to show that it satisfies the vector space axioms. We will focus on the key axioms, as the others follow similarly.

Let $\phi_1, \phi_2 \in \mathrm{Bil}(U \times V, W)$. We need to show that $\phi_1 + \phi_2$ is also a bilinear map, where the addition is defined pointwise:

$$(\phi_1 + \phi_2)(u, v) = \phi_1(u, v) + \phi_2(u, v) \quad \forall u \in U, v \in V.$$

We must verify bilinearity. For any scalar $\lambda \in \mathbb{F}$ and vectors $u, u' \in U$ and $v, v' \in V$:

$$\begin{aligned}
(\phi_1 + \phi_2)(\lambda u, v) &= \phi_1(\lambda u, v) + \phi_2(\lambda u, v) = \lambda\phi_1(u, v) + \lambda\phi_2(u, v) = \lambda(\phi_1(u, v) + \phi_2(u, v)) = \lambda(\phi_1 + \phi_2)(u, v) \\
(\phi_1 + \phi_2)(u, \lambda v) &= \phi_1(u, \lambda v) + \phi_2(u, \lambda v) = \lambda\phi_1(u, v) + \lambda\phi_2(u, v) = \lambda(\phi_1(u, v) + \phi_2(u, v)) = \lambda(\phi_1 + \phi_2)(u, v) \\
(\phi_1 + \phi_2)(u + u', v) &= \phi_1(u + u', v) + \phi_2(u + u', v) = \phi_1(u, v) + \phi_1(u', v) + \phi_2(u, v) + \phi_2(u', v) \\
&= (\phi_1 + \phi_2)(u, v) + (\phi_1 + \phi_2)(u', v) \\
(\phi_1 + \phi_2)(u, v + v') &= \phi_1(u, v + v') + \phi_2(u, v + v') = \phi_1(u, v) + \phi_1(u, v') + \phi_2(u, v) + \phi_2(u, v') \\
&= (\phi_1 + \phi_2)(u, v) + (\phi_1 + \phi_2)(u, v')
\end{aligned}$$

Thus, $\phi_1 + \phi_2$ is bilinear.

Let $\phi \in \mathrm{Bil}(U \times V, W)$ and $\lambda \in \mathbb{F}$. We need to show that $\lambda\phi$ is also a bilinear map, where the scalar multiplication is defined pointwise:

$$(\lambda\phi)(u, v) = \lambda\phi(u, v) \quad \forall u \in U, v \in V.$$

Again, we verify bilinearity:

$$\begin{aligned}
(\lambda\phi)(\mu u, v) &= \lambda\phi(\mu u, v) = \lambda\mu\phi(u, v) = \mu(\lambda\phi)(u, v) \\
(\lambda\phi)(u, \mu v) &= \lambda\phi(u, \mu v) = \lambda\mu\phi(u, v) = \mu(\lambda\phi)(u, v) \\
(\lambda\phi)(u + u', v) &= \lambda\phi(u + u', v) = \lambda(\phi(u, v) + \phi(u', v)) = (\lambda\phi)(u, v) + (\lambda\phi)(u', v) \\
(\lambda\phi)(u, v + v') &= \lambda\phi(u, v + v') = \lambda(\phi(u, v) + \phi(u, v')) = (\lambda\phi)(u, v) + (\lambda\phi)(u, v')
\end{aligned}$$

Thus, $\lambda\phi$ is bilinear.

The zero map $\zeta : U \times V \to W$ defined by $\zeta(u, v) = 0_W$ for all $u \in U, v \in V$ is clearly bilinear and serves as the zero element in $\mathrm{Bil}(U \times V, W)$.

The remaining axioms (associativity of addition, existence of additive inverses, distributivity) follow directly from the corresponding properties of vector addition and scalar multiplication in W. Therefore, $\mathrm{Bil}(U \times V, W)$ is a vector space. ◻

Theorem 2.16. *Given a bilinear map $\phi : \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}$, there exists a unique matrix A such that*

$$\phi(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}.$$

which we denote it by $\phi_A := \phi$. The set of all bilinear maps of $\mathbb{F}^m \times \mathbb{F}^n$ into $\mathbb{F}$ is a vector space, denoted by $\operatorname{Bil}(\mathbb{F}^m \times \mathbb{F}^n, \mathbb{F})$ and the correspondence

$$A \to \phi_A$$

is an isomorphism between $\operatorname{Mat}_{m\times n}(\mathbb{F})$ and $\operatorname{Bil}(\mathbb{F}^m \times \mathbb{F}^n, \mathbb{F})$.

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the standard basis for $\mathbb{F}^m$ and $\mathbf{f}_1, \ldots, \mathbf{f}_n$ be the standard basis for $\mathbb{F}^n$. Define the entries of the matrix $A \in \operatorname{Mat}_{m\times n}(\mathbb{F})$ as $a_{ij} = \phi(\mathbf{e}_i, \mathbf{f}_j)$ for $1 \le i \le m$ and $1 \le j \le n$.

Now, let $\mathbf{u} = \sum_{i=1}^m u_i \mathbf{e}_i \in \mathbb{F}^m$ and $\mathbf{v} = \sum_{j=1}^n v_j \mathbf{f}_j \in \mathbb{F}^n$ be arbitrary vectors. By the bilinearity of ϕ:

$$\phi(\mathbf{u}, \mathbf{v}) = \phi\left(\sum_{i=1}^m u_i \mathbf{e}_i, \sum_{j=1}^n v_j \mathbf{f}_j\right) = \sum_{i=1}^m \sum_{j=1}^n u_i v_j \phi(\mathbf{e}_i, \mathbf{f}_j) = \sum_{i=1}^m \sum_{j=1}^n u_i v_j a_{ij} = \mathbf{u}^t A \mathbf{v}.$$

Thus, the matrix A with entries $a_{ij} = \phi(\mathbf{e}_i, \mathbf{f}_j)$ satisfies the desired property.

Suppose there are two matrices A and B such that $\phi(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v} = \mathbf{u}^t B \mathbf{v}$ for all $\mathbf{u} \in \mathbb{F}^m$ and $\mathbf{v} \in \mathbb{F}^n$. Then $\mathbf{u}^t (A - B)\mathbf{v} = 0$ for all $\mathbf{u}$ and $\mathbf{v}$. Choosing $\mathbf{u} = \mathbf{e}_i$ and $\mathbf{v} = \mathbf{f}_j$, we get $(A - B)_{ij} = 0$ for all i and j. Therefore, $A - B = 0$, which implies $A = B$.

We have already shown in the previous lemma that $\operatorname{Bil}(\mathbb{F}^m \times \mathbb{F}^n, \mathbb{F})$ is a vector space.

The correspondence $A \to \phi_A$ is a linear map. Let $A, B \in \operatorname{Mat}_{m\times n}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Then $(\lambda A) \to \phi_{\lambda A}$, and

$$\phi_{\lambda A}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t (\lambda A)\mathbf{v} = \lambda(\mathbf{u}^t A \mathbf{v}) = \lambda \phi_A(\mathbf{u}, \mathbf{v}).$$

Also, $(A + B) \to \phi_{A+B}$, and

$$\phi_{A+B}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t (A + B)\mathbf{v} = \mathbf{u}^t A \mathbf{v} + \mathbf{u}^t B \mathbf{v} = \phi_A(\mathbf{u}, \mathbf{v}) + \phi_B(\mathbf{u}, \mathbf{v}).$$

The correspondence is injective (one-to-one) because of the uniqueness of A, and it is surjective (onto) because of the existence of A for any bilinear map ϕ. Therefore, the correspondence $A \to \phi_A$ is an isomorphism.

Thus, we have shown that there is a one-to-one correspondence between matrices in $\operatorname{Mat}_{m\times n}(\mathbb{F})$ and bilinear maps in $\operatorname{Bil}(\mathbb{F}^m \times \mathbb{F}^n, \mathbb{F})$, and this correspondence is an isomorphism. □

Exercises:

Here are some exercises on bilinear maps, ranging from basic to more challenging:

1. **Verification of Bilinearity:** Determine if the following functions are bilinear maps:

   (a) $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

(b) $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(\mathbf{x}, \mathbf{y}) = (x_1 y_1, x_2 y_2)$.

(c) $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $h(\mathbf{x}, \mathbf{y}) = x_1 + y_1$.

(d) $k : P_2(\mathbb{R}) \times P_2(\mathbb{R}) \to \mathbb{R}$ defined by $k(p(x), q(x)) = \int_0^1 p(x)q(x)dx$, where $P_2(\mathbb{R})$ is the space of polynomials of degree at most 2.

2. **Matrix Representation:** Find the matrix A associated with the following bilinear maps:

   (a) $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\mathbf{x}, \mathbf{y}) = 2x_1 y_1 - x_1 y_2 + 3x_2 y_1 + y_2 x_2$

   (b) $g : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $g(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + x_3 y_1$.

3. **Bilinear Map from a Matrix:** Given the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, find the associated bilinear map $\phi_A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$.

1. **Symmetric Bilinear Forms:** Determine if the bilinear forms in Exercise 1(a) and 2(a) are symmetric.

2. **Change of Basis:** Let $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a bilinear map with matrix representation A with respect to the standard basis. Find the matrix representation of ϕ with respect to a new basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$.

3. **Bilinear Maps and Linear Transformations:** Let $T : U \to V$ and $S : W \to Z$ be linear transformations. If $\phi : V \times W \to \mathbb{F}$ is a bilinear map, show that the composition $\psi : U \times Z \to \mathbb{F}$ defined by $\psi(u, z) = \phi(T(u), S(z))$ is also a bilinear map.

4. **Isomorphism:** Let U, V be finite-dimensional vector spaces over $\mathbb{F}$. Prove that the set of bilinear maps from $U \times V$ to $\mathbb{F}$, denoted by $\mathrm{Bil}(U \times V, \mathbb{F})$, is isomorphic to the vector space of linear transformations from U to V^*, the dual space of V.

1. **Rank of a Bilinear Form:** Define the rank of a bilinear form $\phi : U \times V \to \mathbb{F}$ as the rank of its matrix representation with respect to any bases of U and V. Show that the rank is well-defined (i.e., it doesn't depend on the choice of bases).

2. **Tensor Product (Introduction):** Let U, V, W be vector spaces over $\mathbb{F}$. A tensor product of U and V is a vector space $U \otimes V$ together with a bilinear map $\otimes : U \times V \to U \otimes V$ such that for any bilinear map $\phi : U \times V \to W$, there exists a unique linear map $\tilde{\phi} : U \otimes V \to W$ such that $\phi = \tilde{\phi} \circ \otimes$. (This is a universal property definition). While a full exploration of tensor products is beyond the scope of basic exercises, try to understand the definition and its implications. Think about how it relates to bilinear maps. We will see tensors in more detail in Chap. 11.

2.6.2 The dual space

The dual space, denoted V^*, is a vector space formed by all linear functionals on V. A *linear functional* is simply a linear map from V to the underlying field $\mathbb{F}$. While this might seem abstract at first, the dual space provides a powerful new way to understand vectors and linear transformations. It allows us to "see" vectors not just as arrows or tuples of numbers, but also as functions that act on other vectors. This perspective is essential in many areas of mathematics and its applications, including optimization, functional analysis, and machine learning (especially in areas like kernel methods and regularization). In this section, we will define the dual space, explore its properties, and see how it connects back to the vector spaces we are already familiar with. We will also introduce the concept of a dual basis, which allows us to represent linear functionals in a concrete way.

Let V be a vector space over the field $\mathbb{F}$.

Definition 2.13. *The* ***dual space*** *of V is the vector space (over $\mathbb{F}$)*

$$V^* := \mathcal{L}(V, \mathbb{F})$$

of all linear maps $L : V \longrightarrow \mathbb{F}$. Elements of the dual space are called ***functionals****.*

Example 2.27. *Let $V = \mathbb{F}^n$. The simple examples of functionals are* ***coordinate functions***

$$\phi_i(x_1, \dots, x_n) = x_i$$

We leave it to the reader to verify that these are functionals.

Theorem 2.17. *Let V be a vector space of finite dimension. Then, $\dim V = \dim V^*$.*

Theorem 2.18. *Let V be a vector space of finite dimension. Then, $\dim V = \dim V^*$.*

Proof. Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Let $n = \dim V$, and let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be a basis for V. We want to show that the dual space V^* also has dimension n.

Define the linear functionals $\phi_i \in V^*$ for $i = 1, 2, \dots, n$ as follows:

$$\phi_i(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

We claim that the set $\mathcal{B}^* = \{\phi_1, \phi_2, \dots, \phi_n\}$ forms a basis for V^*. If we can show this, then $\dim V^* = n$, and the theorem will be proved.

Suppose we have a linear combination of the ϕ_i that equals the zero functional:

$$\sum_{i=1}^{n} c_i \phi_i = 0,$$

where $c_i \in \mathbb{F}$. This means that for all $v \in V$, we have $\sum_{i=1}^{n} c_i \phi_i(v) = 0$. In particular, we can evaluate this at the basis vectors v_j for $j = 1, 2, \dots, n$:

$$\sum_{i=1}^{n} c_i \phi_i(v_j) = 0.$$

Since $\phi_i(v_j) = \delta_{ij}$ (the Kronecker delta), this simplifies to $c_j = 0$. This holds for all $j = 1, 2, \ldots, n$. Therefore, all the coefficients c_i must be zero, which means that the set $\mathcal{B}^*$ is linearly independent.

Let $\phi \in V^*$ be an arbitrary linear functional. For any vector $v = \sum_{j=1}^n x_j v_j \in V$, we have:

$$
\begin{aligned}
\phi(v) &= \phi\left(\sum_{j=1}^n x_j v_j\right) \\
&= \sum_{j=1}^n x_j \phi(v_j) \\
&= \sum_{j=1}^n x_j \phi\left(\sum_{i=1}^n \phi_i(v_j) v_i\right) \\
&= \sum_{j=1}^n x_j \sum_{i=1}^n \phi_i(v_j)\phi(v_i) \\
&= \sum_{i=1}^n \phi(v_i) \sum_{j=1}^n x_j \phi_i(v_j) \\
&= \sum_{i=1}^n \phi(v_i)\phi_i\left(\sum_{j=1}^n x_j v_j\right) = \sum_{i=1}^n \phi(v_i)\phi_i(v)
\end{aligned}
$$

This shows that $\phi = \sum_{i=1}^n \phi(v_i)\phi_i$, which means that any linear functional ϕ can be written as a linear combination of the ϕ_i in $\mathcal{B}^*$. Thus, $\mathcal{B}^*$ spans V^*.

Since $\mathcal{B}^*$ is linearly independent and spans V^*, it is a basis for V^*. Therefore, $\dim V^* = n = \dim V$. $\square$

Definition 2.14. *The basis $\{\phi_1, \ldots, \phi_n\}$ of V^* is called the **dual basis**.*

The dual space is a very important concept in linear algebra. Below we give a few more examples of functionals which are important in different areas of mathematics.

Example 2.28. *Let V be a vector space over $\mathbb{F}$ with scalar product $\langle \cdot, \cdot \rangle$. Fix an element $u \in V$. The map $V \longrightarrow \mathbb{F}$, such that*

$$v \longrightarrow \langle v, u \rangle$$

is a functional. $\square$

Example 2.29. *Let V be a vector space of continuous real-valued functions on the interval $[0,1]$. Define $\delta : V \longrightarrow \mathbb{R}$, such that $\delta(f) = f(0)$. Then δ is a functional called the **Dirac functional**.* $\square$

Theorem 2.19. *Let V be a finite dimensional vector space over $\mathbb{F}$ with a non-degenerate scalar product $\langle \cdot, \cdot \rangle$. The map*

$$
\begin{aligned}
\Phi : V &\longrightarrow V^* \\
v &\mapsto L_v
\end{aligned}
\tag{2.10}
$$

where $L_v(u) = \langle u, v\rangle$ for all $u \in V$, is an isomorphism.

Proof. To prove that Φ is an isomorphism, we need to show that it is a linear map, injective (one-to-one), and surjective (onto).

Let $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$. We need to show that $\Phi(\alpha v_1 + \beta v_2) = \alpha\Phi(v_1) + \beta\Phi(v_2)$. This means showing that $L_{\alpha v_1+\beta v_2} = \alpha L_{v_1} + \beta L_{v_2}$. For any $u \in V$:

$$\begin{aligned}
L_{\alpha v_1+\beta v_2}(u) &= \langle u, \alpha v_1 + \beta v_2\rangle \\
&= \langle u, \alpha v_1\rangle + \langle u, \beta v_2\rangle \quad \text{(by linearity of the inner product)} \\
&= \alpha\langle u, v_1\rangle + \beta\langle u, v_2\rangle \\
&= \alpha L_{v_1}(u) + \beta L_{v_2}(u) \\
&= (\alpha L_{v_1} + \beta L_{v_2})(u).
\end{aligned}$$

Since this holds for all $u \in V$, we have $L_{\alpha v_1+\beta v_2} = \alpha L_{v_1} + \beta L_{v_2}$, so $\Phi(\alpha v_1 + \beta v_2) = \alpha\Phi(v_1) + \beta\Phi(v_2)$. Thus, Φ is linear.

Suppose $\Phi(v_1) = \Phi(v_2)$, which means $L_{v_1} = L_{v_2}$. Then, for all $u \in V$, we have $L_{v_1}(u) = L_{v_2}(u)$, so $\langle u, v_1\rangle = \langle u, v_2\rangle$, or $\langle u, v_1 - v_2\rangle = 0$. Since the inner product is non-degenerate, if $\langle u, v_1 - v_2\rangle = 0$ for all $u \in V$, then $v_1 - v_2 = 0$, which implies $v_1 = v_2$. Therefore, Φ is injective.

Since V is finite-dimensional, V^* is also finite-dimensional, and $\dim V = \dim V^*$. We have shown that Φ is a linear map from V to V^* and that it is injective. For finite-dimensional vector spaces, an injective linear map between spaces of the same dimension is also surjective. Therefore, Φ is surjective.

Since Φ is linear, injective, and surjective, it is an isomorphism. □

Theorem 2.20 (Bilinear Forms and Dual Spaces)**.** *Let V be a finite-dimensional vector space. There is a one-to-one correspondence between bilinear forms on $V \times V$ and linear maps from V to V^*.*

Proof. We will establish this one-to-one correspondence by defining maps in both directions and showing they are inverses of each other.

Let $B : V \times V \to \mathbb{F}$ be a bilinear form on V. Define a map $T_B : V \to V^*$ as follows:

$$T_B(v)(w) = B(v, w) \quad \text{for all } v, w \in V.$$

We need to show that $T_B(v)$ is a linear functional for each $v \in V$ and that T_B is a linear map. For any $w_1, w_2 \in V$ and $\alpha, \beta \in \mathbb{F}$:

$$\begin{aligned}
T_B(v)(\alpha w_1 + \beta w_2) &= B(v, \alpha w_1 + \beta w_2) \\
&= \alpha B(v, w_1) + \beta B(v, w_2) \quad \text{(by bilinearity of } B) \\
&= \alpha T_B(v)(w_1) + \beta T_B(v)(w_2).
\end{aligned}$$

Thus, $T_B(v)$ is a linear functional, so $T_B(v) \in V^*$.

For any $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$, we need to show that $T_B(\alpha v_1 + \beta v_2) = \alpha T_B(v_1) + \beta T_B(v_2)$. This means showing that for all $w \in V$:

$$\begin{aligned} T_B(\alpha v_1 + \beta v_2)(w) &= B(\alpha v_1 + \beta v_2, w) \\ &= \alpha B(v_1, w) + \beta B(v_2, w) \quad \text{(by bilinearity of } B\text{)} \\ &= \alpha T_B(v_1)(w) + \beta T_B(v_2)(w) \\ &= (\alpha T_B(v_1) + \beta T_B(v_2))(w). \end{aligned}$$

Thus, $T_B(\alpha v_1 + \beta v_2) = \alpha T_B(v_1) + \beta T_B(v_2)$, and T_B is a linear map.

Let $T : V \to V^*$ be a linear map. Define a map $B_T : V \times V \to \mathbb{F}$ as follows:

$$B_T(v, w) = T(v)(w) \quad \text{for all } v, w \in V.$$

We need to show that B_T is a bilinear form.

For any $v_1, v_2, w \in V$ and $\alpha, \beta \in \mathbb{F}$:

$$\begin{aligned} B_T(\alpha v_1 + \beta v_2, w) &= T(\alpha v_1 + \beta v_2)(w) \\ &= (\alpha T(v_1) + \beta T(v_2))(w) \quad \text{(by linearity of } T\text{)} \\ &= \alpha T(v_1)(w) + \beta T(v_2)(w) \\ &= \alpha B_T(v_1, w) + \beta B_T(v_2, w). \end{aligned}$$

Similarly, for any $v, w_1, w_2 \in V$ and $\alpha, \beta \in \mathbb{F}$:

$$\begin{aligned} B_T(v, \alpha w_1 + \beta w_2) &= T(v)(\alpha w_1 + \beta w_2) \\ &= \alpha T(v)(w_1) + \beta T(v)(w_2) \quad \text{(because } T(v) \in V^* \text{ is a linear functional)} \\ &= \alpha B_T(v, w_1) + \beta B_T(v, w_2). \end{aligned}$$

Thus, B_T is bilinear.

To show that these maps are inverses of each other we have for any $v, w \in V$:

$$B_{T_B}(v, w) = T_B(v)(w) = B(v, w).$$

Also, for any $v, w \in V$:

$$T_{B_T}(v)(w) = B_T(v, w) = T(v)(w).$$

Therefore, the maps $B \to T_B$ and $T \to B_T$ are inverses of each other, establishing a one-to-one correspondence between bilinear forms on $V \times V$ and linear maps from V to V^*. □

Here are some further problems related to dual spaces, building on the concepts from your existing questions:

Exercises:

1. **Dual Basis Calculation:** Let $V = \mathbb{R}^3$. Given the basis $\mathcal{B} = \{(1,0,0), (1,1,0), (1,1,1)\}$ for V, find the dual basis $\mathcal{B}^*$ for V^*.

2. **Annihilator:** Let W be a subspace of a finite-dimensional vector space V. The *annihilator* of W, denoted W^0, is the set of all linear functionals $\phi \in V^*$ such that $\phi(w) = 0$ for all $w \in W$. Prove that W^0 is a subspace of V^*.

3. **Annihilator and Dimension:** Let V be a finite-dimensional vector space and W be a subspace of V. Prove that $\dim(W^0) = \dim(V) - \dim(W)$.

4. **Dual of a Subspace:** Let W be a subspace of V. Is (W^*) naturally isomorphic to a subspace of V^*? If so, how?

1. **Double Dual:** Let V be a finite-dimensional vector space. The *double dual* of V is $(V^*)^*$, denoted V^{**}. Define a natural map $\iota : V \to V^{**}$ by $\iota(v)(\phi) = \phi(v)$ for all $v \in V$ and $\phi \in V^*$. Prove that ι is a linear isomorphism. (This shows that V and V^{**} are "naturally" isomorphic.)

2. **Dual of a Linear Transformation:** Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Define the *dual map* (or *transpose*) $T^* : W^* \to V^*$ by $T^*(\psi) = \psi \circ T$ for all $\psi \in W^*$.
   (a) Prove that T^* is a linear transformation.
   (b) If A is the matrix representation of T with respect to bases $\mathcal{B}_V$ and $\mathcal{B}_W$, what is the matrix representation of T^* with respect to the dual bases $\mathcal{B}_W^*$ and $\mathcal{B}_V^*$?

3. **Relationship between Range and Annihilator:** Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Prove that $(\mathrm{Im}(T))^0 = \mathrm{Ker}(T^*)$. Also show that $\mathrm{Im}(T^*) = (\mathrm{Ker}(T))^0$.

1. **Reflexivity:** A vector space V is called *reflexive* if the natural map $\iota : V \to V^{**}$ (defined in problem 5) is an isomorphism. Prove that every finite-dimensional vector space is reflexive. (Note: Infinite-dimensional vector spaces are not always reflexive.)

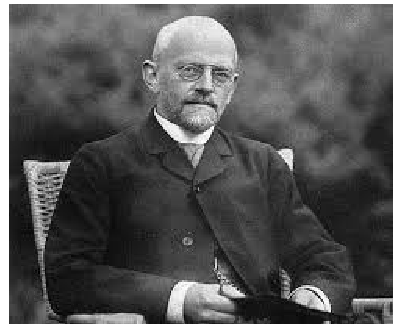
2. **Bilinear Forms and Dual Spaces:** Let V be a finite-dimensional vector space. Show that there is a one-to-one correspondence between bilinear forms on $V \times V$ and linear maps from V to V^*.

Exercises:

214. *Let* $V = \mathrm{Mat}_{n\times n}(\mathbb{R})$*. Describe* V^**.*

215. *Let* $V = \mathbb{R}^{2n}$*. Describe* V^**.*

David Hilbert



(23 January 1862 – 14 February 1943) was a German mathematician and one of the most influential and universal mathematicians of the 19th and early 20th centuries. Hilbert discovered and developed a broad range of fundamental ideas in many areas, including invariant theory, the calculus of variations, commutative algebra, algebraic number theory, the foundations of geometry, spectral theory of operators and its application to integral equations, mathematical physics, and foundations of mathematics (particularly proof theory).

Hilbert adopted and warmly defended Georg Cantor's set theory and transfinite numbers. A famous example of his leadership in mathematics is his 1900 presentation of a collection of problems that set the course for much of the mathematical research of the 20th century.

Hilbert's 1888 work on invariant functions led to his famous finiteness theorem. Two decades earlier, Paul Gordan had shown the finiteness of generators for binary forms through intricate calculations, but struggles emerged when extending this to functions with more variables. To tackle what was known as Gordan's Problem, Hilbert took a novel approach, leading to the Hilbert's Basis Theorem. This theorem proved the existence of a finite set of generators for invariants in any number of variables, though in an abstract, non-constructive manner relying on the law of excluded middle.

Hilbert submitted his findings to the Mathematische Annalen, where Gordan, the expert on invariants, rejected it, criticizing its exposition as insufficiently comprehensive, famously saying, "Das ist nicht Mathematik. Das ist Theologie." However, Felix Klein recognized the work's importance, ensuring its publication. Encouraged, Hilbert extended his method in a second paper, estimating the maximum degree of generators. Klein hailed this as "the most important work on general algebra" published by the Annalen. Gordan later acknowledged the merit in Hilbert's approach.

Despite these successes, Hilbert's non-constructive proof method stirred controversy, aligning with Kronecker's constructivist criticisms. This discord persisted, influencing the intuitionist school led by Brouwer, which opposed Hilbert's use of the Law of Excluded Middle over infinite sets. This philosophical divide even affected Hilbert personally, as his student Weyl was drawn to intuitionism, prompting Hilbert's retort, "Taking the Principle of the Excluded Middle from the mathematician ... is the same as ... prohibiting the boxer the use of his fists."

Chapter 3

Linear Maps Between Finite Dimensional Vector Spaces

This chapter will guide you through the process of constructing matrix representations for linear maps. We will start by defining the matrix associated with a linear map with respect to given bases. Then, we will show how to use this matrix to compute the transformation of a vector. We will explore how the matrix representation changes when we change bases, and we will discuss the connection between the properties of a linear map and the properties of its matrix representation. Finally, we will illustrate these concepts with examples, showing how linear maps and their matrix representations can be used to solve problems in various contexts.

By the end of this chapter, you will be able to:

- Construct the matrix representation of a linear map with respect to given bases.
- Use the matrix representation to compute the transformation of a vector.
- Understand how the matrix representation changes when the bases are changed.
- Apply these concepts to solve problems involving linear maps.

This chapter is not just about manipulating matrices; it's about understanding the deep interplay between linear maps and matrices. This understanding will empower you to tackle a wide range of problems in mathematics, science, and engineering.

3.1 Matrices Associated to Linear Maps

One of the most powerful tools in linear algebra is the ability to represent linear maps as matrices, and vice-versa. This connection allows us to translate the abstract world of linear transformations into the concrete world of matrix operations, enabling efficient computation, analysis, and manipulation. In this section, we will explore how to construct the matrix associated with a given linear map.

Let U and V be finite-dimensional vector spaces over the field $\mathbb{F}$, and let $L: U \to V$ be a linear map. Let $\mathcal{B}_1 = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}_2 = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be bases for U and V, respectively. The vectors $L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)$ can be expressed as linear combinations of the basis vectors in $\mathcal{B}_2$:

$$\begin{aligned}
L(\mathbf{u}_1) &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{m1}\mathbf{v}_m \\
L(\mathbf{u}_2) &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{m2}\mathbf{v}_m \\
&\vdots \\
L(\mathbf{u}_n) &= a_{1n}\mathbf{v}_1 + a_{2n}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_m
\end{aligned}$$

The scalars $a_{ij} \in \mathbb{F}$ are the coordinates of $L(\mathbf{u}_i)$ with respect to the basis $\mathcal{B}_2$. The matrix associated with the linear map L with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$, denoted by $M_{\mathcal{B}_1}^{\mathcal{B}_2}(L)$ or simply $M(L)$ when the bases are clear from context, is the $m \times n$ matrix $A = [a_{ij}]$, where a_{ij} is the coefficient of $\mathbf{v}_i$ in the expansion of $L(\mathbf{u}_j)$. That is,

$$M(L) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The j-th column of $M(L)$ consists of the coordinates of $L(\mathbf{u}_j)$ with respect to the basis $\mathcal{B}_2$. That is, the columns of $M(L)$ are formed by the vectors $L(\mathbf{u}_1)_{\mathcal{B}_2}, L(\mathbf{u}_2)_{\mathcal{B}_2}, \ldots, L(\mathbf{u}_n)_{\mathcal{B}_2}$.

Now, let $\mathbf{x} \in U$ be any vector. We can express $\mathbf{x}$ in terms of the basis $\mathcal{B}_1$ as $\mathbf{x} = x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ represents the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}_1$. Then,

$$\begin{aligned}
L(\mathbf{x}) &= x_1 L(\mathbf{u}_1) + \cdots + x_n L(\mathbf{u}_n) \\
&= x_1 \sum_{i=1}^{m} a_{i1}\mathbf{v}_i + x_2 \sum_{i=1}^{m} a_{i2}\mathbf{v}_i + \cdots + x_n \sum_{i=1}^{m} a_{in}\mathbf{v}_i \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \mathbf{v}_i
\end{aligned}$$

Thus, the coordinates of $L(\mathbf{x})$ with respect to the basis $\mathcal{B}_2$, denoted by $L(\mathbf{x})_{\mathcal{B}_2}$, are given by the matrix-vector product:

$$L(\mathbf{x})_{\mathcal{B}_2} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = M(L)\mathbf{x}$$

Therefore, for any linear map $L : U \to V$, there is a matrix $M(L)$ such that $L(\mathbf{x})_{\mathcal{B}_2} = M(L)\mathbf{x}$. We can write $M(L)$ as:

$$M(L) = \left[L(\mathbf{u}_1)_{\mathcal{B}_2} \mid L(\mathbf{u}_2)_{\mathcal{B}_2} \mid \cdots \mid L(\mathbf{u}_n)_{\mathcal{B}_2} \right]$$

where each $L(\mathbf{u}_i)_{\mathcal{B}_2}$ is the coordinate vector of $L(\mathbf{u}_i)$ with respect to the basis $\mathcal{B}_2$ of V.

Example 3.1. *Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map given by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2x-3y \\ x-3y \end{bmatrix}$. Find the matrix associated with L with respect to the standard bases.*

Solution: *The standard basis for $\mathbb{R}^2$ is $\mathfrak{B}_1 = \{\mathbf{i}, \mathbf{j}\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then $L(\mathbf{i}) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $L(\mathbf{j}) = \begin{bmatrix} -1 \\ -3 \\ -3 \end{bmatrix}$ with respect to the standard basis of $\mathbb{R}^3$. Hence, the associated matrix of L is $M = \begin{bmatrix} 1 & -1 \\ 2 & -3 \\ 1 & -3 \end{bmatrix}$. with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.* □

Example 3.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

for some fixed angle $\theta \in \mathbb{R}$.

The reader should show that T is a linear map. It is an exercise in trigonometry to show that this map rotates every point of $\mathbb{R}^2$ by the angle θ. What is the matrix associated to T with respect to the standard basis of $\mathbb{R}^2$?

Solution: We have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Then, the associated matrix is:

$$A := M(T) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Prove that geometrically the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates every point of the plane by an angle θ counterclockwise. Then $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$. since rotating n-times by θ is the same as rotating by the angle $n\theta$. □

We now will see an example when neither of the bases $\mathfrak{B}_1$, $\mathfrak{B}_2$ is a standard basis.

Example 3.3. *Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the map between sets given by*

$$T(x, y, z) = (x + y, y + z, x - y, y - z)$$

Consider now $\mathbb{R}^3$ and $\mathbb{R}^4$ as $\mathbb{R}$-vector spaces and fix bases

$$\mathfrak{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}, \quad \text{and} \quad \mathfrak{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

of $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. Find the associated matrix of L with respect to $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

Solution: It can be easily checked that this is a linear map if we consider $\mathbb{R}^3$ and $\mathbb{R}^4$ as vector spaces over $\mathbb{R}$. Then

$$T(\mathbf{u}_1) = [2,2,0,0]^t =: \mathbf{w}_1, \quad T(\mathbf{u}_2) = [3,1,1,1]^t =: \mathbf{w}_2, \quad T(\mathbf{u}_3) = [4,2,2,0]^t =: \mathbf{w}_3$$

Now we need to express the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ with respect to the basis $\mathfrak{B}_2$. Each one of them must be expressed as

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 + r_4\mathbf{v}_4 = (r_1 + r_2 + 2r_3, 2r_2 + 3r_3, 2r_3, r_1 + r_3 + 2r_4).$$

Thus we have (with respect to $\mathfrak{B}_2$)

$$\mathbf{w}_1 = \left[1,1,0,-\frac{1}{2}\right]^t, \quad \mathbf{w}_2 = \left[\frac{9}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right]^t, \quad \mathbf{w}_3 = \left[\frac{3}{2}, -\frac{1}{2}, 1, 0\right]^t$$

The associated matrix is

$$M_{\mathfrak{B}_1}^{\mathfrak{B}_2} = \begin{bmatrix} 1 & \frac{9}{4} & \frac{3}{2} \\ 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

□

The following theorem makes precise the relation between matrices and linear maps. Let U and V be vector spaces over $\mathbb{F}$ and $\mathfrak{B}_1$, $\mathfrak{B}_2$ their bases respectively. From now on when there is no confusion for a linear map $f : U \to V$ we will simply use M_f instead of $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(f)$.

Theorem 3.1. *Let U and V be vector spaces over $\mathbb{F}$ and $\mathfrak{B}_1$, $\mathfrak{B}_2$ their respective bases. For any $f, g \in \mathcal{L}(U,V)$ the following hold:*

(i) $M_{f+g} = M_f + M_g$.

(ii) $M_{rf} = rM_f$, *for any scalar $r \in \mathbb{F}$.*

(iii) $M_{f \circ g} = M_f \cdot M_g$.

Proof. The proof is straight forward from the properties of matrix addition and multiplication by a scalar. The third item is true since

$$M_{f\circ g} = M_{f(g(\mathbf{x}))} = M_{f M_g \mathbf{x}} = M_f \cdot \left(M_g \mathbf{x}\right) = (M_f \cdot M_g)\mathbf{x}$$

□

The following theorem shows that not only to every linear map we can associate a matrix but that the converse also holds for finite dimensional vector spaces.

Theorem 3.2 (Isomorphism Theorem of Linear Algebra)**.** *Let U and V be vector spaces over $\mathbb{F}$ of dimension n and m respectively. Fix bases $\mathfrak{B}_1$, $\mathfrak{B}_2$ of U and V. Further, let $\mathcal{L}(U,V)$ be the space of linear maps $T: U \to V$. Then*

$$\Phi : \mathcal{L}(U,V) \longrightarrow \mathrm{Mat}_{m\times n}(\mathbb{F})$$

such that $\Phi(T) = M(T)$, is an isomorphism.

Proof. The previous theorem shows that Φ is a linear map. First we show that Φ is injective. Let $f, g \in \mathcal{L}(U,V)$ such that $\Phi(f) = \Phi(g)$. Thus, $M(f) = M(g)$. Hence, for every $\mathbf{x} \in U$ we have

$$M(f)\,\mathbf{x} = M(g)\,\mathbf{x}$$

which means that $f(x) = g(x)$. Therefore, $f = g$ and Φ is injective.

Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{F})$. Define $L_A : U \longrightarrow V$ such that $L_A(\mathbf{x}) = A\,\mathbf{x}$. Then, $L_A \in \mathcal{L}(U,V)$. Hence, Φ is surjective.

□

Some special linear maps $L: U \to V$ are the ones when the corresponding matrix M_L is a diagonal matrix. Such maps are called **diagonal linear maps**.

Example 3.4. *Consider the linear map $L: \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}.$$

Geometrically, such map shrinks the x-coordinate by a and the y-coordinate by b.

We will discuss diagonal maps in more detail in the coming lectures. Moreover, we will see that every linear map $L: U \to V$, under certain conditions, by picking the right bases for U and V can be diagonalized.

Exercises:

216. *Check whether the map $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $T(x,y,z) = (x-2y, y-x, x+y)$ is linear. If it is linear then find its associated matrix.*

217. *Find the associated matrix for the linear map $T: \mathbb{R}^3 \to \mathbb{R}^4$ such that $T(x,y,z) = (x-y+2z, y+z, 3x-2y-z, 7y+z)$ and find a basis for $\ker(T)$.*

218. *Find the associated matrix for the linear map $T: \mathbb{R}^4 \to \mathbb{R}^4$ such that*

$$T(x,y,z,w) = (x-y+z, 2x-2y+2z, x+y-z-b, 2x-w)$$

and find a basis for $\ker(T)$.

219. *Find the standard matrix representation of the rotation of the xy-plane counterclockwise about the origin with an angle:*

(i) $45°$

(ii) $60°$

(iii) $15°$

220. *Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation counter clockwise by the angle $\theta = \frac{\pi}{3}$. Find $T(0,1)$, $T(1,1)$, $T(-1,1)$.*

221. *Find the rank and nullity, and bases for the column space, row space, and the nullspace of the matrix*
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & 1 & 1 & 2 \\ -1 & 3 & 4 & 3 \\ -1 & 3 & 4 & 3 \end{bmatrix}$$

222. *Let $\mathcal{L}^1(\mathbb{R})$ be the vector space of differentiable* *functions from* $\mathbb{R}$ *to* $\mathbb{R}$*. Let*

$$V := \textit{Span}\,(\sin x, \cos x)$$

and $D : \mathcal{L}^1(\mathbb{R}) \to \mathcal{L}^1(\mathbb{R})$ *the differentiation map. The restriction of this map to* V *gives a linear map* $D_V : V \to V$*. Find the matrix representation of* D_V *for* $\mathfrak{B}_1 = \mathfrak{B}_2 = \{\sin x, \cos x\}$*.*

223. *Let* P_n *denote the vector space over* $\mathbb{R}$ *of polynomials with coefficients in* $\mathbb{R}$ *and degree* $\leq n$*. Differentiation of polynomials is a linear map on this space. Find its matrix representation for*

$$\mathfrak{B}_1 = \mathfrak{B}_2 = \{1, x, \dots, x^n\}.$$

224. *Let* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ *and* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $T(\mathbf{x}) = \mathbf{u} + \mathbf{x}$*. Find the matrix representation of* T *with respect to the standard basis of* $\mathbb{R}^2$*.*

225. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be the transformation which rotates every point counterclockwise by the angle* θ*. Find its matrix representation with respect to the standard basis.*

226. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be the transformation of the plane which sends every point to its symmetric point with respect to the x-axis (i.e.,* $T(x, y) = T(x, -y)$*). Find the matrix representation of* T *with respect to the standard basis.*

227. *Find the standard matrix representation for the reflection of the xy-plane with respect to the line* $y = x + 2$*.*

228. *Check whether the map* $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ *such that* $T(x, y, z) = (x + 2, y - x, x + y)$ *is linear. If it is linear then find its associated matrix.*

229. *Find the associated matrix with respect to the standard bases to the map* $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ *such that* $T(x, y, z) = (x, y, x + y + z)$*.*

230. *Find the associated matrix with respect to the standard bases to the map* $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *such that* $T(x, y) = (x + y, 3y, 7x + 2y)$*.*

231. *Find the associated matrix with respect to the standard bases to the map* $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ *such that* $T(x_1, \dots, x_5) = (x_1, x_2, x_3, x_4, x_5)$*.*

3.2 Nullspace and rank of a matrix

Let A be a $m \times n$ matrix over $\mathbb{F}$. Consider all the rows R_i of A. These are vectors in $\mathbb{F}^n$. The span of row vectors of A is called a **row space of** A. Similarly the column vectors of A are vectors in $\mathbb{F}^m$ and the span of column vectors is called the **column space of** A. As before the **nullspace** of A, denoted by $\mathbf{Null}(A)$, will be the solution set of $A\mathbf{x} = \mathbf{0}$.

Theorem 3.3. *Let A be an $m \times n$ matrix. The dimension of the row space is the same as the dimension of the column space. This common dimension is equal to the number of pivots in the row-echelon form* $\mathrm{rref}(A)$ *of A.*

Proof. We use the previous procedure to find the dimension in both cases. This dimension is the number of pivots.

□

This common dimension is called the **rank of** A and is denoted by **rank** (A). The dimension of the nullspace is called the **nullity** of A and is denoted by **null** (A).

Theorem 3.4 (Rank-Nullity Theorem)**.** *Let A be an $m \times n$ matrix and* $\mathrm{ref}(A)$ *its row-echelon form*

(i) ***rank*** (A) = *number of pivots of* $\mathrm{ref}(A)$

(ii) ***null*** (A)= *number of columns without a pivot*

Moreover,

$$\textbf{\textit{rank}}\ (A) + \textbf{\textit{null}}\ (A) = n$$

Proof. (i) The rank of A is defined as the dimension of its row space (or column space). The row space of A is unchanged by elementary row operations, so the row space of A has the same dimension as the row space of $\mathrm{ref}(A)$. The nonzero rows of $\mathrm{ref}(A)$ are linearly independent, and they form a basis for the row space of $\mathrm{ref}(A)$. Therefore, the dimension of the row space of $\mathrm{ref}(A)$ (and thus the row space of A) is equal to the number of nonzero rows in $\mathrm{ref}(A)$, which is the number of pivot columns. Thus, **rank** (A) is equal to the number of pivot columns in $\mathrm{ref}(A)$.

(ii) The nullspace of A is the set of all vectors $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} = \mathbf{0}$. Solving the homogeneous system $A\mathbf{x} = \mathbf{0}$ is equivalent to solving the system $\mathrm{ref}(A)\mathbf{x} = \mathbf{0}$. Each column of A corresponds to either a pivot variable or a free variable in the solution to this system. The number of free variables is equal to the number of columns without a pivot in $\mathrm{ref}(A)$. Each free variable corresponds to a parameter in the general solution to the homogeneous system. The solutions to $\mathrm{ref}(A)\mathbf{x} = \mathbf{0}$ (and therefore $A\mathbf{x} = \mathbf{0}$) can be written as linear combinations of vectors, where each vector corresponds to a free variable. These vectors form a basis for the nullspace of A. Thus, the dimension of the nullspace of A, which is (A), is equal to the number of columns without pivots in $\mathrm{ref}(A)$.

Finally, we have n columns in the matrix A. Each column corresponds to either a pivot variable or a free variable. The number of pivot variables is **rank** (A), and the number of free variables is (A). Therefore, the total number of columns is the sum of these two:

$$\mathbf{rank}\ (A) + (A) = n.$$

This completes the proof.

□

Algorithm: Finding a Basis for the Nullspace of a Matrix

Input: An $m \times n$ matrix A.
Output: A basis for Null(A).

Steps:

1. **Reduce A to Row-Echelon Form:**

   Use Gaussian elimination (row operations) to transform A into its reduced row-echelon form, denoted as ref(A).

2. **Identify Free Variables:**

   Determine the columns in ref(A) that do not have a leading 1 (pivot). These columns correspond to the free variables in the system $Ax = 0$.

3. **Express Solutions in Terms of Free Variables:**

   Solve the system ref$(A)x = 0$. Write the solutions in parametric form, where the free variables are the parameters. Express each basic variable (corresponding to pivot columns) in terms of the free variables.

4. **Construct Basis Vectors:**

   For each free variable, create a vector by setting that free variable to 1 and all other free variables to 0. The entries of the vector are the corresponding values of the basic variables from the parametric solution. The set of vectors constructed in this way forms a basis for **Null**$(()A)$.

Let us see an example.

Example 3.5. *Let's find a basis for the nullspace of the matrix:*

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 2 & 1 \end{bmatrix}$$

Reduce to row-echelon form:

$$\text{ref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables are x_2 and x_4. From ref(A)*, we have*

$$\begin{aligned} x_1 &= -2x_2 - x_4 \\ x_3 &= x_4 \end{aligned}$$

The general solution is:

$$x = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The set

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbf{Null}(A)$.

Thus, the row-echelon form of a matrix has all the information that we need about the column space of the matrix, the row space, and the nullspace. In the next example we see how we can find a basis for each one of such spaces.

Example 3.6. *Find the rank, nullity, a basis for the row space, a basis for the column space, and a basis for the nullspace of the matrix*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: *We start by finding the reduced row-echelon form of A.*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\mathbf{rank}\,(A) = 2$ *and* $\mathbf{null}\,(A) = 1$*. A basis for the column space is* $\mathfrak{B}_1 = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$*. To find a basis of the row-space we use the rows from H which contains pivots. So we have* $\mathfrak{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$*. To find a basis for the nullspace we have to solve the system* $\mathrm{ref}(A)\mathbf{x} = \mathbf{0}$*. The augmented matrix is:*

$$[\mathrm{rref}(A) \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, x_3 *is a free variable and* $x_2 + x_3 = 0$ *and* $x_1 + x_3 = 0$*. The solution is*

$$\mathbf{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

So a basis for the nullspace is $\mathfrak{B}_3 = \left\{ [-1, -1, 1]^t \right\}$. $\square$

3.2.1 Finding a basis for the row-space, column-space, and nullspace of a matrix.

Given a $m \times n$ matrix A, we would like to find the bases of spaces associated with it. We have the following algorithm.

Algorithm 4. ***Finding a basis for spaces of a matrix***

Input: An $m \times n$ matrix A
Output: A basis for the row-space, column-space, and nullspace $\mathbf{Null}(A)$ *of* A

i) Find a reduced row-echelon form $\mathbf{ref}\,(A)$ *of* A
ii) The columns of A corresponding to the columns of $\mathbf{ref}\,(A)$ *with pivots, form a basis for the column space.*
iii) The nonzero rows of $\mathbf{ref}\,(A)$ *form a basis for the row space.*
iv) Use back substitution to solve $\mathbf{ref}\,(A)\mathbf{x} = \mathbf{0}$ *and determine* $\mathbf{Null}(A)$.

Example 3.7. *Find bases for the spaces associated with* $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{bmatrix}$.

Solution: *A reduced row-echelon form is* $\mathbf{ref}\,(A) = \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$. *A basis for the column space is* $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$. *The rank of A is* $\mathbf{rank}\,(A) = 3$ *and* $\mathbf{null}\,(A) = 1$. *Thus, there is one free variable which we denote by x_4. Solving* $\mathrm{rref}(A)\mathbf{x} = \mathbf{0}$ *we have*

$$\mathbf{x} = \begin{bmatrix} -\frac{3}{2}x_4 \\ -\frac{1}{2}x_4 \\ \frac{1}{2}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} = x_4 \left[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, 1 \right]^t$$

A basis of the nullspace $\mathbf{Null}(()A)$ *is* $\mathfrak{B} = \left\{ \left[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, 1 \right]^t \right\}$. *For a basis of the row space we take all three rows of H.* ◻

Example 3.8. *For the matrix* $A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ -2 & 1 & 1 & 2 \\ 3 & -1 & 2 & 1 \end{bmatrix}$ *find its* $\mathbf{rank}\,(A)$, $\mathbf{null}\,(A)$, *and bases for* $\mathbf{Col}\,(A)$, $\mathbf{Row}\,(A)$, $\mathbf{Null}(A)$.

Solution: *The reduced row-echelon form of A is* $\mathbf{ref}\,(A) = \begin{bmatrix} 1 & 0 & 0 & -\frac{6}{23} \\ 0 & 1 & 0 & \frac{9}{23} \\ 0 & 0 & 1 & \frac{25}{23} \end{bmatrix}$. *Then,* $\mathbf{rank}\,(A) = 3$ *and* $\mathbf{null}\,(A) = 1$. *For the basis of the column space we have* $\left\{ \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$. *For the basis of the* *row-space we take all three rows of A since each one of them contains a pivot. Next we find a basis for the nullspace. Hence, we have to solve the system $H\mathbf{x} = \mathbf{0}$. The solution is*

$$\mathbf{x} = \begin{bmatrix} -\frac{6}{23} \\ \frac{9}{23} \\ \frac{25}{23} \\ 1 \end{bmatrix} \cdot x_4 = \begin{bmatrix} -6 \\ 9 \\ 25 \\ 23 \end{bmatrix} \cdot t,$$

for some free variable t. Hence, a basis is $\mathfrak{B} = \left\{[-6,9,25,23]^t\right\}$. □

The next theorem relates some of the previous topics to this section.

Theorem 3.5. *Let A be an $n \times n$ matrix. The following are equivalent:*

(i) *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.*
(ii) *A is row equivalent to I_n.*
(iii) *A is invertible.*
(iv) *The column vectors of A form a basis for $\mathbb{R}^n$.*

Proof. We will demonstrate that each statement implies each of the others, thus proving their equivalence:

(i) ⇒ (ii): If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$, then A must have a pivot in every row and every column when reduced to row-echelon form; see Lem. 1.14. Thus, the row-echelon form of A must be I_n, meaning A is row equivalent to I_n.

(ii) ⇒ (iii): If A is row equivalent to I_n, then there exists an elementary matrix E such that $EA = I_n$. Since E is invertible (as elementary matrices are invertible), this means A has an inverse, namely $A^{-1} = E$. Therefore, A is invertible.

(iii) ⇒ (i): If A is invertible, then for any $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$. Hence for every $\mathbf{b}$, there is exactly one $\mathbf{x}$ (since $A^{-1}\mathbf{b}$ is unique), thus $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.

(iii) ⇒ (iv): If A is invertible, then:

- The columns of A are linearly independent because if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$, so no non-trivial linear combination of the columns equals zero.

- The columns also span $\mathbb{R}^n$ because, for any $\mathbf{b} \in \mathbb{R}^n$, we can write $\mathbf{b} = A(A^{-1}\mathbf{b})$, meaning any vector in $\mathbb{R}^n$ can be expressed as a linear combination of the columns of A.

Thus, the columns of A form a basis for $\mathbb{R}^n$.

(iv) ⇒ (iii): If the columns of A form a basis for $\mathbb{R}^n$, then:

- There are n linearly independent columns in A, implying that A has full rank (n).

- Since A has n columns and they span $\mathbb{R}^n$, A must map $\mathbb{R}^n$ onto $\mathbb{R}^n$ in a one-to-one fashion, which means A is invertible.

(iv) ⇒ (i): If the columns of A form a basis for $\mathbb{R}^n$, then:

- There's a unique representation of any vector $\mathbf{b}$ as a combination of the basis vectors (columns of A), which means $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.

(i) ⇒ (iv): We've already shown (i) ⇒ (ii) ⇒ (iii) ⇒ (iv), so this is covered by transitivity.

This completes the proof of the equivalence of all four statements. □

The following result is quite useful when checking for inverses.

Corollary 3.1. *Let A be an $n\times n$ matrix. Then A is invertible if and only if* $\textbf{\textit{rank}}\,(A) = n$.

Exercise 48. *Find the rank, a basis for the row space, and a basis for the column space, a basis for the nullspace for the following matrices.*

$$\begin{bmatrix} 2 & 3 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Exercises:

232. *Let A be a square matrix. Show that*

$$\textbf{\textit{null}}\,(A) = \textbf{\textit{null}}\,(A^t).$$

233. *Let A, B be matrices such that the product AB is defined. Show that* $\textbf{\textit{rank}}\,(AB) \leq \textbf{\textit{rank}}\,(A)$.

234. *Give an example of two matrices A, B such that*

$$\textbf{\textit{rank}}\,(AB) < \textbf{\textit{rank}}\,(A).$$

235. *Let A be an $m\times n$ matrix. Prove that*

$$\textbf{\textit{rank}}\,(AA^t) = \textbf{\textit{rank}}\,(A).$$

236. *Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent column vectors in $\mathbb{R}^3$ and A an invertible 3×3 matrix. Prove that the vectors $A\mathbf{u}$ and $A\mathbf{v}$ are linearly independent.*

237. *Generalize the above problem to $\mathbb{R}^n$. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be linearly independent column vectors in $\mathbb{R}^n$ and A an invertible $n\times n$ matrix. Prove that the vectors $A\mathbf{u}_1, \dots, A\mathbf{u}_n$ are linearly independent.*

238. *Let $\mathbf{u}$ and $\mathbf{v}$ be column vectors in $\mathbb{R}^3$ and A an invertible 3×3 matrix. Prove that if vectors $A\mathbf{u}$ and $A\mathbf{v}$ are linearly independent then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.*

239. *Generalize the above problem to $\mathbb{R}^n$. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be column vectors in $\mathbb{R}^n$ and A an invertible $n\times n$ matrix. Prove that if vectors $A\mathbf{u}_1, \dots, A\mathbf{u}_n$ are linearly independent then $\mathbf{u}_1, \dots, \mathbf{u}_n$ are linearly independent.*

240. *Let*

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some angle θ. Take any vector $\mathbf{u} \in \mathbb{R}^2$ and compare it with the vector $A\mathbf{u}$. What happens geometrically?

241. *Let A be as in the previous exercise and $\{\mathbf{u}, \mathbf{v}\}$ a basis in $\mathbb{R}^2$. Show that $\{A\mathbf{u}, A\mathbf{v}\}$ is a basis for $\mathbb{R}^2$. You might want to look at the nullspace of A.*

3.3 Change of basis

Sometimes we have to deal with two different bases for the same vector space. The above discussion gives a way of finding the coordinates of a vector with respect to a given basis.

Let V be a vector space and $\mathfrak{B}, \mathfrak{B}'$ two bases of V given by

$$\mathfrak{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}, \quad \mathfrak{B}' = \{\mathbf{b}'_1, \dots, \mathbf{b}'_n\},$$

where both $\mathfrak{B}$ and $\mathfrak{B}'$ are ordered bases of dimension n. Consider the linear map $T : V \to V$ defined by $T(\mathbf{b}_i) = \mathbf{b}'_i$ for each $i = 1, \dots, n$. This map T transforms vectors from the basis $\mathfrak{B}$ to the basis $\mathfrak{B}'$.

We denote the associated matrix of T with respect to the basis $\mathfrak{B}$ by $M_{\mathfrak{B}}^{\mathfrak{B}'}$ and call it the **transformation matrix** from $\mathfrak{B}$ to $\mathfrak{B}'$. The matrix $M_{\mathfrak{B}}^{\mathfrak{B}'}$ is constructed by expressing each $\mathbf{b}'_i$ as a linear combination of the vectors in $\mathfrak{B}$. Specifically,

$$M_{\mathfrak{B}}^{\mathfrak{B}'} = \left[\mathbf{b}'_1 \,|\, \cdots \,|\, \mathbf{b}'_n \right],$$

where $\mathbf{b}'_i$ denotes the coordinate vector of $\mathbf{b}'_i$ with respect to the basis $\mathfrak{B}$. This means the i-th column of $M_{\mathfrak{B}}^{\mathfrak{B}'}$ contains the coefficients when $\mathbf{b}'_i$ is written as $\mathbf{b}'_i = \sum_{j=1}^{n} m_{ji} \mathbf{b}_j$, where m_{ji} are the entries of $M_{\mathfrak{B}}^{\mathfrak{B}'}$.

Intuitively, $M_{\mathfrak{B}}^{\mathfrak{B}'}$ provides a way to convert the coordinates of any vector $\mathbf{v} \in V$ from the basis $\mathfrak{B}$ to the basis $\mathfrak{B}'$. If $\mathbf{v}_{\mathfrak{B}}$ is the coordinate vector of $\mathbf{v}$ with respect to $\mathfrak{B}$, then the coordinate vector $\mathbf{v}_{\mathfrak{B}'}$ with respect to $\mathfrak{B}'$ is given by

$$\mathbf{v}_{\mathfrak{B}'} = M_{\mathfrak{B}}^{\mathfrak{B}'} \mathbf{v}_{\mathfrak{B}},$$

assuming the standard convention where the transformation matrix maps old coordinates to new coordinates.

We now provide the following algorithm for computing the transformation matrix.

Algorithm 5. *Input: A vector space V and two bases $\mathfrak{B}_1 = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\mathfrak{B}_2 = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ of V. Output: The transformation matrix $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}$, such that $M_{\mathfrak{B}_1}^{\mathfrak{B}_2} \cdot \mathbf{v}_{\mathfrak{B}_1} = \mathbf{v}_{\mathfrak{B}_2}$ for any $\mathbf{v} \in V$.*

i) Create the matrix

$$A = [\, \mathbf{v}_1 \,|\, \dots \,|\, \mathbf{v}_n \,|\, \mathbf{u}_1 \,|\, \dots \,|\, \mathbf{u}_n \,],$$

where the first n columns are the vectors of $\mathfrak{B}_2$ and the last n columns are the vectors of $\mathfrak{B}_1$, all expressed as coordinates in some common basis (e.g., the standard basis if $V = \mathbb{R}^n$).

ii) Transform A by row operations to the matrix $\left[I \,|\, M_{\mathfrak{B}_1}^{\mathfrak{B}_2} \right]$, where I is the $n \times n$ identity matrix, and $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}$ is the desired transformation matrix.

Example 3.9. *Let $V = \mathbb{R}^2$ and $\mathfrak{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, $\mathfrak{B}_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be two bases of V. Find the transformation matrix $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}$. Given vectors $\mathbf{u}, \mathbf{v}$ with coordinates $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ with respect to the $\mathfrak{B}_1$ basis, find their coordinates with respect to $\mathfrak{B}_2$.*

Solution: *We first create the matrix* $A = [\mathfrak{B}_2|\mathfrak{B}_1] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$. *By row operations we transform A to a row echelon form*

$$\textbf{\textit{ref}}\,(A) = \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

Then $M_{\mathfrak{B}_1}^{\mathfrak{B}_2} = \frac{1}{3} \cdot \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$,

$$\mathbf{u}_{\mathfrak{B}_2} = M_{\mathfrak{B}_1}^{\mathfrak{B}_2} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 10 \\ -11 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_{\mathfrak{B}_2} = M_{\mathfrak{B}_1}^{\mathfrak{B}_2} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

□

Example 3.10. *Let* $\mathbf{u} \in \mathbb{R}^3$ *with coordinates in the standard basis* $\mathbf{u}_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *Find the transformation matrix* $M_{\mathfrak{B}}^{\mathfrak{B}'}$ *from the standard basis to the basis* $\mathfrak{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$, *and use it to find the coordinates of* $\mathbf{u}$ *with respect to* $\mathfrak{B}'$.

Solution: *We first create the matrix*

$$A = [\mathfrak{B}'|\mathfrak{B}] = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix},$$

where the first three columns are the vectors of $\mathfrak{B}'$ *and the last three columns are the standard basis vectors. By row operations, we compute the row echelon form:*

$$\textbf{\textit{ref}}\,(A) = \begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & | & 0 & -1 & 1 \\ 0 & 0 & 1 & | & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}.$$

Then the transformation matrix is

$$M_{\mathfrak{B}}^{\mathfrak{B}'} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -1 & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix},$$

and the coordinates of $\mathbf{u}$ *with respect to* $\mathfrak{B}'$ *are*

$$\mathbf{u}_{\mathfrak{B}'} = M_{\mathfrak{B}}^{\mathfrak{B}'} \mathbf{u}_{\mathfrak{B}} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -1 & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 1 \\ -\frac{3}{2} \end{bmatrix}.$$

□

The next result considers the simplest case when the linear map is from V to itself ($L : V \to V$).

Proposition 3.1. *Let V be a finite-dimensional vector space, and let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases of V. Let $L : V \to V$ be a linear transformation, and let $M_{\mathfrak{B}}(L)$ and $M_{\mathfrak{B}'}(L)$ be the matrices representing L with respect to bases $\mathfrak{B}$ and $\mathfrak{B}'$ respectively. Let $M := M_{\mathfrak{B}}^{\mathfrak{B}'}$ be the change of basis matrix from $\mathfrak{B}$ to $\mathfrak{B}'$, which is invertible since $\mathfrak{B}$ and $\mathfrak{B}'$ are bases of the same space. Then,*

$$M_{\mathfrak{B}'}(L) = M^{-1} \cdot M_{\mathfrak{B}}(L) \cdot M.$$

$$\begin{tikzcd}
V_{\mathfrak{B}} \arrow[r, "M=M_{\mathfrak{B}}^{\mathfrak{B}'}"] \arrow[d, "M_{\mathfrak{B}}(L)"'] & V_{\mathfrak{B}'} \arrow[d, "M_{\mathfrak{B}'}(L)"] \\
V_{\mathfrak{B}} \arrow[r, "M=M_{\mathfrak{B}}^{\mathfrak{B}'}"'] & V_{\mathfrak{B}'}
\end{tikzcd}$$

Figure 3.1: Commutative diagram for the change of basis and matrix representation of a linear transformation

Proof. Let $\mathfrak{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ and $\mathfrak{B}' = \{\mathbf{b}_1', \mathbf{b}_2', \dots, \mathbf{b}_n'\}$. The change of basis matrix $M = M_{\mathfrak{B}}^{\mathfrak{B}'}$ is defined as

$$M = \left[(\mathbf{b}_1')_{\mathfrak{B}} \,|\, (\mathbf{b}_2')_{\mathfrak{B}} \,|\, \dots \,|\, (\mathbf{b}_n')_{\mathfrak{B}} \right],$$

where $(\mathbf{b}_i')_{\mathfrak{B}}$ denotes the coordinate vector of $\mathbf{b}_i'$ with respect to the basis $\mathfrak{B}$.

Let $\mathbf{v} \in V$. We can express $\mathbf{v}$ in terms of $\mathfrak{B}$ and $\mathfrak{B}'$ as

$$\mathbf{v}_{\mathfrak{B}} = \sum_{i=1}^{n} r_i \mathbf{b}_i \quad \text{and} \quad \mathbf{v}_{\mathfrak{B}'} = \sum_{i=1}^{n} s_i \mathbf{b}_i',$$

where $\mathbf{v}_{\mathfrak{B}} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ and $\mathbf{v}_{\mathfrak{B}'} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ are the coordinate vectors with respect to $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. The coordinate vectors are related by the change of basis: since $M\mathbf{v}_{\mathfrak{B}'} = \mathbf{v}_{\mathfrak{B}}$, it follows that $\mathbf{v}_{\mathfrak{B}'} = M^{-1}\mathbf{v}_{\mathfrak{B}}$.

Applying the linear transformation L to $\mathbf{v}$ expressed in both bases, we get:

$$L(\mathbf{v}) = \sum_{i=1}^{n} r_i L(\mathbf{b}_i) = M_{\mathfrak{B}}(L)\mathbf{v}_{\mathfrak{B}},$$

where $M_{\mathfrak{B}}(L)$ is the matrix of L with respect to $\mathfrak{B}$. Similarly,

$$L(\mathbf{v}) = \sum_{i=1}^{n} s_i L(\mathbf{b}'_i) = M_{\mathfrak{B}'}(L)\mathbf{v}_{\mathfrak{B}'},$$

where $M_{\mathfrak{B}'}(L)$ is the matrix of L with respect to $\mathfrak{B}'$.

Substituting $\mathbf{v}_{\mathfrak{B}'} = M^{-1}\mathbf{v}_{\mathfrak{B}}$ into the second equation, we obtain:

$$M_{\mathfrak{B}'}(L)\mathbf{v}_{\mathfrak{B}'} = M_{\mathfrak{B}'}(L)M^{-1}\mathbf{v}_{\mathfrak{B}}.$$

Since $L(\mathbf{v}) = M_{\mathfrak{B}}(L)\mathbf{v}_{\mathfrak{B}}$ and this must equal $M_{\mathfrak{B}'}(L)\mathbf{v}_{\mathfrak{B}'}$ when expressed in $\mathfrak{B}'$ coordinates, we have:

$$M_{\mathfrak{B}'}(L)M^{-1}\mathbf{v}_{\mathfrak{B}} = M_{\mathfrak{B}}(L)\mathbf{v}_{\mathfrak{B}}.$$

This equation holds for all $\mathbf{v} \in V$, and thus for all $\mathbf{v}_{\mathfrak{B}}$. Therefore, $M_{\mathfrak{B}'}(L)M^{-1} = M_{\mathfrak{B}}(L)$, and rearranging gives the desired result:

$$M_{\mathfrak{B}'}(L) = M^{-1}M_{\mathfrak{B}}(L)M.$$

This completes the proof. $\square$

Example 3.11. *Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the map defined by*

$$T(x,y,z) = (x - y + 2z, y + z, 3x - 2y - z, 7y + z).$$

Check if T is linear. If it is linear, find the associated matrix for the linear map and find a basis for ker (T).

Solution: *First, we verify linearity. For $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ and scalar $c \in \mathbb{R}$,*

$$T(\mathbf{u}+\mathbf{v}) = T(u_1+v_1, u_2+v_2, u_3+v_3) = ((u_1+v_1) - (u_2+v_2) + 2(u_3+v_3), \dots) = T(\mathbf{u}) + T(\mathbf{v}),$$

and $T(c\mathbf{u}) = cT(\mathbf{u})$, which holds by direct computation (omitted for brevity). Thus, T is linear.

The associated matrix $M(T)$ is found by applying T to the standard basis vectors of $\mathbb{R}^3$:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\-2\\7\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\-1\\1\end{bmatrix}.$$

Thus,

$$M(T) = \begin{bmatrix}1 & -1 & 2\\0 & 1 & 1\\3 & -2 & -1\\0 & 7 & 1\end{bmatrix},$$

a 4×3 matrix. Its reduced row-echelon form is:

$$\textbf{\textit{ref}}\,(M) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix},$$

indicating a rank of 3 (since there are 3 pivot columns). By the rank-nullity theorem, the nullity is $3 - rank = 0$, *so the only solution to* $M(T)\mathbf{x} = \mathbf{0}$ *is* $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. *Hence,* $\ker(T) = \{\mathbf{0}\}$. □

Example 3.12. *Let* $V := \text{Span}\,(e^x, xe^x)$. *Find the transformation matrix from basis* $\mathfrak{B}_1 := \{e^x, xe^x\}$ *to basis* $\mathfrak{B}_2 = \{2xe^x, 4e^x\}$.

Solution: *A vector* $\mathbf{v} \in V$ *can be expressed in* $\mathfrak{B}_1$ *coordinates as*

$$\mathbf{v} = ae^x + bxe^x, \quad \mathbf{v}_{\mathfrak{B}_1} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

We need to express $\mathbf{v}$ *in terms of* $\mathfrak{B}_2 = \{2xe^x, 4e^x\}$. *Since* $2xe^x$ *and* $4e^x$ *are scalar multiples of the original basis vectors, we solve for the coefficients:*

$$\mathbf{v} = c(2xe^x) + d(4e^x).$$

Equating to the $\mathfrak{B}_1$ *expression,* $ae^x + bxe^x = d(4e^x) + c(2xe^x)$, *we match coefficients:*

- *Coefficient of* e^x: $a = 4d \implies d = \frac{a}{4}$.
- *Coefficient of* xe^x: $b = 2c \implies c = \frac{b}{2}$.

Thus,

$$\mathbf{v}_{\mathfrak{B}_2} = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \frac{b}{2} \\ \frac{a}{4} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \end{bmatrix} \mathbf{v}_{\mathfrak{B}_1}.$$

Hence, the transformation matrix from $\mathfrak{B}_1$ *to* $\mathfrak{B}_2$ *is*

$$M = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \end{bmatrix}.$$

□

We can generalize the above theorem to any map $T : U \to V$.

Theorem 3.6. *Let* U *and* V *be* $\mathbb{F}$*-vector spaces with bases* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *respectively, and let*

$$T : U \to V$$

be a linear map with $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)$ *as its associated matrix. Let* $M_{\mathfrak{B}_1}^{\mathfrak{B}_1'}$ *(respectively* $M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}$*) be the matrix corresponding to the change of basis from* $\mathfrak{B}_1$ *to* $\mathfrak{B}_1'$ *for* U *(respectively from* $\mathfrak{B}_2$ *to* $\mathfrak{B}_2'$ *for* V*). Then,*

$$M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T) = (M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1} M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T) M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}.$$

Proof. Let $\mathbf{u} \in U$. The coordinate vector of $\mathbf{u}$ with respect to the basis $\mathfrak{B}_1'$ is related to its coordinate vector with respect to the basis $\mathfrak{B}_1$ by:

$$\mathbf{u}_{\mathfrak{B}_1'} = (M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1} \mathbf{u}_{\mathfrak{B}_1},$$

$$\begin{tikzcd}
U_{\mathfrak{B}_1} \arrow[r, "M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)"] \arrow[d, "M_{\mathfrak{B}_1}^{\mathfrak{B}'_1}"'] & V_{\mathfrak{B}_2} \arrow[d, "M_{\mathfrak{B}_2}^{\mathfrak{B}'_2}"] \\
U_{\mathfrak{B}'_1} \arrow[r, "M_{\mathfrak{B}'_1}^{\mathfrak{B}'_2}(T)"'] & V_{\mathfrak{B}'_2}
\end{tikzcd}$$

Figure 3.2: Commutative diagram illustrating the change of basis and matrix representation of a linear transformation $T : U \to V$

since $M_{\mathfrak{B}_1}^{\mathfrak{B}'_1} \mathbf{u}_{\mathfrak{B}'_1} = \mathbf{u}_{\mathfrak{B}_1}$.

The coordinate vector of $T(\mathbf{u})$ with respect to the basis $\mathfrak{B}_2$ is given by:

$$(T(\mathbf{u}))_{\mathfrak{B}_2} = M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)\mathbf{u}_{\mathfrak{B}_1}.$$

The coordinate vector of $T(\mathbf{u})$ with respect to the basis $\mathfrak{B}'_2$ is related to its coordinate vector with respect to the basis $\mathfrak{B}_2$ by:

$$(T(\mathbf{u}))_{\mathfrak{B}'_2} = M_{\mathfrak{B}_2}^{\mathfrak{B}'_2}(T(\mathbf{u}))_{\mathfrak{B}_2}.$$

Substituting the expression for $(T(\mathbf{u}))_{\mathfrak{B}_2}$, we get:

$$(T(\mathbf{u}))_{\mathfrak{B}'_2} = M_{\mathfrak{B}_2}^{\mathfrak{B}'_2} M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)\mathbf{u}_{\mathfrak{B}_1}.$$

Now, using the change of basis for $\mathbf{u}$, substitute $\mathbf{u}_{\mathfrak{B}_1} = M_{\mathfrak{B}_1}^{\mathfrak{B}'_1}\mathbf{u}_{\mathfrak{B}'_1}$:

$$(T(\mathbf{u}))_{\mathfrak{B}'_2} = M_{\mathfrak{B}_2}^{\mathfrak{B}'_2} M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T) M_{\mathfrak{B}_1}^{\mathfrak{B}'_1}\mathbf{u}_{\mathfrak{B}'_1}.$$

By definition, the matrix $M_{\mathfrak{B}'_1}^{\mathfrak{B}'_2}(T)$ satisfies:

$$(T(\mathbf{u}))_{\mathfrak{B}'_2} = M_{\mathfrak{B}'_1}^{\mathfrak{B}'_2}(T)\mathbf{u}_{\mathfrak{B}'_1}.$$

Comparing the expressions, we conclude that:

$$M_{\mathfrak{B}'_1}^{\mathfrak{B}'_2}(T) = M_{\mathfrak{B}_2}^{\mathfrak{B}'_2} M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T) M_{\mathfrak{B}_1}^{\mathfrak{B}'_1},$$

which, noting that $\mathbf{u}_{\mathfrak{B}_1} = M_{\mathfrak{B}_1}^{\mathfrak{B}'_1}\mathbf{u}_{\mathfrak{B}'_1}$ implies the inverse relationship, corrects to:

$$M_{\mathfrak{B}'_1}^{\mathfrak{B}'_2}(T) = (M_{\mathfrak{B}_1}^{\mathfrak{B}'_1})^{-1} M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T) M_{\mathfrak{B}_2}^{\mathfrak{B}'_2}.$$

□

Algorithm 6. *Input: Vector spaces U, V, bases $\mathfrak{B}_1, \mathfrak{B}_2$, linear map T with matrix $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)$, new bases $\mathfrak{B}_1', \mathfrak{B}_2'$. Output: The matrix $M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T)$.*

i) Compute $M_{\mathfrak{B}_1}^{\mathfrak{B}_1'}$ using row operations on the augmented matrix $[\mathfrak{B}_1' \mid \mathfrak{B}_1]$ reduced to $[I \mid M_{\mathfrak{B}_1}^{\mathfrak{B}_1'}]$.

ii) Compute $(M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1}$ using row operations on the augmented matrix $[M_{\mathfrak{B}_1}^{\mathfrak{B}_1'} \mid I]$ reduced to $[I \mid (M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1}]$.

iii) Compute $M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}$ using row operations on the augmented matrix $[\mathfrak{B}_2' \mid \mathfrak{B}_2]$ reduced to $[I \mid M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}]$.

iv) Compute the product $M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T) = (M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1} \cdot M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T) \cdot M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}$ by sequential matrix multiplication or further row operations:

First, compute $(M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1} \cdot M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)$ by row operations on $[M_{\mathfrak{B}_1}^{\mathfrak{B}_1'} \mid M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)]$ reduced to $[I \mid (M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1} \cdot M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)]$.

Then, compute the result from the previous step multiplied by $M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}$ by row operations on $[I \mid \text{result} \cdot M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}]$, but since the left is I, the right is the product directly.

To compute $M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T)$ using row operations directly, form the augmented matrix

$$[\mathfrak{B}_1' \mid (T(\mathfrak{B}_1'))_{\mathfrak{B}_2'}],$$

where $(T(\mathfrak{B}_1'))_{\mathfrak{B}_2'}$ are the coordinates of $T(\mathbf{b})$ for each $\mathbf{b} \in \mathfrak{B}_1'$ with respect to $\mathfrak{B}_2'$. Reduce the left part to the identity matrix; the right part will be $M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T)$.

To find the coordinates $(T(\mathbf{b}))_{\mathfrak{B}_2'}$, compute $T(\mathbf{b})$ in the standard basis (assuming $\mathfrak{B}_1 = \mathfrak{B}_2 =$ standard), then solve $M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}\left[c\right] = T(\mathbf{b})$, or use row operations on $[\, M_{\mathfrak{B}_2}^{\mathfrak{B}_2'} \mid T(\mathfrak{B}_1')]$ reduced to [I | coordinates].

This method is efficient for small dimensions and reinforces the concept of basis changes. Let us just verify the above theorem with the following example.

Example 3.13. *Let $U = \mathbb{R}^3$ and $V = \mathbb{R}^3$. Define the linear map $T : U \to V$ by*

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ 2x-z \\ y+3z \end{bmatrix}.$$

Denote by $\mathfrak{B}_1 = \mathfrak{B}_2 = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ the standard bases for U and V, and

$$\mathfrak{B}_1' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathfrak{B}_2' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

bases for U and V, respectively. We want to compute $M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T)$.

Solution: Step 1: Find $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)$.

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\0\end{bmatrix},\qquad T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix},\qquad T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\\3\end{bmatrix},$$

so

$$M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)=\begin{bmatrix}1&1&0\\2&0&-1\\0&1&3\end{bmatrix}.$$

Step 2: Change-of-basis matrices.

$$M_{\mathfrak{B}_1}^{\mathfrak{B}_1'}=\begin{bmatrix}1&-1&0\\1&1&0\\0&0&1\end{bmatrix},\qquad (M_{\mathfrak{B}_1}^{\mathfrak{B}_1'})^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\-\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix},$$

$$M_{\mathfrak{B}_2}^{\mathfrak{B}_2'}=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix},\qquad (M_{\mathfrak{B}_2}^{\mathfrak{B}_2'})^{-1}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}.$$

Step 3: Apply the theorem (correct order).

$$M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T)=(M_{\mathfrak{B}_2}^{\mathfrak{B}_2'})^{-1}M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)M_{\mathfrak{B}_1}^{\mathfrak{B}_1'}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\2&0&-1\\0&1&3\end{bmatrix}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&2&1\\1&-3&-4\\1&1&3\end{bmatrix}.$$

Step 4: Direct check (optional). *Solve* $(M_{\mathfrak{B}_2}^{\mathfrak{B}_2'})\mathbf{x}=T(\mathbf{b})$ *for each* $\mathbf{b}\in\mathfrak{B}_1'$*:*

$$T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\2\\1\end{bmatrix}\Rightarrow\mathbf{x}=\begin{bmatrix}0\\1\\1\end{bmatrix},$$
$$T\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\-2\\1\end{bmatrix}\Rightarrow\mathbf{x}=\begin{bmatrix}2\\-3\\1\end{bmatrix},$$
$$T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\\3\end{bmatrix}\Rightarrow\mathbf{x}=\begin{bmatrix}1\\-4\\3\end{bmatrix},$$

which reproduces the columns of $M_{\mathfrak{B}_1'}^{\mathfrak{B}_2'}(T)$ *above.* □

Exercises:

242. *Let $\mathfrak{B}_1 = \{1, x, x^2, x^3\}$ be a basis for P_3. Show that*

$$\mathfrak{B}_2 = \{2x-1, x^2-x+1, x^3-x, x^3+x-2\}$$

is also a basis. Find the transformation matrix $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}$.

Hint: Use the determinant of the change-of-basis matrix to check linear independence.

243. *Let $V :=$ Span (e^x, e^{-x}). Find the coordinates of $f(x) = \sinh x$ and $g(x) = \cosh x$ with respect to $\mathfrak{B} = \{e^x, e^{-x}\}$.*

Hint: Express $\sinh x$ and $\cosh x$ using e^x and e^{-x}.

244. *Let $\mathfrak{B}_1 := \{\mathbf{i}, \mathbf{j}\}$ be the standard basis of $\mathbb{R}^2$, and let $\mathbf{u}, \mathbf{v}$ be the vectors obtained by rotating $\mathbf{i}, \mathbf{j}$ counterclockwise by angle θ, respectively. Clearly $\mathfrak{B}_2 := \{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$. Find $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}$.*

Hint: Use rotation matrices with $\cos\theta$ and $\sin\theta$.

245. *Let $V = M_{2\times 2}(\mathbb{R})$ with basis $\mathfrak{B}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Define $\mathfrak{B}_2 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$. Find the transformation matrix $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}$.*

Hint: Express each matrix in $\mathfrak{B}_2$ as a linear combination of $\mathfrak{B}_1$.

246. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$. Find the matrix $M_{\mathfrak{B}_1}^{\mathfrak{B}_2}(T)$ where $\mathfrak{B}_1 = \{\mathbf{i}, \mathbf{j}\}$ and $\mathfrak{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, and interpret the result in the context of machine learning feature scaling.*

Hint: Compute T on $\mathfrak{B}_1$ vectors and adjust for $\mathfrak{B}_2$ coordinates; consider how shear affects data alignment.

3.4 Linear transformations in geometry

Linear transformations are fundamental to understanding geometric operations in various fields, including computer graphics, physics, and engineering. This chapter explores how linear algebra provides the tools to describe and analyze these transformations mathematically. Real-world examples include:

i) Computer graphics where linear transformations are used to manipulate images, render 3D objects, and create animations. For example, rotating, scaling, and shearing images can be achieved through matrix multiplications representing linear transformations.

ii) Physics where linear transformations describe rotations, reflections, and dilations of physical objects. They are essential in areas like mechanics, optics, and crystallography.

iii) Engineering where linear transformations are used in robotics to model movements and in computer-aided design (CAD) to manipulate designs. They are also crucial in signal processing and data analysis.

3.4.1 Linear transformations in $\mathbb{R}^n$

Let us consider again one of the questions that was raised in **??** and more specifically in **??**. So what kind of transformations of $\mathbb{R}^n$ will preserve most (or all) of geometric properties of the objects and in the same time keep the algebraic structure of $\mathbb{R}^n$?

There are two algebraic operations in $\mathbb{R}^n$, namely the vector addition and the scalar multiplication. How should a map look like, which preserves both of these operations? Do such maps preserve the geometric properties of the objects?

If $L : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear map, then implicitly we are implying that $\mathbb{R}^n$ and $\mathbb{R}^m$ are vector spaces. Hence, elements of the $\mathbb{R}^n, \mathbb{R}^m$ are vectors. Therefore the notation

$$L\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

If $L : \mathbb{R}^n \mapsto \mathbb{R}^m$ is considered as a map among sets $\mathbb{R}^n$ and $\mathbb{R}^m$, then the notation $L(x_1, \dots, x_n) = (y_1, y_2, \dots, y_m)$ must be used. Both notations are used in the literature. We will stick to the column vectors notation whenever possible.

Example 3.14. *Let $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise notation of vectors on the plane by the angle θ. Prove that this is a linear map.*

Example 3.15. *The only constant linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the zero map. In other words, if a liner map $L : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $(L(\mathbf{u}) = \mathbf{u}_0$, where $\mathbf{u}_0$ is a constant vector, then $\mathbf{u}_0 = \mathbf{0}$.*

A natural question is to consider what happens to the shape of an object under a linear map. Is the geometrical shape preserved?

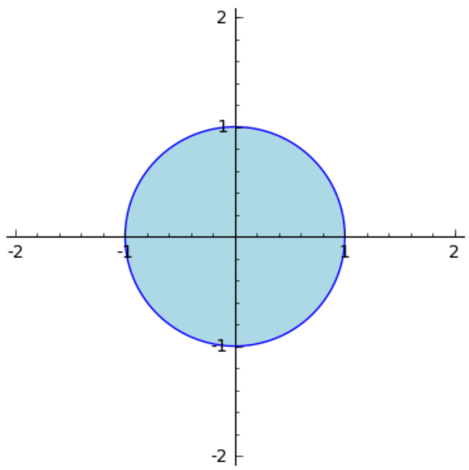
Example 3.16. *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by*

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+2y \\ 2x+3y \end{bmatrix}$$

Let $C : x^2+y^2 = 1$ be the unit circle. The image $L(C)$ is the set of all points such that

$$L(C): \quad (2x+2y)^2 + (2x+3y)^2 = 1$$

This is an ellipse with equation $8x^2+20xy+13y^2 = 1$ as shown Fig. 3.5.



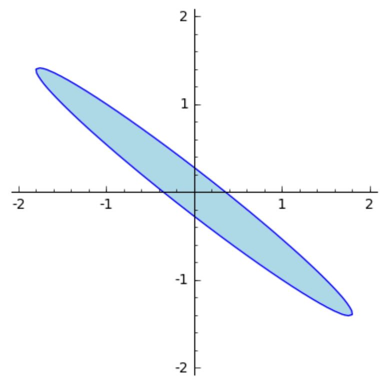


Figure 3.3: Mapping the unit circle to an ellipse

While graphing the above ellipse might be complicated and requires methods that we will learn in the coming chapters, you should be able to easily verify the following.

Example 3.17. *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by*

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$

Let $C : x^2+y^2 = 36$ be the circle with center at the origin and radius $r = 6$. The image $L(C)$ is the set of all points such that

$$L(C): \quad 4x^2 + 9y^2 = 36$$

This is an ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Exercise 49. *Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ a nonzero linear transformation. Prove that every line L is transformed to a line L'.*

3.4.2 Scalings: scalar matrices

A **scaling** is a linear transformation which scales the unit vectors. In other words,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} rx \\ ry \end{bmatrix},$$

for some scalar $r > 0$. The corresponding matrix is $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = rI$. When $r > 1$ it is called a **dilation** and when $r < 1$ a **contraction**.

3.4.3 Rotations

We already have seen what happens to a rotation with an angle θ counterclockwise around the origin. It is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which equivalently says that it is a matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a^2 + b^2 = 1$. A rotation combined with a scaling has a matrix

$$\begin{bmatrix} x \\ y \end{bmatrix} \to r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Lemma 3.1. *A matrix of the form* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *represents a rotation by* θ *combined with a scaling* $r > 0$*, where* r *and* θ *are the polar coordinates of the vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.

3.4.4 Shears

A **horizontal shear** is given by the matrix $\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$. Hence we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ry \\ y \end{bmatrix}$$

A **vertical** shear by the matrix $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ and the transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

Exercise 50. *Let* $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$*. Find what happens to the ellipse* $x^2 + 4y^2 = 1$ *under the shear transformation given by* A*.*

3.4.5 Projections

Let us consider now a problem that we have already seen in Fig. 3.6, finding a projection of a vector $\mathbf{v}$ over a vector $\mathbf{u}$. We already know the formula for $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$. Is this a linear map? Can we find its matrix if that's the case?

Consider vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ as in the Fig. 3.6. The **projection vector** of $\mathbf{v}$ on $\mathbf{u}$, denoted by $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ is the vector obtained by dropping a perpendicular from the vertex of $\mathbf{v}$ on the line determined by $\mathbf{u}$.

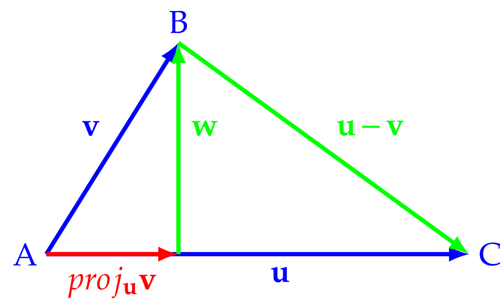


Figure 3.4: The projection of $\mathbf{v}$ onto $\mathbf{u}$

We found its formula in Eq. (1.32)

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\cdot\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}^2}\,\mathbf{u}$$

Let us try to express this in terms of the coordinates of $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ when the unit vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is given. So we have

$$\begin{aligned}\operatorname{proj}_{\mathbf{u}}(\mathbf{x}) &= (\mathbf{x}\cdot\mathbf{u})\,\mathbf{u} = (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2x_1 + u - 1u_2x_2 \\ u_1u_2x_1 + u_2^2x_2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\mathbf{x}\end{aligned}$$

Hence, the projection is a linear map since it is given by multiplication by a matrix.

Consider now if we have a line L going through the origin with equation $y = ax$. A directional vector for L is $\mathbf{w} = \begin{bmatrix} 1 \\ a \end{bmatrix}$, which we can normalize as

$$\mathbf{u} = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\sqrt{a^2+1}}\begin{bmatrix} 1 \\ a \end{bmatrix}. \tag{3.1}$$

Then we have the linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{a^2+1}\begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

In the above discussion it was not necessary to assume that the vector $\mathbf{u}$ be a unit.

Exercise 51. *For any given vector* $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ *the projection map* $T(\mathbf{x}) = proj_{\mathbf{w}}(\mathbf{x})$ *is a linear map with matrix*

$$P = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}.$$

Example 3.18. *Find the matrix P of the projection map onto the line L generated by $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.*

From the Lemma above we have

$$P = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Example 3.19. *Given a line L with equation $y = ax + b$. Find the formula for the projection of a point $P(x, y)$ onto this line. Is this map linear?*

3.4.6 Reflections

We continue our discussion of the previous section but now with the goal of finding the symmetric point of B with respect to the line AC. First we consider the case when the point A is the point (0,0) in $\mathbb{R}^2$. So the problem is the same as before but now we want to find the vector $\mathrm{ref}_{\mathbf{u}}\mathbf{v}$ as in Fig. 3.7.

Consider vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ as in the Fig. 3.6. The **reflection vector** of $\mathbf{v}$ with respect to $\mathbf{u}$, denoted by $\mathrm{ref}_{\mathbf{u}}\mathbf{v}$ is the vector obtained by reflecting the vector $\mathbf{v}$ with respect to the line determined by $\mathbf{u}$.

Hence,

$$\begin{aligned} \mathrm{ref}_{\mathbf{u}}\mathbf{v} &= \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) - \mathbf{w} = \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) - \big(\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})\big) \\ &= 2\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) - \mathbf{v} = 2P\mathbf{v} - \mathbf{v} = (2P - I_2)\,\mathbf{v} \end{aligned}$$

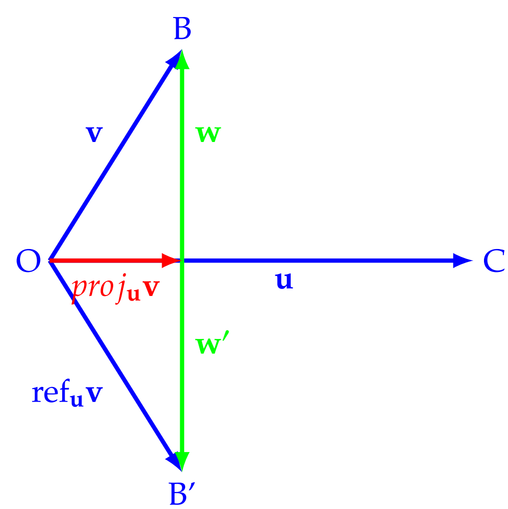


Figure 3.5: Reflection of $\mathbf{v}$ with respect to $\mathbf{u}$

Let us try to express this in terms of the coordinates of $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ when the unit vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is given. So we have the matrix of the reflection as

$$S = 2P - I_2 = \begin{bmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{bmatrix}$$

Consider now, as in the case of projections, the line L with equation $y = ax$. Then the unit vector $\mathbf{u}$ is given by Eq. (3.1). Thus the matrix S becomes

$$S = 2P - I_2 = \frac{1}{a^2 + 1} \begin{bmatrix} 1 - a^2 & 2a \\ 2a & a^2 - 1 \end{bmatrix}$$

Lemma 3.2. *The reflection with respect to a line L going through the origin with equation $y = ax$ is a linear map given by the matrix*

$$S = \frac{1}{a^2 + 1} \begin{bmatrix} 1 - a^2 & 2a \\ 2a & a^2 - 1 \end{bmatrix}$$

Can we generalize this solution to a general line? Let L be a line in $\mathbb{R}^2$ with equation

$$y = ax + b. \tag{3.2}$$

Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that it takes every point $P(x, y)$ to its reflection P'. Determine explicit formulas for this map and check whether it is linear. The following is a high school problem in analytic geometry.

Lemma 3.3. *The reflection map* $\mathrm{ref}_L \mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^2$ *with respect to a general line* $L : \ y = ax + b$ *is given by*

$$(x, y) \to \left(\frac{1}{a^2+1}(1-a^2)x + 2ay - 2ab, \frac{1}{a^2+1} 2ax + (a^2-1)y + 2b \right)$$

Proof. Let $P(x_1, y_1)$ be a point. The line L' going through P and perpendicular to L has equation

$$y = -\frac{1}{a}x + \left(y_1 + \frac{x_1}{a} \right). \tag{3.3}$$

The point of intersection has x-coordinate given by

$$\left(a + \frac{1}{a} \right) x \ = \ -b + y_1 + \frac{x_1}{a}.$$

So we have $x = \frac{x_1 + a y_1 - ab}{a^2+1}$. If we denote $Q(x_2, y_2)$ the reflection point then $x = \frac{x_1 + x_2}{2}$. Therefore,

$$x_2 = 2x - x_1 = 2\frac{x_1 + a y_1 - ab}{a^2+1} - x_1 = \frac{(1-a^2)x_1 + 2a y_1 - 2ab}{a^2+1}$$

Substituting x_2 in Eq. (3.3) we get

$$y_2 = \frac{2a x_1 + (a^2-1) y_1 + 2b}{a^2+1}$$

This completes the result. ◻

Remark 3.1. *Notice that the above map is not linear. There is a way to extend this map to a map* $T' : \ \mathbb{R}^3 \to \mathbb{R}^3$ *such that* T' *is linear, but we will consider that later.*

3.4.7 Linear Geometric Transformations in $\mathbb{R}^3$

Linear geometric transformations in $\mathbb{R}^3$ involve operations that transform vectors in 3-dimensional space while preserving certain geometric properties like the origin, straight lines, and ratios along lines. Here are the key types of transformations:

Scaling

- **Uniform Scaling**: Multiplies all coordinates by the same scalar k. The transformation matrix is:
$$\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$
- **Non-uniform Scaling**: Different scales for different axes k_x, k_y, k_z:
$$\begin{pmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{pmatrix}$$

Defining a Rotation in $\mathbb{R}^3$ as a Linear Transformation

A rotation in $\mathbb{R}^3$ that is also a linear transformation must be about an axis that passes through the origin. This is because all linear transformations fix the origin. We begin by specifying a unit vector $\mathbf{u} = (u_1, u_2, u_3)$ that represents the direction of the **axis of rotation**. Since the axis goes through the origin, this vector defines the entire axis. We choose $\mathbf{u}$ such that

- $\mathbf{u}$ is a unit vector, meaning $\|\mathbf{u}\| = 1$ or $u_1^2 + u_2^2 + u_3^2 = 1$.
- The direction of $\mathbf{u}$ determines the orientation of the axis in space.

We specify an **angle** θ which represents the amount of rotation about the axis on a plane perpendicular to $\mathbf{u}$. We'll assume a counterclockwise rotation when looking down the axis from a point further along it.

The rotation matrix R corresponding to this rotation is given by the Rodrigues' rotation formula:
$$R = I + \sin(\theta)K + (1 - \cos(\theta))K^2$$
where I is the 3×3 identity matrix and K is the skew-symmetric matrix associated with the axis vector $\mathbf{u}$:
$$K = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Expanded on K^2: Since K is skew-symmetric, K^2 is symmetric, and you can compute it as:
$$K^2 = \begin{pmatrix} -u_2^2 - u_3^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & -u_1^2 - u_3^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & -u_1^2 - u_2^2 \end{pmatrix}$$
This matrix K^2 has the effect of projecting vectors onto the plane perpendicular to $\mathbf{u}$ and then scaling them by a factor related to θ.

For any vector $\mathbf{v}$ in $\mathbb{R}^3$, the transformed vector after rotation around $\mathbf{u}$ by angle θ would be $\mathbf{v}' = R\mathbf{v}$. This method allows for an elegant and precise mathematical description of rotations in 3D space, useful in fields like computer graphics, robotics, and physics.

Exercise 52. *Show that*

- ***Around x-axis by angle*** θ *(counterclockwise looking towards positive x):*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

- ***Around y-axis by angle*** θ *(counterclockwise looking towards positive y):*

$$\begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

- ***Around z-axis by angle*** θ *(counterclockwise looking towards positive z):*

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection

Consider now a reflection to a plane. This is the symmetric of a point A with respect to a plane P, not to be confused with mirror reflection of light. For example the reflection of the point $P(5,1,3)$ with respect to the $x = y$ plane in Fig. 3.8 is the point $P_{x=y}(1,5,3)$. However, its reflection with respect to the $z = 0$ plane is the point $P_{z=0}(5,1,-3)$.

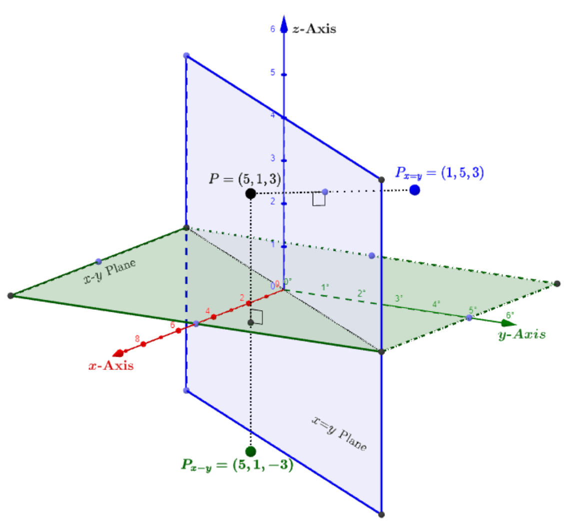


Figure 3.6: Reflection of a point with respect to a plane

Exercise 53. *Let P be a plane in $\mathbb{R}^3$ going through the origin. Then P has equation*

$$ax + by + cz = 0$$

Find the formulas for the reflection map $ref_P \mathbf{x}$ with respect to the plane P. Show that this is a linear map. Find its matrix.

Exercise 54. *Prove the following formulas for reflections in these spacial cases:*

- ***Reflection across the xy-plane:***
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
- ***Reflection across the xz-plane:***
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- ***Reflection across the yz-plane:***
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Shear

Shearing transformations shift one coordinate in proportion to another. For example, shearing along x by y:

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here, k determines how much the x-coordinate is shifted due to y.

Exercises:

247. *Let L be a line in $\mathbb{R}^3$ such that it contains the unit vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$. Find the matrix of the linear transformation $T(\mathbf{x}) = proj_L(\mathbf{x})$. What is the trace of this matrix?*

248. *Let L be a line in $\mathbb{R}^3$ such that it contains the unit vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$. Find the matrix of the linear transformation $T(\mathbf{x}) = ref_L \mathbf{x}$.*

Carl Gustav Jacobi (1804-1851)

Carl Gustav Jacob Jacobi (10 December 1804 – 18 February 1851) was a German mathematician, who made fundamental contributions to elliptic functions, algebraic geometry, dynamics, differential equations, and number theory. His name is occasionally written as Carolus Gustavus Iacobus Iacobi in his Latin books, and his first name is sometimes given as Karl.

One of Jacobi's greatest accomplishments was his theory of elliptic functions and their relation to the elliptic theta function. This was developed in his great treatise Fundamenta nova theoriae functionum ellipticarum (1829), and in later papers in Crelle's Journal. Theta functions are of great importance in mathematical physics because of their role in the inverse problem for periodic and quasi-periodic flows. The equations of motion are integrable in terms of Jacobi's elliptic functions in the well-known cases of the pendulum, the Euler top, the symmetric Lagrange top in a gravitational field and the Kepler problem (planetary motion in a central gravitational field).

He also made fundamental contributions in the study of differential equations and to rational mechanics, notably the Hamilton – Jacobi theory.

It was in algebraic development that Jacobi's peculiar power mainly lay, and he made important contributions of this kind to many areas of mathematics, as shown by his long list of papers in Crelle's Journal and elsewhere from 1826 onwards. One of his maxims was: 'Invert, always invert' ('man muss immer umkehren'), expressing his belief that the solution of many hard problems can be clarified by re-expressing them in inverse form.

In his 1835 paper, Jacobi proved the following basic result classifying periodic (including elliptic) functions: If a univariate single-valued function is multiply periodic, then such a function cannot have more than two periods, and the ratio of the periods cannot be a real number. He discovered many of the fundamental properties of theta functions, including the functional equation and the Jacobi triple product formula, as well as many other results on q-series and hypergeometric series.

The solution of the Jacobi inversion problem for the hyperelliptic Abel map by Weierstrass in 1854 required the introduction of the hyperelliptic theta function and later the general Riemann theta function for algebraic curves of arbitrary genus. The complex torus associated to a genus g algebraic curve, obtained by quotienting $\mathbf{C}^g$ by the lattice of periods is referred to as the Jacobian variety. This method of inversion, and its subsequent extension by Weierstrass and Riemann to arbitrary algebraic curves, may be seen as a higher genus generalization of the relation between elliptic integrals and the Jacobi, or Weierstrass elliptic functions

Jacobi was the first to apply elliptic functions to number theory, for example proving of Fermat's two-square theorem and Lagrange's four-square theorem, and similar results for 6 and 8 squares. His other work in number theory continued the work of Gauss: new proofs of quadratic reciprocity and introduction of the Jacobi symbol; contributions to higher reciprocity laws, investigations of continued fractions, and the invention of Jacobi sums.

He was also one of the early founders of the theory of determinants; in particular, he invented the Jacobian determinant formed from the n^2 differential coefficients of n given functions of n independent variables, and which has played an important part in many analytical investigations. In 1841 he reintroduced the partial derivative ∂ notation of Legendre, which was to become standard. Students of vector fields and Lie theory often encounter the Jacobi identity, the analog of associativity for the Lie bracket operation.

Chapter 4

Determinants, eigenvalues, eigenvectors

The theory of determinants was developed in the 17-th and 18-th centuries. It started mainly with Cramer and continued further with Bezout, Vandermonde, Laplace, Cauchy, et al. With the development of modern algebra and new concepts that came with it as multilinear forms, permutation groups, etc, the concept of the determinant was put in a firm foundation.

4.1 Multilinear Forms

Let $V_1, \dots, V_n$, and W be vector spaces over a field $\mathbb{F}$. A map

$$\phi : V_1 \times \cdots \times V_n \to W$$

is called **multilinear** if for all $i = 1, \dots n$ and $r \in \mathbb{F}$ the following hold:

(i) $\phi(v_1, \dots, v_{i-1}, \mathbf{v} + \mathbf{u}, v_{i+1}, \dots, v_n) = \phi(v_1, \dots, v_{i-1}, \mathbf{v}, v_{i+1}, \dots, v_n) + \phi(v_1, \dots, v_{i-1}, \mathbf{u}, v_{i+1}, \dots, v_n)$

(ii) $\phi(v_1, \dots, v_{i-1}, r\mathbf{v}, v_{i+1}, \dots, v_n) = r\phi(v_1, \dots, v_{i-1}, \mathbf{v}, v_{i+1}, \dots, v_n)$

where $\mathbf{v}, \mathbf{u} \in V_i$.

A multilinear map $\phi : V^n \to W$ (where V^n denotes the Cartesian product $V \times \cdots \times V$ of n copies of V) is called an **n-multilinear function**. If $W = \mathbb{F}$, the map ϕ is called a **multilinear form**.

Definition 4.1. *An n-multilinear function $\phi : V^n \to W$ is called **alternating** if for all $i = 1, \dots, n$ the following holds:*

$$\mathbf{v}_i = \mathbf{v}_{i+1} \implies \phi(v_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}, \dots, v_n) = 0.$$

*It is called **symmetric** if interchanging any two arguments (or coordinates) does not change the value of the function ϕ.*

Exercise 55. *Show that a 2-multi-linear map is a bilinear map.*

Solution: Let $\phi : V_1 \times V_2 \to W$ be a 2-multilinear map. We want to show that ϕ is bilinear. A map is bilinear if it is linear in each argument.

Let $\mathbf{v}_1, \mathbf{u}_1 \in V_1$ and $r \in \mathbb{F}$. We need to show that

(a) $\phi(\mathbf{v}_1 + \mathbf{u}_1, \mathbf{v}_2) = \phi(\mathbf{v}_1, \mathbf{v}_2) + \phi(\mathbf{u}_1, \mathbf{v}_2)$

(b) $\phi(r\mathbf{v}_1, \mathbf{v}_2) = r\phi(\mathbf{v}_1, \mathbf{v}_2)$

These follow directly from the definition of 2-multilinearity (with $n = 2$ and $i = 1$):

(a) $\phi(\mathbf{v}_1 + \mathbf{u}_1, \mathbf{v}_2) = \phi(\mathbf{v}_1, \mathbf{v}_2) + \phi(\mathbf{u}_1, \mathbf{v}_2)$ (by property (i) of multilinearity)

(b) $\phi(r\mathbf{v}_1, \mathbf{v}_2) = r\phi(\mathbf{v}_1, \mathbf{v}_2)$ (by property (ii) of multilinearity)

Let $\mathbf{v}_2, \mathbf{u}_2 \in V_2$ and $r \in \mathbb{F}$. We need to show that

(a) $\phi(\mathbf{v}_1, \mathbf{v}_2 + \mathbf{u}_2) = \phi(\mathbf{v}_1, \mathbf{v}_2) + \phi(\mathbf{v}_1, \mathbf{u}_2)$

(b) $\phi(\mathbf{v}_1, r\mathbf{v}_2) = r\phi(\mathbf{v}_1, \mathbf{v}_2)$

These also follow directly from the definition of 2-multilinearity (with $n = 2$ and $i = 2$):

(a) $\phi(\mathbf{v}_1, \mathbf{v}_2 + \mathbf{u}_2) = \phi(\mathbf{v}_1, \mathbf{v}_2) + \phi(\mathbf{v}_1, \mathbf{u}_2)$ (by property (i) of multilinearity)

(b) $\phi(\mathbf{v}_1, r\mathbf{v}_2) = r\phi(\mathbf{v}_1, \mathbf{v}_2)$ (by property (ii) of multilinearity)

Since ϕ is linear in both its arguments, it is bilinear. □

4.1.1 The Signature of a Permutation

A **transposition** is a permutation that swaps two elements and leaves the others fixed. Any permutation can be written as a composition (product) of transpositions. This decomposition is not unique, but the *number* of transpositions in any decomposition of a given permutation is either always even or always odd.

Definition 4.2. *The* ***signature*** *(or sign) of a permutation σ, denoted $sgn(\sigma)$, is defined as:*

$$sgn(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ can be written as a product of an even number of transpositions} \\ -1, & \text{if } \sigma \text{ can be written as a product of an odd number of transpositions} \end{cases}$$

A permutation is called ***even*** *if its signature is 1, and* ***odd*** *if its signature is -1.*

Proposition 4.1. *Let ϕ be an n-multi-linear alternating function on V. Then,*

(i) *the value of ϕ on an n-tuple is negated if two adjacent components are interchanged.*

(ii) *for each $\sigma \in S_n$,*

$$\phi(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = sgn(\sigma)\phi(v_1, \dots, v_n)$$

(iii) *if $\mathbf{v}_i = \mathbf{v}_j$ for any $i \neq j$ then $\phi(v_1, \dots, v_n) = 0$.*

(iv) *if $\mathbf{v}_i$ is replaced by $\mathbf{v}_i + \lambda\, \mathbf{v}_j$, in $(v_1, \dots, v_n)$ for any $i \neq j$ and $\lambda \in \mathbb{F}$, then the value $\phi(v_1, \dots, v_n)$ is not changed.*

Proof. Let $\phi : V^n \to W$ be an n-multi-linear alternating function.

(i) Let's interchange two adjacent vectors $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$:

$$
\begin{aligned}
\phi(v_1,\dots,v_{i+1},\, v_i,\, \dots,v_n) &= \phi(v_1,\dots,\, v_i + (v_{i+1} - v_i\,)\,,\, v_{i+1} - (v_{i+1} - v_i)\,,\dots,v_n) \\
&= \phi(v_1,\dots,v_i,\, v_{i+1},\, \dots,v_n) \;-\; \phi(v_1,\dots,v_i,\, (v_{i+1} - v_i),\, \dots,v_n) \\
&\quad + \phi(v_1,\dots,v_{i+1},\, v_{i+1},\, \dots,v_n) \;+ \phi(v_1,\dots,v_{i+1},\, v_i,\, \dots,v_n) \\
&\quad - \phi(v_1,\dots,v_{i+1},\, v_{i+1},\, \dots,v_n) \\
&= -\phi(v_1,\dots,v_i,\, v_{i+1},\, \dots,v_n)
\end{aligned}
$$

The terms involving $\phi(\dots,\mathbf{v}_{i+1},\mathbf{v}_{i+1},\dots)$ are zero because ϕ is alternating.

(ii) Any permutation $\sigma \in S_n$ can be written as a product of transpositions. Let $\sigma = \tau_1\tau_2\cdots\tau_k$, where each τ_j is a transposition. Then

$$
\begin{aligned}
\phi(v_{\sigma(1)},\dots,v_{\sigma(n)}) &= \phi(v_{\tau_1(\tau_2(\dots\tau_k(1)))},\dots,v_{\tau_1(\tau_2(\dots\tau_k(n)))}) \\
&= \operatorname{sgn}(\tau_1)\phi(v_{\tau_2(\dots\tau_k(1))},\dots,v_{\tau_2(\dots\tau_k(n))}) \\
&= \operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2)\phi(v_{\tau_3(\dots\tau_k(1))},\dots,v_{\tau_3(\dots\tau_k(n))}) \\
&\dots \\
&= \operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2)\cdots\operatorname{sgn}(\tau_k)\phi(v_1,\dots,v_n) \\
&= \operatorname{sgn}(\sigma)\phi(v_1,\dots,v_n)
\end{aligned}
$$

because the sign of a product of permutations is the product of their signs.

(iii) If $\mathbf{v}_i = \mathbf{v}_j$ for $i \neq j$, we can swap adjacent vectors to bring $\mathbf{v}_i$ and $\mathbf{v}_j$ next to each other. By (i), each swap negates the value of ϕ. After a certain number of swaps, $\mathbf{v}_i$ and $\mathbf{v}_j$ will be adjacent. Since ϕ is alternating, $\phi(\dots,\mathbf{v}_i,\mathbf{v}_i,\dots) = 0$.

(iv)

$$
\begin{aligned}
\phi(\dots,\mathbf{v}_i + \lambda\mathbf{v}_j,\dots,\mathbf{v}_j,\dots) &= \phi(\dots,\mathbf{v}_i,\dots,\mathbf{v}_j,\dots) + \lambda\phi(\dots,\mathbf{v}_j,\dots,\mathbf{v}_j,\dots) \\
&= \phi(\dots,\mathbf{v}_i,\dots,\mathbf{v}_j,\dots) + \lambda\cdot 0 \\
&= \phi(\dots,\mathbf{v}_i,\dots,\mathbf{v}_j,\dots)
\end{aligned}
$$

$\square$

Proposition 4.2. *Assume that ϕ is an n-multi-linear alternating function on V and that for some $\mathbf{v}_1,\dots,\mathbf{v}_n \in V$, $\mathbf{w}_1,\dots,\mathbf{w}_n \in V$ we have*

$$
\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + \cdots + a_{n1}\mathbf{v}_n \\
&\dots \\
\mathbf{w}_n &= a_{1n}\mathbf{v}_1 + \cdots + a_{nn}\mathbf{v}_n
\end{aligned}
$$

Then,

$$
\phi(\mathbf{w}_1,\dots,\mathbf{w}_n) = \sum_{\sigma\in S_n} sgn(\sigma) a_{\sigma(1)\,1}\cdots a_{\sigma(n)\;n}\,\phi(\mathbf{v}_1,\dots,\mathbf{v}_n)
$$

Proof. We proceed by substituting the expressions for $\mathbf{w}_i$ into $\phi(\mathbf{w}_1,\ldots,\mathbf{w}_n)$ and using the multilinearity of ϕ:

$$\begin{aligned}\phi(\mathbf{w}_1,\ldots,\mathbf{w}_n) &= \phi\left(\sum_{i_1=1}^{n} a_{i_1 1}\mathbf{v}_{i_1},\ldots,\sum_{i_n=1}^{n} a_{i_n n}\mathbf{v}_{i_n}\right)\\ &= \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} a_{i_1 1}a_{i_2 2}\ldots a_{i_n n}\phi(\mathbf{v}_{i_1},\mathbf{v}_{i_2},\ldots,\mathbf{v}_{i_n})\end{aligned}$$

Now, because ϕ is alternating, if any two indices i_j and i_k are equal (where $j \neq k$), the term $\phi(\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_n})$ will be zero. Thus, the only terms that survive in the sum are those where the indices $i_1, i_2, \ldots, i_n$ are all distinct. This means that the set $\{i_1, i_2, \ldots, i_n\}$ must be a permutation of the set $\{1,2,\ldots,n\}$. We can therefore rewrite the sum using permutations $\sigma \in S_n$:

$$\phi(\mathbf{w}_1,\ldots,\mathbf{w}_n) = \sum_{\sigma\in S_n} a_{\sigma(1)1}a_{\sigma(2)2}\ldots a_{\sigma(n)n}\phi(\mathbf{v}_{\sigma(1)},\mathbf{v}_{\sigma(2)},\ldots,\mathbf{v}_{\sigma(n)})$$

Since ϕ is alternating, we know that swapping two arguments negates the result. Therefore, if we reorder the vectors in $\phi(\mathbf{v}_{\sigma(1)},\ldots,\mathbf{v}_{\sigma(n)})$ to be in the standard order $(\mathbf{v}_1,\ldots,\mathbf{v}_n)$, we pick up a factor of $\operatorname{sgn}(\sigma)$, the sign of the permutation σ:

$$\begin{aligned}\phi(\mathbf{w}_1,\ldots,\mathbf{w}_n) &= \sum_{\sigma\in S_n} a_{\sigma(1)1}a_{\sigma(2)2}\ldots a_{\sigma(n)n}\operatorname{sgn}(\sigma)\phi(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)\\ &= \left(\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)a_{\sigma(1)1}a_{\sigma(2)2}\ldots a_{\sigma(n)n}\right)\phi(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)\end{aligned}$$

The term in the parentheses is precisely the definition of the determinant of the matrix $A = [a_{ij}]$. Thus, we have

$$\phi(\mathbf{w}_1,\ldots,\mathbf{w}_n) = \det(A)\phi(\mathbf{v}_1,\ldots,\mathbf{v}_n)$$

where $\det(A) = \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)a_{\sigma(1)1}\cdots a_{\sigma(n)n}$. □

4.1.2 Determinants

The determinant is a crucial function associated with square matrices, providing a single number that reveals key properties of the matrix and the linear transformation it represents. We define the determinant function by focusing on its essential characteristics: how it scales with respect to changes in the matrix's columns, and how it reflects the effect of column swaps. Specifically, we require the determinant to be a multilinear function of the columns, meaning it behaves linearly with respect to each column individually. The alternating property captures the idea that swapping two columns negates the determinant, reflecting the change in orientation. Finally, we normalize the determinant by requiring it to be 1 for the identity matrix, ensuring a consistent scale. These properties, taken together, uniquely specify the determinant and allow us to compute it using Leibniz's formula.

Definition 4.3. *An $n \times n$ **determinant function on** $\mathbb{F}$ is called any function*

$$det\ : \mathrm{Mat}_{n\times n}(\mathbb{F}) \to \mathbb{F}$$

which satisfies the following:

(i) *it is a n-multi-linear alternating form on $\mathbb{F}^n$, where n-tuples are $(A^1, \ldots A^n)$ n-columns of matrices A in $\mathbb{F}^n$.*

(ii) *$det\ (I_n) = 1$*

So we have

$$\begin{aligned} \mathrm{Mat}_{n\times n}(\mathbb{F}) &\to \mathbb{F}^n \to \mathbb{F} \\ A = [\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \ldots \,|\mathbf{v}_n] &\to (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \to \phi(\mathbf{v}_1, \ldots, \mathbf{v}_n) \end{aligned}$$

The following is known as the **Leibniz's formula**.

Theorem 4.1 (Leibniz's formula)**.** *There is a unique $n \times n$ determinant function on $\mathbb{F}$ and it can be computed for any $n \times n$ matrix $A = [a_{i,j}]$ by*

$$det\ (A) = \sum_{\sigma \in S_n} sgn(\sigma)\ \cdot\ \prod_{i=1}^{n} a_{\sigma(i),i},$$

where $sgn(\sigma)$ is the sign of the permutation $\sigma \in S_n$.

Proof. We will prove both existence and uniqueness.

Existence:

We define the function $D : \mathrm{Mat}_{n\times n}(\mathbb{F}) \to \mathbb{F}$ by

$$D(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}.$$

We need to show that D is an n-multilinear alternating form and that $D(I_n) = 1$.

1. **n-multilinearity:** Consider the k-th column of A, say $\mathbf{a}_k = (a_{1k}, a_{2k}, \ldots, a_{nk})^T$. We want to show that D is linear with respect to this column. Let $\mathbf{a}_k = \alpha\mathbf{u} + \beta\mathbf{v}$ for some vectors $\mathbf{u} = (u_1, \ldots, u_n)^T$ and $\mathbf{v} = (v_1, \ldots, v_n)^T$ and scalars $\alpha, \beta \in \mathbb{F}$. Then $a_{ik} = \alpha u_i + \beta v_i$ for all i.

$$\begin{aligned} D(A) &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(k),k} \cdots a_{\sigma(n),n} \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1),1} \cdots (\alpha u_{\sigma(k)} + \beta v_{\sigma(k)}) \cdots a_{\sigma(n),n} \\ &= \alpha \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1),1} \cdots u_{\sigma(k)} \cdots a_{\sigma(n),n} + \beta \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1),1} \cdots v_{\sigma(k)} \cdots a_{\sigma(n),n} \end{aligned}$$

This shows that D is linear with respect to the k-th column. Since k was arbitrary, D is n-multilinear.

2. **Alternating property:** Suppose two columns of A are identical, say the k-th and l-th columns ($k \neq l$). Then $a_{ik} = a_{il}$ for all i. Consider the sum for $D(A)$. We can pair up terms corresponding to permutations σ and τ, where τ is obtained from σ by swapping the k-th and l-th entries. Then $\operatorname{sgn}(\tau) = -\operatorname{sgn}(\sigma)$. The terms in the sum corresponding to σ and τ are

$$\begin{aligned}\operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{\sigma(i),i} + \operatorname{sgn}(\tau)\prod_{i=1}^{n} a_{\tau(i),i} &= \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{\sigma(i),i} - \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{\tau(i),i} \\ &= \operatorname{sgn}(\sigma)\left(\prod_{i=1}^{n} a_{\sigma(i),i} - \prod_{i=1}^{n} a_{\tau(i),i}\right) = 0,\end{aligned}$$

because the only difference between the products is the order of the factors a_{ik} and a_{il}, which are equal. Since all terms cancel in pairs, $D(A) = 0$.

3. $D(I_n) = 1$**:** When $A = I_n$, the only permutation σ for which the product $\prod_{i=1}^{n} a_{\sigma(i),i}$ is non-zero is the identity permutation, for which $\operatorname{sgn}(\sigma) = 1$ and the product is 1. All other terms are zero because $a_{ij} = 0$ if $i \neq j$. Thus, $D(I_n) = 1$.

Uniqueness:

Suppose D' is another $n \times n$ determinant function. Since D' is n-multilinear and alternating, by the previous proposition, we have:

$$D'(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n} D'(I_n) = D(A)D'(I_n).$$

Since $D'(I_n) = 1$, we have $D'(A) = D(A)$ for all A. Thus, the determinant function is unique. □

Example 4.1. *Let* $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. *Compute its determinant by the above formula.*

Solution: The permutation group is $S_2 = \{\sigma_1 = id, \sigma_2 = (12)\}$. Then

$$\det A = \xi(\sigma_1)a_{1,1}a_{2,2} + \xi(\sigma_2)a_{\sigma_2(1),1}a_{\sigma_2(2),2} = 1 \cdot 2 \cdot 2 + (-1) \cdot 1 \cdot 3 = 1$$

□

Exercise 56. *Let* $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. *Compute its determinant by the Leibnitz formula.*

Corollary 4.1. *The determinant is an n-multi-linear function on the rows of* $A \in \operatorname{Mat}_{n\times n}(\mathbb{F})$. *Moreover, det* $(A) = $ *det* (A^t).

4.1.3 Expansion by minors

Definition 4.4. *Let $A = [a_{ij}]$ be an $n \times n$ matrix. For each (i, j) let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained by deleting its i-th row and j-column. Then, $A_{i,j}$ is called a* ***minor*** *of A, and*

$$\bar{a}_{i,j} = (-1)^{i+j} det\left(A_{i,j}\right)$$

is called a ***cofactor*** *of A.*

Theorem 4.2. *Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then for a fixed $i = 1, \ldots n$ the* ***determinant*** *of A is defined to be:*

$$det\,(A) := \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{i,j} \cdot det\,(A_{i,j}) = \sum_{j=1}^{n} a_{i,j} \cdot \bar{a}_{i,j}$$

and is independent on the choice of i.

Remark 4.1. *In most elementary linear algebra books the determinant is defined as in Thm. 4.2. For the rest of these lectures we will use Thm. 4.2 as the main way to compute the determinant of a matrix.*

The definition of the determinant as above is called the **expansion by minors** along the i-th row.

Example 4.2. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. For $i = 1$ we have the determinant*

$$det\,(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1)^2 a \cdot d + (-1)^3 b \cdot c = ad - bc.$$

For $i = 2$ we have

$$det\,(A) = (-1)^3 c \cdot b + (-1)^4 d \cdot a = ad - bc.$$

Hence, no matter what row that we pick we get the same result. ◻

The above theorem allows us to pick the row or column with more zeroes when we compute the determinant of a matrix. The determinant of a matrix A

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \ldots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \ldots & a_{3,n} \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ a_{m,1} & a_{m,2} & a_{m,3} & \ldots & a_{m,n} \end{bmatrix} \quad \textit{is denoted by} \quad \det\,(A) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \ldots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \ldots & a_{3,n} \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ a_{m,1} & a_{m,2} & a_{m,3} & \ldots & a_{m,n} \end{vmatrix}$$

Example 4.3 (Laplace's formula). *Let $A = \left[a_{i,j}\right]$ be a 3×3 matrix Then its determinant is*

$$\begin{aligned} det\,(A) &= a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \\ &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{2,1}a_{3,2}a_{1,3} - a_{3,1}a_{2,2}a_{1,3} - a_{3,2}a_{2,3}a_{1,1} - a_{2,1}a_{1,2}a_{3,3} \end{aligned}$$

◻

Example 4.4. *Compute the determinant of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 4 & 5 \\ 0 & 2 & 1 & 2 & 0 \end{bmatrix}$$

Solution: Since the second row has three zeroes we expand along that row. So we have

$$\det(A) = 2 \cdot \begin{vmatrix} 1 & 0 & 4 & 0 \\ 2 & 2 & 1 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 1 & 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 & 0 & 4 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & 1 & 2 \end{vmatrix}$$

We let

$$A_1 := \begin{bmatrix} 1 & 0 & 4 & 0 \\ 2 & 2 & 1 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

Then

$$\det(A_1) = 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 2 & 4 & 5 \\ 1 & 2 & 0 \end{vmatrix} + 4 \cdot \begin{vmatrix} 2 & 2 & 2 \\ 1 & 2 & 5 \\ 0 & 1 & 0 \end{vmatrix} = (5+8-8-20) + 4(2-2\cdot 5) = -15-32 = -47 \tag{4.1}$$

$$\begin{aligned} \det(A_2) &= \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 2 & 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 2 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 2 \end{vmatrix} - 4 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix} \\ &= (4+16+1-4-4-4) - 2(8+1-4-8) - 4(2+4-8-1) \\ &= 9 - 2\cdot(-3) - 4\cdot(-3) = 27 \end{aligned}$$

Hence, $\det(A) = 2\cdot(-47) - 27 = -121$.

□

Lemma 4.1. *$\det(A) = \det(A^t)$*

Proof. Let $A = [a_{ij}]$ be given. We prove the Lemma by induction. For $n = 1$ the proof is trivial. Assume that the lemma holds for $n < r$. We want to show that it holds for $n = r$. The determinant of A is

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{r+1} a_{1r}|A_{1r}|$$

Denote by $B := A^t$. Then

$$\det(B) = b_{11}|B_{11}| - b_{21}|B_{21}| + \cdots + (-1)^{r+1} b_{r1}|B_{1r}|.$$

However, $a_{1j} = b_{j1}$ and $B_{j1} = A^t_{1j}$. By the induction hypothesis we have $|A_{1j}| = |B_{j1}|$. Hence $\det(A) = \det(B) = \det(A^t)$.

□

Remark 4.2. *The determinant of a triangular matrix is the product of its diagonal entries.*

We illustrate with an upper triangular matrix.

Example 4.5. *Let A be a triangular matrix*

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & 0 & a_{3,3} & \dots & a_{3,n} \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & a_{m,n} \end{bmatrix}$$

Solution: We find the determinant by expanding along the first column. It is obvious that $\det(A) = \prod_{i=1}^{n} a_{i,i}$.

□

We now see some properties of determinants.

Lemma 4.2. *Let A be an $n \times n$ matrix. The row operations have the following effect on the determinant:*

(i) *If $R_i \longleftrightarrow R_j$ is performed on a matrix A then the determinant of the resulting matrix A' is*

$$det\,(A') = -det\,(A)$$

(ii) *If two rows of A are the same then*

$$det\,(A) = 0$$

(iii) *If $R_i \to rR_i$ then the determinant of the resulting matrix A' is*

$$det\,(A') = r \cdot det\,(A)$$

(iv) *The operation $R_j \to rR_i + R_j$ does not change the determinant.*

Proof. i) We proceed by induction. The proof for $n = 2$ is trivial. Assume that the property holds for all matrices of size smaller then n. Let B denote the matrix obtained after performing the operation $R_i \longleftrightarrow R_j$ on A. Compute the determinant by expansion along the s-th row, where $s \neq i$ and $s \neq j$. Then

$$\det(A) = a_{s1}|A_{s1}| - a_{s2}|A_{s2}| + \dots + (-1)^{s+n} a_{sn}|A_{sn}|.$$

For each $1 \leq r \leq n$ we have

$$(-1)^{s+r}|A_{sr}| = -(-1)^{s+r}|B_{rs}|.$$

Thus, by induction hypothesis, $|B_{rs}| = -|A_{sr}|$. Hence, $\det(B) = -\det(A)$.

Part ii) is an immediate consequence of part i) and iii) is immediate fromf the definition. To prove iv) let B denote the matrix obtained after performing the operation $R_j \to rR_i + R_j$ on

A. Then,

$$\begin{aligned}
\det(B) &= b_{j1}|B_{j1}| + \cdots + (-1)^{j+n} b_{jn}|B_{jn}| \\
&= (ra_{i1} + a_{j1})|B_{j1}| + \cdots + (-1)^{j+n}(ra_{in} + a_{jn})|B_{jn}| \\
&= \left(ra_{i1}|B_{j1}| + \cdots + (-1)^{j+n} ra_{in}|B_{jn}|\right) + \left(a_{j1}|B_{j1}| + \cdots + (-1)^{j+n} a_{jn}|B_{jn}|\right) \\
&= r\det(C) + \det(A)
\end{aligned}$$

where C is obtained by interchanging the rows of A. Hence, $\det(C) = 0$ and $\det(B) = \det(A)$. ☐

Theorem 4.3. *A matrix A is invertible if and only if $\det(A) \neq 0$.*

Proof. Let A be given. Compute the row echelon form of A. Then $\det(A) = r \cdot \det(H)$, for some constant $r \neq 0$. The matrix A is invertible if and only if H has pivots in every row. Since H is triangular then its determinant is the product of this pivots. Hence, A is invertible if and only if $\det(H) \neq 0$. Therefore, A is invertible if and only if $\det(A) \neq 0$. ☐

Lemma 4.3. *Let $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{F})$. If $\det(A) = 0$ then $\det(AB) = 0$.*

Theorem 4.4. *Let $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{F})$. Then*

$$\det(AB) = \det(A) \cdot \det(B)$$

Proof. First we assume that A is diagonal. Then, to obtain the matrix AB, each row of B is multiplied by $A_{i,i}$. Hence,

$$\det(AB) = (a_{11} \cdots a_{nn}) \cdot \det(B) = \det(A) \cdot \det(B).$$

Without loss of generality assume that A is invertible (otherwise the theorem is true from Lem. 4.3). Then, A can be converted in a diagonal form D by row operations (no multiplying by constants is allowed). Thus, $D = EA$ for some elementary matrix E where E corresponds to row interchanges and row-additions. Hence, $\det(A) = (-1)^r \cdot \det(D)$, for some r. Then, $E(AB) = (EA)B = DB$. Therefore, we have

$$\det(AB) = (-1)^r \cdot \det(DB) = (-1)^r \cdot \det(D) \cdot \det(B) = \det(A) \cdot \det(B).$$

This completes the proof. ☐

Example 4.6. *Find the determinant of the matrix AB when*

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 9 & 2 & 4 & 0 \\ 12 & 10 & 2 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 21 & -7 & 2 & 0 \\ 13 & 2 & 31 & 2 \end{bmatrix}$$

Solution: Since both are triangular matrices and $\det(AB) = \det(A) \cdot \det(B)$ we have $\det(AB) = (1 \cdot 2 \cdot 4 \cdot 5) \cdot (3 \cdot 1 \cdot 2 \cdot 2) = 480$. ☐

4.1.4 Computation of Determinants

Computing the determinant by cofactor expansion (expansion of minors) is a computationally expensive process, especially for larger matrices, as it is recursive. While conceptually important, it's rarely the most efficient method for practical calculations. We can significantly speed up the computation by leveraging elementary row operations.

The following is an immediate corollary of Lem. 4.2

Corollary 4.2. *Let $A \in M_{n\times n}(\mathbb{F})$ and $rref(A)$ be its row-echelon form. Then*

$$det\,(A) = c \cdot det\,(rref(A))$$

for some constant $c \in \mathbb{F}$. Moreover, $det\,(A) = 0$ if and only if $det\,(rref(A)) = 0$.

Then we have the following lgorithm.

Algorithm 1 Computing the Determinant using Row Operations

Input: A square matrix $A \in M_{n\times n}(\mathbb{F})$
Output: The determinant of A, $\det(A)$
1. Reduction to Row-Echelon Form: Reduce A to row-echelon form, denoted $\mathrm{rref}(A)$, using only elementary row operations of the following types:

- Row addition: Adding a multiple of one row to another row.
- Row interchange: Swapping two rows.
- (Avoid) Scalar multiplication of a row: Multiplying a row by a non-zero scalar. While permissible, we want to track the effect of scalar multiplication separately.

2. Determinant Calculation:
if during the reduction, a row of all zeros is encountered **then return** $\det(A) = 0$
else
Let $p_1, p_2, \ldots, p_n$ be the pivots (leading entries) of $\mathrm{rref}(A)$.
Let r be the total number of row interchanges performed.
Let $s_1, s_2, ..., s_k$ be the non-zero scalars by which rows were multiplied, if any. **return** $\det(A) = (-1)^r \cdot \left(\prod_{i=1}^n p_i\right) \cdot \left(\prod_{j=1}^k \frac{1}{s_j}\right)$ (Note: if no scalar multiplication, the second product is 1)
end if

Example 4.7. *Let* $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$.

1. *Subtract 2 times row 1 from row 2:* $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 8 & 7 & 9 \end{bmatrix}$ *(no change in determinant).*
2. *Subtract 4 times row 1 from row 3:* $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix}$ *(no change in determinant).*

3. Subtract 3 times row 2 from row 3: $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ *(no change in determinant).*

The row-echelon form is upper triangular with pivots 2, 1, and 2. No row interchanges were performed ($r = 0$). Thus, det (A) $= (-1)^0 \cdot (2 \cdot 1 \cdot 2) = 4$.

Recall from Sec. 1.4.4 that a linear system $A\mathbf{x} = \mathbf{0}$ is called a **homogenous system**. Then we have the following.

Lemma 4.4. *A homogenous system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution if and only if det $(A) = 0$.*

Proof. The homogenous system $A\mathbf{x} = \mathbf{0}$ have a nonzero solution if and only if **ref** (A) has a row of all zeroes, which is equivalent with det **ref** $(A) = 0$. ◻

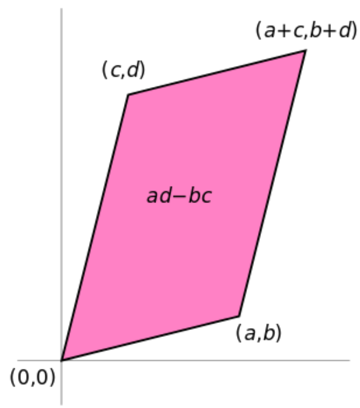


Figure 4.1: The area of the parallelogram

Exercise 57. *Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, say $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Let us form the matrix*

$$M = [\mathbf{u} \,|\, \mathbf{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

Then det $(M) = u_1 v_2 - u_2 v_1$. The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is $A = \|\mathbf{u}\| \cdot \|\mathbf{w}\|$, where $\mathbf{w}$ is the height vector given in Eq. (5.4). Hence, we have

$$A = \|u\| \cdot \left\| \mathbf{v} - proj_{\mathbf{u}}(\mathbf{v}) \right\| = \|\mathbf{u}\| \cdot \left\| \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u}^2} \mathbf{u} \right\|$$

So we have

$$\begin{aligned} \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u}^2} \mathbf{u} &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \frac{u_1 v_1 + u_2 v_2}{u_1^2 + u_2^2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{u_1^2 + u_2^2} \begin{bmatrix} -u_2(u_1 v_2 - u_2 v_1) \\ u_1(u_1 v_2 - u_2 v_2) \end{bmatrix} \\ &= \frac{u_1 v_2 - u_2 v_2}{u_1^2 + u_2^2} \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = \frac{\det M}{\|u\|^2} \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \end{aligned}$$

Then

$$\left\| \mathbf{v} - proj_{\mathbf{u}}(\mathbf{v}) \right\| = \frac{det\ M}{\|u\|^2} \sqrt{(u_2^2 + u_1^2)} = \frac{det\ M}{\|u\|}.$$

Substituting above we have

$$A = \|\mathbf{u}\| \cdot \frac{det\ M}{\|u\|} = det\ M.$$

We illustrate in Fig. 4.1.

Exercise 58. *Compute the area of the parallelogram determined by points $P(1,2)$, $Q(3,5)$, and $R(-2,9)$.*

Solution: The area is $A = |\det M|$, where M is

$$M = \left[\overrightarrow{PQ} \mid \overrightarrow{PR} \right] = \begin{bmatrix} 2 & -3 \\ 3 & 7 \end{bmatrix}$$

Hence, area is $A = |\det M| = 23$. □

Corollary 4.3. *Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ are parallel if and only if $det\ (A) = 0$, where $A = [\mathbf{u} | \mathbf{v}]$.*

We have a similar result in $\mathbb{R}^3$.

Lemma 4.5. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $A = [\mathbf{u} | \mathbf{v} | \mathbf{w}]$. Prove that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is $V = |det\ (A)|$.*

Proof. The area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} \times \mathbf{v}\|$. The height h of the parallelepiped is the projection of $\mathbf{w}$ onto the normal vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$, thus

$$h = \left| \mathbf{w} \cdot \left(\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} \right) \right| = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|}.$$

The volume V is the product of the base area and the height:

$$V = \|\mathbf{u} \times \mathbf{v}\| \cdot h = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|.$$

The scalar triple product $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ is defined as $\det([\mathbf{u} \mid$
$v \mid \mathbf{w}])$. Therefore,

$$V = |\det([\mathbf{u}\ | v \mid \mathbf{w}])|.$$

□

More generally, in $\mathbb{R}^n$, the absolute value of the determinant of an $n \times n$ matrix formed by n vectors represents the n-dimensional volume of the parallelepiped (or hyperparallelepiped) spanned by those vectors.

Exercises:

249. *Let A be a $(n \times n)$ invertible matrix. Show that*

$$det\ (A^{-1}) = \frac{1}{det(A)}$$

250. *Find the determinants of*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

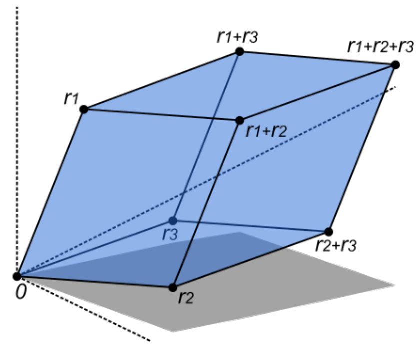


Figure 4.2: The volume of the parallelepiped

251. *Find the determinants of*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

252. *Find the determinants of*

$$A = \begin{bmatrix} 5 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 0 & 4 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 2 & 4 & -1 & 2 \end{bmatrix}$$

and use the result to find det (A^{-1}) *and* det (B^{-1}).

253. *Let A be a matrix such that* det $(A) \neq 0$. *Does the system $Ax = b$ have any solutions?*

254. *Let A be given as*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

What is the condition on a, b, c, d such that A has an inverse? Find the inverse.

255. *Let C be an invertible matrix. Prove that*

$$\det (A) = \det (C^{-1}AC).$$

256. *The determinant of an $n \times n$ matrix A is* det $(A) = 3$. *Find* det $(2A)$, det $(-A)$, *and* det (A^3).

257. *Let A be an $n \times n$ matrix. If every row of A adds to 0 prove that* det $(A) = 0$.

258. *Let A be an $n \times n$ matrix. If every row of A adds to 1 prove that* det $(A - I) = 0$. *Does this imply that* det $(A) = 0$?

259. *Let A be an $n \times n$ matrix with integer entries. Show that A^{-1} exists and has integer entries if and only if $|A| = \pm 1$.*

260. *Using raw operations compute the determinant of*

$$M = \begin{bmatrix} 7 & 0 & 0 & -2 \\ 0 & 6 & -3 & 0 \\ 0 & -3 & 6 & 0 \\ -2 & 0 & 0 & 7 \end{bmatrix}$$

and show that it is det $M = 1215$.

261. *Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^3$ such that*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Find the volume of the parallelepiped defined by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

4.2 Eigenvalues, eigenvectors, and eigenspaces

In linear algebra, diagonal matrices hold a special appeal due to their remarkable simplicity. Computations involving diagonal matrices are significantly easier than those with general matrices. For instance, the determinant of a diagonal matrix is simply the product of its diagonal entries; solving a linear system with a diagonal coefficient matrix involves only division; and powers of a diagonal matrix are obtained by simply raising the diagonal entries to the desired power. These computational advantages make diagonal matrices highly desirable in various applications.

This raises a natural and crucial question: Can a given matrix be "transformed" into a diagonal matrix? If so, we could potentially simplify many computations by working with this diagonal form. The concepts of eigenvalues and eigenvectors, which we introduce in this section, are central to answering this question. They provide the key to understanding when and how a matrix can be diagonalized (or, more generally, brought into a simpler, canonical form). Moreover, eigenvalues and eigenvectors have a deep geometric significance, revealing intrinsic properties of the linear transformation represented by the matrix. Their importance extends far beyond just diagonalization, playing a crucial role in areas like differential equations, stability analysis, and many other applications, as we will see in the next section.

When we have a matrix, we can think of it as representing a linear transformation. This transformation takes vectors as input and spits out other vectors as output. In general, when we apply this transformation to a vector, both the direction and the magnitude of the vector change.

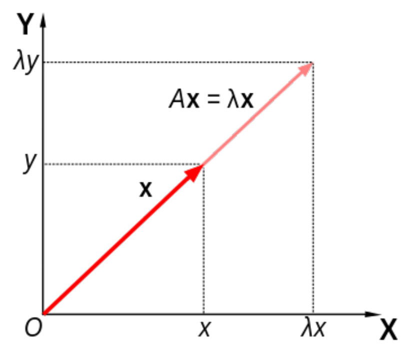


Figure 4.3: Eigenvalues and eigenvectors

However, there are some special vectors, called eigenvectors, which only change in magnitude (or stay the same) when the transformation is applied. Their direction remains unchanged. It's like they're "eigen" to the transformation, meaning they belong or are characteristic to it.

Imagine stretching or squashing space along certain axes. Eigenvectors point along these special axes, the ones that don't rotate. The eigenvalue associated with an eigenvector tells us how much the vector is stretched or squashed in that direction.

Visualizing this can be tricky in 3D, but in 2D, imagine a vector that, when transformed, stays on the same line through the origin, though it might get longer or shorter. That's an eigenvector!

Why do we care? Well, these eigenvectors reveal a lot about the transformation itself. They give us a set of "natural" coordinates for understanding what the transformation does. They are also crucial in many applications, like solving differential equations and analyzing vibrations, because they represent stable states or modes of a system.

Let A be an $n \times n$ matrix. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** if there exists a nonzero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda \mathbf{v}$$

The vector $\mathbf{v}$ is called an **eigenvector** corresponding to λ.

Proposition 4.3. *The following are equivalent:*

(i) *λ is an eigenvalue of A*

(ii) *$det\ (\lambda I - A) = 0$*

Proof. In order to compute such eigenvalues and eigenvectors we notice that

$$A\mathbf{v} = \lambda \mathbf{v} \Longrightarrow (A - \lambda I)\mathbf{v} = 0$$

Hence, an eigenvalue is a scalar λ for which the system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a non trivial solution. The system has a nontrivial solution if and only if $\det (A - \lambda I) = 0$; see Lem. 4.4. Thus, we want to find λ such that $\det (A - \lambda I) = 0$. This completes the proof. $\square$

Let $A = [a_{i,j}]$ be a given matrix. Then the above equation can be written as

$$\det (A - \lambda I) = \begin{vmatrix} a_{1,1} - \lambda & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - \lambda & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} - \lambda & \dots & a_{3,n} \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ & & \cdot & & \cdot \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} - \lambda \end{vmatrix}$$

Computing this determinant we get a polynomial in λ of degree at most n. This is called the **characteristic polynomial** of A, which we denote by **char** (A, λ). Finding the eigenvalues of A is equivalent to finding the roots of the polynomial **char** (A, λ).

Corollary 4.4. *λ is an eigenvalue if and only if it is a root of the characteristic polynomial.*

Recall from algebra that a polynomial of degree n can have at most n roots. Hence an $n \times n$ matrix can have at most n eigenvalues.

The multiplicity of an eigenvalue as a root of the characteristic polynomial is called the **algebraic multiplicity of the eigenvalue**.

Example 4.8. *Prove that the eigenvalues of a triangular matrix are entries in the main diagonal.*

Example 4.9. *Prove that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same eigenvalues with the same multiplicities.*

4.2.1 Eigenspaces

For a fixed eigenvalue λ the corresponding eigenvectors are given by the solutions of the system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

Equivalently we have called such a space the nullspace of the coefficient matrix $(A - \lambda I)$.

Definition 4.5. *If λ is an eigenvalue of A, the set*

$$E_\lambda := \{\mathbf{v} \in V \mid A\,\mathbf{v} = \lambda\,\mathbf{v}\}$$

is called the ***eigenspace*** *of A corresponding to λ. The dimension of the eigenspace is called the* ***geometric multiplicity of the eigenvalue*** *λ which we will denote by geom (λ).*

Lemma 4.6. *Let A be a square matrix and λ any eigenvalue of A. The following are true:*
i) geom $(\lambda) =$ ***null*** *$(A - \lambda I)$*
ii) geom $(\lambda) \leq$ alg (λ).

Proof. Part i) is simply by definition of the geometric multiplicity.
Part ii)

□

Finding the eigenvalues requires solving a polynomial equation which can be difficult for high degree polynomials. Once the eigenvalues are found then we use the linear system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

to find a basis for the corresponding eigenspace E_λ. A basis of E_λ is usually called an **eigenbasis**.

We illustrate below.

Example 4.10. *Find the characteristic polynomial and the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.*

Solution: *The characteristic polynomial is*

$$\textbf{\textit{char}}\ (A, \lambda) = det\ (A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 5 \cdot 2 = \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6)$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 6$. Both of them have algebraic multiplicity 1. If $\lambda_1 = -1$ the system becomes:

$$\begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

and its solution is $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Its eigenspace is $E_{\lambda_1} = \langle \mathbf{v}_1 \rangle$. It has dimension 1 and therefore the geometric multiplicity of $\lambda_1 = -1$ is also 1. For $\lambda_2 = 6$ the system becomes:

$$\begin{bmatrix} -5 & 2 \\ 5 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

and its solution is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \frac{5}{2} \end{bmatrix}$. *Its eigenspace is* $E_{\lambda_2} = \langle \mathbf{v}_2 \rangle$. *This eigenspace also has dimension 1 and therefore the geometric multiplicity of* $\lambda_2 = 6$ *is also 1.*

□

Example 4.11. *Find the eigenvalues and their multiplicities for the matrix*

$$A := \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Solution: *The characteristic polynomial is*

$$\textbf{\textit{char}}\,(A,x) = (x-1)^2\,(x-2)\,(x+2)$$

Hence there are three eigenvalues, namely $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 2$. *The eigenvalue* $\lambda_1 = 1$ *has algebraic multiplicity 2 and the others have algebraic multiplicity 1.*

To find the geometric multiplicities for $\lambda_1, \lambda_2, \lambda_3$ *we have to find their corresponding eigenvectors. By solving the corresponding systems we have*

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\frac{5}{3} \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 9 \\ 17 \\ 4 \\ 1 \end{bmatrix}$$

Thus the geometric multiplicities for $\lambda_1, \lambda_2, \lambda_3$ *are respectively 1, 1, 1.*

□

Next we will see an example when the algebraic and geometric multiplicities are the same for each eigenvalue.

Example 4.12. *Find the eigenvalues and their multiplicities for the matrix*

$$A := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Solution: *The characteristic polynomial is*

$$\textbf{\textit{char}}\,(A,x) = (x-1)^2\,(x-2)\,(x+2)$$

Hence there are three eigenvalues, namely $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 2$. *The eigenvalue* $\lambda_1 = 1$ *has algebraic multiplicity 2 and the others have algebraic multiplicity 1.*

To find the geometric multiplicities for $\lambda_1, \lambda_2, \lambda_3$ *we have to find their corresponding eigenvectors. By solving the corresponding systems we have:*

For $\lambda = 1$ the eigenvectors are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence the geometric multiplicity of $\lambda_1 = 1$ is 2.

For λ_2 and λ_3 the eigenvectors are respectively $\mathbf{v}_2$ and $\mathbf{v}_3$ as below:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ \frac{5}{2} \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence, the geometric multiplicity for λ_2 and λ_3 is 1.

□

Example 4.13. *Find the eigenvalues and the corresponding eigenbasis for the matrix*

$$M = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 4 & -1 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ -4 & 0 & -4 & -4 & -1 \end{bmatrix}$$

Solution: The characteristic polynomial is

$$\textbf{char}\,(M,\lambda) = (\lambda+1)^2\,(\lambda-3)^3$$

Denote by $\lambda_1 = 3$ and $\lambda_2 = -1$. The reader should determine an eigenbasis for each. For $\lambda_1 = 3$, the geometric multiplicity is 3 and an eigenbasis $\mathfrak{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda_2 = -1$ the geometric multiplicity is 2 and an eigenbasis $\mathfrak{B}_2 = \{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

□

We will see in the next chapter that the above two examples illustrate two classes of matrices. We will learn how to deal with each of these classes separately.

Exercises:

262. *If A has eigenvalues $\lambda_1, \dots, \lambda_n$ then show that A^2 has eigenvalues $\lambda_1^2, \dots, \lambda_n^2$.*

263. *If A and B are $n \times n$ matrices, then show that AB and BA have the same eigenvalues.*

264. *Let A be a diagonal $n \times n$ matrix such that $\det(A) \neq 0$. Assume that all entries in the diagonal are distinct. How many distinct eigenvalues has A and what are their multiplicities?*

265. *Let A be a 2 by 2 matrix with trace T and determinant D. Find a formula that gives the eigenvalues of A in terms of T and D.*

266. *Let A and B be given as below:*

$$A = \begin{bmatrix} 5 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 0 & 4 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 2 & 4 & -1 & 2 \end{bmatrix}$$

Find their eigenvalues. In each case compute the sum and product of eigenvalues and compare it with the trace and determinant of the matrix.

267. *Prove that a square matrix is invertible if and only if no eigenvalue is zero.*

268. *Let A be a 3 by 3 matrix. Can you find a formula which determines the eigenvalues of A if you know the trace and determinant of A?*

269. *Find the characteristic polynomial, eigenvalues, and eigenvectors of the matrix*

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

270. *Compute the eigenvalues and their multiplicities of the matrix A^3, where A is as in the previous example.*

271. *Find the eigenvalues and their algebraic and geometric multiplicities for each of the matrices*

$$A = \begin{bmatrix} 5 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 0 & 4 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 2 & 4 & -1 & 2 \end{bmatrix}$$

272. *Let A be a diagonal $n \times n$ matrix given by*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

What are its eigenvalues and their multiplicities?

273. *Find all the eigenvalues of a $n \times n$ matrix with all diagonal entries equal to a and all other entries equal to b.*

274. *If A is a symmetric matrix prove that the algebraic multiplicity for each eigenvalue is equal to its geometric multiplicity.*

Nils Abel (1802-1829)

Niels Henrik Abel, (born August 5, 1802, island of Finney, near Stavanger, Norway–died April 6, 1829, Froland), Norwegian mathematician, a pioneer in the development of several branches of modern mathematics.

Abel's father was a poor Lutheran minister who moved his family to the parish of Gjerstad, near the town of Risor in southeast Norway, soon after Niels Henrik was born. In 1815 Niels entered the cathedral school in Oslo, where his mathematical talent was recognized in 1817 with the arrival of a new mathematics teacher, Bernt Michael Holmboe, who introduced him to the classics in mathematical literature and proposed original problems for him to solve. Abel studied the mathematical works of the 17th-century Englishman Sir Isaac Newton, the 18th-century German Leonhard Euler, and his contemporaries the Frenchman Joseph-Louis Lagrange and the German Carl Friedrich Gauss in preparation for his own research.

Abel's father died in 1820, leaving the family in straitened circumstances, but Holmboe contributed and raised funds that enabled Abel to enter the University of Christiania (Oslo) in 1821. Abel obtained a preliminary degree from the university in 1822 and continued his studies independently with further subsidies obtained by Holmboe.

Abel's first papers, published in 1823, were on functional equations and integrals; he was the first person to formulate and solve an integral equation. His friends urged the Norwegian government to grant him a fellowship for study in Germany and France. In 1824, while waiting for a royal decree to be issued, he published at his own expense his proof of the impossibility of solving algebraically the general equation of the fifth degree, which he hoped would bring him recognition. He sent the pamphlet to Gauss, who dismissed it, failing to recognize that the famous problem had indeed been settled.

Abel spent the winter of 1825–26 with Norwegian friends in Berlin, where he met August Leopold Crelle, civil engineer and self-taught enthusiast of mathematics, who became his close friend and mentor. With Abel's warm encouragement, Crelle founded the Journal für die Reine und Angewandte Mathematik ("Journal for Pure and Applied Mathematics"), commonly known as Crelle's Journal. The first volume (1826) contains papers by Abel, including a more elaborate version of his work on the quintic equation. Other papers dealt with equation theory, calculus, and theoretical mechanics. Later volumes presented Abel's theory of elliptic functions, which are complex functions (see complex number) that generalize the usual trigonometric functions.

In 1826 Abel went to Paris, then the world centre for mathematics, where he called on the foremost mathematicians and completed a major paper on the theory of integrals of algebraic functions. His central result, known as Abel's theorem, is the basis for the later theory of Abelian integrals and Abelian functions, a generalization of elliptic function theory to functions of several variables. However, Abel's visit to Paris was unsuccessful in securing him an appointment, and the memoir he submitted to the French Academy of Sciences was lost.

Abel returned to Norway heavily in debt and suffering from tuberculosis. He subsisted by tutoring, supplemented by a small grant from the University of Christiania and, beginning in 1828, by a temporary teaching position. His poverty and ill health did not decrease his production; he wrote a great number of papers during this period, principally on equation theory and elliptic functions. Among them are the theory of polynomial equations with Abelian groups. He rapidly developed the theory of elliptic functions in competition with the German Carl Gustav Jacobi. By this time Abel's fame had spread to all mathematical centres, and strong efforts were made to secure a suitable position for him by a group from the French Academy, who addressed King Bernadotte of Norway-Sweden; Crelle also worked to secure a professorship for him in Berlin.

4.3 Similar matrices, diagonalizing matrices, eigendecomposition

In this section we will study the concept of similarity of matrices. We will determine necessary and sufficient conditions for a matrix to be similar to a diagonal matrix. When this is possible we will provide an algorithm for determining this diagonal matrix.

Definition 4.6. *Two matrices A and B are called* ***similar*** *if there exists a matrix C such that*

$$A = CBC^{-1}.$$

Two similar matrices A and B are denoted by $A \sim B$.

Exercise 59. *The similarity relation is an equivalence relation.*

For a given square matrix A consider the following problem. Determine a matrix C such that $C^{-1}AC$ is a diagonal matrix. The following theorem is the main result of this section.

Theorem 4.5. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ be a square matrix and $\lambda_1, \dots, \lambda_s$ all distinct eigenvalues of A. Suppose that for all $i = 1, \dots, s$, the algebraic multiplicity of λ_i equals its geometric multiplicity, say*

$$alg.\ mult.(\lambda_i) = geom.\ mult.(\lambda_i) = e_i.$$

Then there exists an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$. The matrix D is given by

$$D = \begin{bmatrix} \lambda_1 &&&&&&&& \\ & \ddots &&&&&&& \\ && \lambda_1 &&&&&& \\ &&& \lambda_2 &&&&& \\ &&&& \ddots &&&& \\ &&&&& \lambda_2 &&& \\ &&&&&& \ddots && \\ &&&&&&& \lambda_s & \\ &&&&&&&& \ddots & \\ &&&&&&&&& \lambda_s \end{bmatrix}$$

where each λ_i appears e_i times on the diagonal, and

$$C = \left[\mathbf{v}_{1,1} \quad \dots \quad \mathbf{v}_{1,e_1} \quad \mathbf{v}_{2,1} \quad \dots \quad \mathbf{v}_{2,e_2} \quad \dots \quad \mathbf{v}_{s,1} \quad \dots \quad \mathbf{v}_{s,e_s} \right],$$

where $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,e_i}$ is a basis for the eigenspace E_{λ_i}.

Proof. For each eigenvalue λ_i, since the geometric multiplicity equals the algebraic multiplicity (e_i), we can find e_i linearly independent eigenvectors associated with λ_i. Let these eigenvectors be $\mathbf{v}_{i,1}, \mathbf{v}_{i,2}, \dots, \mathbf{v}_{i,e_i}$. These vectors form a basis for the eigenspace E_{λ_i}.

Form the matrix C by placing the eigenvectors as columns:

$$C = \left[\mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \dots, \mathbf{v}_{1,e_1}, \mid \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,e_2}, \dots, \mid \mathbf{v}_{s,1}, \dots, \mathbf{v}_{s,e_s} \right]$$

Form the diagonal matrix D with the eigenvalues on the diagonal, repeated according to their algebraic multiplicity.

$$D = \begin{bmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_s & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_s \end{bmatrix}$$

We want to show that $AC = CD$. Consider the product AC. The j-th column of AC is given by A times the j-th column of C. If the j-th column of C is an eigenvector $\mathbf{v}_{i,k}$ corresponding to eigenvalue λ_i, then $A\mathbf{v}_{i,k} = \lambda_i \mathbf{v}_{i,k}$.

Now consider the product CD. The j-th column of CD is given by C times the j-th column of D. If the j-th column of D has λ_i on the diagonal, then the j-th column of CD will be λ_i times the j-th column of C, which is the eigenvector $\mathbf{v}_{i,k}$. Thus, the j-th column of AC is the same as the j-th column of CD. Since this holds for all columns, we have $AC = CD$.

Since the eigenvectors corresponding to distinct eigenvalues are linearly independent, and we have a total of $e_1 + e_2 + \cdots + e_s = n$ linearly independent eigenvectors (because the geometric multiplicity equals the algebraic multiplicity for each eigenvalue), the matrix C formed by these eigenvectors is invertible.

Since $AC = CD$ and C is invertible, we can multiply both sides by C^{-1} on the left to get

$$C^{-1}AC = C^{-1}CD = D.$$

Therefore, A is similar to the diagonal matrix D. $\square$

We call the matrix C above the **transitional matrix** of A associated with D. We illustrate the above theorem with the following two examples.

Example 4.14. *Let A be the 4×4 matrix as follows*

$$A = \begin{bmatrix} 2 & 1 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

Find a diagonal matrix similar to A and the transitional matrix.

Solution: The characteristic polynomial of A is

$$\textbf{char}\,(A, \lambda) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - \lambda - 1)$$

The eigenvalues are $1 \pm \mathbf{i}, \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ and their algebraic multiplicity is 1. We now find the geometric multiplicity for each one of the eigenvalues.

Let $\lambda = 1 + \mathbf{i}$. Then we solve the system $A - (1+\mathbf{i})I_n = \mathbf{0}$. The solution space has dimension 1 and a basis for it is $\mathbf{v}_1$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1+\mathbf{i} \\ 1 \\ 0 \end{bmatrix}$$

Similarly, if $\lambda = 1 - \mathbf{i}$ then the eigenvector is:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1-\mathbf{i} \\ 1 \\ 0 \end{bmatrix}$$

If $\lambda_3 = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $\lambda_4 = \frac{1}{2} + \frac{\sqrt{5}}{2}$, then the corresponding eigenvectors are

$$\mathbf{v}_3 = \begin{bmatrix} -\frac{13}{2} + \frac{5}{2}\sqrt{5} \\ \\ 1 \\ \\ 6 - 3\sqrt{5} \\ \\ \frac{15}{2} - \frac{7}{2}\sqrt{5} \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} -\frac{13}{2} - \frac{5}{2}\sqrt{5} \\ \\ 1 \\ \\ 6 - 3\sqrt{5} \\ \\ \frac{15}{2} - \frac{7}{2}\sqrt{5} \end{bmatrix}$$

Hence, since the algebraic multiplicity of each eigenvalue is the same with the geometric multiplicity then A is similar to

$$D = \begin{bmatrix} 1+\mathbf{i} & 0 & 0 & 0 \\ 0 & 1-\mathbf{i} & 0 & 0 \\ 0 & 0 & \frac{1}{2} + \frac{\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}$$

The transitional matrix in this case is $C = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$. □

Lemma 4.7. *Similar matrices have the same characteristic polynomial. Therefore, they have the same eigenvalues with the same algebraic multiplicities.*

Proof. Let $A \sim B$, say

$$A = CBC^{-1}$$

for some invertible matrix C. Then,

$$\begin{aligned} \mathbf{char}\,(A, \lambda) &= \det(A - \lambda I) = \det(A - \lambda I) \cdot \det(C) \cdot \det(C^{-1}) \\ &= \det\left(C(A - \lambda I)C^{-1}\right) = \det\left(CAC^{-1} - \lambda CIC^{-1}\right) \\ &= \det\left(CAC^{-1} - \lambda I\right) = \det(B - \lambda I) = \mathbf{char}\,(B, \lambda). \end{aligned}$$

Thus, the characteristic polynomial is the same. Hence, A and B have the same eigenvalues. ◻

The converse of the lemma is not true. In other words there are matrices with the same characteristic polynomial which are not similar.

Example 4.15. *Consider the matrices* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Let us try to diagonalize A. Its characteristic polynomial is*

$$\textbf{\textit{char}}\,(A, \lambda) = (\lambda - 1)^2.$$

Hence, there is only one eigenvalue $\lambda = 1$ *of algebraic multiplicity 2. For this eigenvalue* $\lambda = 1$ *we have*

$$A - \lambda I = A - 1 \cdot I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which is already in row-echelon form. Since there is only one pivot then the **null** $(A - I) = 1$ *and therefre the geometric myltiplicity of* $\lambda = 1$ *is 1. Hence, this matrix can not be diagonalized, which means it can not be similar to a diagonal matrix. Therefore, A can not be similar to I.* ◻

Remark 4.3. *In the last chapter of this book we will learn how to determine all similarity classes of matrices for a given characteristic polynomial.*

Lemma 4.8. *Let A be a* $n \times n$ *matrix and* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *its eigenvalues (not necessarily distinct) such that the algebraic and geometric multiplicity are the same. Then,*

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Proof. Since the algebraic and geometric multiplicities of the eigenvalues of A are equal, A is diagonalizable. This means there exists an invertible matrix P such that A can be written as:

$$A = PDP^{-1}$$

where D is a diagonal matrix whose diagonal entries are the eigenvalues of A:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

The trace of a matrix has the property that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Using this property, we can write:

$$\operatorname{tr}(A) = \operatorname{tr}(PDP^{-1}) = \operatorname{tr}(P^{-1}PD) = \operatorname{tr}(ID) = \operatorname{tr}(D)$$

Since I is the identity matrix, $P^{-1}P = I$. The trace of a diagonal matrix is simply the sum of its diagonal entries. Therefore,

$$\operatorname{tr}(D) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Thus, we have shown that:

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Therefore, the trace of matrix A is equal to the sum of its eigenvalues. ◻

Notice that this result is true even if we drop the assumption that algebraic and geometric multiplicities are the same; see Thm. 5.7.

4.3.1 Diagonalizing matrices

We want to consider the following: Given a matrix A, find a diagonal matrix D such that A is similar to D. Further, find the matrix C which conjugates A and D. The theorem above provides an algorithm for how this can be done.

Algorithm 7. *Input: An $n \times n$ matrix A.*
Output: Matrices C and D such that

$$D = C^{-1} A C$$

if A is diagonalizable, otherwise display `'A is not diagonalizable'`.

`Step:`*i) Compute the eigenvalues of A and their algebraic multiplicities.*
`Step:`*ii) For each eigenvalue λ_1, compute the geometric multiplicity of λ_i and the corresponding eigenvectors*

$$\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,s}$$

`Step:`*iii) Create the matrix D and C as in the previous theorem.*

Example 4.16. *Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then*

$$C = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

You can verify that $A = CDC^{-1}$.

Example 4.17. *Let A be the 4×4 matrix given below*

$$A := \begin{bmatrix} 9 & 0 & 0 & 0 \\ -2 & 1 & -3 & -4 \\ -6 & 0 & 6 & 0 \\ 4 & 4 & 3 & 11 \end{bmatrix}$$

Find out if this matrix is diagonalizable and in that case find a diagonal matrix D similar to A and the transitional matrix C associated to D.
Solution: *The characteristic polynomial of A is*

$$\textbf{\textit{char}}\,(A, x) = (x-3)(x-6)(x-9)^2.$$

Thus, the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 6$, and $\lambda_3 = 9$ with algebraic multiplicities 1, 1, and 2 respectively. The corresponding eigenvectors of $\lambda_1, \lambda_2, \lambda_3$ are respectively $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{w}_1, \mathbf{w}_2$ as below

$$\mathbf{v}_1 := \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 := \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{w}_1 := \begin{bmatrix} 2 \\ 1 \\ -4 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 := \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Hence, the geometric multiplicities are respectively 1,1, and 2. Therefore the matrix A is diagonalizable and C and D are

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 0 & 2 & 1 \\ -2 & 1 & 1 & 0 \\ 0 & -3 & -4 & -2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

□

Example 4.18. *Let A be a 3 by 3 matrix as below*

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Check if A is similar to a diagonal matrix.
Solution: *Then* **char** $(A,\lambda) = (\lambda-2)^2(\lambda-3)$. *For the eigenvalue* $\lambda = 2$, *the algebraic multiplicity is 2 and the eigenspace is given by*

$$E_2 = \left\{ t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mid t \in \mathbb{Q} \right\}$$

The geometric multiplicity is 1, hence A is not similar to the diagonal matrix of eigenvalues. □

Exercise 60. *Let A be similar to a diagonal matrix D such that* $A = CDC^{-1}$. *Prove that for every positive integer n,* $A^n = CD^nC^{-1}$.

Example 4.19. *Diagonalize the matrix*

$$M = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Solution: The characteristic polynomial of M is

$$\textbf{char}\,(M,\lambda) = (\lambda-1)^3(\lambda-2)^2.$$

For $\lambda = 1$ the corresponding eigenbasis is $\mathfrak{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\mathbf{v}_3\}$ and for $\lambda = 2$ the eigenbasis is $\mathfrak{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ such that

$$C = \left[\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3 | \mathbf{w}_1 | \mathbf{w}_2 \right] = \begin{bmatrix} 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

The reader can check that $D = C^{-1} M C$. □

Remark 4.4. *Perhaps it is worth mentioning that to check that $D = C^{-1} M C$ it is easier to check that $CD = MC$. In this way we don't have to compute C^{-1}.*

In later chapters we will see how we can pick the transitional matrix C differently.

Remark 4.5. *It is often not a simple task to factorize the characteristic polynomial, especially if it is of high degree. Moreover, solving high degree algebraic equations it can be quite difficult. In most exercises we will give matrices whose characteristic polynomial can be easily factored or give the characteristic polynomial in factored form.*

Example 4.20. *Diagonalize the matrix*

$$M = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Solution: The characteristic polynomial of M is **char** $(M, \lambda) = (\lambda - 1)(\lambda - 2)^4$. The eigenbasis for $E_{\lambda=1}$ is $\mathfrak{B}_1 = \{\mathbf{v}_1\}$ and the eigenbasis for $E_{\lambda=2}$ is $\mathfrak{B}_2 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ and we get C and D as

$$C = \left[\mathbf{v}_1 | \mathbf{u}_1 | \mathbf{u}_2 | \mathbf{u}_3 | \mathbf{u}_4 \right] = \begin{bmatrix} -1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

The reader can now verify that $D = C^{-1} M C$. □

4.3.2 Eigendecomposition

The expression of a square matrix A as

$$A = CDC^{-1}$$

where D is a diagonal matrix formed from the eigenvalues of A, and the columns of C are the corresponding eigenvectors, is called the **eigendecomposition** (or **spectral decomposition**) of A.

An $n \times n$ matrix A has n complex eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with algebraic multiplicity), which form the diagonal entries of D. The corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form the columns of C: $C = [\mathbf{v}_1 \; \mathbf{v}_2 \; \cdots \; \mathbf{v}_n]$. This decomposition is equivalent to the equation $AC = CD$.

C is invertible if and only if A has n linearly independent eigenvectors (i.e., each eigenvalue's geometric multiplicity equals its algebraic multiplicity). A sufficient (but not necessary) condition for this is that all eigenvalues are distinct. In this case, A is diagonalizable.

Eigenvectors are often normalized to have length 1. This does not change the eigendecomposition, as any scalar multiple of an eigenvector is still an eigenvector.

Exercises:

275. *Let A be a $n \times n$ matrix with characteristic polynomial*

$$\textbf{\textit{char}}\,(A, \lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0.$$

Show that $\operatorname{tr}(A) = (-1)^{n-1} \cdot a_{n-1}$.

276. *Diagonalize (if possible) the matrix:*

$$A = \begin{bmatrix} 3 & 1 & 4 & 2 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

277. *Let*

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ -1 & 0 & -1 & 0 \\ 5 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 & 4 & 2 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}.$$

Determine if A and B are similar.

278. *Let*

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ -1 & 0 & -1 & 0 \\ 5 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -10 & -2 & 2 & 3 \\ 11 & 7 & -5 & 1 \\ -15 & -2 & 5 & 4 \\ -15 & -4 & 5 & 3 \end{bmatrix}.$$

Determine if A and B are similar.

279. *Let* $A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$. *Find its eigenvalues and their geometric and algebraic multiplicities. For each eigenspace determine a basis. Find the matrices C and D such that $CA = DC$.*

280. *Let*

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 2 & 6 & 9 \end{bmatrix}$$

Find the eigenvalues of A and determine a basis for each eigenspace. Determine matrices C and D such that $D = C^{-1}AC$. Compute A^{11}.

281. *Let A be the 4 by 4 matrix*

$$A := \begin{bmatrix} -2 & -5 & -2 & -1 \\ \frac{3}{2} & \frac{7}{2} & \frac{3}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{-3}{2} & -1 \\ \frac{-5}{2} & \frac{-7}{2} & \frac{-1}{2} & 1 \end{bmatrix}$$

Show that $D = C^{-1}AC$ where

$$C := \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \end{bmatrix}, \qquad D := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Compute A^6.

282. *Compute A^r for* $A = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$, *where r is a positive integer.*

4.4 Cramer's rule and adjoint matrices

Until now we have solved linear systems using the Gauss method. In this section we will see a different method which implies a formula for solving linear systems. Let a linear system $A \cdot \mathbf{x} = \mathbf{b}$ be given where

$$A = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n} \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & a_{m,n} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ x_m \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \\ \\ \\ b_m \end{bmatrix}.$$

For each $k = 1, \dots, n$, we define the matrix B_k to be the matrix obtained by replacing the $\mathbb{F}$-column of A by the vector $\mathbf{b}$ as below:

$$B_k = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & b_1 & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & b_2 & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & b_3 & \dots & a_{3,n} \\ & & \cdot & \dots & \cdot & \dots & \\ & & \cdot & \dots & \cdot & \dots & \\ & & \cdot & \dots & b_i & \dots & \\ & & \cdot & \dots & \cdot & \dots & \\ & & \cdot & \dots & \cdot & \dots & \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & b_n & \dots & a_{m,n} \end{bmatrix}$$

Theorem 4.6 (Cramer). *If A is an invertible matrix then the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by*

$$x_k = \frac{det\ (B_k)}{det\ (A)}, \qquad for \qquad k = 1, \dots, n.$$

Proof. The solution is $\mathbf{x} = A^{-1}\mathbf{b}$. Expand $\det B_k$ in cofactors of the k-th column. We have

$$\det B_k = b_1 A_{1k} + \dots + b_n A_{nk}.$$

Multiplying by $\frac{1}{\det A}$, this is exactly the k-th component of the vector $\mathbf{x}$. $\square$

Example 4.21. *Solve the following system using Cramer's rule*

$$\begin{cases} 2x + 3y = 5 \\ 5x - y = 7 \end{cases}$$

Solution: *Then* $A = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 5 & 3 \\ 7 & -1 \end{bmatrix}$, *and* $B_2 = \begin{bmatrix} 2 & 5 \\ 5 & 7 \end{bmatrix}$. *Hence,*

$$det\ (A) = -17, \qquad det\ (B_1) = -26, \qquad det\ (B_2) = -11$$

and $x_1 = \frac{26}{17}$ *and* $x_2 = \frac{11}{17}$. $\square$

We now illustrate with a linear system with five equations and five unknowns.

Example 4.22. *Solve the linear system $A\mathbf{x} = \mathbf{b}$, where A is as in Exa. 4.4, and $\mathbf{b} = [1,0,0,-1,0]^t$.*

Solution: *As shown in Exa. 4.4 the determinant of A is $\det(A) = -121$. Further, we compute*

$$det\,(B_1) = -61,\ det\,(B_2) = -14,\ det\,(B_3) = 44,\ det\,(B_4) = -8,\ det\,(B_5) = 28$$

Then, the solution of the system is $\mathbf{x} = \left[\frac{61}{121}, \frac{14}{121}, -\frac{4}{11}, \frac{8}{121}, -\frac{28}{121}\right]^t$. □

4.4.1 Adjoints of matrices

The existence of the inverse of a matrix depends on whether or not the determinant of the matrix is 0. Naturally one would like to find a formula for the inverse in terms of the determinant. The concept of the adjoint (or conjugate transpose) of a matrix plays a fundamental role in linear algebra, particularly in inner product spaces, optimization, and functional analysis. It generalizes the notion of the transpose for real matrices and arises naturally in various applications

Definition 4.7. *Let A be a $n \times n$ matrix with entries in $\mathbb{C}$ given by $A = [a_{i,j}]$. For each entry $a_{i,j}$ the corresponding cofactor is denoted by $c_{i,j}$. Create the matrix $C = [c_{i,j}]$. Let*

$$\bar{C} := [\,\bar{c}_{i,j}\,],$$

*where $\bar{C}_{i,j} = [\bar{c}_{i,j}]$ contains the conjugates of elements $c_{i,j}$; see **??**. The matrix*

$$adj\,(A) := (\bar{C})^t$$

is called the ***adjoint*** *of A.*

Hence, given a complex matrix $A \in \mathbb{C}^{m\times n}$, its **adjoint** (also called the Hermitian transpose or conjugate transpose) is denoted by A^* and is defined as

$$A^* = \overline{A}^T, \tag{4.2}$$

where $\overline{A}$ represents the entrywise complex conjugate of A, and A^T is the usual transpose. For a real matrix $A \in \mathbb{R}^{m\times n}$, the adjoint reduces to the ordinary transpose: $A^* = A^T$.

Example 4.23. *Find the adjoint of the matrix $A = \begin{bmatrix} i+1 & 2 & i-1 \\ 0 & 2i & 0 \\ i & 1 & -1 \end{bmatrix}$.*

Solution: *Then we find $C = \begin{bmatrix} -2i & 0 & 2 \\ -1-i & 0 & 1-i \\ 4 & 0 & -2+2i \end{bmatrix}$. Hence,*

$$\bar{C} = \begin{bmatrix} 2i & 0 & 2 \\ -1+i & 0 & 1+i \\ 4 & 0 & -2-2i \end{bmatrix} \quad \text{and} \quad adj\,(A) = \begin{bmatrix} -2i & -1-i & 4 \\ 0 & 0 & 0 \\ 2 & 1-i & -2+2i \end{bmatrix}$$

□

Remark 4.6. *Notice that if the matrix has entries in $\mathbb{R}$ then it is not necessary to take the conjugates of $c_{i,j}$ since the conjugates of real numbers are the numbers themselves. That is why in most textbooks which treat only the matrices with entries from $\mathbb{R}$ the definition of the adjoint does not contain taking conjugates.*

Lemma 4.9. *The adjoint operation satisfies several important properties:*

1. ***Involution Property:*** $(A^*)^* = A$.
2. ***Reversal of Products:*** $(AB)^* = B^* A^*$ *for any conformable matrices* A, B.
3. ***Sum Property:*** $(A + B)^* = A^* + B^*$.
4. ***Scalar Multiplication:*** $(\lambda A)^* = \overline{\lambda} A^*$ *for any scalar* $\lambda \in \mathbb{C}$.
5. ***Self-Adjoint Matrices:*** *If* $A^* = A$, *the matrix is called* Hermitian *and has real eigenvalues.*
6. ***Unitary Matrices:*** *If* $A^* A = AA^* = I$, *then* A *is* unitary, *meaning it preserves inner products and norms.*

Theorem 4.7. *Let A be an invertible matrix and* adj(A) *its adjoint. Then*

$$A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = det\ (A) \cdot I_n$$

Proof. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The adjoint of A, denoted adj(A), is the transpose of the cofactor matrix of A. The cofactor C_{ij} of the element a_{ij} is given by $(-1)^{i+j} M_{ij}$, where M_{ij} is the determinant of the submatrix obtained by deleting the i-th row and j-th column of A. Thus, $\mathrm{adj}(A) = [C_{ji}]$.

We want to show that $A \cdot \mathrm{adj}(A) = \det(A) \cdot I_n$. The (i, j)-entry of the product $A \cdot \mathrm{adj}(A)$ is given by:

$$(A \cdot \mathrm{adj}(A))_{ij} = \sum_{k=1}^{n} a_{ik} C_{jk}$$

We consider two cases:

Case 1: $i = j$. Then the (i,i)-entry of $A \cdot \mathrm{adj}(A)$ is:

$$(A \cdot \mathrm{adj}(A))_{ii} = \sum_{k=1}^{n} a_{ik} C_{ik}$$

This is the cofactor expansion of the determinant of A along the i-th row. Therefore,

$$(A \cdot \mathrm{adj}(A))_{ii} = \det(A)$$

Case 2: $i \neq j$. Then the (i, j)-entry of $A \cdot \mathrm{adj}(A)$ is:

$$(A \cdot \mathrm{adj}(A))_{ij} = \sum_{k=1}^{n} a_{ik} C_{jk}$$

This sum represents the cofactor expansion of the determinant of a matrix A' where the j-th row of A' is identical to the i-th row of A, and all other rows are the same as in A. Since A' has two identical rows, its determinant is zero. Therefore,

$$(A \cdot \text{adj}(A))_{ij} = 0$$

Combining these two cases, we have:

$$(A \cdot \text{adj}(A))_{ij} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This means that $A \cdot \text{adj}(A) = \det(A) \cdot I_n$.

Similarly, we can show that $\text{adj}(A) \cdot A = \det(A) \cdot I_n$. The (i, j)-entry of $\text{adj}(A) \cdot A$ is:

$$(\text{adj}(A) \cdot A)_{ij} = \sum_{k=1}^{n} C_{ki} a_{kj}$$

If $i = j$, this is the cofactor expansion along the i-th *column*, which also equals $\det(A)$. If $i \neq j$, this represents the determinant of a matrix with two identical columns, which is 0. Therefore, $\text{adj}(A) \cdot A = \det(A) \cdot I_n$.

Thus, we have shown that $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A) \cdot I_n$. □

From the above theorem we conclude that for a given matrix A such that $\det(A) \neq 0$ we have

$$A^{-1} = \frac{1}{\det(A)} \text{ adj}(A)$$

Example 4.24. *Find the adjoint of A* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution: *Check that the adjoint is*

$$adj\,(A) = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Notice that $det\,(A) = 0$ *so this matrix does not have an inverse.* □

Exercises:

283. *Compute the inverse of the matrix*

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

using the adjoint method.

284. *Show that if* A *is an invertible* 3×3 *matrix, then* $A^{-1} = \frac{1}{det\,(A)}\, adj\,(A)$.

285. *Let*

$$A = \begin{bmatrix} 2 & i \\ -i & 3 \end{bmatrix}.$$

Find its adjoint matrix A^*.

286. *Determine whether the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is invertible using the determinant criterion.

287. *Prove that for any square matrix A, $\det(\operatorname{adj}(A)) = (\det A)^{n-1}$.*

288. *Prove that if A is an $n \times n$ invertible matrix, then the solution to $A\mathbf{x} = \mathbf{b}$ given by Cramer's rule is unique.*

289. *Show that if A is a singular $n \times n$ matrix, then at least one of the determinant expressions in Cramer's rule is undefined.*

290. *Prove that if A is an invertible $n \times n$ matrix, then $\operatorname{adj}(A)$ satisfies the equation $A \cdot \operatorname{adj}(A) = (\det A)I_n$.*

291. *Let A be a 3×3 matrix. Show that $\operatorname{adj}(A)$ satisfies the matrix equation $A \cdot \operatorname{adj}(A) = (\det A)I_3$ explicitly by computing both sides.*

292. *If A is a diagonal matrix, show that its adjugate matrix is also diagonal, and find a general formula for $\operatorname{adj}(A)$ in terms of the diagonal entries of A.*

293. *Prove that for any invertible $n \times n$ matrix A, we have $\operatorname{adj}(A^{-1}) = (\operatorname{adj}(A))^{-1}$.*

294. *Show that if A is an upper triangular or lower triangular matrix, then $\operatorname{adj}(A)$ is also upper triangular or lower triangular, respectively.*

295. *If A is an $n \times n$ matrix with integer entries and $\det(A) = \pm 1$, prove that all entries of $\operatorname{adj}(A)$ are also integers.*

296. *Let A be an $n \times n$ matrix. Prove that if A is invertible, then $\det(\operatorname{adj}(A)) = (\det A)^{n-1}$.*

297. *For an arbitrary $n \times n$ matrix A, prove that $\operatorname{adj}(\operatorname{adj}(A)) = (\det A)^{n-2}A$.*

4.5 Resultants

The study of resultants is fundamental in polynomial algebra, as it provides a powerful criterion for detecting common roots between two polynomials. Given two polynomials $f(x)$ and $g(x)$, their resultant encodes information about whether they share a nontrivial common factor. This concept is particularly useful in elimination theory, where we seek to eliminate variables from systems of polynomial equations, and in computational algebra, where resultants appear in algorithms for solving polynomial systems. The construction of the Sylvester matrix offers an explicit and systematic way to compute the resultant, leveraging linear algebra techniques to transform a problem in polynomial factorization into a problem of matrix determinants. This approach not only deepens our understanding of polynomial structure but also has far-reaching applications in areas such as algebraic geometry, number theory, and symbolic computation.

Let $f(x)$ and $g(x)$ be polynomials of degree n and m given as follows:

$$
\begin{aligned}
f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\
g(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0
\end{aligned} \tag{4.3}
$$

with coefficients in $\mathbb{F}$. Consider vector spaces P_n and P_m of polynomials of degree $\leq n$ and $\leq m$ respectively with coefficients in $\mathbb{F}$. Let $P_n \times P_m$ be the direct product as in Def. 2.11 and define

$$
\begin{aligned}
\varphi : P_n \times P_m &\to P_{n+m} \\
(p, q) &\to fq + gp
\end{aligned}
$$

Lemma 4.10. *φ is a linear map and the corresponding matrix is then $(n+m)$-square matrix*

$$
M_\varphi = \begin{bmatrix}
a_n & & & & & b_m & & & & \\
a_{n-1} & a_n & & & & b_{m-1} & b_m & & & \\
a_{n-2} & a_{n-1} & \cdot & & & b_{m-2} & b_{m-1} & \cdot & & \\
\cdot & a_{n-2} & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & a_n & \cdot & \cdot & \cdot & \cdot & b_m \\
a_1 & \cdot & \cdot & \cdot & a_{n-1} & b_0 & \cdot & \cdot & \cdot & b_{m-1} \\
a_0 & a_1 & \cdot & \cdot & a_{n-2} & & b_0 & \cdot & \cdot & \cdot \\
 & a_0 & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\
 & & \cdot & & \cdot & & & & \cdot & \cdot \\
 & & & \cdot & \cdot & & & & & \cdot \\
 & & & & a_0 & & & & & b_0
\end{bmatrix} \tag{4.4}
$$

Proof. The reader can check that φ is a well defined map. Consider (p_1, q_1) and $(p_2, , q_2)$ points in $P_n \times P_m$. Then

$$
\begin{aligned}
\varphi\Big((p_1, q_1) + (p_2, q_2) \Big) &= \varphi(p_1 + p_2, q_1 + q_2) = f(q_1 + q_2) + g(p_1 + p_2) \\
&= (fq_1 + gp_1) + (fq_2 + gp_2) = \varphi(p_1, q_1) + \varphi(p_2, q_2)
\end{aligned}
$$

Similarly for the scalar multiplication. Hence φ is linear.

From Lem. 2.7 we have $\dim(P_n \times P_m) = n+m$. Hence, there is a $(n+m)$-square matrix associated to φ. Pick a basis $\mathfrak{B}_1 = \{x^{n-1}, \ldots, x, 1\}$ for P_n and $\mathfrak{B}_2 = \{x^{m-1}, \ldots, x, 1\}$ for P_m.

□

The matrix M_φ is called the **Sylvester matrix** of $f(x)$ and $g(x)$ and denoted by $\mathrm{Syl}(f, g, x)$.

Definition 4.8. *The **resultant** of $f(x)$ and $g(x)$, denoted by $Res(f, g, x)$, is*

$$Res(f, g, x) := det\,(Syl(f, g, x)).$$

The following is a basic fact in the algebra of polynomials.

Lemma 4.11. *The polynomials $f(x)$ and $g(x)$ have a common factor in $\mathbb{F}[x]$ if and only if* $\mathrm{Res}(f, g, x) = 0$.

The proof is part of a course in computational algebra.

Example 4.25. *Consider the polynomials:*

$$f(x) = x^2 + ax + b$$

$$g(x) = x + c$$

We will compute the resultant $\mathrm{Res}(f, g, x)$ *using the Sylvester matrix.*

Step 1: Construct the Sylvester Matrix

The Sylvester matrix is formed by writing down the coefficients of $f(x)$ and $g(x)$ in a structured way.

For the given polynomials:

$$f(x) = 1x^2 + ax + b = [1, a, b]$$

$$g(x) = 1x + c = [1, c]$$

The Sylvester matrix is:

$$S = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$$

Step 2: Compute the Determinant

$$\mathrm{Res}(f, g, x) = det \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$$

Expanding the determinant:

$$\mathrm{Res}(f, g, x) = (1 \cdot 1) \cdot c - (0 \cdot a) \cdot b = c - ac + b.$$

Thus, the resultant is:

$$\mathrm{Res}(f, g, x) = c - ac + b.$$

This resultant tells us that $f(x)$ and $g(x)$ share a common root if and only if $\mathrm{Res}(f, g, x) = 0$.

4.5.1 The Discriminant

Let $f(x)$ be a degree n polynomial given as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the coefficients a_i are in $\mathbb{R}$. The discriminant of a polynomial encodes important information about its roots, especially their multiplicities.

Geometric Interpretation

Intuitively, the discriminant measures how "clustered" or "spread out" the roots of $f(x)$ are. If $f(x)$ has distinct roots, they contribute nonzero factors to the product defining the discriminant. However, when two or more roots coincide, a factor in the product vanishes, causing the discriminant to be zero. This provides a powerful way to detect whether a polynomial has multiple roots without explicitly solving for them.

From the Fundamental Theorem of Algebra, every degree n polynomial has precisely n complex roots, which we denote by

$$\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}.$$

The **discriminant** Δ_f of $f(x)$ is defined as

$$\Delta_f := \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

This product runs over all pairs of distinct roots, capturing how "spread apart" the roots are.

Lemma 4.12. *The polynomial $f(x)$ has a multiple root if and only if $\Delta_f = 0$.*

Proof. If $f(x)$ has a multiple root α_k, then there exists some $j \neq k$ such that $\alpha_j = \alpha_k$, making one of the terms in the product zero. Conversely, if $\Delta_f = 0$, then at least one term $(\alpha_i - \alpha_j)^2$ must be zero, which implies that some roots coincide. □

While the discriminant is naturally defined in terms of the roots, it can also be expressed purely in terms of the coefficients of $f(x)$. A key result in classical algebra states that:

Lemma 4.13. *The discriminant of a polynomial*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is given by

$$\boxed{\Delta_f := \frac{(-1)^{\frac{n(n-1)}{2}}}{a_n} \operatorname{Res}(f, f', x).}$$

Proof. The proof follows from the properties of resultants and is usually covered in an abstract algebra or computational algebra course. It relies on the fact that the resultant of $f(x)$ and its derivative $f'(x)$ is zero precisely when $f(x)$ and $f'(x)$ share a common root, which happens if and only if $f(x)$ has a multiple root. □

Example 4.26. *Let us compute the discriminant of a quadratic polynomial*

$$f(x) = ax^2 + bx + c.$$

First, compute the derivative:

$$f'(x) = 2ax + b.$$

The roots of $f(x)$ are given by the quadratic formula:

$$\alpha_1, \alpha_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Applying the definition of the discriminant:

$$\Delta_f = (\alpha_1 - \alpha_2)^2 = \left(\frac{2\sqrt{b^2 - 4ac}}{2a} \right)^2 = \frac{(b^2 - 4ac)^2}{4a^2}.$$

From the formula using the resultant, we obtain:

$$\Delta_f = b^2 - 4ac.$$

Thus, for a quadratic polynomial, the discriminant is precisely the well-known expression under the square root in the quadratic formula.

Example 4.27. *Find the discriminant of a cubic polynomial:*

$$f(x) = x^3 + px + q.$$

Using the formula:

$$\Delta_f = -4p^3 - 27q^2.$$

This expression determines when the cubic polynomial has a multiple root.

Exercises:

Exercise 61. *Find the discriminant of $f(x) = x^2 + bx + c$, $f(x) = x^3 + bx + c$, and $f(x) = x^4 + bx^2 + c$.*

Exercise 62. *Determine whether the polynomial $f(x) = 6x^4 - 23x^3 - 19x + 4$ has multiple roots in $\mathbb{C}$.*

Exercise 63. *Show that for a monic polynomial of degree n, the discriminant is a homogeneous polynomial of degree $2(n-1)$ in its coefficients.*

4.5.2 Elimination Theory

Resultants have a fundamental application in **elimination theory**, which concerns the process of eliminating variables from a system of polynomial equations. This is particularly useful in solving nonlinear algebraic systems, computing implicit equations, and understanding geometric projections.

Basic Idea

Assume we have a system of two nonlinear algebraic equations in three variables:

$$F_1(x,y,z) = 0 \quad \text{and} \quad F_2(x,y,z) = 0.$$

In many situations, we are interested in eliminating one of the variables—say, z—to obtain an equation purely in terms of x and y. Geometrically, this corresponds to determining the projection of the intersection of two surfaces in $\mathbb{R}^3$ onto the xy-plane.

Since both F_1 and F_2 are polynomials in z, we can compute their resultant:

$$\text{Res}(F_1, F_2, z) = 0.$$

This resultant is a polynomial in x and y alone, and its zero set describes the projection of the intersection onto the xy-plane. More generally, elimination theory provides methods for systematically removing variables from polynomial systems to derive useful implicit relationships.

Generalization to More Variables

The method of elimination extends beyond two equations and three variables. Given a system of n polynomial equations in $n+1$ variables:

$$F_1(x_1,\dots,x_n,x_{n+1}) = 0, \quad \dots, \quad F_n(x_1,\dots,x_n,x_{n+1}) = 0,$$

we can iteratively compute resultants to eliminate x_{n+1}, then x_n, and so on, eventually obtaining a single equation in a reduced set of variables.

This process underlies many algorithms in computational algebra, including Gröbner basis methods, and plays a role in algebraic geometry.

Applications of Elimination Theory

Elimination theory has applications across mathematics, engineering, and computer science. Some key areas include:

- **Geometric Modeling:** Implicitization of parametric curves and surfaces, such as converting a Bézier or B-spline representation into a polynomial equation.
- **Kinematics and Robotics:** Solving equations for the motion of mechanical systems by eliminating intermediate variables.
- **Computer Vision:** Computing epipolar constraints in stereo vision problems using polynomial equations from projective geometry.
- **Artificial Intelligence:** In symbolic AI and automated reasoning, elimination methods can be used for solving constraint systems and algebraic inference.
- **Cryptography:** Some cryptographic protocols, such as multivariate public key cryptosystems, rely on solving systems of polynomial equations where elimination techniques are useful.

Example: Eliminating a Variable

Consider the system:

$$F_1(x,y,z) = x + y + z - 1 = 0, \quad F_2(x,y,z) = x^2 + y^2 + z^2 - 1 = 0.$$

To eliminate z, we compute:

$$\mathrm{Res}(F_1, F_2, z).$$

Solving for z in F_1, we substitute $z = 1 - x - y$ into F_2:

$$x^2 + y^2 + (1 - x - y)^2 - 1 = 0.$$

Expanding and simplifying:

$$x^2 + y^2 + 1 - 2x - 2y + x^2 + 2xy + y^2 - 1 = 0.$$

$$2x^2 + 2y^2 + 2xy - 2x - 2y = 0.$$

Dividing by 2:

$$x^2 + y^2 + xy - x - y = 0.$$

This equation describes the projection of the intersection of the two surfaces onto the xy-plane.

Exercises

Exercise 64. *Use resultants to eliminate z from the system:*

$$F_1(x,y,z) = xz + y^2 - 1, \quad F_2(x,y,z) = z^2 + xy - 2.$$

Exercise 65. *Find the implicit equation in x and y for the parametric curve:*

$$x = t^2 + 1, \quad y = 2t + 3.$$

(Hint: Use elimination to eliminate t.)

Exercises:

298. *Let the curve*

$$A + By + Cx + Dy^2 + Exy + x^2 = 0$$

be given. It passes through the points $(x_1, y_1), \ldots, (x_5, y_5)$*. Determine the* A, B, C, D*, and* E*.* This was the original problem that Cramer was concerned with when he discovered his formula.

299. *Using Cramer's rule solve the system* $A\mathbf{x} = \vec{b}$ *where*

$$A = \begin{bmatrix} 5 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 0 & 4 & -1 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

300. *Let* A *be the following matrix.*

$$A := \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

301. *Find the adjoint of*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

and use the result to find A^{-1} *and* B^{-1}*.*

302. *Find the adjoint of*

$$A = \begin{bmatrix} 5 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 0 & 4 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 2 & 4 & -1 & 2 \end{bmatrix}$$

and use the result to find A^{-1} *and* B^{-1}*.*

303. *Determine if the matrix*

$$A := \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

is invertible.

304. *Let*

$$\begin{aligned} F(t) &= u(1+t^2) - t^2 \\ G(t) &= v(1+t^2) - t^3 \end{aligned} \tag{4.5}$$

Find $Res(F, G, t)$*.*

305. *Let*

$$\begin{aligned} f(x) &= x^5 - 3x^4 - 2x^3 + 3x^2 + 7x + 6 \\ g(x) &= x^4 + x^2 + 1 \end{aligned} \tag{4.6}$$

Find $Res(f, g, x)$*.*

306. *Find* b *such that*

$$f(x) = x^4 - bx + 1$$

has a double root in $\mathbb{C}$*.*

307. *Find* p *such that*

$$f(x) = x^3 - px + 1$$

has a double root in $\mathbb{C}$*.*

Review exercises

308. *Let A and B be two matrices which have the same eigenvalues. Are A and B necessarily similar? Explain your answer.*

309. *For any matrix M, show that* $M = 0$ *if and only if* $\operatorname{tr}(M^t M) = 0$.

310. *Prove that if A is similar to a diagonal matrix, then A is similar to* A^t.

311. *Let M be a square matrix such that the sum of entries in each row is equal to 1. Prove that if* $MM^t = M^t M$ *then the sums of entries in each column is equal to 1.*

312. *Let* A, B, C, D *be* $n \times n$ *matrices such that the matrix* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *has rank n. Show that det* $(AD) =$ *det* (BC).

313. *Let A and B be* $n \times n$ *matrices such that* ***rank*** $(AB) =$ ***rank*** (B). *Prove that for any matrices X and Y the following holds:* $ABX = ABY \implies BX = BY$.

314. *Let A be an* $n \times n$ *matrix such that* ***rank*** $(A) =$ ***rank*** (A^2). *Prove that* ***rank*** $(A) =$ ***rank*** (A^i), *for any positive integer i.*

315. *Let A and B be* $n \times n$ *matrices such that* $AB = 0$. *Prove that* ***rank*** $(A) +$ ***rank*** $(B) \leq n$.

316. *Find the eigenvalues and their algebraic and geometric multiplicities for the matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

Determine if the matrix is diagonalizable and if so find the matrices C and D such that $A = CDC^{-1}$.

317. *Find the eigenvalues and their algebraic and geometric multiplicities for the matrix*

$$B = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ 1 & 4 & -1 & 2 \end{bmatrix}$$

Determine if the matrix is diagonalizable and if so find the matrices C and D such that $A = CDC^{-1}$.

318. *Diagonalize the matrices* $M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$ *and* $N = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$. *Are they similar?*

319. *Find two 3 by 3 matrices with the same characteristic polynomial which are not similar to each other.*

320. *Let A be a matrix as follows*

$$A = \begin{pmatrix} -1 & 0 & 6 & 2 & -8 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & 3 & 2 & -6 \\ 0 & 0 & 2 & 3 & -6 \\ 0 & 0 & 2 & 2 & -5 \end{pmatrix}$$

(i) *Find the characteristic polynomial of A.*
(ii) *Find the eigenvalues and their algebraic multiplicities*
(iii) *Compute a basis for each eigenspace.*
(iv) *Determine matrices C and D such that* $A = CDC^{-1}$.
(v) *Find* A^{12}. *Complete the multiplication and give an exact answer*

321. *Find the sum of the eigenvalues of the matrix A*

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 4 & 0 & -1 & 0 \\ 0 & 2 & 0 & 3 & -1 \end{bmatrix}$$

322. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ -5x_1 + 13x_2 - x_3 \\ -7x_1 + 16x_2 + x_3 \end{bmatrix}$$

(i) *Fid a basis for the kernel ker(T)*
(ii) *Fid a basis for the image* $\mathrm{Img}(T)$
(iii) *Determine if T is injective, surjective, bijective*

323. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for $\mathbb{R}^3$ such that $\mathfrak{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, $\mathfrak{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{u}_1 = [3,1,1]^t$, $\mathbf{u}_2 = [5,2,0]^t$. $\mathbf{u}_3 = [1,1,1]^t$ and $\mathbf{v}_1 = [1,0,-1]^t$, $\mathbf{v}_2 = [-1,2,2]^t$, $\mathbf{v}_3 = [0,1,2]^t$*

(i) *Find the change of basis matrix $M_{\mathfrak{B}}^{\mathfrak{B}'}$.*
(ii) *Given the vector $\boldsymbol{X}$ with coordinates of $\mathfrak{B}$ as $\mathbf{x}_{\mathfrak{B}} = [1,1,1]^t$ find its coordinates in $\mathfrak{B}'$. In other words, find $\mathbf{x}_{\mathfrak{B}'} =?$.*

Paul Gordan: The King of Invariant Theory

Paul Albert Gordan (April 27, 1837 – December 21, 1912) was a towering figure in 19th-century German mathematics, renowned especially for his mastery of invariant theory. His epithet, "the king of invariant theory," speaks to his profound contributions and his dominant influence in this field.

Figure 4.4: Paul Albert Gordan

Born in Breslau, Germany (now Wroclaw, Poland), Gordan's academic journey began at the University of Königsberg, where he studied under the tutelage of the eminent Carl Gustav Jacobi. He earned his Ph.D. from the University of Breslau in 1862 and subsequently embarked on a distinguished professorial career at the University of Erlangen-Nuremberg. He remained there for the rest of his life, contributing significantly to establishing Erlangen as a global center for mathematical research.

Gordan's most celebrated achievement was proving the finite generation of the ring of invariants of binary forms of a fixed degree. This was a monumental task, achieved through intricate and laborious calculations, a hallmark of Gordan's approach to mathematics. He was a master of computational techniques, often undertaking complex calculations by hand. A testament to this dedication is his computation of all 70 invariants of binary sextics – a feat of remarkable perseverance and skill.

His collaboration with Alfred Clebsch led to the development of the now-famous Clebsch-Gordan coefficients, which are crucial in representation theory and quantum mechanics. These coefficients arise in the decomposition of tensor products of representations and have far-reaching applications in physics and other areas.

Gordan's influence extended beyond his own research. He played a key role in making Erlangen a leading mathematical center, working alongside Felix Klein and Max Noether. This trio fostered a vibrant intellectual environment that attracted mathematicians from around the world.

One of Gordan's most significant legacies is his role as the doctoral advisor to Emmy Noether, one of the most important mathematicians of the 20th century. He recognized her exceptional talent and guided her early research, even though her later work in abstract algebra eventually diverged significantly from his own computational focus.

A well-known anecdote, often repeated, involves Gordan and David Hilbert's groundbreaking proof of Hilbert's basis theorem. This theorem, which drastically generalized Gordan's work on invariants, demonstrated the existence of a finite basis for invariants in a much broader context using non-constructive methods. The quote attributed to Gordan, "This is not mathematics; this is theology," reflects the initial shock and perhaps skepticism that some mathematicians felt towards Hilbert's abstract, existence-based approach, which contrasted sharply with the constructive, computational methods prevalent at the time.

However, the historical accuracy and intended meaning of this quote are debated. The earliest known reference to it appears long after the events and Gordan's death. Furthermore, the narrative of Gordan as being opposed to Hilbert's work is largely a myth. In reality, Gordan recognized the power of Hilbert's methods, used them in his own research, and even supported Hilbert's publications. It's likely that the quote, if indeed Gordan uttered it, was meant as a humorous or nuanced observation, not as a categorical rejection of Hilbert's approach. Gordan himself acknowledged the significance of Hilbert's work, and the two mathematicians maintained a professional respect for each other.

Paul Gordan's legacy is multifaceted. He was a master of classical invariant theory, a key figure in the development of Erlangen's mathematical school, and a mentor to one of the most influential mathematicians of the 20th century. While the anecdote about Hilbert persists, it's crucial to understand it in the context of the evolving mathematical landscape of the late 19th century and to recognize Gordan's own contributions to and acceptance of the new, more abstract mathematics that was emerging.

Chapter 5

Inner Spaces and Orthogonality

In this chapter we will study the important concept of inner product in a vector space. We give the most general definition of the inner product and briefly look at Hermitian products. The rest of the chapter is focused on orthogonal and orthonormal bases we will study the Gram-Schmidt orthogonalization process. In the last section a brief introduction to dual spaces is given.

5.1 Inner products

Let V be a vector space over the field $\mathbb{F}$. Recall that in this book $\mathbb{F}$ denotes one of the fields $\mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$. For $\alpha \in \mathbb{C}$ the complex conjugate of α is denoted by $\bar{\alpha}$. Let $f(\mathbf{u}, \mathbf{v})$ be a function given as below

$$\begin{aligned} f: \quad & V \times V \longrightarrow \mathbb{F} \\ & (\mathbf{u}, \mathbf{v}) = f(\mathbf{u}, \mathbf{v}) \end{aligned} \tag{5.1}$$

The function f is called an **inner product** (**scalar product**) if the following properties hold for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r \in \mathbb{F}$:

(i) $f(\mathbf{u}, \mathbf{v}) = \overline{f(\mathbf{v}, \mathbf{u})}$,

(ii) $f(\mathbf{u}, \mathbf{v} + \mathbf{w}) = f(\mathbf{u}, \mathbf{v}) + f(\mathbf{u}, \mathbf{w})$

(iii) $f(r\mathbf{u}, \mathbf{v}) = rf(\mathbf{u}, \mathbf{v})$.

We denote inner products with $\langle \mathbf{u}, \mathbf{v} \rangle$ instead of $f(\mathbf{u}, \mathbf{v})$. An inner product is called **non-degenerate** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, for all $\mathbf{v} \in V$ implies that $\mathbf{u} = 0$. A vector space V with an inner product is called an **inner space**. We give some examples of inner products and inner spaces.

Example 5.1. *Let $V = \mathbb{C}$ be a vector space over $\mathbb{C}$. Show that $\langle u, zv \rangle = \bar{z} \langle u, v \rangle$, where $\bar{z}$ is the complex conjugate; see* **??**.

Solution: *Indeed, $\langle u, zv \rangle = \overline{\langle zv, u \rangle} = \overline{z\langle u, v \rangle} = \bar{z} \langle u, v \rangle$.* □

Example 5.2. *Let $V = \mathbb{R}^n$ and consider $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix}$ and two vectors in $\mathbb{R}^n$. The dot product* *studied before is defined as*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$$

We leave it as an exercise for the reader to show that this is an inner product in V. □

Let us see some other examples of inner products different from the dot product.

Exercise 66. *Let $V = \mathbb{R}^n$ and $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a symmetric matrix. Prove that the following is an inner product on V,*

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^t A \mathbf{v}.$$

Example 5.3. *Let $C_{[0,1]}$ be the* ***space of real continuous functions*** *$f : [0,1] \longrightarrow \mathbb{R}$. For $f, g \in V$ define*

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t)\, dt \tag{5.2}$$

Using properties of integrals it is easy to verify that this is an inner product. □

Example 5.4. *Let $f(x) = \sin x$ and $g(x) = \cos x$ in $C_{[0,1]}$ with the inner product defined in Eq. (5.2). Compute $\langle f, g \rangle$.*

Solution: *We have*

$$\langle f, g \rangle = \int_0^1 \sin x \cos x \, dx = \frac{1}{2} \int_0^1 \sin(2x)\, dx = \frac{1}{4}(-\cos 2 + \cos 0) = \frac{1 - \cos 2}{4}$$

Definition 5.1. *Let V be a vector space and $\langle \cdot, \cdot \rangle$ an inner product on V. Let $\mathbf{u} \in V$. We call $\mathbf{v}$* ***orthogonal*** *to $\mathbf{u}$ if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, sometimes denoted by $\mathbf{u} \perp \mathbf{v}$. For a set $U \subset V$ its* ***orthogonal set*** *$U^\perp$ is defined as*

$$U^\perp := \{ \mathbf{v} \in V \mid \mathbf{v} \perp \mathbf{u}, \text{ for all } \mathbf{u} \in U, \}$$

If U is a subspace of V then $U^\perp$ is called the ***orthogonal complement*** *of U.*

Exercise 67. *Take $V = \mathbb{R}^3$ and U as the subspace given by the equation of the line $\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\mathbf{v}$, for $t \in \mathbb{R}$. Find $U^\perp$.*

5.1.1 Positive definite inner products

Notice that for any vector space V over $\mathbb{C}$, the inner product has the property

$$\langle \mathbf{u}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

Hence $\langle \mathbf{u}, \mathbf{u} \rangle \in \mathbb{R}$ and therefore the following definition makes sense.

Definition 5.2. *An inner product is* ***positive definite*** *if the following hold:*

(i) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ *for all* $\mathbf{u} \in V$.

(ii) $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ *if and only if* $\mathbf{u} \neq 0$.

Ann inner product is called **positive semidefinite** if $\langle u,u\rangle \geq 0$ if and only if $u \neq 0$. The **norm** of an element $v \in V$ is defined to be

$$\|\mathbf{v}\| := \sqrt{\langle \mathbf{v},\mathbf{v}\rangle}.$$

For a positive definite inner product such norm is always $\|\mathbf{v}\| > 0$ for $\mathbf{v} \neq \mathbf{0}$. In the case of the Euclidean product is the magnitude of the vector that we studied in Chap. 1. The following gives an example of an inner product which is not positive definite.

Example 5.5. *Let $V = \mathbb{R}^2$ and $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Define the following $\langle \mathbf{u},\mathbf{v}\rangle := \mathbf{u}^t A\mathbf{v}$ and verify that this is an inner product. Take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then,*

$$\langle \mathbf{v}_1,\mathbf{v}_1\rangle = \mathbf{v}_1^t A\mathbf{v}_1 = \begin{bmatrix} 1, & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1.$$

and

$$\langle \mathbf{v}_2,\mathbf{v}_2\rangle = \mathbf{v}_2^t A\mathbf{v}_2 = \begin{bmatrix} 1, & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0.$$

Hence, this inner product is not positive definite. □

Exercise 68. *Consider the inner space $C_{[0,1]}$ as in* **??**. *Is the inner product defined in Eq. (5.2) positive definite?*

5.1.2 Real inner spaces

Notice that if V is a vector space over $\mathbb{R}$ then the definition of the inner product is a function

$$\langle \mathbf{u},\mathbf{v}\rangle : V \times V \longrightarrow \mathbb{R}$$

such that the following properties hold for every $\mathbf{u},\mathbf{v},\mathbf{w} \in V$ and $r \in \mathbb{R}$:

(i) $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v},\mathbf{u}\rangle$,
(ii) $\langle \mathbf{u},\mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{u},\mathbf{w}\rangle$
(iii) $\langle r\,\mathbf{u},\mathbf{v}\rangle = r\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{u},r\,\mathbf{v}\rangle$.

The most common inner product of Euclidean spaces $\mathbb{R}^n$ is the dot product

Exercise 69. *Let $V = \mathbb{R}^n$ and the inner product is the Euclidean product. As a review of Chap. 1 prove the following for any $\mathbf{u},\mathbf{v} \in V$.*

(i) $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$
(ii) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
(iii) $\|\mathbf{u}\| = 0$ *if and only if* $\mathbf{u} = 0$.
(iv) $|\langle \mathbf{u},\mathbf{v}\rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$

Similarly to the discussion in Chap. 1 for vectors **u** and **v** in $\mathbb{R}^2$ (see Eq. (5.4) and Eq. (1.33)) we can try to get a projection formula for any inner product $\langle \cdot, \cdot \rangle$. The **projection vector** of **v** on **u**, denoted by $\text{proj}_{\mathbf{u}}(\mathbf{v})$ is the vector

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) \;=\; \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||}\cdot\frac{\mathbf{u}}{||\mathbf{u}||} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

If we want a vector perpendicular to **u** we have

$$\boxed{\mathbf{w} = \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}.} \tag{5.3}$$

as in Eq. (1.33). Everything works fine for the dot product or even for an inner product such that $\langle \mathbf{u}, \mathbf{u} \rangle \neq 0$ for $\mathbf{u} \neq \mathbf{0}$. To avoid dividing by zero, we have to take an inner product which is positive definite.

Exercise 70. *Let* **u** *and* **v** *be vectors in an inner space with a positive definite inner product* $\langle \cdot, \cdot \rangle$. *Take*

$$\mathbf{w} = \mathbf{v} - \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\right)\mathbf{u}.$$

Prove that **w** *is orthogonal to* **u**.

Let W be a subspace of V. Then $W^\perp$ is defined as

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle, \;\text{ for all }\; \mathbf{w} \in W.\}$$

and is called the **orthogonal complement** of W with respect to the inner products $\langle \cdot, \cdot \rangle$. The orthogonal complement of a k-dimensional subspace is an $(n-k)$-dimensional subspace as proved below.

Theorem 5.1. *Let V be a finite dimensional vector space over $\mathbb{R}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$. If W is a subspace of V then*

$$V = W \oplus W^\perp.$$

Moreover,

$$\dim V = \dim W + \dim W^\perp.$$

Proof. See [Lan87, pg. 140] □

Exercise 71. *Prove that the double orthogonal complement is the original subspace. In other words.*

$$\left(w^\perp\right)^\perp = W.$$

Exercise 72. *Let A be a $m \times n$ matrix and* ***Row*** *A,* ***Col*** *A, and* ***Null***(A) *are the row space, column space, and null space of A. Prove that*

(i) $(\boldsymbol{Row}\ A)^\perp = \boldsymbol{Null}(A)$

(ii) $(\boldsymbol{Col}\ A)^\perp = \boldsymbol{Null}(A^t)$.

Consider a subspace V of $\mathbb{R}^n$. For any vector $\mathbf{x} \in \mathbb{R}^n$ there is a unique vector $\mathbf{w} \in V$ such that $\mathbf{x} - \mathbf{w} \in V^\perp$. This vector $\mathbf{w}$ is called the **orthogonal projection** of $\mathbf{x}$ onto V and is denoted by $\text{proj}_V(\mathbf{x})$. When $n = 2, 3$ it corresponds to the geometric projection. Next we see a generalization of Lem. 1.11

Lemma 5.1. *Let V be a subspace of $\mathbb{R}^n$ with an orthonormal basis $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. Then*

$$proj_V(\mathbf{x}) = (\mathbf{v}_1 \cdot \mathbf{x}) \cdot \mathbf{v}_1 + \cdots + (\mathbf{v}_m \cdot \mathbf{x}) \cdot \mathbf{v}_m. \tag{5.4}$$

Moreover, the transformation $T(\mathbf{x}) = proj_V(\mathbf{x})$ is linear.

Proof. We need to show that $\mathbf{u} := \mathbf{x} - \text{proj}_V(\mathbf{x}) \in V^\perp$. Since the dot product is linear, it is enoguh to prove that $\mathbf{x} - \text{proj}_V(\mathbf{x})$ is orthogonal with every element of $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. Then, since $(\mathbf{v}_i, \mathbf{v}_j) = 0$ for all $i \neq j$ we have

$$\begin{aligned}(\mathbf{u}, \mathbf{v}_i) &= \Big(\big(\mathbf{x} - \text{proj}_V(\mathbf{x})\big), \mathbf{v}_i\Big) = (\mathbf{x}, \mathbf{v}_i) - [(\mathbf{v}_1, \mathbf{x})\,\mathbf{v}_1 + \cdots + (\mathbf{v}_i, \mathbf{x})\,\mathbf{v}_i + \cdots + (\mathbf{v}_m, \mathbf{x})\mathbf{v}_m]\,\mathbf{v}_i \\ &= (\mathbf{x}, \mathbf{v}_i) - ((\mathbf{v}_i, \mathbf{x})\,\mathbf{v}_i, \mathbf{v}_i) = (\mathbf{x}, \mathbf{v}_i) - (\mathbf{v}_i, \mathbf{x})(\mathbf{v}_i, \mathbf{v}_i) = (\mathbf{x}, \mathbf{v}_i) - (\mathbf{v}_i, \mathbf{x})\,\|\mathbf{v}_i\|^2 = (\mathbf{x}, \mathbf{v}_i) - (\mathbf{v}_i, \mathbf{x}) = 0,\end{aligned}$$

because $\|\mathbf{v}_i\| = 1$. The transformation $T(\mathbf{x}) = \text{proj}_V(\mathbf{x})$ is linear since the dot product is linear. In other words, it is easy to check that $\text{proj}_V(\mathbf{x} + \mathbf{y}) = \text{proj}_V(\mathbf{x}) + \text{proj}_V(\mathbf{y})$ and $\text{proj}_V(r\mathbf{x}) = r\text{proj}_V(\mathbf{x})$ for any $x, \mathbf{y} \in \mathbb{R}^n$ and $r \in \mathbb{R}$. This completes the proof. $\square$

There is a nice application of the above result. Let $V = \mathbb{R}^n$ and $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ an orthonormal basis of V. Since the projection of $\mathbf{x}$ onto V is $\mathbf{x}$ itself we have:

Corollary 5.1. *Let $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered orthonormal basis of $\mathbb{R}^n$. Then every $\mathbf{x} \in \mathbb{R}^n$ is written as*

$$\mathbf{x} = (\mathbf{v}_1 \cdot \mathbf{x}) \cdot \mathbf{v}_1 + \cdots + (\mathbf{v}_m \cdot \mathbf{x}) \cdot \mathbf{v}_n.$$

Notice that for a general basis $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ to find the coordinates $x_1, \dots, x_n$ of $\mathbf{x} \in \mathbb{R}^n$ we assume that $\mathbf{x}$ is written as a linear combination

$$\mathbf{x} = r_1 \mathbf{v}_1 + \cdots r_n \mathbf{v}_n,$$

and solve the linear system for $r_1, \dots, r_n$. However, if $\mathfrak{B}$ is orthonormal we have

$$r_i = \langle \mathbf{v}_i, \mathbf{x} \rangle = \mathbf{v}_i \cdot \mathbf{x},$$

which is much easier.

5.1.3 Complex inner spaces, Hermitian products

Let V be a vector space over $\mathbb{C}$. The inner product in this case is called a **Hermitian product**. It is a function on V such that

$$\langle \mathbf{u}, \mathbf{v} \rangle : V \times V \longrightarrow \mathbb{C}$$

such that the following properties hold for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r \in \mathbb{C}$:

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$,
(ii) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
(iii) $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \bar{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$.

We leave the following to the reader:

Exercise 73. *Prove the following for the Hermitian product:*
(i) $||\mathbf{u}|| \geq 0$
(ii) $||\mathbf{u}|| = 0$ *if and only if* $\mathbf{u} = 0$.
(iii) $||\alpha \mathbf{u}|| = |\alpha| \, ||\mathbf{u}||$
(iv) $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$

Lemma 5.2 (The Schwartz inequality)**.** *The following holds*

$$|\langle u, \mathbf{v} \rangle| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$$

for the Hermitian product.

Example 5.6. *Let $\mathbb{F} \subset \mathbb{C}$ and $V = \mathbb{F}^n$. For any*

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n) \end{bmatrix}$$

define

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n$$

Show that this is a Hermitian product. This particular product we will call the ***Euclidean inner product****.* □

Notice that for the Euclidean inner product $\langle \cdot, \cdot \rangle$

$$\langle \mathbf{u}, \mathbf{u} \rangle = u_1 \bar{u}_1 + \cdots + u_n \bar{u}_n = ||u_1||^2 + \cdots + ||u_n||^2$$

The norm of $\mathbf{u} \in V$ is defined as

$$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{||u_1||^2 + \cdots + ||u_n||^2}$$

Example 5.7. *Let V be the* ***space of complex continuous functions***

$$f : [0,1] \longrightarrow \mathbb{C}$$

For $f, g \in V$ we define

$$\langle f, g \rangle = \int_0^1 f(t) \cdot \overline{g(t)} \, dt$$

Using properties of complex integrals show that this is an inner product. □

Consider now the **space of square-integrable functions** $L^2(\mathbb{R})$ already considered in Chap. 2. Define

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \overline{f(x)}\, g(x)\, \mathrm{dx}$$

where f and g are square-integrable functions $\overline{f(x)}$ is the complex conjugate of $f(x)$.

A **Hilbert space** is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

Example 5.8. ***(Fourier series)*** *Let V be the space of continuous complex-valued functions*

$$f : [-\pi, \pi] \longrightarrow \mathbb{C}$$

For $f, g \in V$ we define

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \cdot \overline{g(t)}\, dt$$

For any integer n define

$$f_n(t) = e^{n \cdot it}.$$

Prove that:

(i) *if $m \neq n$ then $\langle f_n, f_m \rangle = 0$*
(ii) $\langle f_n, f_n \rangle = 2\pi$
(iii) $\frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$

The quantity $\frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle}$ is called the ***Fourier coefficient*** *with respect to f.* □

Let A be a matrix with real entries. Then its characteristic polynomial **char** (A, x) has real coefficients. Since complex eigenvalues occur in pairs via the conjugate, for every polynomial with real coefficients, consider such a pair $\alpha \pm i\beta$ as eigenvalues of A. Then their corresponding eigenvectors are $\mathbf{v} \pm i\mathbf{w}$, respectively.

Lemma 5.3. *If $\lambda = \alpha \pm i\beta$ are eigenvalues of A, then their corresponding eigenvectors are $\mathbf{v} \pm i\mathbf{w}$ respectively, for some $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Moreover,*

$$\langle \mathbf{v} + i\mathbf{w}, \mathbf{v} - i\mathbf{w} \rangle = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

Proof. Exercise. □

Exercises:

324. *Let V be the space of real continuous functions*

$$f : [0, 1] \longrightarrow \mathbb{R}$$

For $f, g \in V$ we define

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t)\, dt$$

Given $f(x) = x^3$, find $g(x) \in V$ such that g is orthogonal to f.

325. *Let V be the vector space as in the previous exercise and W the set all polynomials in V. Is W is a subspace of V? Given a polynomial*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

Can you find $g(x) \in V$ *such that* $\langle f, g \rangle = 0$ *?*

326. *Let* $V := \mathrm{Mat}_n(\mathbb{R})$*. Define the inner product of matrices* M *and* N *as*

$$\langle M, N \rangle = tr(MN)$$

Show that this is an inner product and it is non-degenerate.

327. *Let* $V := \mathrm{Mat}_n(\mathbb{R})$*. Let* A, B *be any matrices in* V *such that*

$$A := \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad \text{and} \quad B := \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

Is the following

$$\langle A, B \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

an inner product on V*?*

328. *Let* P_2 *denote the space of polynomials in* $\mathbb{F}[x]$ *and degree* ≤ 2*. Let* $f, g \in P_2$ *such that*

$$f(x) = a_2 x^2 + a_1 x + a_0,$$

and

$$g(x) = b_2 x^2 + b_1 x + b_0.$$

Define

$$\langle f, g \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2.$$

Prove that this is an inner product on P_2*.*

329. *Let* P_2 *be equipped with the inner product as in the above example. Describe all the polynomials of norm 1.*

330. *Let* $V := \mathcal{L}([0,1], \mathbb{R})$ *be the space of real continuous functions on* $[0,1]$ *with the inner product*

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t)\, dt$$

Is this a non-degenerate inner product?

331. *Let* $V := \mathcal{L}([0,1], \mathbb{R})$ *be the space of real continuous functions on* $[0,1]$ *with the inner product*

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t)\, dt$$

Describe the norm associated to this inner product and all functions of norm 1.

5.2 Orthogonal bases, Gram-Schmidt orthogonalization

Let V be a finite dimensional vector space over $\mathbb{F}$ with an inner product $\langle \cdot, \cdot \rangle$. We assume throughout this section that *the inner product is positive definite*. The **norm** of an element $\mathbf{v} \in V$ is defined as

$$\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Then for every nonzero vector $\mathbf{v}$ we have $\|\mathbf{v}\| > 0$. Hence we can normalize each element $\mathbf{v} \in V$ by $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ such that $\|\mathbf{u}\| = 1$. A set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of vectors is called an **orthogonal** if for any $i \neq j$ we have

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0.$$

If in addition, all vectors have norm one then they are called an **orthonormal**.

Lemma 5.4. *Orthonormal vectors in $\mathbb{R}^n$ are linearly independent.*

Proof. Exercise ◻

Exercise 74. *Is the above Lemma true for any inner space?*

Exercise 75 (Pythagorean theorem)**.** *Consider vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Prove that*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Is this true for in any inner space?

Theorem 5.2. *Let V be a finite dimensional vector space with a positive definite inner product. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent then there is an orthogonal set $\mathbf{w}_1, \dots, \mathbf{w}_n$ such that*

$$\mathit{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathit{Span}\,(\mathbf{w}_1, \dots \mathbf{w}_n)$$

Proof. Let us fix an ordering on $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ say $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. The vector $\mathbf{w}_2 = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{v}_2)$ is orthogonal to $\mathbf{v}_1$Moreover, $\operatorname{Span}\,(\mathbf{v}_1, \mathbf{v}_2) = \operatorname{Span}\,(\mathbf{v}_1, \mathbf{w}_2)$.

Denote by $V_2 = \operatorname{Span}\,(\mathbf{v}_1, \mathbf{v}_2)$. Then, $\mathbf{w}_3 = \mathbf{v}_3 - \operatorname{proj}_{V_2}(\mathbf{w}_3)$ is orthogonal to V_2. In general, we let $V_i := \operatorname{Span}\,(\mathbf{w}_1, \dots, \mathbf{w}_i)$ and take $\mathbf{w}_{i+1} = \mathbf{v}_{i+1} - \operatorname{proj}_{V_i}(\mathbf{v}_{i+1})$ and we have $\mathbf{w}_{i+1} \perp V_i$. Hence, we take the following set of vectors

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 \\
\mathbf{w}_2 &= \mathbf{v}_2 - \left(\frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \right) \mathbf{w}_1 \\
\mathbf{w}_3 &= \mathbf{v}_3 - \left(\frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \right) \mathbf{w}_2 - \left(\frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \right) \mathbf{w}_1 \\
&\dots\dots \\
\mathbf{w}_{i+1} &= \mathbf{v}_{i+1} - \left(\frac{\langle \mathbf{v}_{i+1}, \mathbf{w}_i \rangle}{\langle \mathbf{w}_i, \mathbf{w}_i \rangle} \right) \mathbf{w}_i - \cdots - \left(\frac{\langle \mathbf{v}_{i+1}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \right) \mathbf{w}_1
\end{aligned} \tag{5.5}$$

to get the desired orthogonal set. ◻

Let $\mathfrak{B} = \{v_1, \dots, v_n\}$ be a basis for V. Then $\mathfrak{B}$ is called an **orthogonal basis** if for any $i \neq j$ we have

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0.$$

If in addition, for all $i = 1, \dots, n$, $\|\mathbf{v}_i\| = 1$ then $\mathfrak{B}$ is called an **orthonormal basis**. Then we have the following:

Corollary 5.2. *Every finite dimensional inner space with a positive definite inner product has an orthogonal basis.*

The proof of the above theorem is constructive and provides and algorithm to find an orthogonal space (and therefore orthonormal) of any inner space.

Algorithm 8. *Gram-Schmidt Algorithm:*

Input: A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of vectors.
Output: An orthogonal set of vectors $W = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ such that

$$\text{Span}\,(v_1, \dots, v_n) = \text{Span}\,(w_1, \dots w_n)$$

(i) *Fix an ordering of the set S, say $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.*
(ii) *Let $\mathbf{w}_1 := \mathbf{v}_1$*
(iii) *Compute all $\mathbf{w}_i$'s using the recursive formula Eq. (5.5), for all $i = 1, \dots, n-1$.*
(iv) *The set $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is the required W.*

Let us see some examples.

Example 5.9. *Let $V = \mathbb{R}^3$ and the inner product on V is the dot product . Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. Find an orthogonal basis of Span $(\mathbf{v}_1, \mathbf{v}_2)$.*
Solution: *Let $\mathbf{w}_1 = \mathbf{v}_1$. Then*

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{9}{14} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{14} \\ \frac{5}{7} \\ -\frac{13}{14} \end{bmatrix}$$

Clearly $\mathbf{w}_1 \perp \mathbf{w}_2$. □

Example 5.10. *Let V be the space of real continuous functions $f : [0,1] \longrightarrow \mathbb{R}$. For $f, g \in V$ we define*

$$\langle f, g \rangle = \int_0^1 f(x) \cdot g(x)\, dx$$

As shown above, this is an inner product. Let $f(x) = x$ and $g(x) = x^2$. Since both are continuous then $f, g \in V$. Find an orthogonal basis of Span (f, g).
Solution: *Let $\mathbf{w}_1 = f$. Then*

$$\mathbf{w}_2 = g - \frac{\langle g, f \rangle}{\langle f, f \rangle} f = x^2 - \left(\frac{\int_0^1 x^3\, dx}{\int_0^1 x^2\, dx} \right) x = x^2 - \frac{3}{4} x$$

The reader should check whether $\mathbf{w}_1 \perp \mathbf{w}_2$. □

Example 5.11. *Let V be the space of real continuous functions. Given $f(x) = x^3$, find $g(x) \in V$ such that g is orthogonal to f.*

Solution: *Take $S = \{f, 1\}$. We want to find an orthogonal set W such that $f \in W$. Let $w_1 = f$. Then*

$$w_2 = 1 - \frac{\langle 1, f\rangle}{\langle f, f\rangle} f = 1 - \frac{\int_0^1 x^3\, dx}{\int_0^1 x^6\, dx} x^3 = 1 - \frac{7}{4} x^3$$

The reader can check that $\langle f, w_2\rangle = 0$. □

5.2.1 Orthonormal basis

Can we modify the Gram-Schmidt to get an orthonormal basis? The answer is "yes" since we have a positive definite inner product. In the view of the orthogonal projections on subspaces (see Eq. (5.4)) we can rewrite the Gram Schmidt algorithm as

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \quad \mathbf{w}_i = \frac{1}{\left\|\mathbf{v}_i - \mathrm{proj}_{V_{i-1}}(\mathbf{v}_i)\right\|} \left(\mathbf{v}_i - \mathrm{proj}_{V_{i-1}}(\mathbf{v}_i)\right), \quad \textit{for } j = 2, \dots, n,$$

where $V_{i-1} = \mathrm{Span}\,(\mathbf{v}_1, \cdots, \mathbf{v}_{i-1})$. In the coming lectures we will see how to construct an *orthonormal basis* in the case of inner products which are not necessarily positive-definite. This is the celebrated *Sylvester's theorem*.

Exercises:

332. *Can we find an orthonormal basis for the inner space of Exa. 5.11? Is the inner product positive definite?*

333. *Find an orthonormal basis for the nullspace **Null**(A) of the matrix* $A := \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}$

334. *Find an orthonormal basis for the nullspace of the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$.

335. *Find an orthogonal basis for the nullspace of the matrix*

$$A = \begin{bmatrix} 3 & 1 & 0 & -1 \\ 4 & 0 & 0 & 3 \\ -4 & 2 & 2 & -3 \\ 2 & -4 & 0 & 7 \end{bmatrix}$$

336. *Let V be the space of real continuous functions. Given $f(x) = x^2$ and $g(x) = e^x$, find an orthogonal set $W = \{\mathbf{w}_1, \mathbf{w}_2\}$ such that Span $(f, g) =$ Span $(\mathbf{w}_1, \mathbf{w}_2)$.*

337. *In the space of real continuous functions find a function $g(x)$ which is orthogonal to $f(x) = \sin x$.*

338. *Show that the following identity holds for any inner product*

$$\|\mathbf{u} + \mathbf{v}\| + \|\mathbf{u} - \mathbf{v}\| = 2\|\mathbf{u}\| + 2\|\mathbf{v}\|$$

339. *Let $V = \mathbb{R}^4$ and the inner product on V is the dot product . For*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

find an orthonormal basis for Span $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$.

340. *Let P_2 denote the space of polynomials in $\mathbb{F}[x]$ and degree ≤ 2. Let $f, g \in P_2$ such that $f(x) = a_2x^2 + a_1x + a_0$ and $g(x) = b_2x^2 + b_1x + b_0$. Define the inner product*

$$\langle f, g \rangle = a_0b_0 + a_1b_1 + a_2b_2.$$

Let f_1, f_2, f_3, f_4 be given as below

$$\begin{aligned} f_1 &= x^2 + 3 \\ f_2 &= 1 - x \\ f_3 &= 2x^2 + x + 1 \\ f_4 &= x + 1. \end{aligned} \tag{5.6}$$

Find an orthogonal basis of Span (f_1, f_2, f_3, f_4).

341. *Find an orthogonal basis for the subspace Span $(1, \sqrt{x}, x)$ of the vector space $C_{[0,1]}$ of continuous functions on $[0,1]$, where $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$.*

342. *Find an orthonormal basis for the plane $x + 7y - z = 0$.*

343. *The matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has characteristic polynomial

$$\textbf{\textit{char}}\,(A, \lambda) = (\lambda + 1)^2 (\lambda - 1)^3$$

Determine an orthogonal basis for each eigenvalue. Diagonalize A using the orthogonal eigenbasis that you found for each eigenvalue.

344. *The matrix*

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. Determine their algebraic multiplicities and if there are any other eigenvalues. Find an orthonormal basis for each eigenspace. Using such orthonormal bases determine C and D such that $D = C^{-1}AC$ and D is diagonal.

5.3 Orthogonal transformations and orthogonal matrices

Let us start with some examples from the geometry of $\mathbb{R}^2$. Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}.$$

The corresponding matrix for this map is $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. If we have the ellipse

$$x_1^2 + \frac{x_2^2}{4} = 1,$$

then it is transformed to the unit circle $x_1^2 + x_2^2 = 1$. So linear transformations change shapes of objects in $\mathbb{R}^2$.

How should a linear transformation be such that it preserves shapes? Obviously, it has to preserve distances. This motivates the following:

Definition 5.3. *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called **orthogonal** if it preserves the length:*

$$\|T(\mathbf{x})\| = \|\mathbf{x}\|,$$

*for all $\mathbf{x} \in \mathbb{R}^n$. The corresponding matrix of an orthogonal map is called an **orthogonal matrix**.*

Example 5.12. *Perhaps the simplest orthogonal maps are* rotations. *Obviously a rotation preserves the length of every vector. For example, the rotation by θ on $V = \mathbb{R}^2$ is an orthogonal map with orthogonal matrix*

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Proposition 5.1. *Orthogonal transformations preserve orthogonality. In other words, if $\mathbf{u}$ is orthogonal to $\mathbf{v}$ then $T(\mathbf{u})$ is orthogonal to $T(\mathbf{v})$.*

Proof. Using Exe. 75 it is enough to show that

$$\|T(\mathbf{u}) + T(\mathbf{v})\|^2 = \|T(\mathbf{u})\|^2 + \|T(\mathbf{v})\|^2.$$

We have

$$\|T(\mathbf{u}) + T(\mathbf{v})\|^2 = \|T(\mathbf{u}+\mathbf{v})\|^2 = \|(\mathbf{u}+\mathbf{v})\|^2 = \|u\|^2 + \|\mathbf{v}\|^2 = \|T(\mathbf{u})\|^2 + \|T(\mathbf{v})\|^2$$

□

Theorem 5.3. *The following are true:*

(i) *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if the image of the standard basis is an orthonormal basis.*

(ii) *An $n \times n$ matrix is orthogonal if and only if its columns form an orthonormal basis for $\mathbb{R}^n$.*

Proof. Let $\mathfrak{B} = \{e_1, \dots, e_n\}$ be the standard basis. Then, by Prop. 5.1 the set $\{T(e_1), \dots, T(e_n)\}$ is an orthogonal basis. Since T preserves norms, then it is an orthonormal basis.

An orthogonal matrix is the matrix of an orthogonal transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Then the matrix is given by

$$[T(\mathbf{v}_1) \,|\, T(\mathbf{v}_2) \,|\, \dots, |T(\mathbf{v}_n)]$$

Since $v_1, \dots, v_n$ are orthonormal, then $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$ are orthonormal. ◻

Example 5.13. *The matrix A is orthogonal since its column vectors are all orthogonal to each other and they all have length one. However, the matrix B is not orthogonal, because even though its columns are all orthogonal to each other, they don't have norm one.*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Next we have the following theorem.

Theorem 5.4. *A square matrix A is orthogonal if and only if $A^{-1} = A^t$.*

Proof. Let $A = [\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n]$ be a given orthogonal matrix. From Thm. 5.3 its columns form an orthonormal basis. Then,

$$A^t A = \begin{bmatrix} - & \mathbf{v}_1^t & - \\ - & \mathbf{v}_2^t & - \\ & \vdots & \\ - & \mathbf{v}_n^t & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_1 \cdot \mathbf{v}_n \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_2 \cdot \mathbf{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_n \cdot \mathbf{v}_1 & \mathbf{v}_n \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_n \cdot \mathbf{v}_n \end{bmatrix} = I,$$

since $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal set. ◻

Another property of orthogonal matrices is the following, the proof of which is left as an exercise.

Proposition 5.2. *Let Q be an orthogonal matrix. For all $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^n$ we have that $\mathbf{u} \cdot \mathbf{v} = (Q\mathbf{u}) \cdot (Q\mathbf{v})$.*

Summarizing all properties of orthogonal matrices we have the following.

Corollary 5.3. *Let $Q \in \mathrm{Mat}_{n \times n}(\mathbb{R})$. Then the following statements are equivalent.*

(i) *Q is an orthogonal matrix*
(ii) *The transformation $T(\mathbf{x}) = Q\mathbf{x}$ preserves norms (in other words $\|Q\mathbf{x}\| = \|\mathbf{x}\|$).*
(iii) *The columns of Q form an orthonormal basis of $\mathbb{R}^n$*
(iv) *$Q^t Q = I_n$. In other words, $Q^{-1} = Q^t$.*
(v) *Q preserves the dot product. In other words $\mathbf{u} \cdot \mathbf{v} = (Q\mathbf{u}) \cdot (Q\mathbf{v})$.*

Proof. Combine all the results proved above to show the equivalence of these statements. □

Since we will be using the transpose more often in the coming lectures let's recall some of its properties. The reader should be able to verify them easily.

Exercise 76. *The following are true:*
(i) $(A+B)^t = A^t + B^t$
(ii) $(rA)^t = rA^t$
(iii) $AB)^t = B^t A^t$
(iv) $\textbf{\textit{rank}}\ A^t = \textbf{\textit{rank}}\ A$
(v) $\left(A^t\right)^{-1} = \left(A^{-1}\right)^t$.

5.3.1 Orthogonal projections

We have seen orthogonal projections before. Let us now consider how to find the matrix of an orthogonal projection.

Theorem 5.5. *Let V be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ and let Q be the matrix*

$$Q = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_m \\ | & | & & | \end{bmatrix}.$$

The matrix of the orthogonal projection onto V is $P = QQ^t$. Moreover, the matrix of an orthogonal transformation is symmetric.

Proof. The projection of $\mathbf{x}$ onto V is given by

$$\mathrm{proj}_V(\mathbf{x}) = (\mathbf{v}_1 \cdot \mathbf{x})\,\mathbf{v}_1 + \cdots + (\mathbf{v}_m \cdot \mathbf{x})\,\mathbf{v}_m = \mathbf{v}_1 \mathbf{v}_1^t \mathbf{x} + \cdots + \mathbf{v}_m \mathbf{v}_m^t \mathbf{x} = \left[\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_m\right] \begin{bmatrix} -\mathbf{v}_1^t- \\ \vdots \\ -\mathbf{v}_m^t- \end{bmatrix} \mathbf{x} = QQ^t \mathbf{x}$$

□

Let us go back to the case of the line.

Example 5.14. *Consider a line L in $\mathbb{R}^2$ with equation $y = ax + b$. Find the matrix of the orthogonal projection onto L.*

Proof. We have noted before that if the line doesn't pass through the origin, the projection is not even a linear map. However, let's just pretend that we don't even know that. Notice that a directional vector for L is

$$\mathbf{u} = \begin{bmatrix} -b/a \\ b \end{bmatrix} = \frac{b}{a} \begin{bmatrix} -1 \\ a \end{bmatrix}.$$

Its norm is $\|\mathbf{u}\| = \frac{b}{a}\sqrt{a^2+1}$. Hence, $\{\mathbf{v}\}$ is a orthonormal basis for L, where

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{a^2+1}} \begin{bmatrix} -1 \\ a \end{bmatrix}.$$

Notice that there is no b anymore in this vector. That's because this is a vector not on the original line, but on the line parallel to L which goes through the origin.

Then from Thm. 5.5 the matrix P is

$$P = \left(\frac{1}{\sqrt{a^2+1}} \begin{bmatrix} -1 \\ a \end{bmatrix} \right) \left(\frac{1}{\sqrt{a^2+1}} \begin{bmatrix} -1 & a \end{bmatrix} \right) = \frac{1}{a^2+1} \begin{bmatrix} 1 & -a \\ -a & a^2 \end{bmatrix}.$$

Let us check how this will work with the directional vector $\mathbf{u} \in \mathcal{L}$. We have

$$P\mathbf{u} = \frac{1}{a^2+1} \begin{bmatrix} 1 & -a \\ -a & a^2 \end{bmatrix} \cdot \left(\frac{b}{a} \begin{bmatrix} -1 \\ a \end{bmatrix} \right) = \frac{b}{a(a^2+1} \begin{bmatrix} -(a^2+1) \\ a(a^2+1) \end{bmatrix} = \frac{b}{a} \begin{bmatrix} -1 \\ a \end{bmatrix} = \mathbf{u},$$

as expected. □

We have already seen the above example; see Sec. 1.2.3 and Sec. 3.5.5.

Exercise 77. *Find the matrix of the orthogonal projection onto the subspace V of $\mathbb{R}^4$ such that $V = \text{Span}\,(\mathbf{u}, \mathbf{v})$, where*

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Notice that $\mathbf{u}$ and $\mathbf{v}$ already form an orthonormal basis for V.

Solution: Since the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthonormal we have

$$P = QQ^t = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

□

Exercise 78 (Projection onto a plane). *For given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $U = \text{Span}\,(\mathbf{u}, \mathbf{v})$, in Lem. 1.10 we determined a formula for the projection map $\text{proj}_U(\mathbf{w})$ onto U.*

Given a plane P in $\mathbb{R}^3$ going through the origin, say with equation $ax+by+cz=0$. Find the matrix of the orthogonal map onto P.

Consider now the reflection to a plane as discussed in Exe. 53. Instead of considering a point P, we consider the vector $\mathbf{u} = \vec{OP}$. When the plane goes through the origin, the reflection $\mathbf{v} := \vec{OP'}$ of $\mathbf{u}$ has the same length as $\mathbf{u}$. Hence, reflection to a plane must be an orthogonal map. Let's find its corresponding orthogonal matrix.

Example 5.15 (Reflection to a plane). *Let P be given a plane in $\mathbb{R}^3$ with equation*

$$ax+by+cz=0.$$

Consider the reflection map $T : \mathbb{R}^3 \to \mathbb{R}^3$ which takes a point to its reflection with respect to the given plane. Is this map linear? If that is the case determine its corresponding matrix. Prove your answers.

Solution: The normal vector of the plane P is $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Let A be a point and $\mathbf{v} := \vec{OA}$. We know how to find the projection $\text{proj}_P \mathbf{v}$ of $\mathbf{v}$ on the plane; see Sec. 3.5.5. Then the vector perpendicular from A to the plane is

$$\mathbf{w} = \mathbf{v} - \text{proj}_P \mathbf{v}$$

The symmetric point A' of the point A with respect to the plane P is represented by the vector

$$\vec{OA'} = \mathbf{v} - 2\left(\mathbf{v} - \text{proj}_P \mathbf{v}\right)$$

Show that this is

$$\vec{OA'} = I_3 - 2\frac{\mathbf{u}\mathbf{u}^t}{a^2+b^2+c^2}.$$

If $\vec{OA} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then

$$\vec{OA'} = \begin{bmatrix} (1-2a^2)x - 2aby - 2acz \\ -2abx + (1-2b^2)y - 2bcz \\ -2acx - 2bcy + (1-2c^2)z \end{bmatrix} = \begin{bmatrix} 1-2a^2 & -2ab & -2ac \\ -2ab & 1-2b^2 & -2bc \\ -2ac & -2bc & 1-2c^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Since this map is given through a multiplication by a matrix then it must be linear. □

This transformation is sometimes called **Householder transformation** and is widely used in optics, computer vision, etc.

Exercise 79. *Find the matrix of the reflection transformation to the plane with equation*

$$2x + 3y + 5z = 0$$

Find the reflections of points $P(1,1,1)$ *and* $Q = (2,1,0)$.

5.3.2 Unitary matrices

Next we will see the concept of *orthogonality* for matrices with complex entries. Recall that over the complex numbers $\mathbb{C}$ the dot product becomes the Hermitian product. The complex transpose of a matrix A is the transpose of the conjugate of A, denoted by $A*$.

A complex square matrix $U \in \text{Mat}_{n\times n}(\mathbb{C})$ is called **unitary** if its conjugate transpose $U*$ is also its inverse, that is, if

$$U^* U = UU^* = I,$$

where I is the identity matrix. The general expression of a 2 by 2 unitary matrix is

$$U = \begin{bmatrix} a & b \\ -e^{i\varphi} b^* & e^{i\varphi} a^* \end{bmatrix}, \qquad |a|^2 + |b|^2 = 1,$$

which depends on 4 real parameters (the phase of a, the phase of b, the relative magnitude between a and b, and the angle φ). The determinant of such a matrix is $\det(U) = e^{i\varphi}$. The subgroup of those elements U with $\det U = 1$ is called the special unitary group $SU(2)$. The matrix U can also be written in this alternative form:

$$U = e^{i\varphi/2}\begin{bmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ -e^{-i\varphi_2}\sin\theta & e^{-i\varphi_1}\cos\theta \end{bmatrix},$$

which, by introducing $\varphi_1 = \psi + \Delta$ and $\varphi_2 = \psi ? \Delta$, takes the following factorization:

$$U = e^{i\varphi/2}\begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{bmatrix}.$$

This expression highlights the relation between 2 by 2 unitary matrices and 2 by 2 orthogonal matrices of angle θ.

Exercises:

345. *Prove the linearity of the reflection to a plane using only the geometry of vectors in* $\mathbb{R}^3$.

346. *Find the orthogonal projection of* $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ *onto the plane* V *in* $\mathbb{R}^3$ *with equation*

$$x - y + 2z = 0.$$

347. *Check whether the matrix is orthogonal*

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & -5 \\ 2 & 2 & 0 \end{bmatrix}$$

348. *For a field* $\mathbb{F}$ *the space* $\mathrm{Mat}_{n\times n}(\mathbb{F})$ *of* $n \times n$ *matrices with entries in* $\mathbb{F}$ *is a vectors space. Consider the map* $T : \mathrm{Mat}_{n\times n}(\mathbb{F}) \to \mathrm{Mat}_{n\times n}(\mathbb{F})$ *given by*

$$T(A) = A^t.$$

Is this a linear map? Prove your answer.

5.4 QR-factorization

In previous lectures, we have seen that diagonalizing matrices requires finding the eigenvalues, which can be a difficult problem, especially for large or non-symmetric matrices. The QR-factorization offers an alternative approach, enabling numerical approximations of eigenvalues and solutions to systems of equations with greater stability. However, every time there is approximation involved, an error analysis is required.

In an inner space V, if the inner product $\langle \cdot, \cdot \rangle$ is positive definite, we can always obtain an orthonormal basis.

Lemma 5.5. *A finite dimensional vector space V with a positive definite inner product has an* orthonormal basis.

Proof. Apply the Gram-Schmidt process to any basis of V. Since the inner product is positive definite, all norms are nonzero, ensuring the process yields an orthonormal basis. □

The orthogonalization process can be represented via a matrix form. Let $M \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ with linearly independent column vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$. Then from the basis

$$\mathfrak{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$$

we can get an orthonormal basis

$$\mathfrak{B}' = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$$

by using first the Gram-Schmidt orthogonalization and then normalizing each element. Thus,

$$\begin{aligned}
\mathbf{w}_1 &= \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 \\
\mathbf{w}_2 &= \frac{1}{\left\|\mathbf{v}_2 - \mathrm{proj}_{\mathbf{v}_1}(\mathbf{v}_2)\right\|}(\mathbf{v}_2 - \mathrm{proj}_{\mathbf{v}_1}(\mathbf{v}_2)) \\
&\vdots \\
\mathbf{w}_i &= \frac{1}{\left\|\mathbf{v}_i - \mathrm{proj}_{V_{i-1}}(\mathbf{v}_i)\right\|}(\mathbf{v}_i - \mathrm{proj}_{V_{i-1}}(\mathbf{v}_i)) \\
&\vdots \\
\mathbf{w}_m &= \frac{1}{\left\|\mathbf{v}_m - \mathrm{proj}_{V_{m-1}}(\mathbf{v}_m)\right\|}(\mathbf{v}_m - \mathrm{proj}_{V_{m-1}}(\mathbf{v}_m))
\end{aligned} \tag{5.7}$$

Using Gram-Schmidt, each original vector $\mathbf{v}_i$ can be written as a linear combination of the orthonormal vectors $\mathbf{w}_1, \dots, \mathbf{w}_i$. This follows because the algorithm ensures that

$$\mathrm{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_i) = \mathrm{Span}\,(\mathbf{w}_1, \dots, \mathbf{w}_i)$$

at each step, and $\mathbf{w}_{i+1}, \dots, \mathbf{w}_m$ are orthogonal to $\mathbf{v}_i$ by construction. Specifically, we express

$$\mathbf{v}_i = \sum_{j=1}^{i} r_{ji}\mathbf{w}_j,$$

where the coefficients come from the projections and normalization: $r_{ji} = \mathbf{w}_j \cdot \mathbf{v}_i$ for $j < i$, and

$$r_{ii} = \left\| \mathbf{v}_i - \operatorname{proj}_{V_{i-1}}(\mathbf{v}_i) \right\|,$$

for $i > 1$ (with $r_{11} = \|\mathbf{v}_1\|$). This relationship is the key to representing M in a factored form, as we will see in the theorem.

Example 5.16. *Consider* $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *with columns* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Applying Gram-Schmidt:*

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \frac{\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)}{\left\| \mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2) \right\|},$$

where $proj_{\mathbf{v}_1}(\mathbf{v}_2) = (\mathbf{w}_1 \cdot \mathbf{v}_2)\mathbf{w}_1 = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, *so*

$$\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *Now, express* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *as linear combinations of the* $\mathbf{w}_1$, $\mathbf{w}_2$.

$$\mathbf{v}_1 = r_{11}\mathbf{w}_1 = \|\mathbf{v}_1\|\mathbf{w}_1 = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{v}_2 = r_{12}\mathbf{w}_1 + r_{22}\mathbf{w}_2,$$

where $r_{12} = \mathbf{w}_1 \cdot \mathbf{v}_2 = 1$, $r_{22} = \left\| \mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2) \right\| = \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = 1$, *so*

$$\mathbf{v}_2 = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Notice that if we create this matrix with $r_{1,1}, r_{1,2}, r_{2,2}$ *and put* $r_{2,1}$ *then*

$$M = [\mathbf{w}_1 \mid \mathbf{w}_2] \begin{bmatrix} r_{1,1} & r_{1,2} \\ 0 & r_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Next we will generalize this to a theorem called the QR-decomposition and the QR-algorithm, which is considered one of the main algorithms of numerical linear algebra. Its main application is to numerically compute the eigenvalues for invertible matrices.

5.4.1 The QR-decomposition

In many problems, we need to break down a matrix into simpler parts that are easier to work with. The QR-decomposition does this by writing any matrix with independent columns as a product

$$A = QR,$$

where Q is an orthogonal matrix and R an upper triangular matrix. This form is powerful because orthogonal matrices preserve lengths and angles, making computations stable, while the triangular matrix simplifies solving equations. Whether we're finding eigenvalues, solving systems, or handling numerical approximations, QR-decomposition provides a reliable tool that connects geometry and algebra, as we will see in some of the applications at the end of this section.

Theorem 5.6. *Let $M \in \mathrm{Mat}_{n\times m}(\mathbb{R})$, say $M = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_m]$ with linearly independent column vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$. Then*

$$M = QR,$$

where Q is an orthogonal matrix and R an upper triangular matrix. Moreover,

$$Q = [\mathbf{w}_1 \mid \mathbf{w}_2 \mid \cdots \mid \mathbf{w}_m] \quad \text{and} \quad R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1,m} \\ 0 & r_{22} & \dots & r_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_{m,m} \end{bmatrix} \tag{5.8}$$

where $\mathbf{w}_1, \dots, \mathbf{w}_m$ are orthonormal vectors obtained by the Gram-Schmidt algorithm on the set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, and r_{ij} are given by

$$r_{1,1} = \|\mathbf{v}_1\|, \quad r_{i,i} = \left\|\mathbf{v}_i - proj_{V_{i-1}}(\mathbf{v}_i)\right\| \text{ for } i > 1, \quad r_{i,j} = \mathbf{w}_i \cdot \mathbf{v}_j \text{ for } i < j,$$

with $V_{i-1} =$ Span $(\mathbf{v}_1, \cdots, \mathbf{v}_{i-1})$. This factorization is unique.

Proof. The vectors $\mathbf{w}_1, \dots, \mathbf{w}_m$ form an orthonormal basis for the column space of M, obtained by applying the Gram-Schmidt algorithm to $\mathbf{v}_1, \dots, \mathbf{v}_m$. We define $Q = [\mathbf{w}_1 \mid \cdots \mid \mathbf{w}_m]$ and $R = [r_{ij}]$, where $r_{ij} = \mathbf{w}_j \cdot \mathbf{v}_i$ for $j \leq i$ and $r_{ij} = 0$ for $j > i$, making R upper triangular. The equation $M = QR$ holds because each column of QR is $\sum_{j=1}^{i} r_{ji}\mathbf{w}_j = \mathbf{v}_i$, as shown above. Multiplying QR confirms this: the i-th column is $\sum_{j=1}^{m} r_{ji}\mathbf{w}_j$, but $r_{ji} = 0$ for $j > i$, matching $\mathbf{v}_i$. The factorization is unique because Gram-Schmidt, with a fixed order of $\mathbf{v}_1, \dots, \mathbf{v}_m$, yields a unique orthonormal basis, and requiring $r_{ii} > 0$ fixes the scaling. $\square$

Hence we have the following:

Corollary 5.4. *Any real square matrix A may be decomposed as $A = QR$, where Q is an orthogonal matrix and R is an upper triangular matrix. If A is invertible, then the factorization is unique if we require the diagonal elements of R to be positive.*

Proof. For a square matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, Theorem 5.6 guarantees $A = QR$ with Q orthogonal ($Q^t Q = I_n$) and R upper triangular. If A is invertible, its columns are linearly independent, and the Gram-Schmidt process is unique up to signs of the $\mathbf{w}_i$. Requiring $r_{ii} > 0$ fixes these signs, ensuring uniqueness. $\square$

Example 5.17. *Let us see how this will work for a general matrix $M \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$. Say $M = [\mathbf{v}_1 \mid \mathbf{v}_2]$. Then*

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \quad \mathbf{w}_2 = \frac{\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)}{\left\|\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)\right\|}$$

Hence, r_{11}, r_{12}, and r_{22} are given by

$$r_{11} = \|\mathbf{v}_1\|, \quad r_{12} = \mathbf{w}_1 \cdot \mathbf{v}_2, \quad r_{22} = \left\|\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)\right\|$$

Then we have

$$M = [\mathbf{v}_1 \mid \mathbf{v}_2] = [\mathbf{w}_1 \mid \mathbf{w}_2]\begin{bmatrix} \|\mathbf{v}_1\| & \mathbf{w}_1 \cdot \mathbf{v}_2 \\ 0 & \left\|\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)\right\| \end{bmatrix}$$

Let us see now a more concrete example.

Example 5.18. *Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, $\mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and*

$$r_{11} = \sqrt{2}, \quad r_{12} = \mathbf{w}_1 \cdot \mathbf{v}_2 = \frac{1}{\sqrt{2}}, \quad r_{22} = \left\|\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)\right\| = \frac{1}{\sqrt{2}}.$$

Thus we have,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Check: $QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = M.$

Example 5.19. *Find a QR-factorization of the matrix*

$$A = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Solution: First we find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for the column space and then we have

$$Q = [\mathbf{w}_1 \mid \mathbf{w}_2 \mid \mathbf{w}_3] = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

Then we have

$$r_{1,1} = \|\mathbf{v}_1\| = 2, \quad r_{2,2} = \left\|\mathbf{v}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{v}_2)\right\| = 1, \quad r_{3,3} = \left\|\mathbf{v}_3 - \text{proj}_{V_2}(\mathbf{v}_3)\right\|$$

where $V_2 = $ Span $(\mathbf{v}_1, \mathbf{v}_2)$. Since V_2 has an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$, then from Eq. (5.4) we have

$$\text{proj}_{V_2}(\mathbf{v}_3) = (\mathbf{w}_1 \cdot \mathbf{v}_3) \cdot \mathbf{w}_1 + (\mathbf{w}_2 \cdot \mathbf{v}_3) \cdot \mathbf{w}_2 = \mathbf{w}_1 - 2\mathbf{w}_2 = \begin{bmatrix} -1/2 \\ 3/2 \\ 3/2 \\ -1/2 \end{bmatrix}.$$

Then

$$\mathbf{v}_3 - \text{proj}_{V_2}(\mathbf{v}_3) = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 3/2 \\ 3/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

and $r_{3,3} = 1$. Also, $r_{1,2} = \mathbf{w}_1 \cdot \mathbf{v}_2 = 1$, $r_{1,3} = \mathbf{w}_1 \cdot \mathbf{v}_3 = 1$, $r_{2,3} = \mathbf{w}_2 \cdot \mathbf{v}_3 = -2$. Hence,

$$R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Corollary 5.5. *The matrix R in the QR-factorization of A is given by*

$$R = Q^t A.$$

Proof. Since Q is orthogonal, $Q^t Q = I$. Then

$$Q^t A = Q^t(QR) = (Q^t Q)R = IR = R.$$

□

Exercise 80. *Prove that the QR-factorization of*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix}$$

is $A = QR$ where

$$Q = \begin{bmatrix} 1/10 & -1/2\sqrt{2} & 0 \\ \frac{7}{10} & 0 & 1/2\sqrt{2} \\ 1/10 & 1/2\sqrt{2} & 0 \\ \frac{7}{10} & 0 & -1/2\sqrt{2} \end{bmatrix}, \quad R = \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Exercise 81. *Prove that the QR-factorization of*

$$A = \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix}$$

is $A = QR$ where

$$Q = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 9 & 1/3 \\ 0 & 6 & 2/3 \\ 0 & 0 & 7/3 \end{bmatrix}$$

5.4.2 Solving Linear Systems with QR-Decomposition

One key application of QR-decomposition is solving linear systems of the form $A\mathbf{x} = \mathbf{b}$, where $A \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ has linearly independent columns, $\mathbf{x} \in \mathbb{R}^m$, and $\mathbf{b} \in \mathbb{R}^n$. If $A = QR$, where $Q \in \mathrm{Mat}_{n\times m}$ has orthonormal columns ($Q^tQ = I_m$) and $R \in \mathrm{Mat}_{m\times m}$ is upper triangular, we can transform the system into a simpler one. Substitute $A = QR$ into the equation to get $QR\mathbf{x} = \mathbf{b}$. Since $Q^tQ = I_m$, multiply both sides by Q^t:

$$Q^tQR\mathbf{x} = Q^t\mathbf{b} \quad \Rightarrow \quad R\mathbf{x} = Q^t\mathbf{b}.$$

The system $R\mathbf{x} = Q^t\mathbf{b}$ is easy to solve because R is upper triangular: we use back substitution starting from the last row. This method is numerically stable because Q^t preserves lengths, avoiding the amplification of errors that can occur when solving $A\mathbf{x} = \mathbf{b}$ directly with elimination. For square, invertible A ($n = m$), this gives the exact solution. For $n > m$ (overdetermined systems), it provides the least squares solution, as we'll explore later.

Example 5.20. *Solve the system $A\mathbf{x} = \mathbf{b}$, where*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: *First, find the QR-decomposition of A. The columns are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Apply Gram-Schmidt:*

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_2 = \frac{\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)}{\left\|\mathbf{v}_2 - proj_{\mathbf{v}_1}(\mathbf{v}_2)\right\|},$$

and we get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$$

Now, compute $Q^t\mathbf{b} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ *and solve* $R\mathbf{x} = Q^t\mathbf{b}$, *where*

$$\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Thus, $\mathbf{x} = \begin{bmatrix} 2 \\ \frac{1}{3} \end{bmatrix}$. □

Is it faster to use row operations or QR-decomposition to solve $A\mathbf{x} = \mathbf{b}$? For a single system with a square, invertible A ($n = m$), row operations (Gaussian elimination) are typically faster, requiring about $\frac{2}{3}n^3$ operations, while QR-decomposition via Gram-Schmidt needs around $2n^3$ operations. However, if A is fixed and you solve for multiple $\mathbf{b}$ vectors, QR-decomposition is more efficient: compute Q and R once (costing $2n^3$), then each solution takes only n^2 operations ($Q^t\mathbf{b}$ and back substitution), compared to $\frac{2}{3}n^3$ per system with row operations. Moreover, QR is preferred for numerical stability or overdetermined systems ($n > m$), despite the initial cost.

5.4.3 Computing eigenvalues: QR-algorithm

Finding eigenvalues of a matrix A by solving $\mathbf{char}\,(A, \lambda) = 0$ is impractical for large n due to the lack of general formulas for polynomials of degree $n \geq 5$. The QR-algorithm offers a numerically stable iterative method to approximate these eigenvalues.

Assume as above that A has full rank. From Thm. 5.6 we can write A as

$$A = QR$$

for some orthogonal matrix Q and an upper triangular matrix R. Notice that

$$Q^{-1}AQ = Q^{-1}(QR)Q = RQ$$

Thus A is similar to RQ and therefore they have the same eigenvalues. Instead of computing the eigenvalues of A, we can compute the eigenvalues of $A_2 := RQ$. Let

$$A_2 = Q_2R_2$$

be the QR-decomposition of A_2. Again the eigenvalues of A_2 are the same as eigenvalues of $A_3 := R_2Q_2$. We can continue this process by letting

$$A_{i+1} := R_iQ_i.$$

Then all matrices $A, A_2, A_3, \ldots, A_{i+1}$ are similar and have the same eigenvalues, since

$$A_{i+1} = R_iQ_i = Q_i^{-1}Q_iR_iQ_i = Q_i^{-1}A_iQ_i = Q_i^tA_iQ_i,$$

so all the A_i are similar and hence they have the same eigenvalues. Remarkably, the sequence of matrices

$$A, A_2, A_3, \dots, A_i, \dots$$

converges to an upper triangular matrix with these eigenvalues on the main diagonal.

The algorithm is numerically stable because it proceeds by orthogonal similarity transforms. The eigenvalues of a triangular matrix are listed on the diagonal, and the eigenvalue problem is solved.

Lemma 5.6. *The QR-factorization is used to numerically approximate the eigenvalues of A.*

Proof. Given $A = Q_1R_1$, define $A_2 = R_1Q_1 = Q_1^t A Q_1$, which is similar to A and thus has the same eigenvalues. Iterating, $A_{i+1} = R_iQ_i = Q_i^t A_i Q_i$ remains similar to A. Each step applies an orthogonal transformation, preserving the spectrum. For a full-rank matrix A, the off-diagonal entries of A_i diminish as i increases, converging to an upper triangular form where the diagonal entries are the eigenvalues (or blocks for complex pairs). This convergence occurs because repeated QR-factorizations shift A toward a form where eigenvectors align with the basis, and orthogonality ensures numerical errors do not grow exponentially, unlike polynomial root-finding. □

Example 5.21. *Estimate the eigenvalues of*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

using the QR-algorithm.

Solution: First we find the QR-decomposition of $A = Q_1R_1$, which is

$$Q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad R_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}$$

Then

$$A_2 = R_1Q_1 = \frac{1}{5} \begin{bmatrix} 7 & 9 \\ 4 & -2 \end{bmatrix}$$

Find the QR-decomposition of A_2, which is

$$Q_2 = \frac{1}{\sqrt{65}} \begin{bmatrix} 7 & 4 \\ 4 & -7 \end{bmatrix} \quad R_2 = \frac{1}{\sqrt{65}} \begin{bmatrix} 13 & 10 \\ 0 & 10 \end{bmatrix}$$

Then A_3 is

$$A_3 = \frac{1}{13} \begin{bmatrix} 27 & -5 \\ 8 & -14 \end{bmatrix} \approx \begin{bmatrix} 2.08 & -0.38 \\ 0.62 & -1.08 \end{bmatrix}$$

So we are already getting closer to the actual eigenvalues, which are 2 and -1.

Example 5.22. *Estimate the eigenvalues of*

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 10^{-4} \\ 1 & 10^{-4} & 2 \end{bmatrix},$$

where small entries test numerical stability.

Solution: The characteristic polynomial $\det(A - \lambda I)$ is cubic, and small terms like 10^{-4} make direct root-finding prone to rounding errors. Using the QR-algorithm:

- Step 1: $A = Q_1 R_1$. Gram-Schmidt on columns $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 10^{-4} \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 10^{-4} \\ 2 \end{bmatrix}$:

- $Q_1 \approx \begin{bmatrix} 0.943 & -0.041 & 0.029 \\ 0.236 & 0.999 & -0.013 \\ 0.236 & -0.102 & 0.999 \end{bmatrix}$, $R_1 \approx \begin{bmatrix} 4.243 & 1.179 & 1.179 \\ 0 & 2.723 & -0.278 \\ 0 & 0 & 1.943 \end{bmatrix}$.

- $A_2 = R_1 Q_1 \approx \begin{bmatrix} 4.236 & 0.936 & 0.827 \\ 0.935 & 3.000 & 0.001 \\ 0.827 & 0.001 & 1.764 \end{bmatrix}$.

- Step 2: $A_2 = Q_2 R_2$, repeating yields $A_3 \approx \begin{bmatrix} 4.309 & 0.765 & 0.594 \\ 0.765 & 2.999 & 0.002 \\ 0.594 & 0.002 & 1.692 \end{bmatrix}$.

After a few iterations (e.g., 5–10 in practice), off-diagonal entries shrink, and the diagonal stabilizes near 4.31, 3, and 1.69 (true eigenvalues, via numerical solvers). Unlike direct methods, QR avoids amplifying the 10^{-4} terms, maintaining accuracy.

Remark 5.1. *If some of the eigenvalues of a real matrix A are not real, the QR-algorithm converges to a block upper triangular matrix where the diagonal blocks are either 1×1 (the real eigenvalues) or 2×2 (each providing a pair of conjugate complex eigenvalues of A).*

Exercises:

349. *Can a matrix have more than one QR-factorization?*

350. *Estimate the eigenvalues of the matrix A up to two decimal places using the QR-algorithm.*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

351. *Estimate the eigenvalues of the matrix A up to two decimal places using the QR-algorithm.*

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \\ 3 & -7 & 8 \end{bmatrix}$$

352. *Suppose $A\mathbf{x} = \mathbf{b}$ is a linear system, where A is a matrix of size $m \times n$, $\mathbf{x}$ of size $n \times 1$, and $\mathbf{b}$ of size $m \times 1$. Let $A = QR$ be a QR-factorization of A. Demonstrate a simple way of solving the linear system $A\mathbf{x} = \mathbf{b}$.*

353. *Perform two iterations of the QR-algorithm* *on*

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

*Estimate the eigenvalues from A_3 and compare with the exact values (solve **char** $(A, \lambda) = 0$).*

354. *For*

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

compute $A_2 = R_1 Q_1$ using the QR-algorithm. How do the off-diagonal entries change, and what does this suggest about convergence?

355. *Apply one iteration of the QR-algorithm to*

$$A = \begin{bmatrix} 2 & 1 & 10^{-3} \\ 1 & 3 & 1 \\ 10^{-3} & 1 & 4 \end{bmatrix}.$$

Compare the diagonal of A_2 to A's, noting the effect of the small 10^{-3} entries on stability.

356. *Consider*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

Perform three iterations of the QR-algorithm. Are the eigenvalues real? How does the matrix form indicate this?

357. *For a 3×3 matrix A with eigenvalues 2, 3, and 4, explain why the QR-algorithm converges to a triangular matrix with these values on the diagonal, assuming no complex eigenvalues. What role does orthogonality play?*

5.5 Schur decomposition

Above we learnt about the QR-factorization, but that worked only for matrices in $\mathrm{Mat}_{m\times n}(\mathbb{R})$. What about matrices with complex coefficients? The modification of the above factorization for matrices with complex entries is called Schur decomposition.

Lemma 5.7 (Schur). *Any $A \in \mathrm{Mat}_n(\mathbb{C})$ can be expressed as*

$$A = QUQ^{-1}$$

where Q is a unitary matrix and U is an upper triangular matrix.

Proof. The proof relies on the existence of eigenvalues and eigenvectors for any square matrix over $\mathbb{C}$, and proceeds by induction on the matrix size n.

- **Base Case ($n = 1$):** If A is a 1×1 matrix, say $A = [\lambda]$, then take $Q = [1]$ (unitary, since $1^* = 1$ and $1 \cdot 1 = 1$) and $U = [\lambda]$ (upper triangular). Clearly, $A = QUQ^{-1}$ holds.

- **Inductive Step:** Assume the result holds for all $(n-1)\times(n-1)$ matrices. Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Since A has at least one eigenvalue λ (by the Fundamental Theorem of Algebra applied to its characteristic polynomial), there exists a corresponding eigenvector $\mathbf{v}$ with $\|\mathbf{v}\| = 1$. Extend $\mathbf{v}$ to an orthonormal basis $\{\mathbf{v}, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ for $\mathbb{C}^n$ using the Gram-Schmidt process with the standard Hermitian inner product.

  Define $Q_1 = [\mathbf{v} \mid \mathbf{w}_2 \mid \cdots \mid \mathbf{w}_n]$, a unitary matrix (since its columns are orthonormal). Compute the conjugate transpose $Q_1^* = [\mathbf{v}^* \mid \mathbf{w}_2^* \mid \cdots \mid \mathbf{w}_n^*]$, where each row is the conjugate transpose of the corresponding column. Then:

  $$Q_1^* A Q_1 = Q_1^* A[\mathbf{v} \mid \mathbf{w}_2 \mid \cdots \mid \mathbf{w}_n].$$

  Since $A\mathbf{v} = \lambda\mathbf{v}$ and $\{\mathbf{v}, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is orthonormal:

  - $Q_1^*(A\mathbf{v}) = Q_1^*(\lambda\mathbf{v}) = \lambda Q_1^*\mathbf{v} = \lambda[1, 0, \dots, 0]^T$,
  - For $j = 2, \dots, n$, the column $Q_1^*(A\mathbf{w}_j)$ produces a vector in $\mathbb{C}^n$.

  Thus:

  $$Q_1^* A Q_1 = \begin{bmatrix} \lambda & \mathbf{a}^* \\ \mathbf{0} & B \end{bmatrix},$$

  where $\mathbf{a}^* \in \mathbb{C}^{1\times(n-1)}$ and $B \in \mathrm{Mat}_{n-1}(\mathbb{C})$. By the induction hypothesis, $B = Q_2 U_2 Q_2^{-1}$, where Q_2 is unitary and U_2 is upper triangular. Define:

  $$Q = Q_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2 \end{bmatrix}, \quad U = \begin{bmatrix} \lambda & \mathbf{a}^* Q_2 \\ \mathbf{0} & U_2 \end{bmatrix}.$$

  - **Check Q is unitary:** Since Q_1 and $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2 \end{bmatrix}$ are unitary (the latter because $Q_2^* Q_2 = I_{n-1}$), their product Q is unitary.

– **Check U is upper triangular:** U has zeros below the diagonal, as U_2 is upper triangular.

Verify:

$$QUQ^{-1} = Q_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2 \end{bmatrix} \begin{bmatrix} \lambda & \mathbf{a}^* Q_2 \\ \mathbf{0} & U_2 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2^{-1} \end{bmatrix} Q_1^{-1}.$$

Compute the inner product:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2 \end{bmatrix} \begin{bmatrix} \lambda & \mathbf{a}^* Q_2 \\ \mathbf{0} & U_2 \end{bmatrix} = \begin{bmatrix} \lambda & \mathbf{a}^* Q_2 \\ \mathbf{0} & Q_2 U_2 \end{bmatrix},$$

$$\begin{bmatrix} \lambda & \mathbf{a}^* Q_2 \\ \mathbf{0} & Q_2 U_2 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2^{-1} \end{bmatrix} = \begin{bmatrix} \lambda & \mathbf{a}^* \\ \mathbf{0} & Q_2 U_2 Q_2^{-1} \end{bmatrix} = \begin{bmatrix} \lambda & \mathbf{a}^* \\ \mathbf{0} & B \end{bmatrix},$$

$$Q_1 \begin{bmatrix} \lambda & \mathbf{a}^* \\ \mathbf{0} & B \end{bmatrix} Q_1^{-1} = Q_1 (Q_1^* A Q_1) Q_1^{-1} = A.$$

Thus, $A = QUQ^{-1}$, completing the induction.

□

U is called a **Schur form** of A. Since U is similar to A, it has the same eigenvalues and since it is triangular, its eigenvalues are the diagonal entries of U.

The Schur decomposition implies that there exists a nested sequence of A-invariant subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = C_n,$$

and that there exists an ordered orthonormal basis (for the standard Hermitian product) such that the first i vectors in the basis span V_i for each i occurring in the nested sequence.

Remark 5.2. *Now we can say that for matrices of full rank (all columns are linearly independent) we have an effective way of computing the eigenvalues via the QR-algorithm. What about matrices which don't have full rank? In Thm. 6.11 we will explain how this is done via the Singular Value Decomposition.*

Theorem 5.7. *Let A be an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues (not necessarily distinct). Then,*

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Proof. From s Schur decomposition we have

$$A = UTU^*$$

where T is an upper triangular matrix. The diagonal entries of T are the eigenvalues of A (including repetitions according to algebraic multiplicity). Hence,

$$\operatorname{tr}(A) = \operatorname{tr}(UTU^*) = \operatorname{tr}(U^*UT) = \operatorname{tr}(IT) = \operatorname{tr}(T)$$

Since $U^*U = I$ (because U is unitary).

The trace of an upper triangular matrix is simply the sum of its diagonal entries. As the diagonal entries of T are the eigenvalues of A, we have:

$$\mathrm{tr}(T) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Therefore, the trace of matrix A is equal to the sum of its eigenvalues. $\square$

Exercises:

358. *For* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$*, compute the Schur decomposition* $A = QUQ^{-1}$*. Verify that* Q *is unitary and* U *is upper triangular, and check that the eigenvalues of* A *appear on the diagonal of* U*.*

359. *Consider* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$*. Find its Schur decomposition and confirm that the diagonal entries of* U *are the eigenvalues of* A*.*

360. *Let* A *be a* 2×2 *matrix with complex eigenvalues (e.g.,* $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$*). Show that its Schur decomposition yields an upper triangular* U *with these eigenvalues on the diagonal.*

361. *If* A *is normal (*$AA^* = A^*A$*), prove that* U *in its Schur decomposition is diagonal. What does this imply about normal matrices?*

362. *Using Schur decomposition, prove that for any square matrix* A *and positive integer* k*,* $\mathrm{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k$*, where* λ_i *are the eigenvalues of* A*.*

5.6 Sylvester's theorem

Let V be a finite dimensional vector space over $\mathbb{R}$ and $\langle \cdot, \cdot \rangle$ an inner product on V. By the previous section we can find an orthogonal basis $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of V. Since the inner product is not necessarily positive definite then $\langle \mathbf{v}_i, \mathbf{v}_i \rangle$ could be ≤ 0. Denote

$$c_i := \langle \mathbf{v}_i, \mathbf{v}_i \rangle$$

for $i = 1, \dots n$. We can reorder the basis $\mathfrak{B}$ such that

$$c_1, \dots, c_p > 0, \qquad c_{p+1}, \dots, c_{p+s} < 0, \qquad c_{p+s+1}, \dots, c_{p+s+r} = 0,$$

where $p + s + r = n$. Sylvester's theorem says that the numbers p, s, r don't depend on the choice of the orthogonal basis $\mathfrak{B}$. So we have an inner space V with inner product $\langle \cdot, \cdot \rangle$, not necessarily positive definite. We can still get an **orthonormal basis** on V as follows. Let

$$\mathbf{v}_i' := \begin{cases} \mathbf{v}_i, & \text{if} \quad c_i = 0 \\ \dfrac{\mathbf{v}_i}{\sqrt{c_i}}, & \text{if} \quad c_i > 0 \\ \dfrac{\mathbf{v}_i}{\sqrt{-c_i}}, & \text{if} \quad c_i < 0 \end{cases} \tag{5.9}$$

Then the set $\mathfrak{B}'$ is a basis of V such that

$$\langle \mathbf{v}_i, \mathbf{v}_i \rangle = \pm 1, \quad or \quad 0.$$

Such basis is called an **orthonormal basis** of V. Let

$$\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$$

be an orthogonal basis of V such that

$$c_1, \dots, c_p = 1, \qquad c_{p+1}, \dots, c_{p+s} = -1, \qquad c_{p+s+1}, \dots, c_{p+s+r} = 0$$

where $p + s + r = n$ and $c_i := \langle \mathbf{v}_i, \mathbf{v}_i \rangle$.

Theorem 5.8. *The number p, r, s are uniquely determined by the inner product and do not depend on the choice of the orthogonal basis $\mathfrak{B}$.*

Proof. Fix an orthogonal basis

$$\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$$

of V, where $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = c_i \delta_{ij}$ and $c_1, \dots, c_p = 1$, $c_{p+1}, \dots, c_{p+s} = -1$, $c_{p+s+1}, \dots, c_{p+s+r} = 0$. In this basis, the inner product is represented by the diagonal matrix

$$D = \operatorname{diag}(c_1, \dots, c_n),$$

with p positive entries, s negative entries, and r zeros along the diagonal. Now consider another orthogonal basis

$$\mathfrak{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_n\},$$

where $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = c'_i \delta_{ij}$, and let $D' = \operatorname{diag}(c'_1, \ldots, c'_n)$ represent the inner product in $\mathfrak{B}'$, with $c'_1, \ldots, c'_{p'} = 1$, $c'_{p'+1}, \ldots, c'_{p'+s'} = -1$, and $c'_{p'+s'+1}, \ldots, c'_{p'+s'+r'} = 0$, where $p' + s' + r' = n$. The change-of-basis matrix P from $\mathfrak{B}$ to $\mathfrak{B}'$ satisfies $\mathbf{w}_i = \sum_j P_{ji} \mathbf{v}_j$, and because the inner product must be consistent across bases, we have $D' = P^T D P$. Since D and D' are congruent—both being real symmetric matrices transformed by an invertible P—Sylvester's Law of Inertia applies, stating that the number of positive, negative, and zero eigenvalues of D and D' must be equal. Here, the eigenvalues of D are its diagonal entries $c_1, \ldots, c_n$, giving p positive, s negative, and r zero eigenvalues, and similarly for D' with p', s', and r'. Thus, $p = p'$, $s = s'$, and $r = r'$, proving that these counts are invariants of the inner product, not the basis chosen.

□

The integer p (resp., s) is sometimes called the **index of positivity** (resp., **negativity**) and the pair (p, s) the **signature** of the inner product. Let us see an example.

Example 5.23. *Let $V = \mathbb{R}^2$ and $A \in \operatorname{Mat}_{2\times 2}(\mathbb{R})$ be a symmetric matrix. Define an inner product on V as follows*

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^t A \mathbf{v}.$$

The reader should show that this is an inner product.

Example 5.24. *Let $V = \mathbb{R}^2$ and $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Then V is an inner space with the inner product*

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^t A \mathbf{v}.$$

Take a basis in V as $\mathfrak{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, This is an orthogonal basis since

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1^t A \mathbf{v}_2 = \begin{bmatrix} 1, & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

What is the signature of this inner product? We have

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \mathbf{v}_1^t A \mathbf{v}_1 = \begin{bmatrix} 1, & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1.$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \mathbf{v}_2^t A \mathbf{v}_2 = \begin{bmatrix} 1, & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0.$$

Hence the signature is $(0, 1)$. □

5.6.1 Decomposition of Inner Spaces

Given a finite-dimensional real vector space V with an inner product $\langle \cdot, \cdot \rangle$, define the sets

$$V^+ = \{ \mathbf{u} \in V \mid \langle \mathbf{u}, \mathbf{u} \rangle > 0 \}$$
$$V^- = \{ \mathbf{u} \in V \mid \langle \mathbf{u}, \mathbf{u} \rangle < 0 \}$$
$$V^0 = \{ \mathbf{u} \in V \mid \langle \mathbf{u}, \mathbf{u} \rangle = 0 \}$$

to capture vectors based on the sign of their inner product. While these sets are not subspaces—since, for example, $2\mathbf{u}$ may not be in V^+ if $\mathbf{u} \in V^+$ and the inner product is indefinite—we can use them to motivate a decomposition into subspaces derived from an orthogonal basis.

Lemma 5.8. *The space V admits a direct sum decomposition $V = V^+ \oplus V^- \oplus V^0$, where V^+, V^-, and V^0 are subspaces spanned by basis vectors with positive, negative, and zero inner products, respectively.*

Proof. From Sylvester's Theorem, V has an orthogonal basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ with $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = c_i \delta_{ij}$, where $c_1, \ldots, c_p > 0$, $c_{p+1}, \ldots, c_{p+s} < 0$, and $c_{p+s+1}, \ldots, c_{p+s+r} = 0$, and $p + s + r = n$. Define subspaces

$$
\begin{aligned}
V^+ &= \text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \\
V^- &= \text{Span}\,\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_{p+s}\} \\
V^0 &= \text{Span}\,\{\mathbf{v}_{p+s+1}, \ldots, \mathbf{v}_{p+s+r}\}.
\end{aligned}
$$

Any $\mathbf{v} \in V$ can be expressed as

$$
\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{v}_i = \sum_{i=1}^{p} v_i \mathbf{v}_i + \sum_{i=p+1}^{p+s} v_i \mathbf{v}_i + \sum_{i=p+s+1}^{p+s+r} v_i \mathbf{v}_i,
$$

where the first sum lies in V^+, the second in V^-, and the third in V^0, showing $V = V^+ + V^- + V^0$. The sum is direct because the basis vectors span disjoint index sets: if $\mathbf{u} \in V^+ \cap V^-$, then

$$
\mathbf{u} = \sum_{i=1}^{p} u_i \mathbf{v}_i = \sum_{i=p+1}^{p+s} w_i \mathbf{v}_i,
$$

and since $\mathcal{B}$ is linearly independent, $\mathbf{u} = 0$; similarly, $V^+ \cap V^0 = \{0\}$ and $V^- \cap V^0 = \{0\}$. With dimensions p, s, and r summing to n, we have $V = V^+ \oplus V^- \oplus V^0$.

$\square$

This decomposition leverages the orthogonal basis to define V^+, V^-, and V^0 as subspaces, distinct from the broader sets initially proposed. For

$$
\mathbf{u} = u^+ + u^- + u^0
$$

with $u^+ \in V^+$, $u^- \in V^-$, and $u^0 \in V^0$, the inner product is

$$
\langle \mathbf{u}, \mathbf{u} \rangle = \langle u^+, u^+ \rangle + \langle u^-, u^- \rangle,
$$

since $\langle u^0, u^0 \rangle = 0$ and orthogonality eliminates cross terms.

Example 5.25. *Consider $V = \mathbb{R}^2$ with $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{v}$. The basis*

$$
\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}
$$

has

$$\left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\rangle = -1, \quad \left\langle \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\rangle = 0, \quad \text{and} \quad \left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\rangle = 0,$$

so $V^- = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$, $V^0 = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, *and* $V^+ = \{0\}$, *forming*

$$V = \{0\} \oplus \mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2.$$

For $\mathbf{v} = (a,b)$, *we write* $\mathbf{v} = 0 + a\begin{bmatrix}1\\0\end{bmatrix} + (b-a)\begin{bmatrix}1\\1\end{bmatrix}$, *and* $\langle \mathbf{v}, \mathbf{v}\rangle = -a^2$, *consistent with* V^-.

Example 5.26. *In* $V = \mathbb{R}^3$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 - u_2v_2 + u_3v_3$, *the standard basis gives* $c_1 = 1$, $c_2 = -1$, $c_3 = 1$, *so*

$$V^+ = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}, \quad V^- = \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \quad V^0 = \{0\}.$$

A vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$ *decomposes as* $\begin{bmatrix}a\\0\\c\end{bmatrix} + \begin{bmatrix}0\\b\\0\end{bmatrix} + \mathbf{0}$, *with*

$$\langle \begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}a\\b\\c\end{bmatrix} \rangle = a^2 - b^2 + c^2,$$

reflecting contributions from V^+ *and* V^-. *This decomposition is a powerful tool for analyzing indefinite inner products, separating positive, negative, and null behaviors into orthogonal components.*

Exercises:

363. *What is the signature of* $\mathbb{R}^n$ *with the usual Euclidean inner product?*

364. *On* $V = \mathbb{R}^2$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 + u_2v_2$, *show that* $s = 0$ *and find* p *and* r.

365. *For* $V = \mathbb{R}^2$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 - u_2v_2$, *find an orthogonal basis and determine the signature* (p,s).

366. *Let* $V = \mathbb{R}^3$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T A\mathbf{v}$, *where* $A = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}$. *Compute the signature* (p,s) *using the standard basis.*

367. *For* $V = \mathbb{R}^2$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 - u_2v_2$, *find bases for the subspaces* V^+, V^-, *and* V^0, *and verify* $V = V^+ \oplus V^- \oplus V^0$.

368. *Let* $V = \mathbb{R}^3$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}\mathbf{v}$. *Determine the subspaces* V^+, V^-, *and* V^0, *and confirm the direct sum.*

369. *For* $V = \mathbb{R}^3$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 - u_2v_2 + u_3v_3$, *find an orthogonal basis and verify that the signature is* $(2,1)$.

370. *On* $V = \mathbb{R}^2$ *with* $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_2 + u_2v_1$, *identify* V^+, V^-, *and* V^0, *and check if the decomposition* *holds as a direct sum.*

371. *Let* P_2 *denote the space of polynomials in* $\mathbb{F}[x]$ *of degree* ≤ 2. *For* $f, g \in P_2$ *with* $f(x) = a_2x^2 + a_1x + a_0$ *and* $g(x) = b_2x^2 + b_1x + b_0$, *define* $\langle f, g \rangle = a_0b_0 + a_1b_1 + a_2b_2$. *Find the signature of this inner product for* P_2.

372. *Given a symmetric matrix* $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, *prove that the signature of* $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$ *matches the number of positive and negative eigenvalues of* A, *assuming* A *is invertible.*

Chapter 6

Symmetric matrices

The theory of symmetric matrices is tied closely with the classical theory of quadratic forms. In the first section we give a brief overview of quadratic forms and how to associate them with matrices. In the next few sections we study in more detail the symmetric matrices. The main focus of this chapter can be summarized in the problem of diagonalizing a quadratic form. For example, given and equation

$$61x^2 + 104xy + 22xz + 52y^2 + 40yz + 61z^2 = 144$$

show that by a change of basis this equation can be written as $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$, which obviously is a lot easier to recognize its shape.

6.1 Quadratic forms

The theory of n-ary forms is one of the oldest and most beautiful parts of mathematics. In this chapter we give a brief introduction to this theory and its relation to linear algebra. As before we take the field of scalar to be $\mathbb{R}$.

6.1.1 Binary quadratic forms

A **binary quadratic form** is a homogenous degree 2 polynomial in two variables, in other words a polynomial of the form

$$f(x,y) = ax^2 + bxy + cy^2$$

and its discriminant defined as $\Delta_f = b^2 - 4ac$. We let the matrices M and $\mathbf{v}$ be defined as

$$M = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}, \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Then, a binary quadratic form is given by

$$f(x,y) = \mathbf{v}^t M \mathbf{v}$$

Hence, *there is a one-to-one correspondence between the binary quadratic forms and symmetric* 2×2 *matrices*. For a given form f its corresponding matrix will be denoted by M_f.

Lemma 6.1. *The discriminant Δ_f of f is zero if and only if $\det M_f = 0$. Moreover, $\Delta_f = -4\det M_f$.*

Remark 6.1. *There are many authors who define binary forms as*

$$f(x,y) = ax^2 + 2bxy + cy^2$$

so that the corresponding matrix is $M_f = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and the discriminant $\Delta_f = ac - b^2$ instead of $b^2 - 4ac$. We will stick with the usual conventions.

Change of coordinates

A **change of coordinates** is any linear map $\mathbb{R}^2 \to \mathbb{R}^2$ for some matrix $M = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{R})$,

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Notice that M is not necessarily an invertible matrix.

Let $f(x,y) = ax^2 + 2bxy + cy^2$ with matrix $M_f = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. We normally denote by f^M (not to be confused by any power of f) the new quadratic form

$$f^M(x,y) := f(\lambda_1 x + \lambda_2 y, \lambda_3 x + \lambda_4 y)$$

Then expanding the expression for $f^M(x,y)$ we have

$$\begin{aligned} f(\lambda_1 x + \lambda_2 y, \lambda_3 x + \lambda_4 y) =& (a\lambda_1^2 + b\lambda_1\lambda_3 + c\lambda_3^2)x^2 + (2a\lambda_1\lambda_2 + b\lambda_1\lambda_4 + b\lambda_2\lambda_3 + 2cy\lambda_3\lambda_4)xy \\ & + (a\lambda_2^2 + b\lambda_2\lambda_4 + c\lambda_4^2)y^2 \end{aligned}$$

which has matrix

$$\begin{aligned} & \begin{bmatrix} a\lambda_1^2 + b\lambda_1\lambda_3 + c\lambda_3^2 & a\lambda_1\lambda_2 + c\lambda_3\lambda_4 + \frac{b}{2}(\lambda_1\lambda_4 + \lambda_2\lambda_3) \\ a\lambda_1\lambda_2 + c\lambda_3\lambda_4 + \frac{b}{2}(\lambda_1\lambda_4 + \lambda_2\lambda_3) & a\lambda_1^2 + b\lambda_1\lambda_3 + c\lambda_3^2 \end{bmatrix} \\ & = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}^t M_f \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} = M^t M_f M. \end{aligned}$$

So if we let $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ then

$$f^M(x,y) = f(\lambda_1 x + \lambda_2 y, \lambda_3 x + \lambda_4 y) = \mathbf{v}^t \left(M^t M_f M \right) \mathbf{v}.$$

Hence, the associated matrix for $f^M(x,y)$ is precisely the matrix $M^t M_f M$. This is a symmetric matrix as expected because of the following easy exercise.

Exercise 82. *If A is a symmetric matrix, then for any matrix B, the matrix B^tAB is symmetric.*

Hence, through the change of coordinates $\mathbf{x} \to M\mathbf{x}$ the binary quadratic form $\mathbf{f}(x,y)$ became $f^M(x,y)$ or in other words

$$f(x,y) = \mathbf{v}^t M \mathbf{v} \longrightarrow f^M(x,y) = \mathbf{v}^t \left(M^t M_f M\right) \mathbf{v}.$$

Recall that we didn't assume that M is invertible. If M is invertible then we call the forms $f(x,y)$ and $g(x,y) = f^M(x,y)$ *equivalent* and with a change of coordinates $\mathbf{x} \to M^{-1}\mathbf{x}$ we can write $f(x,y) = g^{M^{-1}}(x,y)$. We make this precise in the following definition.

Definition 6.1. *Two binary quadratic forms $f(x,y)$ and $g(x,y)$ are called **equivalent over** $\mathbb{R}$ if they are related through an invertible change of coordinates. In other words, if there exists an invertible matrix $M \in GL_2(\mathbb{R})$ such that*

$$f^M(x,y) = g(x,y).$$

So answered the natural question: *if two binary forms f and g are related by a change of coordinates, how are their matrices M_f and M_g related?*

Lemma 6.2. *Two binary quadratic forms $f(x,y)$ and $g(x,y)$ are related through a change of coordinates $M \in \mathrm{Mat}_2(\mathbb{R})$ if and only if their corresponding matrices satisfy,*

$$M_g = M^t M_f M.$$

Moreover, two binary forms are equivalent over $\mathbb{R}$ if and only if there exists $M \in GL_2(\mathbb{R})$ such that $M_g = M^t M_f M$.

Two matrices A and B are called **congruent over** $\mathbb{R}$ if there is an invertible $M \in \mathrm{GL}_2(\mathbb{R})$ such that $A = M^t B M$. We now understand what happens to the binary quadratics and their associated matrices under a change of coordinates. Next we will focus on the geometric point of view and see what happens to the graph of $f(x,y) = d$ under a coordinate change.

Example 6.1. *Given the graph of*

$$8x^2 + 20xy + 13y^2 = 1$$

we want to see what happens to this graph after the coordinate change by the matrix $A = \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix}$. Then we have

$$\begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x - y \\ -x + y \end{bmatrix}.$$

Substituting in the original equation we have $x^2 + y^2 = 1$. This is illustrated in Fig. 6.1.

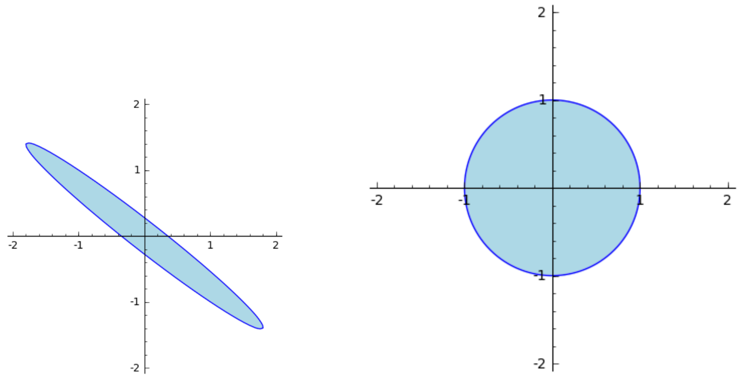


Figure 6.1: Mapping the ellipse to the unit circle

Geometry of binary quadratic forms

Let's see if we can determine the shape of the graph in $\mathbb{R}^2$ for the curve with equation

$$f(x,y) = ax^2 + bxy + cy^2 = d$$

for some fixed constant $d \in \mathbb{R}$. Can we somehow use the matrix M_f to determine the shape of the graph? Without any loss of generality we can assume that $d = 1$ by applying the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \sqrt{d}\, x \\ \sqrt{d}\, y \end{bmatrix}.$$

If the quadratic form would have the shape

$$f(x,y) = \alpha_1 x^2 + \alpha_2 y^2 \tag{6.1}$$

then this would be much easier to graph. We have seen such graphs from high school. We call binary quadratics as in Eq. (6.1) **diagonal**, since their corresponding matrices are diagonal $\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$. So how can we change a binary quadratic to a diagonal quadratic? We know how to diagonalize matrices. So maybe the same procedure can be used to diagonalize quadratics? Let's give it a try.

The characteristic polynomial of A is

$$\textbf{char}\,(A,x) = x^2 - (a+c)x - \frac{b^2}{4} - ac$$

Its eigenvalues are

$$\lambda_1 = \frac{a+c}{2} + \frac{\sqrt{(a-c)^2 + b^2}}{2} \quad \text{and} \quad \lambda_1 = \frac{a+c}{2} - \frac{\sqrt{(a-c)^2 + b^2}}{2}$$

and their corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{(c-a)+\sqrt{(c-a)^2+b^2}} \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{(c-a)-\sqrt{(c-a)^2+b^2}} \\ 1 \end{bmatrix}.$$

Hence, A is diagonalizable as $A = PDP^{-1}$ where P

$$P = \begin{bmatrix} \frac{1}{(c-a)+\sqrt{(c-a)^2+b^2}} & \frac{1}{(c-a)-\sqrt{(c-a)^2+b^2}} \\ 1 & 1 \end{bmatrix}.$$

Since D is given by

$$D = P^{-1}AP$$

we can make this work if somehow $P^{-1} = P^t$ as in Lem. 6.2. But we know exactly about matrices with this property, thanks to Thm. 5.4. They are the orthogonal matrices.

So our next challenge becomes to find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or in other words to choose the eigenvectors in the diagonalization process such that the transition matrix C is orthogonal.

Exercise 83. *Let $f(x,y) = x^2 - 2xy + y^2$. Diagonalize $f(x,y)$. Make each eigenvector a unit vector and determine the transitional matrix C. Graph $f(x,y) = 1$ and then graph the equation $g(x,y) = 1$, where $g(x,y)$ is the diagonal form of $f(x,y)$. Compare the two graphs.*

Positive definite forms

Let us now recall a few things from high school.

Example 6.2. *For a given binary quadratic form*

$$f(x,y) = ax^2 + bxy + cy^2$$

the sign of $f(x,1)$ is determined by the following: $f(x,1)$ has the opposite sign of a in the interval $(-\alpha_1, \alpha_2)$ and it has the sign of a everywhere else.

x		α_1	α_2	
f(x)	a		- a	a

Table 6.1: The sign of the quadratic polynomial

A binary quadratic is called **positive definite** if $f(x,y) > 0$ for every $\mathbf{x} \in \mathbb{R}^2$.

Lemma 6.3. *A binary quadratic form*

$$f(x,y) = ax^2 + bxy + cy^2$$

is a positive definite quadratic if and only if $a > 0$ and $\Delta_f < 0$.

Proof. We assume that $y \neq 0$ and write

$$f(x,y) = ax^2 + bxy + cy^2 = \frac{1}{y^2}\left[a\left(\frac{x}{y}\right)^2 + b\frac{x}{y} + c\right]$$

let us make the substitution $t = \frac{x}{y}$. Then the sign of $f(x,y)$ is the same as the sign of

$$g(t) = at^2 + bt + c.$$

From the above discussion, this is always positive if and only if $a > 0$ and $\Delta_g = \Delta_f < 0$.

For values $y = 0$ we have $f(x,0) = ax^2$, so $f(x,y)$ is not positive definite since for $x = 0$ it is $f(0,0) = 0$. This completes the proof. □

Definition 6.2. *A binary quadratic*

$$f(x,y) = x^2 + bxy + cy^2$$

with integer coefficients is called ***reduced*** *if*

$$\mathbf{h}(f) := max\{|a|,|b|,|c|\}$$

is minimal. The integer $\mathbf{h}(f)$ *is called the* ***height*** *of* $f(x,y)$.

The definition above means that for every $M \in \mathrm{GL}_2(Z)$, the height of $f^M(x,y)$ is bigger or equal to the height of $f(x,y)$. The height of a binary form can be defined in terms of the corresponding matrix M.

6.1.2 n-ary quadratic forms

Next, we generalize the concept of a binary quadratic to that of an n-ary quadratic form.

Definition 6.3. *A* ***quadratic form*** *defined over* $\mathbb{R}$ *is called a function* $\mathbf{q} : \mathbb{R}^n \to \mathbb{R}$*, such that*

$$q(\mathbf{x}) \; = \sum_{i=0,j=0}^{n} a_{i,j}\, x_i x_j, \tag{6.2}$$

where $a_{i,j}$ *are coefficients from* $\mathbb{R}$.

As for binary quadratics, every quadratic form has an **associated matrix** A_q given by $A = \left[a_{i,j}\right]$. Binary forms are the simplest of all quadratic forms. Quadratic forms $\mathbf{q}: \mathbb{R}^3 \to \mathbb{R}$ are called **ternary forms**. A **ternary form** is given by

$$q(\mathbf{x}) = a_{1,1}\, x_1^2 + a_{1,2}\, x_1 x_2 + a_{1,3}\, x_1 x_3 + a_{2,2}\, x_2^2 + a_{2,3}\, x_2 x_3 + a_{3,3}\, x_3^2$$

and has coefficient matrix

$$A = [a_{i,j}] = \begin{bmatrix} a_{1,1} & \frac{1}{2}a_{1,2} & \frac{1}{2}a_{1,3} \\ \frac{1}{2}a_{1,2} & a_{2,2} & \frac{1}{2}a_{2,3} \\ \frac{1}{2}a_{1,3} & \frac{1}{2}a_{2,3} & a_{3,3} \end{bmatrix}$$

As in the case of binary quadratics, even for n-ary forms we have

$$q(\mathbf{x}) = \mathbf{x}^t A_q \, \mathbf{x}$$

Exercise 84. *Prove that for a given form q the matrix A_q is unique.*

Exercise 85. *Prove that the set Q_n of all quadratic forms over $\mathbb{R}$ forms a subspace in the space of all functions from $\mathbb{R}^n$ to $\mathbb{R}$. What is the dimension of this space?*

A quadratic form $\mathbf{q}(\mathbf{x})$ is called **diagonal** (or **canonical**) if it is given by

$$\mathbf{q}(\mathbf{x}) = \sum_{i=1}^{n} a_{i,i} \cdot x_i^2.$$

As for binary quadratics we can diagonalize any quadratic for $\mathbf{q}(\mathbf{x})$ by diagonalizing its associated matrix. Diagonal quadratic forms have diagonal matrices as it can be seen in the following example.

Consider the ternary form

$$\mathbf{q}(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2},$$

which has corresponding matrix

$$A_q = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{bmatrix}$$

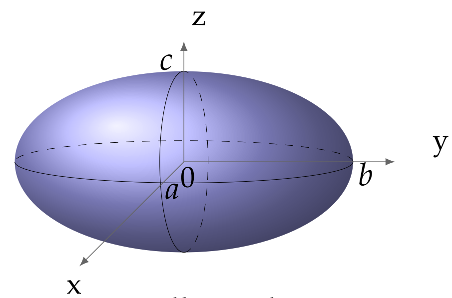


Figure 6.2: Ellipsoid

From Calculus we know that the equation of $\mathbf{q}(x,y,z) = 1$ in $\mathbb{R}^3$ is an ellipsoid as shown in Fig. 6.2.

Next we will describe briefly the steps how to diagonalize a quadratic form. Let $\mathbf{q}(\mathbf{x})$ be a quadratic form. To find its **canonical form** (or **diagonal form**) we perform the following steps:

Algorithm 9. *Input: A quadratic form* $\mathbf{q}(\mathbf{x})$.
Output: A diagonal quadratic form $Q(\mathbf{x})$ *equivalent to* $\mathbf{q}(\mathbf{x})$.

`Step:`*Determine the matrix M associated to* $\mathbf{q}(\mathbf{x})$
`Step:`*Diagonalize the matrix M as* $M = CDC^{-1}$, *where C is the transitional matrix and D is a diagonal matrix.*
`Step:`*The form* $Q(\mathbf{x}) := \mathbf{q}(C\mathbf{x})$ *is a diagonal form and equivalent to* $\mathbf{q}(\mathbf{x})$.

Exercises:

373. *Prove that there is a one to one correspondence between the set of positive definite quadratic forms and the upper half plane*

$$\mathbb{H} = \{z \in \mathbb{C} \mid \Re(z) > 0\}$$

374. *Let*

$$q(x,y) = ax^2 + 2bxy + cy^2$$

be a quadratic form. From methods of multivariable calculus determine the global extrema of this *function. Can you accomplish this via linear algebra methods?*

375. *Transform each of the following quadratic forms into a sum of squares:*

(i) $x^2+2xy+2y^2+4yz+5z^2$
(ii) $x^2-4xy+2xz+4y^2+z^2$
(iii) $xy+yz+zx$
(iv) $x^2-2xy+2xz-2xw+y^2+2yz-4yw+z^2-2w^2$
(v) $x^2+xy+yw$

376. *Transform the following into a diagonal form*

$$\sum_{i=1}^{n} x_i^2+\sum_{i<j} x_i x_j$$

377. *Transform the following into a diagonal form*

$$\sum_{i<j} x_i x_j$$

378. *Show that every principal minor of a positive definite quadratic form is positive.*

379. *A binary form (not necessary quadratic) is a homogenous polynomial*

$$f(x,y)=\sum_{i=0}^{d} a_i x^i y^{d-i}.$$

Is there a way to associate a matrix to $f(x,y)$ when $d>0$?

6.2 Symmetric matrices, Spectral theorem

From Sec. 5.3 we recall that a matrix Q is called orthogonal if its corresponding linear map $\mathbf{x} \to Q\mathbf{x}$ preserves lengths, for all $\mathbf{x} \in \mathbb{R}^n$. As we have seen in Cor 5.3, the following are true for orthogonal matrices:

(i) Q is orthogonal
(ii) $Q^t = Q^{-1}$
(iii) Columns of Q form an orthonormal basis for $\mathbb{R}^n$

The reader should check Sec. 5.3 for details.

Definition 6.4. *A matrix A is called **orthogonally diagonalizable** if there if there is an orthogonal matrix Q and a diagonal matrix D such that*

$$A = Q^t D Q.$$

Expressing a matrix A in the above form would be beneficial for obvious reasons, not only we change the base of the vector space such that A becomes a diagonal matrix, but we do so preserving distances. The natural question is, which matrices are orthogonally diagonalizable? We will answer this question in the remaining of this lecture.

Lemma 6.4. *If A is orthogonally diagonalizable then $A^t = A$.*

Proof. If A is orthogonally diagonalizable then it exists an orthogonal matrix $Q \in GL_n(\mathbb{R})$ such that $A = QDQ^t$, for some diagonal matrix D. Then,

$$A^t = (QDQ^t)^t = (Q^t)^t \cdot D^t \cdot Q^t = A.$$

Hence, $A^t = A$. □

Example 6.3. *Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$. Find Q orthogonal such that $Q^t A Q$ is diagonal.*

Solution: Since for an orthogonal matrix Q we have $Q^t = Q^{-1}$, then we are looking for a matrix such that $Q^{-1}AQ$ is diagonal. We follow the same method as in Sec. 4.3. The characteristic polynomial is

$$\mathbf{char}\,(A, \lambda) = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$$

For $\lambda = 2$ and $\lambda = -3$ we have

$$E_2 = \text{ Span } \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad E_{-3} = \text{ Span } \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} \right)$$

respectively. Then the matrices Q and D are

$$Q = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \qquad \textit{and} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

The matrix Q is not orthogonal, since its columns do not form an orthonormal basis for $\mathbb{R}^2$. We can fix this by taking $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. □

Next, we see how to do this in general.

Theorem 6.1. *Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of a symmetric matrix A belonging to distinct eigenvalues λ_1 and λ_2. Then, $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.*

Proof. Consider the product $\mathbf{v}_1^t A\mathbf{v}_2$. Then we have,

$$\mathbf{v}_1^t A\mathbf{v}_2 = \mathbf{v}_1^t(\lambda_2\mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2)$$

Also,

$$\mathbf{v}_1^t A\mathbf{v}_2 = \mathbf{v}_1^t A^t\mathbf{v}_2 = (A\mathbf{v}_1)^t\mathbf{v}_2 = (\lambda_1\mathbf{v}_1)^t\mathbf{v}_2 = \lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2)$$

Hence, we have

$$\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

Thus, $(\lambda_1 - \lambda_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) = 0$, which implies that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and therefore $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$. □

The above theorem shows that symmetric matrices are special. Indeed, it gets even better.

Theorem 6.2. *If A is a symmetric matrix with real entries, then all its eigenvalues are real.*

Proof. Since complex eigenvalues occur in pairs via the conjugate, consider such a pair $\alpha \pm i\beta$ and the corresponding eigenvectors $v \pm iw$, respectively. Note that

$$(v+iw)^t(v-iw) = \|v\|^2 + \|w\|^2,$$

see Lem. 5.3. Then, we have

$$(v+iw)^t A(v-iw) = (v+iw)^t(\alpha - i\beta)(v-iw) = (\alpha - i\beta)(\|v\|^2 + \|w\|^2)$$

Also,

$$(v+iw)^t A(v-iw) = (A(v+iw))^t(v-iw) = (\alpha + i\beta)(v+iw)^t(v-iw) = (\alpha + i\beta)(\|v\|^2 + \|w\|^2)$$

Hence, $\alpha + i\beta = \alpha - i\beta$ and we are done. □

Next, we consider the main result of this lecture, the so called spectral theorem.

Theorem 6.3 (Spectral theorem). *A matrix A is orthogonally diagonalizable if and only if A is symmetric.*

Proof. Proof. We have already shown that if A is orthogonally diagonalizable, then A is symmetric. Now we prove the converse, that if A is symmetric, then A is orthogonally diagonalizable. We will prove this using induction on n, the size of the matrix.

For $n = 1$, the matrix A is a 1×1 matrix, and the statement is trivially true. We can simply take $Q = [1]$ and $D = A$.

Assume the theorem is true for $k \times k$ symmetric matrices. Now consider an $(k+1) \times (k+1)$ symmetric matrix A. Since A is symmetric, all its eigenvalues are real. Let λ_1 be an eigenvalue of A, and let $\mathbf{v}_1$ be a corresponding eigenvector. We can assume without loss of generality that $\|\mathbf{v}_1\| = 1$. Extend $\mathbf{v}_1$ to an orthonormal basis $\{\mathbf{v}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k+1}\}$ of $\mathbb{R}^{k+1}$. Let $Q_1 = [\mathbf{v}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_{k+1}]$ be the orthogonal matrix formed by these vectors. Then

$$Q_1^t A Q_1 = \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & B \end{bmatrix}$$

where B is a $k \times k$ symmetric matrix. This is because $Q_1^t A Q_1$ is similar to A, and the first column of $Q_1^t A Q_1$ is $Q_1^t A \mathbf{v}_1 = \lambda_1 \mathbf{e}_1$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

By the induction hypothesis, there exists an orthogonal matrix Q_2 such that $Q_2^t B Q_2 = D_2$, where D_2 is a $k \times k$ diagonal matrix. Now let

$$Q = Q_1 \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & D_2 \end{bmatrix}.$$

Then Q is an orthogonal matrix, and

$$\begin{aligned} Q^t A Q &= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2^t \end{bmatrix} Q_1^t A Q_1 \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2^t \end{bmatrix} \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & B \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & Q_2^t B Q_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & D_2 \end{bmatrix} = D. \end{aligned}$$

Thus, $A = QDQ^t$, and A is orthogonally diagonalizable. This completes the proof by induction.

□

6.2.1 Orthogonally Diagonalizing a Matrix

The following algorithm gives a method to orthogonally diagonalize a matrix.

Algorithm 10. ***Orthogonal diagonalization of matrices***

Input: A symmetric matrix A
Output: An orthogonal matrix Q and a diagonal matrix D such that $A = Q^t D Q$.

`Step:`*Compute all eigenvalues*

$$\lambda_1, \dots, \lambda_r,$$

and their multiplicities
`Step:`*For each eigenvalue λ_i determine an orthonormal basis*

$$\mathcal{B} = \{\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,s_i}\}$$

`Step:`*The matrix*

$$Q = \left[\mathbf{v}_{1,1} \,|\, \dots \,|\mathbf{v}_{1,s_1}|\mathbf{v}_{2,1}|\dots|\mathbf{v}_{2,s_2}|\dots|\mathbf{v}_{r,s_r} \right]$$

is the desired orthogonal matrix and the matrix of the eigenvalues is the matrix D.

Notice that the matrix Q is guaranteed to be an orthogonal matrix, since any two eigenvectors in the same eigenspace are orthonormal by construction and two eigenvalues in different eigenspaces are orthogonal from Thm. 6.1. Let us see an example.

Example 6.4. *Orthogonally diagonalize the matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Solution: The characteristic polynomial is **char** $(A, x) = (x-3)(x-1)^3$. For $\lambda_1 = 3$ the corresponding normalized eigenvector is $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda_2 = 1$ we have the following orthonormal eigenbasis

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The transitional orthogonal matrix is

$$Q = [\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3 | \mathbf{v}_4] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Then, $A = Q^t \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} Q$. □

6.2.2 Orthogonally diagonalizing a quadratic form

For any given form $\mathbf{q}(\mathbf{x})$, the corresponding matrix A is a symmetric matrix. Since A is symmetric, then it can be orthogonally diagonalized, say

$$A = Q^t D Q,$$

for some orthogonal matrix Q and a diagonal matrix D. Let $\mathbf{v} = Q\mathbf{x}$. Then we have the following result

Theorem 6.4 (Principal Axes Theorem). *Let $\mathbf{q}(\mathbf{x})$ be a quadratic form and A its corresponding matrix with*

$$A = QDQ^t$$

its orthogonal diagonalization. Then

$$\mathbf{q}(Q\mathbf{x}) = \lambda_1 \mathbf{v}_1^2 + \cdots + \lambda_n \mathbf{v}_n^2,$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

Proof. Let $\mathbf{q}(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$. By the Spectral Theorem there exist matrices Q and D such that Q is orthogonal and D is diagonal with eigenvalues of A as entries in the main diagonal and

$$A = Q^t D Q.$$

Then we have

$$D = Q^{-1} A \left(Q^t\right)^{-1} = Q^t A Q.$$

Let us now compute $\mathbf{q}(Q\mathbf{x})$,

$$\mathbf{q}(Q\mathbf{x}) = (Q\mathbf{x})^t A (Q\mathbf{x}) = \mathbf{x}^t (Q^t A Q)\mathbf{x} = \mathbf{x}^t D\mathbf{x} = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2.$$

This completes the proof. □

The eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are called the **principal axes**. Let us consider a few examples.

Example 6.5. *Find a change of coordinates that transforms the quadratic form*

$$\mathbf{q}(x, y) = 5x^2 + 4xy + 2y^2,$$

into a diagonal form. Sketch the graph of the curve $\mathbf{q}(x) = 1$ before and after the diagonalizing it.

Proof. The corresponding matrix is $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$. Its eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 1$ and the corresponding unit eigenvectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So the matrix for the coordinate change is

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

If we check the change of coordinates we have

$$\mathbf{q}(Q\mathbf{x}) = \mathbf{q}\left(\frac{2x+y}{\sqrt{5}}, \frac{x-2y}{\sqrt{5}} \right) = 6x^2 + y^2,$$

as expected.

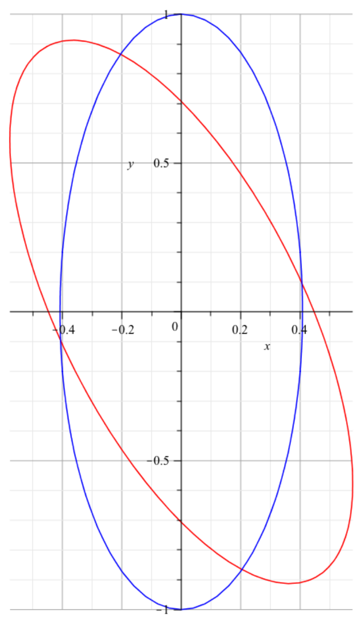


Figure 6.3: The ellipse after the transformation

Example 6.6. *Graph the equation*

$$\mathbf{q}(x,y) = -7x^2 - 12xy + 2y^2 = 1$$

Diagonalize the quadratic form $\mathbf{q}(\mathbf{x})$ *and graph the equation again.*

Proof. The corresponding matrix is $A = \begin{bmatrix} -7 & -6 \\ -6 & 2 \end{bmatrix}$. Its eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -10$ and the corresponding unit eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Normalizing them we have

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

So the matrix for the coordinate change is

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

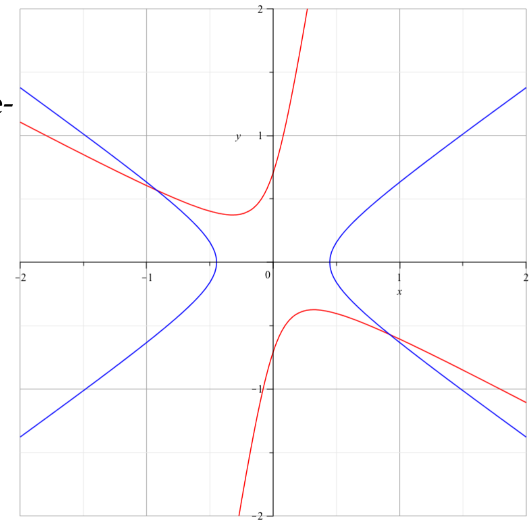


Figure 6.4: The hyperbola after the transformation

If we check the change of coordinates we have

$$\mathbf{q}\,(Q\mathbf{x}) = \mathbf{q}\left(-\frac{1}{2}x + 2y, x + y\right) = 5x^2 - 10y^2.$$

The red graph is the initial one and the blue graph is the graph of the quadratic in the diagonal form. □

Let us see another example.

Exercise 86. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear map such that*

$$T(\mathbf{x}) = A\mathbf{x},$$

for A *a* 2×2 *invertible symmetric matrix. Show that the unit circle is mapped to an ellipse under* T. *Find the lengths of the semi-major and the semi-minor axis of the ellipse in terms of the eigenvalues of* A.

Solution: Since A is invertible, then its eigenvalues λ_1, λ_2 are nonzero and real. Assume that $|\lambda_1| \geq |\lambda_2|$. We denote by $\mathbf{v}_1, \mathbf{v}_2$ the corresponding orthonormal eigenbasis. Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a vector on the unit circle. Then, $\mathbf{x} = \mathbf{v}_1 \cos\theta + \mathbf{v}_2 \sin\theta$. Thus,

$$T(\mathbf{x}) = \cos\theta \cdot T(\mathbf{v}_1) + \sin\theta \cdot T(\mathbf{v}_2) = \cos\theta \cdot (\lambda_1 \mathbf{v}_1) + \sin\theta \cdot (\lambda_2 \mathbf{v}_2)$$

which is on the ellipse with semi-major axis $||\lambda_1 \mathbf{v}_1|| = |\lambda_1|$ and semi-minor axis $||\lambda_2 \mathbf{v}_2|| = |\lambda_2|$. □

Corollary 6.1. *Let C be the curve in $\mathbb{R}^2$ given by the equation*

$$f(x,y) = ax^2 + bxy + cy^2 = 1$$

and $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ *its corresponding matrix. If both eigenvalues of A are positive then C is an ellipse, if they have different signs then C is a hyperbola.*

Proof. The proof is straight forward. The binary form is equivalent to

$$f^Q = \lambda_1 x^2 + \lambda_2 y^2.$$

From high school we know that the corresponding graph is an ellipse if λ_1 and λ_2 have the same sign and a hyperbola if they have different signs. □

Example 6.7. *Find the shape of the equation*

$$\mathbf{q}(\mathbf{x}) = x^2 + 18xy + 6y^2 = 2$$

Proof. The matrix for $\mathbf{q}(\mathbf{x})$ is $A = \begin{bmatrix} 1 & 9 \\ 9 & 6 \end{bmatrix}$ Since its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -3$ and they have different signs then the shape is a hyperbola. □

Exercises:

380. *Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^t$, where*

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

381. *Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^t$, where*

$$A = \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}$$

382. *Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^t$, where*

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

383. *Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^t$, where*

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$$

384. *Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^t$, where*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

385. *Prove that the algebraic multiplicities equal the geometric multiplicities for all the eigenvalues of the following matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

What is the kernel of A?

386. *Find all the eigenvalues and their multiplicities of the following matrix*

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$

387. *Use an orthogonal transformation to transform each of the following into canonical form:*

(i) $2x^2 + y^2 - 4xy - 4xz$

(ii) $x^2 + 2y^2 + 3z^2 - 4xy - 4yz$

(iii) $2x^2 + 2y^2 + 2z^2 + 2w^2 - 4xy + 2xw + 2yz - 4zw$

(iv) $2xy + 2zw$

(v) $x^2 + y^2 + z^2 + w^2 + 2xy - 2xw - 2yz + 2zw$

(vi) $2xy + 2yz - 2yw - 2yz + 2yw + 2zw$

(vii) $x^2 + y^2 + z^2 + w^2 - 2xy + 6xz - 4xw - 4yz + 6yw - 2zw$

(viii) $8xy + 2xw + 2yz + 8yw$

388. *Use an orthogonal transformation to transform the following into a canonical form*

(i) $\sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j$

(ii) $\sum_{i<j} x_i x_j$

389. *Use an orthogonal transformation to transform the following quadratic form into a canonical form*

$$\mathbf{q}(\mathbf{x}) = x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n.$$

390. *Show that a nonsingular matrix* $M \in \mathrm{Mat}\, m \times n(\mathbb{R})$ *can be written as a product of an orthogonal matrix with a symmetric matrix which correspond to a positive definite quadratic form.*

6.3 Quadratic surfaces

A quadratic surface in $\mathbb{R}^3$ is a surface which is given by a degree two polynomial equation $f(x,y,z) = 0$. Hence, a general equation for such surfaces is

$$\boxed{f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0} \tag{6.3}$$

where $a,b,c,d,e,f,g,h,i,j \in \mathbb{R}$ and a,b,c are not simultaneously zero. Without any loss of generality we can assume that Eq. (6.3) is given in the form

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$$

where coefficients a through f are real numbers. Hence, a quadratic surface is the graph of a ternary quadratic form. Now we can use the results from diagonalizing quadratic forms to graph such surfaces.

Consider the curve

$$F(x,y,z) = h.$$

Then this equation can be written in the form

$$\mathbf{x}^t A \mathbf{x} = h,$$

where

$$\mathbf{x} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}, \quad \textit{and} \quad A = \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{e} \\ \mathrm{d} & \mathrm{b} & \mathrm{f} \\ \mathrm{e} & \mathrm{f} & \mathrm{c} \end{bmatrix}$$

A is called the **matrix associated with the quadratic form** $F(x,y,z)$. Sometimes it is useful to rotate the xy-axis such that the equation of the above curve does not have the terms xy, yz, xz. Such quadratic forms are called **diagonal quadratic forms**. This would be equivalent to asking that the associated matrix be diagonal.

By the Principal Axes Theorem, the above equation can be written as

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = h,$$

where λ_1, λ_2, λ_3 are the eigenvalues of A. The **inertia** of A, denoted in (A), is defined as the triple

$$\text{in}\,(A) := (n_1, n_2, n_3)$$

where n_i, $i = 1,2,3$ denotes the number of positive, negative, and zero eigenvalues of A respectively.

Example 6.8. *Let the quadratic form* $\mathbf{q}(x,y,z)$ *be given as below*

$$\mathbf{q}(x,y,z) = 2x^2 + 3y^2 + 3z^2 - 2xy - 2xz - 2yz.$$

Determine its diagonal form.

Proof. The matrix corresponding to $\mathbf{q}(x, y, z)$ is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

The characteristic polynomial is

$$\textbf{char}\,(A, x) = (x-4)(x^2 - 4x + 2)$$

and the eigenvalues

$$\lambda_1 = 4, \ \ \lambda_2 = 2 - \sqrt{2}, \ \ \lambda_3 = 2 + \sqrt{2}.$$

The diagonal form is

$$\mathbf{q}(Q\mathbf{x}) = 4x^2 + (2 - \sqrt{2})y^2 + (2 + \sqrt{2})z^2.$$

□

Next we go over each case for the signature of a ternary quartic and therefore each shape of their graphs. For the rest of this section we assume that $\mathbf{q}(\mathbf{x})$ is a ternary form and A its associated symmetric matrix.

6.3.1 Ellipsoid

Suppose that in $(A) = (3, 0, 0)$. Then the equation can be written as

$$\mathbf{q}(Q\mathbf{x}) = \lambda_1^2 x^2 + \lambda_2^2 y^2 + \lambda_3^2 z^2$$

for eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2$. Hence, the equation $\mathbf{q}(\mathbf{x}) = r$ becomes $\mathbf{q}(Q\mathbf{x}) = r$. For $r > 0$, we usually write this equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

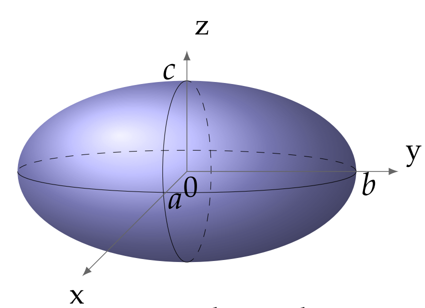


Figure 6.5: Elipsoid

Its graph Fig. 6.5 is called an **elipsoid**. If all eigenvalues of A are equal, we have $a^2 = b^2 = c^2$. In this case we get a sphere. Cross sections of an ellipsoid are ellipses.

If $r < 0$ then the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$$

which is called the **imaginary ellipsoid**. Let P be a plane in $\mathbb{R}^3$. It is obvious that if P is one of the planes xy, xz, or yz then its intersection with an ellipsoid is an ellipse. Is this true in general? Prove your answer.

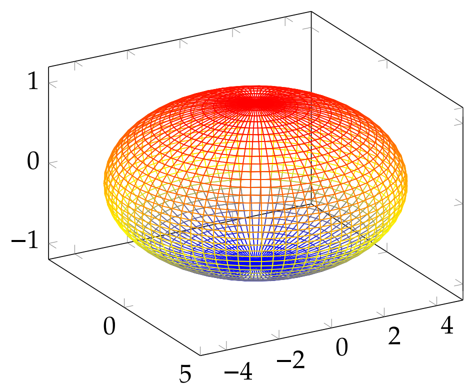
Spheroid

The **spheroid** is a special case of the ellipsoid, when one of the eigenvalues has multiplicity two. This implies that two of the coefficients are equal, say $a^2 = b^2$. Hence, the equation of the spheroid is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \tag{6.4}$$

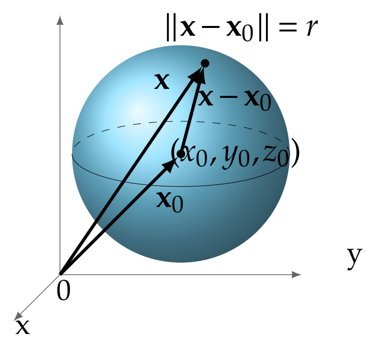
The graph of a spheroid is given in Fig. 6.6. If $c < a$ then the spheroid is called **oblate** and if $c > a$ **prolate**.

Figure 6.6: Spheroid



Spheres

A **sphere** is the case when the quadratic has an eigenvalue of multiplicity three. Hence, the equation is that of the ellipse with $a^2 = b^2 = c^2$. Usually we write $x^2 + y^2 + z^2 = r^2$.



A general sphere can be written in vector equation as

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : ||\mathbf{x} - \mathbf{x}_0|| = r \right\} \tag{6.5}$$

where $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ is a fixed vector in $\mathbb{R}^3$.

6.3.2 Elliptic paraboloid

If in $(A) = (2,0,1)$ then we get an **elliptic paraboloid**. In this case one of the eigenvalues is zero, say $\lambda_3 = 0$. Hence the equation of the surface can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \tag{6.6}$$

The intersection with a plane $z = r > 0$ parallel to the xy-plane is an ellipse. The intersection with the xy-plane is a single point.

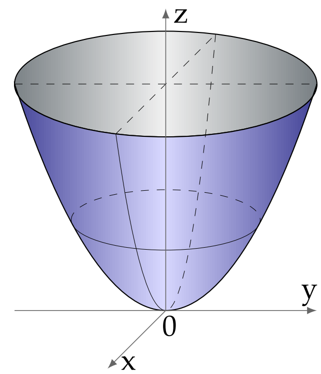


Figure 6.7: Elliptic paraboloid

If $c > 0$ we have a graph as in Fig. 6.7. When $c < 0$, the surface is below the $xy-plane$ pointed downwards. If $a = b$ then the surface is a cylinder.

6.3.3 Hyperbolic paraboloid

Assume that in $(A) = (1,1,1)$. So we have one positive eigenvalue, one negative, and one zero. Thus, we can assume that the surface has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c} \tag{6.7}$$

This is an example of what we call a saddle point in Calculus. In Fig. 6.8 is the graph of the hyperbolic paraboloid $z = y^2 - x^2$, which is a special case for $a = b = 1$ and $c = -1$.

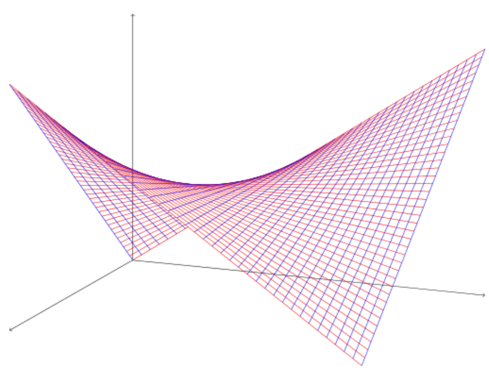


Figure 6.8: Hyperbolic paraboloid

The intersection with the plane $z = z_0$ is $y^2 = -x^2 - z_0$, hence a hyperbola with two sheets. Meanwhile, the intersection with the plane $x = c$ is a curve with equation $z = y^2 - c$, hence a parabola. This justifies the name *hyperbolic paraboloid*.

6.3.4 Hyperboloid of one sheet

Assume that in $(A) = (2,1,0)$. Then we can assume that the surface has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{6.8}$$

Hyperboloid of one sheet every cross section parallel with the xy-plane is an ellipse. Cross sections parallel with the xz and yz planes are hyperbolas; Fig. 6.9, the only cases when this is not true are when $x = \pm a$ and $y = \pm b$; in such cases we get pairs intersecting lines.

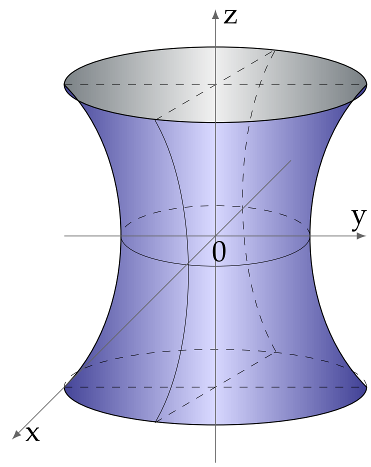


Figure 6.9: Hyperboloid of one sheet

Example 6.9. *Identify the quadratic surface with equation*

$$5x^2 + 16xy + 11y^2 + 20xz - 4yz + 2z^2 = 36.$$

by finding its diagonal form.

Proof. The corresponding matrix is

$$A = \begin{bmatrix} 5 & 8 & 10 \\ 8 & 11 & -2 \\ 10 & -2 & 2 \end{bmatrix}$$

with

$$\textbf{char}\,(A,x) = (x^2 - 81)(x - 18).$$

So the eigenvalues are $\lambda_1 = 18$, $\lambda_2 = 9$,, $\lambda_3 = -9$. The signature is in $(A) = (2,1,0)$ and therefore the surface is a hyperboloid with one sheet.

The normalized eigenvectors are:

$$\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Hence the orthogonal matrix Q is

$$Q = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

Then the change of coordinates is

$$Q\mathbf{x} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y+z \\ 2x-2y-z \\ x+2y-2z \end{bmatrix}$$

Then the diagonalized form is

$$\mathbf{q}(Q\mathbf{x}) = \mathbf{q}(2x+y+z, 2x-2y-z, x+2y-2z) = 18x^2+9y^2-9z^2 = 36.$$

The equation of the surface

$$\frac{x^2}{2}+\frac{y^2}{4}-\frac{z^2}{4}=1.$$

□

Elliptic cone

A special case of the hyperboloid with one sheet is when the constant of the right hand side is $c=0$. The surface in such case has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \tag{6.9}$$

and it is called an **elliptic cone**. Cross intersections with the plane $z=z_0$ are ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{z_0}{c}\right)^2$$

except with the xy- plane when the intersection is a single point. Intersections with planes parallel to the xz or yz planes, are hyperbolas, except the planes xy and yz themselves which intersect the surface along a pair of intersecting lines. The special case when $a^2=b^2$ is called a **circular cone**.

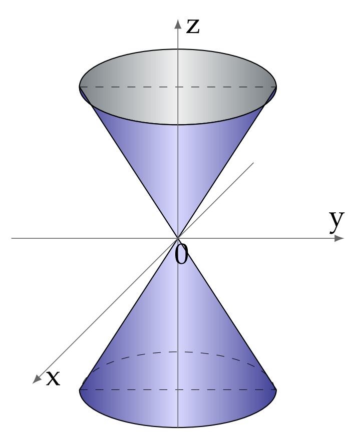


Figure 6.10: Elliptic cone

6.3.5 Hyperboloid of two sheets

In the case when in $(A) = (1,2,0)$ the matrix has one positive eigenvalue and two negative ones. Thus, its equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Every plane parallel with the xy-plane intersects this surface along a hyperbola; Fig. 6.11. With the yz-plane there is no intersection, because for $x = 0$ the equation

$$-\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

has no solutions. With every plane parallel with the yz-plane for which $|x| > a$ the surface intersects along an ellipse.

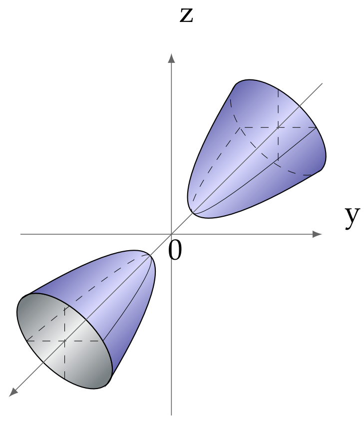


Figure 6.11: Hyperboloid of two sheets

Example 6.10. *Determine the shape of the graph given by the equation*

$$x^2 + 2xy + 2xz + y^2 - 2yz + z^2 + 2x + 6z = -2$$

Solution: *We first consider the quadratic form*

$$\mathbf{q}(\mathbf{x}) = x^2 + 2xy + 2xz + y^2 - 2yz + z^2.$$

The matrix associated to $\mathbf{q}(\mathbf{x})$ *is*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

with characteristic polynomial

$$\textbf{\textit{char}}\,(A,x) = (x+1)(x-2)^2$$

and eigenvalues $\lambda_1 = -1$ *and* $\lambda_2 = \lambda_3 = 2$*. The corresponding eigenbases are*

$$E_1 = \mathit{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } E_2 = \mathit{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Thus, we can take orthonormal bases $\{\mathbf{v}_1\}$ *and* $\{\mathbf{v}_2, \mathbf{v}_3\}$ *as follows*

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

The change of coordinates is

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3]\, \mathbf{x} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Applying this to the initial quadratic equation we have

$$-x^2+2y^2+2z^2+\frac{4}{3}\sqrt{3}x+4\sqrt{2}y-\frac{2}{3}\sqrt{6}z=-2$$

Hence, now we have no xy terms and

$$-\left(x^2-\frac{4}{3}\sqrt{3}x\right)+2\left(y^2+2\sqrt{2}y\right)+2\left(z^2-\frac{1}{3}\sqrt{6}z\right)=-2$$

which becomes

$$-\left(x-\frac{2}{3}\sqrt{3}\right)^2+2(y+2)^2+2\left(z-\frac{1}{6}\right)^2=\frac{35}{9}-2$$

This quadratic has no xy-terms and you can finish it by completing the squares. The equation in the standard form is $-x^2+2y^2+2z^2=-2$*, or*

$$\frac{x^2}{2}-y^2-z^2=1.$$

Hence the graph is a hyperboloid with two sheets.

6.3.6 Parabolic cylinders

Assume that in $(A)=(1,0,2)$. Then the equation in the standard form becomes

$$a^2x^2+by+cz=h.$$

Cross sections of such surfaces are parabolas. Hence, such surfaces are called **parabolic cylinders**.

Example 6.11. *Graph the surface with equation* $z=x^2$

Solution: Since the equation doesn't have any y term, then every plane with equation $y=k$ intersects the graph along the curve $z=x^2$. Hence, it is a parabola.

The graph is presented in Fig. 6.12.

□

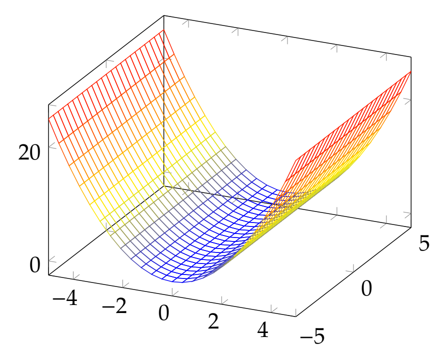
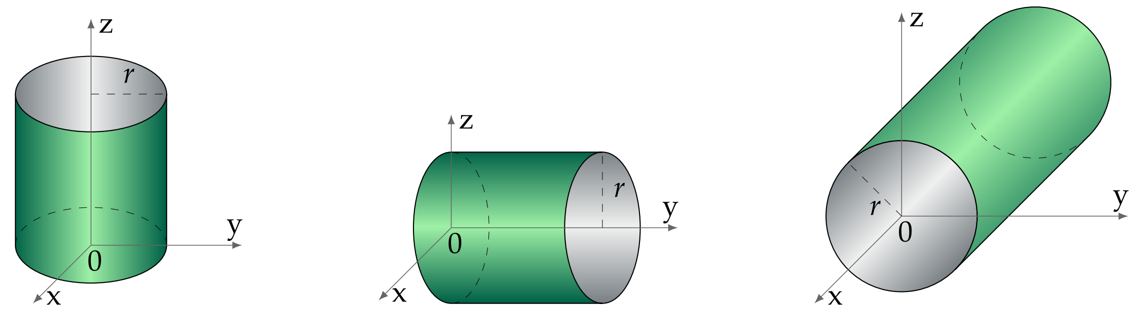


Figure 6.12: The surface $z=x^2$.

Circular cylinders

Perhaps is worth mentioning that a very special case, known from high school, is that of a straight circular cylinder. The inertia is in $(A)=(2,0,1)$ and two of the positive eigenvalues are equal. Hence, we have an equation in the standard form

$$a^2x^2+b^2y^2=r^2.$$



(a) $x^2 + y^2 = r^2$, pÃ«r $z \in \mathbb{R}$ (b) $x^2 + z^2 = r^2$, pÃ«r $y \in \mathbb{R}$ (c) $y^2 + z^2 = r^2$, pÃ«r $x \in \mathbb{R}$

Figure 6.13: Cylinders in $\mathbb{R}^3$

Similarly such equation can be written in any pair of variables, depending on the order of the eigenvalues. We illustrate below.

We summarize in the following theorem.

Theorem 6.5. *Let $F(x,y,z)$ be a ternary quadratic form and A its associated matrix. The following are true:*

(i) *If $in(A) = (3,0,0)$ then the quadratic is an ellipsoid.*
(ii) *If $in(A) = (2,0,1)$ then the quadratic is an elliptic paraboloid.*
(iii) *If $in(A) = (2,1,0)$ then the quadratic is a hyperboloid of one sheet.*
(iv) *If $in(A) = (1,2,0)$ then the quadratic is a hyperboloid of two sheets.*
(v) *If $in(A) = (1,1,1)$ then the quadratic is a hyperbolic paraboloid.*
(vi) *If $in(A) = (1,0,2)$ then the quadratic is a parabolic cylinder.*

Example 6.12. *Determine the change of coordinates to bring the equation*

$$x^2 + y^2 - 2z^2 + 4xy - 2xz + 2yz - x + y + z = 0$$

in the standard form. Write down the standard form and identify the surface.

Solution: We first take

$$\mathbf{q}(\mathbf{x}) = x^2 + y^2 - 2z^2 + 4xy - 2xz + 2yz.$$

The corresponding matrix for this quadratic form is

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

and its characteristic polynomial

$$\textbf{char}\,(A, x) = x(x-3)(x+3).$$

We take the eigenvalues as $\lambda_1 = 0$, $\lambda_2 = -3$, $\lambda_3 = 3$. The corresponding eigenvectors in normal form are

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

Hence, the orthogonal transitional matrix is $Q = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]$. The diagonalized form will be

$$\mathbf{q}'(\mathbf{x}) = 0 \cdot X^2 - 3Y^2 + 3Z^2.$$

The change of coordinates is

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}} \\ \frac{x}{\sqrt{3}} - \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}} \\ \frac{x}{\sqrt{3}} + \frac{y\sqrt{6}}{3} \end{bmatrix}$$

Hence our equation becomes

$$-x + 3y^2 - 3z^2 = 0$$

This is a hyperbolic paraboloid.

□

Exercises:

391. *Use an orthogonal transformation to transform each of the following into canonical form:*

(i) $3x^2 + 4y^2 + 5z^2 + 4xyb - 4yz$
(ii) $2x^2 + 5y^2 + 5z^2 + 4xy - 4xz - 8yz$
(iii) $x^2 - 2y^2 - 2z^2 - 4xy + 4xz + 8yz$
(iv) $5x^2 + 6y^2 + 4z^2 - 4xy - 4xz$
(v) $3x^2 + 6y^2 + 3z^2 - 4xy - 8xz - 4yz$
(vi) $7x^2 + 5y^2 + 3z^2 - 8xy + 8yz$

392. *Let the unit sphere in* $\mathbb{R}^3$ *with equation*

$$x^2 + y^2 + z^2 = 1$$

be given. Using the method of the previous exercise, classify it according to the above list.

393. *Classify the quadratic surface*

$$2x^2 + 4y^2 - 5z^2 + 3xy - 2xz + 4yz = 2.$$

394. *Classify the quadratic surface*

$$x^2 + y^2 - z^2 + 3xy - 5xz + 4yz = 1.$$

395. *Classify the quadratic surface*

$$x^2 + y^2 + z^2 = 1.$$

6.4 Positive definite matrices

We have already seen positive definite quadratics. In this section we will study positive definite and positive semidefinite matrices.

First we recall definitions of positive definite and positive semidefinite for quadratic forms. Let $\mathbf{q}(\mathbf{x})$ be a quadratic form. We say $\mathbf{q}(\mathbf{x})$ is **positive definite** if for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{q}(\mathbf{x}) > 0$. We say that the quadratic form is **positive semidefinite** if $\mathbf{q}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. A quadratic form $\mathbf{q}(\mathbf{x})$ is called **indefinite** if it takes both positive and negative values.

Similarly, a symmetric matrix A is called **positive definite** if the quadratic form $\mathbf{q}(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$ is positive definite and **positive semidefinite** when $\mathbf{q}(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$ is positive semidefinite. A symmetric matrix A is called **indefinite** if the quadratic form $\mathbf{q}(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$ is indefinite. Hence, we have the following:

Lemma 6.5. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a symmetric matrix.*
(i) *A is positive definite if and only if $\mathbf{x}^t A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$*
(ii) *A is positive semidefinite if and only if $\mathbf{x}^t A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$*
(iii) *A is negative definite if and only if $\mathbf{x}^t A\mathbf{x} < 0$ for all $\mathbf{x}^{\mathbb{R}} \in n$*
(iv) *A is negative semidefinite if and only if $\mathbf{x}^t A\mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$*

The following lemma is an easy exercise.

Lemma 6.6. *Let $M \in \mathrm{Mat}_{n\times m}(\mathbb{R})$. The matrix $A = M^t M$ is positive semidefinite. Let*

$$\mathbf{q}(\mathbf{x}) := ||M\mathbf{x}||^2 = \mathbf{x}^t \left(M^t M\right)\mathbf{x}, \quad \textit{for } \mathbf{x} \in \mathbb{R}^n.$$

Then, $\mathbf{q}(\mathbf{x})$ is a positive semidefinite. Moreover, $\mathbf{q}(\mathbf{x})$ is positive definite if and only if $\mathbf{Null}(M) = \{0\}$.

Proof. Notice that for any vector $\mathbf{v} \in \mathbb{R}^n$, we have $||\mathbf{v}||^2 = \mathbf{v}^t \cdot \mathbf{v}$, as multiplication of matrices. Hence,

$$||M\mathbf{x}||^2 = (M\mathbf{x})^t \cdot (M\mathbf{x}) = \mathbf{x}^t (M^t M)\mathbf{x} \geq 0$$

If $\mathbf{Null}(M) = \{0\}$ then $\mathbf{q}(\mathbf{x}) > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$.

□

In Lem. 6.3 we proved when a binary quadratic form is positive definite. Hence, we have the following:

Corollary 6.2. *A symmetric matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if $a > 0$ and $b^2 - ac < 0$.*

However, this doesn't seem quite satisfactory.

Example 6.13. *Prove that $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if and only if its eigenvalues are positive.*

In fact, we can generalize this result to all symmetric matrices.

Theorem 6.6. *The following are true:*
(i) *A symmetric matrix M is positive definite if and only if all of its eigenvalues are positive.*
(ii) *A symmetric matrix M is positive semidefinite if and only if all of its eigenvalues are positive or zero.*

Proof. The proof is rather straightforward. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then in its diagonal form

$$\mathbf{q}(\mathbf{x}) = \mathbf{x}^t A \mathbf{x} = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$$

is positive if all its coefficients are positive. The rest follows. □

Exercise 87. *Prove that the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

is not positive definite.

6.4.1 Principal matrices

Principal matrices (or principal submatrices) arise naturally when studying various properties of a matrix, particularly in the context of positive definiteness. They allow us to break down a larger matrix into smaller, more manageable pieces. The determinants of these principal submatrices provide crucial information about the overall matrix's properties. In the case of positive definiteness, as your theorem states, they provide a necessary and sufficient condition. This is very useful because checking positive definiteness directly can be computationally expensive, but checking the determinants of principal submatrices can be more efficient in some cases. They are also important in other areas like stability theory and network analysis.

Let A be a square matrix. The **principal matrices** of A, denoted by

$$A^{(1)}, A^{(2)}, \cdot A^{(i)}, \cdots, A^{(n)},$$

are the matrices obtained by chopping off all the rows and columns $> i$ of A. We have the following:

Theorem 6.7. *A symmetric $n \times n$ matrix A is positive definite if and only if $\det A^{(m)} > 0$ for all principal submatrices $A^{(m)}$, $m = 1, \ldots, n$.*

Proof. We will prove this by induction on the size of the matrix n.

Base Case ($n = 1$): If A is a 1×1 matrix, then $A = A^{(1)} = [a_{11}]$. A is positive definite if and only if $a_{11} > 0$, which is equivalent to $\det A^{(1)} > 0$.

Inductive Hypothesis: Assume the theorem holds for all $k \times k$ symmetric matrices for some $k \geq 1$.

Inductive Step ($n = k+1$): Let A be a $(k+1) \times (k+1)$ symmetric matrix. We can partition A as follows:

$$A = \begin{bmatrix} a_{11} & w^T \\ w & B \end{bmatrix}$$

where a_{11} is a scalar, w is a $k \times 1$ vector, and B is a $k \times k$ matrix. Note that $A^{(1)} = a_{11}$ and $A^{(m)} = \begin{bmatrix} a_{11} & w_m^T \\ w_m & B_m \end{bmatrix}$ where w_m and B_m are sub-vectors and sub-matrix of w and B respectively.

($\Rightarrow$) Suppose A is positive definite. Then, by definition, $x^T A x > 0$ for all nonzero vectors $x \in \mathbb{R}^{k+1}$. Consider any nonzero vector $y \in \mathbb{R}^m$ for $m \leq k+1$, and let $x = \begin{bmatrix} y \\ 0 \end{bmatrix}$ where 0 is of appropriate size. Then $x^T A x = y^T A^{(m)} y > 0$, since A is positive definite. Thus $A^{(m)}$ is positive definite. By the induction hypothesis, $\det A^{(m)} > 0$ for $m = 1, \ldots, k$. Since A is positive definite, $a_{11} > 0$. We also know that $B - \frac{1}{a_{11}} w w^T$ is positive definite. By the induction hypothesis, we have $\det (B - \frac{1}{a_{11}} w w^T)^{(m)} > 0$ for $m = 1, ..., k$. Now, using the Schur complement, we have:

$$\det A = a_{11} \det (B - \frac{1}{a_{11}} w w^T).$$

Since A is positive definite, we have $\det A > 0$. Also $a_{11} = \det A^{(1)} > 0$. By the induction hypothesis, $\det B = \det A^{(k)} > 0$. Since $A^{(m)}$ is positive definite for $m \leq k$, we also have $\det A^{(m)} > 0$.

($\Leftarrow$) Suppose $\det A^{(m)} > 0$ for all $m = 1, \ldots, k+1$. We want to show that A is positive definite. Since $\det A^{(1)} = a_{11} > 0$, and by the induction hypothesis, B is positive definite. We can write

$$\det A = a_{11} \det (B - \frac{1}{a_{11}} w w^T).$$

Since $\det A > 0$ and $a_{11} > 0$, we must have $\det (B - \frac{1}{a_{11}} w w^T) > 0$. By the induction hypothesis, $B - \frac{1}{a_{11}} w w^T$ is positive definite. Now, for any nonzero vector $x = \begin{bmatrix} x_1 \\ y \end{bmatrix} \in \mathbb{R}^{k+1}$, where $y \in \mathbb{R}^k$, we have:

$$x^T A x = a_{11} x_1^2 + 2 x_1 w^T y + y^T B y = a_{11} \left(x_1 + \frac{w^T y}{a_{11}} \right)^2 + y^T \left(B - \frac{1}{a_{11}} w w^T \right) y$$

Since $a_{11} > 0$ and $B - \frac{1}{a_{11}} w w^T$ is positive definite, $x^T A x > 0$. Thus, A is positive definite.

This completes the proof by induction. □

Example 6.14. *Let A be the matrix given by*

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

Indeed we have

$$A^{(1)} = [1], \; A^{(2)} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Then,

$$\det A^{(1)} = 1, \;\; \det A^{(2)} = 3, \;\; \det A^{(3)} = 8.$$

Therefore, A is positive definite.

Exercise 88 (Sylvester's conditions). *Show that the quadratic form*

$$\mathbf{q}(\mathbf{x}) = \sum_{i,j=1}^{n} a_{i,j} x_i x_j$$

is positive definite if and only if the following inequalities are valid:

$$a_{1,1} > 0, \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} > 0, \ldots, \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix} > 0$$

6.4.2 Min-Max Theorem for Quadratic Forms

The Min-Max Theorem for quadratic forms provides a fundamental connection between the eigenvalues of a real symmetric matrix and the maximum and minimum values of the associated quadratic form on the unit sphere. This theorem has significant applications in various areas, including optimization, eigenvalue problems, and the study of the geometry of quadratic forms. It essentially tells us that the largest and smallest eigenvalues of a matrix correspond to the maximum and minimum values of the quadratic form when the input vector is constrained to have unit length. This connection is crucial for understanding the behavior of quadratic forms and for developing algorithms to compute eigenvalues and eigenvectors.

Theorem 6.8 (Min-Max Theorem for Quadratic Forms). *Let $q(x) = x^T A x$ be a quadratic form, where A is a real symmetric $n \times n$ matrix. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. On the domain $\|\mathbf{x}\| = 1$ the following are true:*

(i) $\lambda_1 \geq \mathbf{q}(\mathbf{x}) \geq \lambda_n$

(ii) $\mathbf{q}(\mathbf{x})$ *achieves a maximum value when* $\mathbf{x}$ *is a unit eigenvector of* λ_1. *Moreover, this maximum value is* $\mathbf{q}(\mathbf{x}) = \lambda_1$.

(iii) $\mathbf{q}(\mathbf{x})$ *achieves a minimum value when* $\mathbf{x}$ *is a unit eigenvector of* λ_n. *Moreover, this minimum value is* $\mathbf{q}(\mathbf{x}) = \lambda_n$.

Proof. Since A is a real symmetric matrix, its eigenvalues are real, and there exists an orthonormal basis of eigenvectors $v_1, v_2, \ldots, v_n$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. That is, $Av_i = \lambda_i v_i$ and $v_i^T v_j = \delta_{ij}$ (Kronecker delta).

Let $\mathbf{x}$ be any vector with $\|\mathbf{x}\| = 1$. We can express $\mathbf{x}$ as a linear combination of the eigenvectors:

$$\mathbf{x} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_i = \mathbf{x}^T v_i$. Because the eigenvectors are orthonormal and $\|\mathbf{x}\| = 1$, we have:

$$1 = \|\mathbf{x}\|^2 = \mathbf{x}^T \mathbf{x} = c_1^2 + c_2^2 + \cdots + c_n^2$$

Now, consider the quadratic form $\mathbf{q}(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$:

$$\begin{aligned}\mathbf{q}(\mathbf{x}) &= (c_1v_1 + \cdots + c_nv_n)^T A(c_1v_1 + \cdots + c_nv_n)\\ &= (c_1v_1 + \cdots + c_nv_n)^T (c_1\lambda_1v_1 + \cdots + c_n\lambda_nv_n)\\ &= c_1^2\lambda_1 + c_2^2\lambda_2 + \cdots + c_n^2\lambda_n\end{aligned}$$

Since $\lambda_1 \geq \lambda_i \geq \lambda_n$ for all i, and $\sum_{i=1}^n c_i^2 = 1$, we have:

$$\mathbf{q}(\mathbf{x}) = \sum_{i=1}^n c_i^2\lambda_i \leq \lambda_1 \sum_{i=1}^n c_i^2 = \lambda_1$$

and

$$\mathbf{q}(\mathbf{x}) = \sum_{i=1}^n c_i^2\lambda_i \geq \lambda_n \sum_{i=1}^n c_i^2 = \lambda_n$$

Thus, $\lambda_1 \geq \mathbf{q}(\mathbf{x}) \geq \lambda_n$.

If $x = v_1$ (the eigenvector corresponding to λ_1), then $c_1 = 1$ and $c_i = 0$ for $i > 1$. In this case:

$$\mathbf{q}(\mathbf{x}) = \sum_{i=1}^n c_i^2\lambda_i = \lambda_1$$

So, the maximum value of $\mathbf{q}(\mathbf{x})$ is λ_1, achieved when $\mathbf{x}$ is a unit eigenvector of λ_1.

Similarly, if $\mathbf{x} = v_n$, then $\mathbf{q}(\mathbf{x}) = \lambda_n$, and the minimum value of $\mathbf{q}(\mathbf{x})$ is λ_n, achieved when $\mathbf{x}$ is a unit eigenvector of λ_n. □

6.5 The Cholesky Factorization

The Cholesky factorization is a cornerstone of numerical linear algebra, providing an elegant and efficient decomposition for a specific class of matrices. It plays a crucial role in solving linear systems, least-squares problems, and Monte Carlo methods, among other applications. This document will explore the Cholesky factorization, its key properties, and its significance.

The central idea behind the Cholesky factorization is to express a given matrix as the product of a lower triangular matrix and its transpose. Formally, we have the following theorem:

Theorem 6.9 (Cholesky Factorization). *Let A be a real, symmetric, positive-definite $n \times n$ matrix. Then there exists a unique lower triangular matrix L with positive diagonal entries such that:*

$$A = LL^T$$

Proof. We will prove this by induction on the size of the matrix n.

Base Case ($n = 1$): If A is a 1×1 matrix, then $A = [a_{11}]$. Since A is positive definite, $a_{11} > 0$. We can simply choose $L = [\sqrt{a_{11}}]$, and then $LL^T = [\sqrt{a_{11}}][\sqrt{a_{11}}] = [a_{11}] = A$.

Inductive Hypothesis: Assume the theorem holds for all $k \times k$ positive-definite matrices for some $k \geq 1$.

Inductive Step ($n = k+1$): Let A be a $(k+1) \times (k+1)$ positive-definite matrix. We can partition A as follows:

$$A = \begin{bmatrix} a_{11} & w^T \\ w & B \end{bmatrix}$$

where a_{11} is a scalar, w is a $k \times 1$ vector, and B is a $k \times k$ matrix. Since A is positive definite, $a_{11} > 0$, and the matrix $B - \frac{1}{a_{11}} w w^T$ is also positive definite.

By the inductive hypothesis, there exists a unique lower triangular matrix L_B with positive diagonal entries such that $B - \frac{1}{a_{11}} w w^T = L_B L_B^T$.

Now, we construct the lower triangular matrix L as follows:

$$L = \begin{bmatrix} l_{11} & 0 \\ v & L_B \end{bmatrix}$$

where $l_{11} = \sqrt{a_{11}}$ and $v = \frac{1}{l_{11}} w$.

Then,

$$\begin{aligned} LL^T &= \begin{bmatrix} l_{11} & 0 \\ v & L_B \end{bmatrix} \begin{bmatrix} l_{11} & v^T \\ 0 & L_B^T \end{bmatrix} \\ &= \begin{bmatrix} l_{11}^2 & l_{11} v^T \\ v l_{11} & v v^T + L_B L_B^T \end{bmatrix} \end{aligned}$$

Substituting our values for l_{11} and v, we get:

$$LL^T = \begin{bmatrix} a_{11} & w^T \\ w & \frac{1}{a_{11}} w w^T + L_B L_B^T \end{bmatrix}$$

Since $L_B L_B^T = B - \frac{1}{a_{11}} w w^T$, we have:

$$LL^T = \begin{bmatrix} a_{11} & w^T \\ w & B \end{bmatrix} = A$$

Thus, we have shown that $A = LL^T$. The uniqueness of L follows from the uniqueness of L_B (by the induction hypothesis) and the fact that the diagonal entries of L are determined by the positive definiteness of A.

This completes the proof by induction. □

Remark 6.2. *The matrix L is referred to as the Cholesky factor of A. It's important to note that the Cholesky factorization exists if and only if the matrix A is symmetric and positive definite. If A is merely positive semidefinite, the decomposition still exists, but L may have zeros on the diagonal, and the uniqueness of L is lost.*

The Cholesky factorization possesses several remarkable properties that contribute to its utility. We encapsulate some of these in the following lemma:

Lemma 6.7.
- *(Uniqueness): If A is positive definite, its Cholesky factor L is unique. This uniqueness is a powerful asset in many applications, ensuring a well-defined decomposition.*
- *(Triangularity): The Cholesky factor L is a lower triangular matrix. This structure significantly reduces computational costs, as operations involving triangular matrices are generally more efficient.*
- *(Positive Diagonal Entries): When A is positive definite, the diagonal entries of its Cholesky factor L are all positive. This property is closely linked to the positive definiteness of A.*
- *(Semidefinite Case): If A is positive semidefinite, the Cholesky factorization still exists, though the resulting L may have zeros on the diagonal, and it's no longer unique.*

6.5.1 Computation

The Cholesky factor L can be computed using the Cholesky algorithm (or square root method). This algorithm directly arises from the equation $A = LL^T$ and calculates the entries of L column by column. Here's a step-by-step description of the procedure:

Cholesky Algorithm (Column-wise):
Given a symmetric, positive-definite matrix A, the Cholesky factor L is computed as follows:

1. **Initialization:** Create a lower triangular matrix L of the same size as A, and initialize all its entries to 0.
2. **Iteration:** For each column $j = 1, 2, \dots, n$:
   (a) **Diagonal element:** Compute the diagonal element l_{jj}:
   $$l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}$$
   (Note: For $j = 1$, the sum is empty, so $l_{11} = \sqrt{a_{11}}$.)
   (b) **Sub-diagonal elements:** For each row $i = j+1, j+2, \dots, n$ below the diagonal in the current column:
   $$l_{ij} = \frac{1}{l_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk} \right)$$

The algorithm proceeds column by column. For each column j, we first compute the diagonal element l_{jj}. Then, we compute the elements l_{ij} below the diagonal in that column. The formulas ensure that the equation $A = LL^T$ is satisfied. The key is that we use previously computed values of L to calculate the current entries.

Example 6.15. *Let's apply the algorithm to the matrix:*

$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

1. ***j = 1:***

   - $l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

2. ***j = 2:***

   - $l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{37 - 6^2} = \sqrt{1} = 1$
   - $l_{21} = \frac{1}{l_{11}} a_{21} = \frac{1}{2} \times 12 = 6$

3. ***j = 3:***

   - $l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{98 - (-8)^2 - 5^2} = \sqrt{9} = 3$
   - $l_{31} = \frac{1}{l_{11}} a_{31} = \frac{1}{2} \times (-16) = -8$
   - $l_{32} = \frac{1}{l_{22}} (a_{32} - l_{31} l_{21}) = \frac{1}{1}(-43 - (-8)(6)) = -43 + 48 = 5$

Thus, we obtain:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$$

The Cholesky factorization finds widespread use in various computational tasks. Some prominent applications include:

- **Solving Linear Systems:** For positive definite matrices A, solving $Ax = b$ is efficiently done by first computing the Cholesky factorization $A = LL^T$, and then solving the two triangular systems $Ly = b$ and $L^T x = y$ using forward and backward substitution.

- **Least Squares Problems:** Cholesky factorization is instrumental in solving normal equations arising in least squares problems.

- **Monte Carlo Methods:** Generating correlated random vectors often relies on the Cholesky decomposition of the covariance matrix.

- **Numerical Optimization:** Certain optimization algorithms employ the Cholesky factorization.

The Cholesky factorization is a powerful and computationally efficient method for decomposing positive-definite (and positive semi-definite) matrices. Its properties and diverse applications underscore its importance in numerical computation.

6.5.2 Inner Products and Positive Definite Matrices

Let M and N be Hermitian positive definite matrices of order m and n respectively. These matrices can be used to define **weighted inner products** (or **inner product induced by M**) on $\mathbb{C}^m$ and $\mathbb{C}^n$ as follows:

$$\langle \mathbf{x}, \mathbf{y} \rangle_M = \mathbf{y}^* M \mathbf{x}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^m$$

and

$$\langle \mathbf{x}, \mathbf{y} \rangle_N = \mathbf{y}^* N \mathbf{x}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$$

Here, $\mathbf{y}^*$ denotes the conjugate transpose of $\mathbf{y}$. It is essential to use the conjugate transpose (rather than just the transpose) when working with complex vectors to ensure that the resulting inner product satisfies the required properties (e.g., conjugate symmetry).

When $M = I$ (the identity matrix), we recover the standard inner product (or dot product):

$$\langle \mathbf{x}, \mathbf{y} \rangle_I = \mathbf{y}^* I \mathbf{x} = \mathbf{y}^* \mathbf{x}$$

Hence the standard inner product can be seen as a special case of a weighted inner product. The reason it's often distinguished is due to its simplicity and its direct geometric interpretation (e.g., relating to lengths and angles in Euclidean space). However, the choice of $M = I$ is essentially a choice of basis. Any positive definite matrix M can define a valid inner product, and in some contexts, these weighted inner products are more natural or relevant.

Why Positive Definiteness is Crucial:

The requirement that M and N be positive definite is *not* arbitrary. It's precisely what guarantees that the defined "inner products" actually satisfy all the axioms of an inner product. These axioms include:

- **Linearity:** $\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle_M = \alpha \langle \mathbf{x}, \mathbf{z} \rangle_M + \beta \langle \mathbf{y}, \mathbf{z} \rangle_M$
- **Conjugate Symmetry:** $\langle \mathbf{x}, \mathbf{y} \rangle_M = \overline{\langle \mathbf{y}, \mathbf{x} \rangle_M}$
- **Positive Definiteness:** $\langle \mathbf{x}, \mathbf{x} \rangle_M > 0$ for $\mathbf{x} \neq \mathbf{0}$

It's the positive definiteness of M that ensures the last property holds. If M were only positive semidefinite, we could have $\langle \mathbf{x}, \mathbf{x} \rangle_M = 0$ for some $\mathbf{x} \neq \mathbf{0}$, violating the inner product axioms.

Remark 6.3. *The term "weighted inner product" is commonly used to describe inner products of the form $\langle \mathbf{x}, \mathbf{y} \rangle_M = \mathbf{y}^* M \mathbf{x}$, where M is a positive definite matrix. However, the use of "weighted" can be somewhat misleading. It might suggest an analogy to weighted averages or sums, where individual components are multiplied by scalar weights. While there is a loose analogy, the matrix M does more than just apply individual weights.*

*The matrix M acts as a transformation that *modifies* the standard inner product. It scales and *mixes* the components of the vectors $\mathbf{x}$ and $\mathbf{y}$ before the inner product is computed. It's not simply a matter of assigning a "weight" to each component; M can introduce correlations between the components. The term "weighted" emphasizes that the contribution of each component is modified by M, but it's crucial to understand that this modification is more complex than a simple scalar multiplication.*

*Some authors prefer the terminology "inner product induced by M" or "inner product defined by M" to avoid the potentially misleading connotations of "weighted." These alternative terms more accurately reflect the role of M as a matrix that *defines* a new inner product space, rather than simply applying weights to an existing one. The matrix M effectively changes the geometry or "metric" of the vector space.*

Therefore, while "weighted inner product" is common parlance, it's important to keep in mind that the "weight" is the matrix M, and its action is a transformation of the vector space, not just a scalar weighting of components.

Exercises:

396. *Show that every principal minor of a positive definite quadratic form is positive.*

6.6 Singular values and singular value decomposition

Consider a transformation $T : \mathbb{R}^n \to \mathbb{R}^M$. We saw that many linear transformations were projections, rotations, reflections or a combination of those. Can we express every transformation as a composition of simpler transformations? If so what would happen to the corresponding matrix of this transformation? This section barely touches in this very important topic.

Let A be an $n \times m$ matrix. Then $A^t A$ is an $m \times m$ symmetric matrix. For any symmetric matrix we can ask if it is positive definite, positive semidefinite, or indefinite. The matrix $A^t A$ is always positive semidefinite; see Lem. 6.6. Hence all eigenvalues of $A^t A$ are positive or zero. Assume that they are

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$$

and denote by

$$\sigma_i = \sqrt{\lambda_i}, \quad i = 1, \dots m.$$

The **singular values of** A are the square roots of the eigenvalues of the $m \times m$ matrix $A^t A$. Usually we write the singular values $\sigma_1, \dots, \sigma_m$ of a matrix in decreasing order

$$\sigma_1 \geq \cdots \geq \sigma_m \geq 0$$

Example 6.16. *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ *be given. Then*

$$M := A^t A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic polynomial of M is

$$\textbf{\textit{char}}\,(M, \lambda) = (\lambda - 1)(\lambda - 4)^2$$

Hence, the singular values of A are $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 2$. □

Exercise 89. *Show that the singular values of the matrix*

$$M = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

are $\sigma_1 = 1$, $\sigma_2 = 4$, $\sigma_3 = 4$. □

The following theorem shows that the number of singular values that are equal to zero is an invariant under any base change.

Theorem 6.10 (Singular values and rank)**.** *If A is an $n \times m$ matrix of rank r, then the singular values $\sigma_1, \dots, \sigma_r$ are nonzero and*

$$\sigma_{r+1} = \cdots = \sigma_m = 0.$$

Proof. □

Example 6.17. *Let be given the matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Find its singular values.

Solution: We have

$$A^t A = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 5 \end{bmatrix}$$

Then,

$$\textbf{char}\,(A^t A, x) = (x-9)(x-1)^3$$

so the eigenvalues are $\lambda_1 = 9$, $\lambda_2 = 1$ and the singular values are $\sigma_1 = 3$, $\sigma_2 = 1$. □

Next we take a non-symmetric matrix

Example 6.18. *Let be given the matrix*

$$A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -3 & 2 \end{bmatrix}$$

Find its singular values.

Solution: We have

$$A^t A = \begin{bmatrix} 7 & 3 & -4 & 7 \\ 3 & 2 & -1 & 4 \\ -4 & -1 & 11 & -7 \\ 7 & 4 & -7 & 14 \end{bmatrix}$$

Then,

$$\textbf{char}\,(A^t A, x) = x^4 - 34x^3 + 253x^2 - 508x + 144$$

This polynomial is irreducible. We can find its eigenvalues numerically and they are

$$24.51545253,\ 6.450415601,\ 2.696419097,\ 0.3377127726,$$

and the singular values

$\sigma_1 = 4.95130816342049$, $\sigma_2 = 2.53976683989367$, $\sigma_3 = 1.64207767698920$, $\sigma_4 = .581130598536912$.

□

6.6.1 Singular value decomposition

Let us get back to our original question. Can we write our linear maps as some kind of composition of simpler linear maps. The next theorem addresses that question.

Theorem 6.11 (Singular Value Decomposition)**.** *Any matrix $A \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ of* **rank** $(A) = r$ *can be written as*

$$A = U\Sigma V^t$$

where U is an orthogonal $n\times n$ matrix, V is an orthogonal $m\times m$ matrix, and Σ is an $n\times m$ matrix whose first r diagonal entries are the nonzero singular values $\sigma_1,\dots,\sigma_r$ of A, while all the other values are zero. If $A \in \mathrm{Mat}_{n\times m}(\mathbb{C})$, then $A = U\Sigma V^*$.

Proof. Let A be a matrix with real entries and **rank** $(A) = r$. We denote its non-zero singular values by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$. Choose an orthonormal basis of eigenvectors $\mathfrak{B} := \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$. Let $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$ for $i = 1,\dots,r$, $V = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_r]$ and $U = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_r]$. Take Σ to be the $n\times m$ diagonal matrix whose first r diagonal entries are the nonzero singular values $\sigma_1,\dots,\sigma_r$ of A, while all the other values are zero. □

From the proof of the theorem we are able to devise a procedure how to compute such singular value decomposition. Let **rank** $(A) = r$ and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$$

the non-zero singular values. Choose an orthonormal basis of eigenvectors

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m.$$

Let

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1,\ \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2, \dots, \mathbf{u}_r = \frac{1}{\sigma_r} A\mathbf{v}_r.$$

Then take

$$V = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_r], \qquad U = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_r]$$

and

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

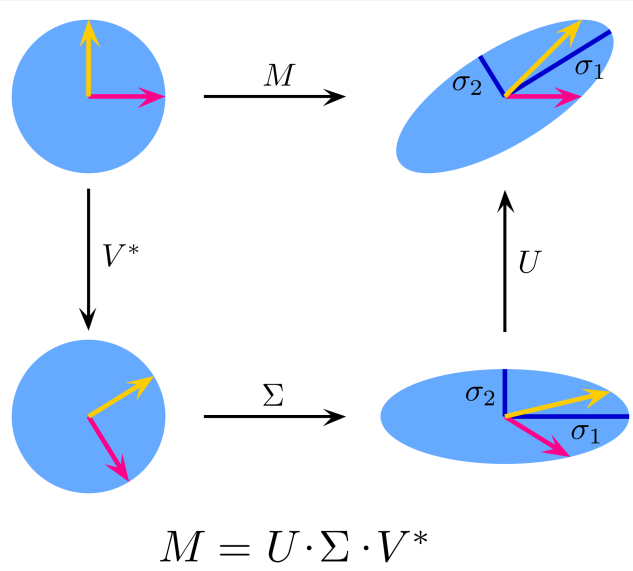


Figure 6.14: Singular value decomposition

Next we see some examples.

Example 6.19. *Find a singular value decomposition for* $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$.

Solution: We have $A^tA = \begin{bmatrix} 85 & -30 \\ -30 & 40 \end{bmatrix}$ with characteristic polynomial

$$\mathbf{char}\,(A^tA, x) = x^2 - 125x + 2500 = (x-100)(x-25).$$

The eigenvalues are $\lambda_1 = 100 \geq \lambda_2 = 25$ and the singular values are $\sigma = 10 \geq \sigma_2 = 5$. The normalized eigenvectors are

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

and

$$V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Then we have

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence, $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Finally $\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$. You should verify whether $A = U\Sigma V^t$. □

Example 6.20. *Find the singular value decomposition of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

6.6.2 Singular values and linear maps

Do the singular values have any geometrical interpretation? We consider the simplest cases.

Theorem 6.12. *Let $L(\mathbf{x}) = A\mathbf{x}$ be an invertible linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. The image of the unit circle under L is an ellipse E. The lengths of the semi-major and semi-minor axes of E are the singular values σ_1 and σ_2 of A, respectively.*

Proof. Exercise. □

Theorem 6.13. *Let $L(\mathbf{x}) = A\mathbf{x}$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then, there exists an orthonormal basis $v_1, \dots, v_m$ or $\mathbb{R}^m$ such that vectors $L(v_1), \dots, L(v_m)$ are orthogonal and their lengths are the singular values $\sigma_1, \dots, \sigma_m$ of A.*

Proof. Exercise. □

The above theorem give a constructive method for the basis $\mathcal{B} = \{v_1, \dots, v_m\}$. We determine all eigenvalues

$$\lambda_1 \geq \cdots \geq \lambda_1 \geq 0$$

for the matrix A^tA and from that an eigenbasis for A^tA.

Exercises:

397. *Find the singular values of*

$$A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}.$$

Explain the results geometrically.

398. *Find the singular values of*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

6.7 Data compression and SVD

Imagine a large digital photograph. It contains a vast amount of data, representing the color and brightness of each tiny point (pixel). Storing and transmitting this data requires significant space and bandwidth. Image compression aims to reduce the data needed to represent the image, making storage and transmission more efficient. Think of it like neatly packing a suitcase—organizing the contents for optimal space utilization.

6.7.1 From Image to Matrix: The Grayscale Case

Consider a grayscale image (black and white with shades of gray). The image is a grid of tiny squares (pixels), each with a brightness value, typically between 0 (black) and 255 (white). We can arrange these values into a matrix. Each row corresponds to a row of pixels, and each column to a column of pixels. The matrix entry at a row and column intersection is the pixel's brightness.

For example, a small 4x4 grayscale image is represented by a 4x4 matrix:

$$\begin{bmatrix} 100 & 120 & 140 & 160 \\ 80 & 100 & 120 & 140 \\ 60 & 80 & 100 & 120 \\ 40 & 60 & 80 & 100 \end{bmatrix}$$

Each number represents the gray level of a corresponding pixel.

6.7.2 From Image to Matrix: The Color Case

Color images are more complex. Instead of one brightness value, we need three: Red, Green, and Blue (RGB). A color image has three matrices, one for each color. Each matrix is like the grayscale example, representing the intensity of that color in each pixel.

6.7.3 How Matrices Help with Compression (The Intuition)

Large image matrices often contain *redundancy*, meaning pixel values are related. Neighboring pixels often have similar colors or brightnesses. Compression techniques exploit this redundancy. Instead of storing every pixel value, we find more efficient representations. This is where techniques like SVD come in. They decompose the matrix, capturing essential information and discarding redundant parts.

6.7.4 Project: Image Compression with Singular Value Decomposition (SVD)

Project Goal:

Understand and implement Singular Value Decomposition (SVD) for image compression, exploring the trade-off between compression ratio and image quality. This project demonstrates a practical application of linear algebra in data compression and introduces fundamental concepts relevant to machine learning.

Introduction:

Digital images are represented as matrices of pixel values. For grayscale images, this is a single matrix, while color images use three matrices (Red, Green, Blue). Singular Value Decomposition (SVD) is a powerful tool for analyzing and manipulating matrices, with important applications in image compression.

SVD for Image Compression: The Big Idea

SVD decomposes a matrix A into three matrices: U, Σ, and V^T, such that $A = U\Sigma V^T$. The singular values (diagonal elements of Σ) represent the "importance" of components in the image. By keeping the k largest singular values, we create a lower-rank approximation, achieving compression. Larger singular values capture more of the image's information.

Python Implementation: Step-by-Step Guide

This section details the specific steps you must take in your Python implementation:

1. **Image Loading:** Use the Pillow (PIL) library to load an image (choose a grayscale image for your initial implementation). Convert the image to a NumPy array. This array will represent your image matrix. You can use code like this:

   Listing 6.1: Image Loading Example

   ```
   from PIL import Image
   import numpy as np

   img = Image.open("your_image.jpg").convert("L") # Grayscale conversion
   img_array = np.array(img)
   ```

2. **SVD Computation:** Use 'numpy.linalg.svd()' to decompose the image matrix.

   Listing 6.2: SVD Example

   ```
   U, S, V = np.linalg.svd(img_array)
   ```

3. **Compression:** The 'S' returned by 'np.linalg.svd()' is a 1D array of singular values. Convert it to a diagonal matrix using 'np.diag(S)'. Choose a value for k (the number of largest singular values to keep). Create a compressed version of Σ by setting all but the top k singular values to zero. Then, create the compressed image matrix:

Listing 6.3: Compression Example

```
S = np.diag(S) # Convert to diagonal matrix
k = 50 # Example value - experiment with this!
S_compressed = np.zeros(S.shape)
S_compressed[:k, :k] = S[:k, :k]
compressed_img = U @ S_compressed @ V # Reconstruct the compressed image
```

4. **Reconstruction:** Reconstruct the compressed image matrix back into an image using the inverse process.

5. **Compression Ratio:** Calculate the compression ratio: $\frac{\text{Original Image Size}}{\text{Compressed Image Size}}$. The compressed image size is related to the number of singular values kept (k).

6. **Image Quality Metrics:** Calculate the Mean Squared Error (MSE) and/or Peak Signal-to-Noise Ratio (PSNR) between the original and compressed images. These metrics will quantify the "loss" due to compression.

Experimentation and Analysis: What to Explore

This is the most important part of the project. Don't just implement the code; *analyze* the results. Here are specific questions to address:

1. **Varying k:** Experiment with different values of k (e.g., 10, 20, 30, 50, 100, etc.). For each k, record the compression ratio and the MSE/PSNR.

2. **Trade-off Analysis:** Create plots showing the relationship between k, compression ratio, and MSE/PSNR. Describe the trade-off you observe. As you increase k, what happens to the compression ratio? What happens to the image quality?

3. **Optimal k?:** Is there a "sweet spot" for k? A value where you get significant compression without sacrificing too much image quality? Justify your answer based on your results.

4. **Image Dependence:** Does the "optimal" k depend on the image itself? (You can explore this further in the bonus challenges).

Bonus Challenges (Optional): Extend to color images (process each RGB channel separately), test on various images, research JPEG/PNG compression, or explore other SVD applications.

Deliverables: Well-documented Python code and a clear, detailed report. The report should include:

- Introduction: Briefly explain SVD and its application to image compression.

- Implementation Details: Describe your Python code, including any challenges you faced. Include your code in an appendix or as a separate file.
- Results: Present your experimental results (tables of data, plots).
- Analysis: Discuss your findings, addressing the questions outlined in the "Experimentation and Analysis" section.
- Conclusion: Summarize your key findings and what you learned from the project.

Grading Rubric: Code correctness and efficiency, report clarity and completeness, analysis depth, visualization quality, and bonus challenge effort.

Review Exercises

399. *True or False?*

(1) ———- *If the system $A\mathbf{x} = \mathbf{y}$ has a unique solution, then A is a square matrix.*
(2) ———- *If A^2 is invertible, then the matrix A is invertible.*
(3) ———- *Let A be a 2 by 2 matrix. Then $\det(2A) = 2\det A$.*
(4) ———- *If the image of an $n \times n$ matrix A is all of $\mathbb{R}^n$, then A must be invertible.*
(5) ———- *Let U and V be subspaces of a vector space W. Then $\dim(U+V) = \dim U + \dim V$.*
(6) ———- *Let U be the set of points $\mathbb{R}^3$ with coordinate $z = 0$. Then U is a subspace of $\mathbb{R}^3$.*
(7) ———- *$T : \mathbb{R}^2 \to \mathbb{R}^2$ is the map projecting any $P \in \mathbb{R}^2$ onto the line $y = x + 1$. Then T is a linear.*
(8) ———- *If A and B are symmetric, then AB must be symmetric.*
(9) ———- *If $A^{-1} = A$, then A must be orthogonal.*
(10) ———- *If A is orthogonal, then A^2 must be orthogonal as well.*
(11) ———- *All nonzero symmetric matrices are invertible.*
(12) ———- *If A is a square matrix such that $A^tA = I$ then A is orthogonal.*
(13) ———- *If $AA^t = A^2$ for a 2×2 matrix A, then A must be symmetric.*
(14) ———- *If A and B are symmetric square matrices, then $ABBA$ is symmetric.*
(15) ———- *If A is orthogonal, then A^t is orthogonal.*
(16) ———- *The entries of an orthogonal matrix are all less than 1.*
(17) ———- *If A is symmetric and S is orthogonal, then $S^{-1}AS$ is symmetric.*
(18) ———- *Let A be a 5×5 matrix. Then $\det(A)^5 = (\det A)^5$.*
(19) ———- *Similar matrices have the same characteristic polynomials.*
(20) ———- *If two matrices have the same characteristic polynomial they must be similar.*
(21) ———- *Similar matrices have the same rank.*
(22) ———- *Similar matrices have the same nullity.*
(23) ———- *Similar matrices have the same eigenvalues with the same algebraic and geometric multiplicities.*
(24) ———- *Similar matrices have the same eigenvectors.*
(25) ———- *If A is similar to B then $\det A = \det B$.*
(26) ———- *If A is similar to B then $\mathrm{tr}(A) = \mathrm{tr}(B)$.*
(27) ———- *The trace of any square matrix is the sum of its eigenvalues.*
(28) ———- *The eigenvalues of any triangular matrix are its diagonal entries.*
(29) ———- *If a matrix is positive definite, then all its eigenvalues must be positive.*
(30) ———- *If A is an invertible symmetric matrix, then A^2 must be positive definite.*

400. *Sketch the curve $x_1^2 + 4x_1x_2 + 4x_2^2 = 1$. Label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.*

401. *Find an orthogonal matrix S and a diagonal matrix D such that $A = S^{-1}DS$, for*

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

402. *Find an SVD (singular value decomposition) for*

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

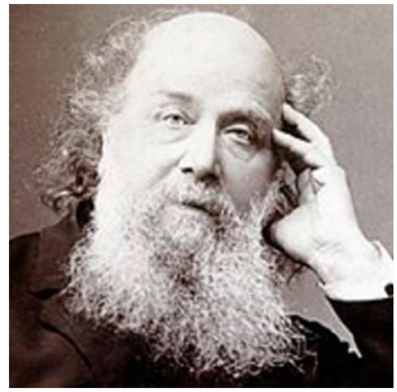
403. *Let $f(x,y,z)$ be given as*

$$f(x,y,z) = x^2 + 2xy + 2xz + y^2 - 2yz + z^2$$

and $c = 2$. Do each of the following tasks. For the items that you choose to use Python you have to show the Python code and the output.

(1) *Find the matrix A associated to $f(x,y,z)$.*
(2) *Determine* ***char*** *(A,λ)*
(3) *Find the eigenvalues and their algebraic multiplicities*
(4) *Determine the inertia of A. Guess the shape of the graph $f(x,y,z) = c$*
(5) *For each eigenvalue find a basis for the corresponding eigenspace*
(6) *Determine the geometric multiplicities for each eigenvalue*
(7) *Determine matrices C and D such that D is diagonal and $A = CDC^{-1}$.*
(8) *Determine a diagonal quadratic form $g(x,y,z)$ equivalent to $f(x,y,z)$*
(9) *Graph $g(x,y,z) = c$*
(10) *Using the Gram-Schmidt algorithm determine an orthonormal basis for each eigenspace*
(11) *Find matrices S and D such that $A = SDS^t$, where S is orthogonal and D is diagonal*
(12) *Determine an orthogonal transformation which transform $f(x,y,z)$ to a diagonal form $h(x,y,z)$*
(13) *Make the algebraic substitutions from the above and determine algebraically $h(x,y,z)$*
(14) *Graph $h(x,y,z) = c$*
(15) *Compare graphs $f(x,y,z) = c$, $g(x,y,z) = c$, and $h(x,y,z) = c$.*
(16) *Determine if A is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.*
(17) *Determine the singular values of A.*
(18) *Determine the singular value decomposition of A.*
(19) *Consider the linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L(\mathbf{x}) = A\mathbf{x}$. Determine and graph the image of the unit sphere S.*
(20) *Do you see any relation between $L(S)$ and singular values of A?*

James Joseph Sylvester (1814 - 1897)



James Joseph was born in London on 3 September 1814, the son of Abraham Joseph, a merchant. James later adopted the surname Sylvester when his older brother did so upon emigration to the United States, a country which at that time required all immigrants to have a given name, a middle name, and a surname.

At the age of 14, Sylvester was a student of Augustus De Morgan at the University of London. His family withdrew him from the University after he was accused of stabbing a fellow student with a knife. Subsequently, he attended the Liverpool Royal Institution.

Sylvester began his study of mathematics at St John's College, Cambridge in 1831, where his tutor was John Hymers. Although his studies were interrupted for almost two years due to a prolonged illness, he nevertheless ranked second in Cambridge's famous mathematical examination, the tripos, for which he sat in 1837. However, Sylvester was not issued a degree, because graduates at that time were required to state their acceptance of the Thirty-nine Articles of the Church of England, and Sylvester could not do so because he was Jewish. For the same reason, he was unable to compete for a Fellowship or obtain a Smith's prize. In 1838, Sylvester became professor of natural philosophy at University College London and in 1839 a Fellow of the Royal Society of London. In 1841, he was awarded a BA and an MA by the University of Dublin (Trinity College). In the same year he moved to the United States to become a professor of mathematics at the University of Virginia, but left after less than four months following a violent encounter with two students he had disciplined. He moved to New York City and began friendships with the Harvard mathematician Benjamin Peirce (father of Charles Sanders Peirce) and the Princeton physicist Joseph Henry. However, he left in November 1843 after being denied appointment as Professor of Mathematics at Columbia College (now University), again for his Judaism, and returned to England.

On his return to England, he was hired in 1844 by the Equity and Law Life Assurance Society for which he developed successful actuarial models and served as de facto CEO, a position that required a law degree. As a result, he studied for the Bar, meeting a fellow British mathematician studying law, Arthur Cayley, with whom he made significant contributions to invariant theory and also matrix theory during a long collaboration. He did not obtain a position teaching university mathematics until 1855, when he was appointed professor of mathematics at the Royal Military Academy, Woolwich, from which he retired in 1869, because the compulsory retirement age was 55. The Woolwich academy initially refused to pay Sylvester his full pension, and only relented after a prolonged public controversy, during which Sylvester took his case to the letters page of The Times.

One of Sylvester's lifelong passions was for poetry; he read and translated works from the original French, German, Italian, Latin and Greek, and many of his mathematical papers contain illustrative quotes from classical poetry. Following his early retirement, Sylvester (1870) published a book entitled The Laws of Verse in which he attempted to codify a set of laws for prosody in poetry.

In 1872, he finally received his B.A. and M.A. from Cambridge, having been denied the degrees due to his being a Jew. In 1876 Sylvester again crossed the Atlantic Ocean to become the inaugural professor of mathematics at the new Johns Hopkins University in Baltimore, Maryland. His salary was $5,000 (quite generous for the time), which he demanded be paid in gold. After negotiation, agreement was reached on a salary that was not paid in gold. In 1878 he founded the American Journal of Mathematics. The only other mathematical journal in the US at that time was the Analyst, which eventually became the Annals of Mathematics. In 1883, he returned to England to take up the Savilian Professor of Geometry at Oxford University. He held this chair until his death, although in 1892 the University appointed a deputy professor to the same chair. He was on the governing body of Abingdon School. Sylvester died at 5 Hertford Street, London on 15 March 1897.

Chapter 7

Optimization

Optimization plays a central role in machine learning and numerical computing. It involves finding the best parameters that minimize (or maximize) a given function, often a loss function in machine learning.

7.1 Gradient Descent

Gradient descent is an iterative optimization algorithm used to minimize functions by moving in the direction of the negative gradient.

Definition: Given a differentiable function $f(\theta)$, gradient descent updates parameters using:

$$\theta_{t+1} = \theta_t - \alpha \nabla f(\theta_t), \tag{7.1}$$

where $\alpha > 0$ is the learning rate.

7.1.1 Batch Gradient Descent

- Computes the gradient over the entire dataset.
- More stable but computationally expensive for large datasets.

7.1.2 Stochastic Gradient Descent (SGD)

- Computes the gradient using a single random data point.
- Faster updates but introduces higher variance in optimization steps.

7.1.3 Mini-Batch Gradient Descent

- Uses a small batch of data to compute gradients.
- Balances efficiency and stability.

7.2 Backpropagation

Backpropagation is a key algorithm for training neural networks, efficiently computing gradients using the chain rule.

Definition: Given a loss function L, backpropagation computes:

$$\frac{\partial L}{\partial W^{(l)}} = \frac{\partial L}{\partial A^{(l+1)}} \cdot \frac{\partial A^{(l+1)}}{\partial Z^{(l+1)}} \cdot \frac{\partial Z^{(l+1)}}{\partial W^{(l)}}. \tag{7.2}$$

It propagates gradients backward through the layers of a neural network, making training feasible for deep networks.

7.3 Convexity

Convexity ensures optimization problems have global minima, simplifying convergence analysis.

7.3.1 Convex Sets

Definition: A set C is convex if for any $x, y \in C$ and $\lambda \in [0,1]$:

$$\lambda x + (1-\lambda)y \in C. \tag{7.3}$$

7.3.2 Convex Functions

Definition: A function f is convex if its domain is convex and for all x, y:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{7.4}$$

Convex functions guarantee efficient optimization due to unique global minima.

7.4 Regularization

Regularization prevents overfitting by adding constraints or penalties to optimization.

7.4.1 L1 Regularization (Lasso)

$$L = L_0 + \lambda \sum |w_i|. \tag{7.5}$$

Encourages sparsity by driving some weights to zero.

7.4.2 L2 Regularization (Ridge)

$$L = L_0 + \lambda \sum w_i^2. \tag{7.6}$$

Prevents large weights, improving generalization.

7.4.3 Dropout

Randomly deactivates neurons during training to improve robustness.

7.5 Optimization Algorithms

Different optimization algorithms improve upon vanilla gradient descent:

7.5.1 Momentum

Accelerates convergence by adding a moving average to updates.

7.5.2 RMSprop

Adapts learning rates using exponentially weighted squared gradients.

7.5.3 Adam (Adaptive Moment Estimation)

Combines momentum and RMSprop, adjusting learning rates adaptively:

$$m_t = \beta_1 m_{t-1} + (1-\beta_1) g_t, \tag{7.7}$$
$$v_t = \beta_2 v_{t-1} + (1-\beta_2) g_t^2. \tag{7.8}$$

Corrects bias using:

$$\hat{m}_t = \frac{m_t}{1-\beta_1^t}, \quad \hat{v}_t = \frac{v_t}{1-\beta_2^t}. \tag{7.9}$$

This chapter has covered key optimization techniques, highlighting their mathematical foundations and importance in machine learning.

Chapter 8

Probability and Statistics for Linear Algebra

8.1 Random Variables and Vectors

In the context of linear algebra, random variables can be scalar or vector-valued. A scalar random variable X maps outcomes from a sample space to real numbers, allowing us to model uncertainty in measurements or predictions. When we extend this to vectors, we get random vectors, like $\mathbf{X} = [X_1, X_2, ..., X_n]^T$, where each component X_i can be treated as a scalar random variable. This vector approach is crucial in multivariate analysis, where we're interested not just in individual outcomes but in how these outcomes relate across dimensions. For instance, in financial models, a vector might represent different asset returns, highlighting their joint behavior rather than just individual stock performance.

8.2 Probability Distributions

Probability distributions describe how probabilities are distributed over the values of random variables. For example, the Gaussian (or normal) distribution is pivotal due to its properties under linear transformations, making it central in linear algebra applications. It's characterized by its mean μ and variance σ^2, with the probability density function $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$. Uniform distributions, where every value in a range has equal probability, also play a role, particularly in simulation or when modeling systems with no particular bias towards any value within a range. Other distributions like the exponential or Poisson might appear in contexts like time between events or counts of events in a fixed interval, respectively, often requiring transformation or linear combination in linear algebra contexts.

8.3 Mean, Variance, and Covariance

The mean of a random variable gives us the expected value, a central tendency measure. For a vector $\mathbf{X}$, the mean vector $\mathbb{E}[\mathbf{X}]$ summarizes the average behavior across dimensions. Variance quantifies the spread of a single variable, while covariance measures how two variables change together. The covariance matrix Σ for a random vector $\mathbf{X}$ encapsulates this relationship for all pairs of components:

$$\Sigma = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T]$$

This matrix is symmetric and positive semi-definite, reflecting the interdependence among variables in a way that's fundamental for operations like principal component analysis or when working with multidimensional data.

8.4 The Multivariate Gaussian Distribution

The multivariate Gaussian distribution extends the univariate Gaussian to vectors, offering a model for data where all components are jointly normally distributed. This distribution is fully described by its mean vector $\boldsymbol{\mu}$ and covariance matrix Σ:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

This form is particularly useful because linear combinations of multivariate Gaussians remain Gaussian, a property exploited in many linear algebra applications from signal processing to statistical inference.

8.5 Maximum Likelihood Estimation (MLE)

Maximum Likelihood Estimation is a method used to estimate the parameters of a statistical model by maximizing the likelihood function. In the context of linear algebra, MLE often translates to solving a system of equations or optimizing a function. For example, when fitting a linear model $y = \mathbf{X}\boldsymbol{\beta} + \epsilon$ where ϵ is Gaussian noise, the MLE for $\boldsymbol{\beta}$ coincides with the least squares solution:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Here, the likelihood maximization problem reduces to minimizing the squared error, linking probability theory with linear algebra through the lens of optimization. This connection is profound as it not only facilitates parameter estimation but also underlies many statistical tests and confidence intervals, crucial in understanding model reliability and significance in applied settings.

Chapter 9

Dimensionality Reduction

Dimensionality reduction is a crucial technique in machine learning and data analysis that aims to reduce the number of variables under consideration while preserving as much information as possible. It helps mitigate the curse of dimensionality, improves computational efficiency, and can aid in data visualization.

9.1 Principal Component Analysis (PCA)

Principal Component Analysis (PCA) is a widely used linear dimensionality reduction technique. It finds a new set of orthogonal basis vectors (principal components) that maximize the variance of the data.

Definition: Given a dataset represented as an $m \times n$ matrix X with mean-centered columns, PCA seeks an orthogonal transformation P such that the transformed data $Y = XP$ has mutually uncorrelated columns ordered by decreasing variance.

Computation: 1. Compute the covariance matrix $C = \frac{1}{m} X^T X$. 2. Compute the eigenvalues and eigenvectors of C. 3. Select the top k eigenvectors to form the transformation matrix P_k. 4. Project data onto the lower-dimensional subspace: $Y = XP_k$.

Example: Consider a dataset with two correlated features. PCA identifies the direction of maximum variance, allowing data to be projected onto a single principal component for better interpretation.

9.2 Linear Discriminant Analysis (LDA)

Linear Discriminant Analysis (LDA) is a supervised dimensionality reduction technique that aims to maximize class separability.

Definition: Given labeled data (X, y), LDA finds a transformation matrix W such that the projected data maximizes the ratio of between-class variance to within-class variance.

Computation: 1. Compute the mean vectors for each class. 2. Compute the within-class scatter matrix S_W and between-class scatter matrix S_B. 3. Solve the generalized eigenvalue problem $S_B w = \lambda S_W w$. 4. Select the top eigenvectors corresponding to the largest eigenvalues. 5. Transform the data using W_k.

Example: In a binary classification problem, LDA finds the best linear boundary to separate the two classes.

9.3 t-SNE and UMAP

9.3.1 t-Distributed Stochastic Neighbor Embedding (t-SNE)

t-SNE is a nonlinear dimensionality reduction technique primarily used for visualization.

Definition: t-SNE models the high-dimensional data structure by minimizing the Kullback-Leibler divergence between probability distributions in high and low dimensions.

Computation: 1. Compute pairwise similarities in the high-dimensional space using Gaussian distributions. 2. Compute pairwise similarities in the low-dimensional space using Student's t-distribution. 3. Minimize the divergence between these distributions using gradient descent.

Example: t-SNE is often used to visualize complex datasets like handwritten digits or word embeddings.

9.3.2 Uniform Manifold Approximation and Projection (UMAP)

UMAP is another nonlinear dimensionality reduction technique that preserves both global and local structures better than t-SNE.

Definition: UMAP constructs a high-dimensional graph representation of the data and optimizes a low-dimensional embedding to approximate it.

Computation: 1. Compute a fuzzy simplicial complex representing data relationships. 2. Optimize a low-dimensional representation using stochastic gradient descent.

Example: UMAP is useful for clustering and exploring large-scale datasets while maintaining structure better than t-SNE.

This chapter provided an overview of several dimensionality reduction techniques, each with unique properties suitable for different applications.

Chapter 10

Linear Models

Historically, the method of least squares was used by Gauss and Legendre to solve problems in astronomy and geodesy. The method was first published by Legendre in 1805 in a paper on methods for determining the orbits of comets. However, Gauss had already used the method of least squares as early as 1801 to determine the orbit of the asteroid Ceres, and he published a paper about it in 1810 after the discovery of the asteroid Pallas. It is in that same paper that Gaussian elimination using pivots is introduced.

10.1 The method of least squares

The method of least squares was first discovered by Gauss in the early 1800's and has been used successively since then in many areas of mathematics and engineering. Consider the following problem:

Problem: *Given a set of data*

x	x_1	x_2	x_3	x_4	$\dots$	x_n
y	y_1	y_2	y_3	y_4	$\dots$	y_n

Find a linear function $y = f(x)$ that best fits this data.

Geometrically two of these points $P_i = (x_i, y_i)$ determine a line. However, we are looking for the line that is "closest" to all the given points. Let us assume that the equation of that line is given by $y = f(x) = ax + b$. Then we have

$$y_i = ax_i + b, \textit{ for } i = 1, \dots n.$$

In matrix notation it becomes

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ax_1+b \\ ax_2+b \\ \cdot \\ \cdot \\ \cdot \\ ax_n+b \end{bmatrix}$$

or we write this as $A\,\mathbf{v} \cong \mathbf{y}$, where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_n & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} ax_1+b \\ ax_2+b \\ \cdot \\ \cdot \\ \cdot \\ ax_n+b \end{bmatrix}.$$

The problem now becomes to determine $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that the **error vector** $A\mathbf{v} - \mathbf{y}$ is minimal. The concept of minimal depends on the type of application. The method of least squares is based on the idea that we require that the magnitude $\left\| A\mathbf{v} - \mathbf{y} \right\|$ is minimal. Denote by $\mathbf{d} := A\mathbf{v} - \mathbf{y}$. Then, $d_i = (ax_i + b) - y_i$. Minimizing $\left\| A\mathbf{v} - \mathbf{y} \right\|$ means minimizing $\left\| A\mathbf{v} - \mathbf{y} \right\|^2$, which means minimizing

$$d_1^2 + d_2^2 + \cdots + d_n^2.$$

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the column vectors of A. The vector $A\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ lies in the space $W =$ Span $(\mathbf{v}_1, \mathbf{v}_2)$. We want to find a vector $\mathbf{v}_0 \in W$ such that the dot product $A\mathbf{v} \cdot (A\mathbf{v}_0 - \mathbf{y}) = 0$ for all $\mathbf{v} \in W$. Then we have

$$\begin{aligned} A\mathbf{v} \cdot (A\mathbf{v}_0 - \mathbf{y}) &= (A\mathbf{v})^t (A\mathbf{v}_0 - \mathbf{y}) = (A\mathbf{v})^t A\mathbf{v}_0 - (A\mathbf{v})^t \mathbf{y} \\ &= \mathbf{v}^t A^t A\mathbf{v}_0 - \mathbf{v}^t A^t \mathbf{y} = \mathbf{v}^t \left(A^t A\mathbf{v}_0 - A^t \mathbf{y} \right) = 0 \end{aligned} \tag{10.1}$$

for all $\mathbf{v} \in W$. Because the dot product is a non-degenerate inner product then

$$A^t A\mathbf{v}_0 - A^t \mathbf{y} = 0$$

and

$$\boxed{\mathbf{v}_0 = (A^t A)^{-1} A^t \mathbf{y}}$$

The matrix $P := (A^t A)^{-1} A^t$ is sometimes called the **projection matrix** of A. Let us see an example.

Example 10.1. *Let the following data be given*

x	*1*	2	2	5
y	2	3	5	7

Find a linear function that best fits the data.

Solution: Then

$$A := \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \\ 5 & 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

We have $A^tA = \begin{bmatrix} 34 & 10 \\ 10 & 4 \end{bmatrix}$ and the least squares solution is

$$\mathbf{v}_0 = (A^tA)^{-1}A^t\mathbf{y} = \frac{1}{6}\begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Hence, the best fitting line to the above data is $y = \frac{7}{6}x + \frac{4}{3}$.

□

As we will see in the next example the least squares method has its limitations. As expected not everything in applications is linear. If we approximate a given data with a linear model then this model might not fit the data very well. In the next example we see that sometimes such an approximation is not close at all to the data.

Example 10.2. *Let the following data be given*

x	1	2	3	4	5
y	2	5	4	7	2

Find a linear function that best fits the data.

Solution: Then

$$A := \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}, \quad \textit{and} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 5 \\ 4 \\ 7 \\ 2 \end{bmatrix}$$

The least squares solution is

$$\mathbf{v}_0 = (A^tA)^{-1}A^t\mathbf{y} = \frac{1}{5}\begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

Hence, the best fitting line to the above data is $y = \frac{x}{5} + \frac{17}{5}$. The graph in Fig. 10.1 presents the graph of the data and of the function. □

In the above examples we found a linear function that best fits a given set of data. However, the method of least squares can be used not only to find linear functions. Next we see how to generalize the method.

10.1.1 The method of least squares for higher degree polynomials

We consider the same problem as in the previous subsection. However, the approximation we want to use is not necessarily linear but a degree n polynomial. It is known that if n points

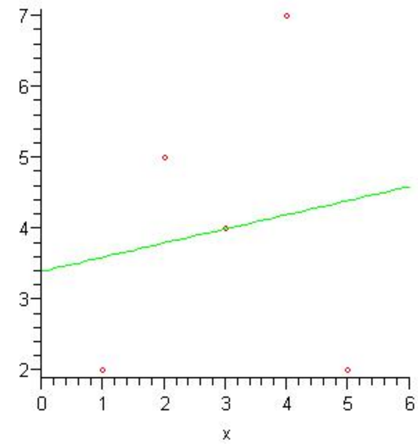


Figure 10.1: Fitting of the above data by the least squares method.

are given on the plane then there is always a degree n polynomial which passes through these points, unless the points are linearly dependent. Thus, for most applications we have r points and want to find a polynomial of degree n that best fits the data for $n < r$. Consider the problem:

Problem: *Given a set of data Find a degree n polynomial*

x	x_1	x_2	x_3	x_4	$\dots$	x_r
y	y_1	y_2	y_3	y_4	$\dots$	y_r

Table 10.1

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

that best fits this data.

We can write this in a matrix form as follows:

$$\begin{bmatrix} x_1^n & \dots & x_1 & 1 \\ x_2^n & \dots & x_2 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_n^n & \dots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ a_0 \end{bmatrix} = \begin{bmatrix} a_n x_1^n + a_{n-1} x_1^{n-1} + \cdots + a_1 x_1 + a_0 \\ a_n x_2^n + a_{n-1} x_2^{n-1} + \cdots + a_1 x_2 + a_0 \\ \cdot \\ \cdot \\ \cdot \\ a_n x_n^n + a_{n-1} x_n^{n-1} + \cdots + a_1 x_n + a_0 \end{bmatrix}$$

As previously we denote this as $A\mathbf{v} = \mathbf{y}$. The least squares solution is

$$\mathbf{v}_0 = (A^t A)^{-1} A^t \mathbf{y}$$

Example 10.3. *Let the following data be given as in the previous example.*

x	1	2	3	4	5
y	2	5	4	7	2

Find a polynomial of degree 2 that best fits the data.

Solution: Then

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix}, \quad and \quad \mathbf{y} = \begin{bmatrix} 2 \\ 5 \\ 4 \\ 7 \\ 2 \end{bmatrix}$$

The least squares solution is

$$\mathbf{v}_0 = (A^t A)^{-1} A^t \mathbf{y} = \left[-\frac{6}{7}, \frac{187}{35}, -\frac{13}{5} \right]^t$$

Hence, the best fitting degree 2 polynomial to the above data is

$$y = -\frac{6}{7}x^2 + \frac{187}{35}x - \frac{13}{5}$$

The graph of Fig. 10.2 presents the graph of the data and of the function. Notice how we get a better approximation than in the linear case.

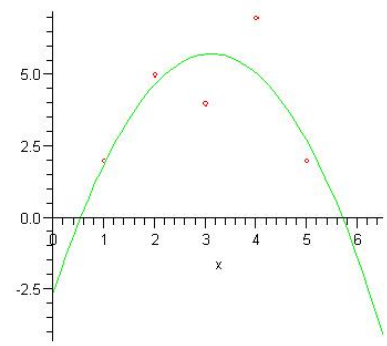


Figure 10.2: Quadratic approximation

□

Example 10.4. *Find degree 3 and 4 polynomials that approximate the data of the previous example.*

Solution: For a degree 3 polynomial we have

$$y = -\frac{1}{3}x^3 + \frac{15}{7}x^2 - \frac{53}{21}x + 3$$

The graph is presented in Fig. 10.3. Compare this with degree 1 and 2 polynomials to see that we get a better fit.

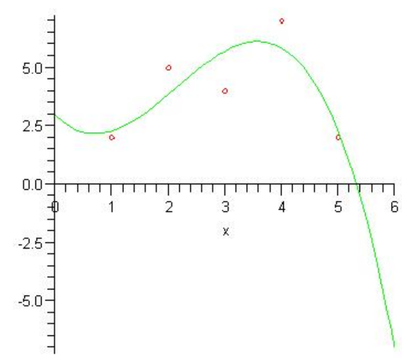


Figure 10.3: Cubic fitting

Since we have four points on the plane, then by using a degree 4 polynomial we are able to find a polynomial that will pass through the points. The least squares method will find this unique solution when it exists. In this case, the degree 4 polynomial that fits the data is

$$y = -\frac{5}{6}x^4 + \frac{29}{3}x^3 - \frac{235}{6}x^2 + \frac{196}{3}x - 33$$

and the graph is presented in Fig. 10.4.

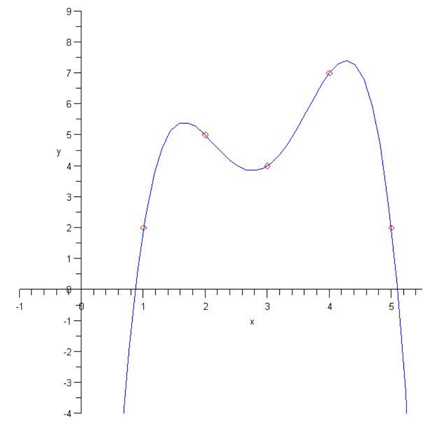


Figure 10.4: Quartic fitting

Remark 10.1. *$(A^tA)^{-1}$ exists if A has independent column vectors. Thus, we have a unique least squares solution if* ***null*** $(A) = 0$. □

The least squares method can be used for many other applications.

10.1.2 Least Squares and the Pseudo-Inverse

The method of least squares can be used for "solving" an overdetermined system of linear equations $A\mathbf{x} = \mathbf{b}$, i.e., a system in which A is a rectangular $m \times n$ matrix with more equations than unknowns (when $m > n$).

Consider the following: Find a vector $\mathbf{x}_0$ which minimizes

$$||A\mathbf{x} - b||^2$$

When such vector $\mathbf{x}_0$ exists we call it a **least squares solution for the Euclidean norm**.

Theorem 10.1. *Every linear system $A\mathbf{x} = b$, where A is an $m \times n$ matrix, has a unique least squares solution $\mathbf{x}_0$ of smallest norm.*

Proof. Let $\mathcal{R}(A)$ denote the column space of A, which is a subspace of $\mathbb{R}^m$. By the Projection Theorem, for any $\mathbf{b} \in \mathbb{R}^m$, there exists a unique vector $p \in \mathcal{R}(A)$ such that $\mathbf{b} - p$ is orthogonal to every vector in $\mathcal{R}(A)$. This vector p is the orthogonal projection of $\mathbf{b}$ onto $\mathcal{R}(A)$.

Since $p \in \mathcal{R}(A)$, there exists at least one vector $\mathbf{x}_0 \in \mathbb{R}^n$ such that $A\mathbf{x}_0 = p$.

Now, let's show that this $\mathbf{x}_0$ minimizes $||A\mathbf{x} - b||^2$. For any other vector $\mathbf{x} \in \mathbb{R}^n$, we have:

$$||A\mathbf{x} - b||^2 = \left\|(A\mathbf{x} - p) + (p - b)\right\|^2$$

Since $p = A\mathbf{x}_0 \in \mathcal{R}(A)$, the vector $A\mathbf{x} - p = A(\mathbf{x} - \mathbf{x}_0)$ is also in $\mathcal{R}(A)$. By the Projection Theorem, $p - b = -(b - p)$ is orthogonal to every vector in $\mathcal{R}(A)$, including $A\mathbf{x} - p$. Therefore, by the Pythagorean theorem:

$$\left\|(A\mathbf{x} - p) + (p - b)\right\|^2 = \left\|A\mathbf{x} - p\right\|^2 + \left\|p - b\right\|^2$$

Since $\left\|A\mathbf{x} - p\right\|^2 \geq 0$, we have:

$$||A\mathbf{x} - b||^2 = \left\|A\mathbf{x} - p\right\|^2 + \left\|p - b\right\|^2 \geq \left\|p - b\right\|^2$$

Equality holds when $\left\|A\mathbf{x}-p\right\|^2 = 0$, which means $A\mathbf{x}-p = \mathbf{0}$, or $A\mathbf{x} = p = A\mathbf{x}_0$. Thus, $\mathbf{x}_0$ is a least squares solution, and the minimum value of $\|A\mathbf{x}-b\|^2$ is $\|A\mathbf{x}_0-b\|^2 = \left\|p-b\right\|^2$.

We have shown that any least squares solution $\mathbf{x}_0$ must satisfy $A\mathbf{x}_0 = p$, where p is the orthogonal projection of $\mathbf{b}$ onto $\mathcal{R}(A)$.

Consider the normal equations:

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

If $A\mathbf{x}_0 = p$, and since $\mathbf{b}-p$ is orthogonal to $\mathcal{R}(A)$, it is orthogonal to the columns of A. This means $A^T(\mathbf{b}-p) = \mathbf{0}$, so $A^T(\mathbf{b}-A\mathbf{x}_0) = \mathbf{0}$, which gives $A^T\mathbf{b}-A^TA\mathbf{x}_0 = \mathbf{0}$, or $A^TA\mathbf{x}_0 = A^T\mathbf{b}$. Thus, any least squares solution must satisfy the normal equations.

Now, let's consider the uniqueness and the smallest norm. Let $\mathcal{N}(A)$ be the null space of A, i.e., $\{\mathbf{z} \in \mathbb{R}^n \mid A\mathbf{z} = \mathbf{0}\}$. If $\mathbf{x}_0$ is a least squares solution ($A\mathbf{x}_0 = p$), then for any $\mathbf{z} \in \mathcal{N}(A)$, $\mathbf{x}_0+\mathbf{z}$ is also a least squares solution because $A(\mathbf{x}_0+\mathbf{z}) = A\mathbf{x}_0 + A\mathbf{z} = p+\mathbf{0} = p$.

To find the least squares solution of smallest norm, we want a solution that is orthogonal to the null space $\mathcal{N}(A)$. By the Fundamental Subspaces Theorem, we know that $\mathcal{R}(A^T) = \mathcal{N}(A)^\perp$. Therefore, the least squares solution of smallest norm must lie in the row space of A, $\mathcal{R}(A^T)$.

Let $\mathbf{x}_0$ be a least squares solution with the smallest norm. Then $\mathbf{x}_0 \in \mathcal{R}(A^T)$, so there exists a vector $\mathbf{y}$ such that $\mathbf{x}_0 = A^T\mathbf{y}$. Since $\mathbf{x}_0$ is a least squares solution, it satisfies the normal equations:

$$A^TA\mathbf{x}_0 = A^T\mathbf{b}$$

Substituting $\mathbf{x}_0 = A^T\mathbf{y}$, we get:

$$A^TA(A^T\mathbf{y}) = A^T\mathbf{b}$$

Now, consider any other least squares solution $\mathbf{x}_1$. We know that $A\mathbf{x}_1 = p = A\mathbf{x}_0$, so $A(\mathbf{x}_1-\mathbf{x}_0) = \mathbf{0}$, which means $\mathbf{x}_1-\mathbf{x}_0 \in \mathcal{N}(A)$. If $\mathbf{x}_1$ also has the smallest norm, then $\mathbf{x}_1$ must also be in $\mathcal{R}(A^T)$. Since $\mathbf{x}_1-\mathbf{x}_0 \in \mathcal{N}(A)$ and $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{R}(A^T) = \mathcal{N}(A)^\perp$, it follows that $\mathbf{x}_1-\mathbf{x}_0$ must be orthogonal to itself, which implies $\mathbf{x}_1-\mathbf{x}_0 = \mathbf{0}$, so $\mathbf{x}_1 = \mathbf{x}_0$. This shows that the least squares solution of smallest norm is unique.

Alternatively, we can argue about the uniqueness within $\mathcal{R}(A^T)$ directly from the normal equations. Consider two least squares solutions $\mathbf{x}_0$ and $\mathbf{x}_1$ in $\mathcal{R}(A^T)$. They both satisfy $A^TA\mathbf{x} = A^T\mathbf{b}$. Let $\mathbf{w} = \mathbf{x}_1-\mathbf{x}_0$. Then $A^TA\mathbf{w} = A^TA\mathbf{x}_1 - A^TA\mathbf{x}_0 = A^T\mathbf{b}-A^T\mathbf{b} = \mathbf{0}$. This means $\mathbf{w} \in \mathcal{N}(A^TA)$. We know that $\mathcal{N}(A^TA) = \mathcal{N}(A)$. Since both $\mathbf{x}_0$ and $\mathbf{x}_1$ are in $\mathcal{R}(A^T)$, their difference $\mathbf{w} = \mathbf{x}_1-\mathbf{x}_0$ is also in $\mathcal{R}(A^T)$. Thus, $\mathbf{w} \in \mathcal{N}(A) \cap \mathcal{R}(A^T) = \{\mathbf{0}\}$, which implies $\mathbf{x}_1 = \mathbf{x}_0$. Therefore, there is a unique least squares solution in $\mathcal{R}(A^T)$, which is the least squares solution of smallest norm.

□

Remark 10.2. *Let A^+ denote the pseudo-inverse of A (cf. Thm. 6.11). Then the least squares solution of smallest norm of the linear system $A\mathbf{x} = \mathbf{b}$, is given by*

$$\mathbf{x}_0 = A^+\mathbf{b}$$

Of course, when A has full rank, then the pseudo-inverse is given by

$$A^+ = (A^tA)^{-1}A^t$$

Example 10.5. *Find the least squares solution to the system*

$$\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 + x_3 = 3 \\ 3x_1 + 2x_2 - 5x_3 = 1 \\ 2x_1 + x_2 - x_3 = 3 \end{cases}$$

Solution: *We have*

$$A\mathbf{x} = \mathbf{y}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 3 & 2 & -5 \\ 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 3 \end{bmatrix}.$$

The least squares solution is

$$\mathbf{v}_0 = (A^t A)^{-1} A^t \mathbf{y} = \begin{bmatrix} \frac{617}{275} \\ \frac{-21}{11} \\ \frac{29}{275} \end{bmatrix}.$$

Then the orthogonal projection of $\mathbf{y}$ *on the column space of* A *is the vector* $A\mathbf{v}_0$ *given by*

$$A\mathbf{v}_0 = \begin{bmatrix} \frac{1142}{275} \\ \frac{179}{55} \\ \frac{331}{275} \\ \frac{123}{55} \end{bmatrix} = \begin{bmatrix} 4.152727273 \\ 3.254545455 \\ 1.203636364 \\ 2.236363636 \end{bmatrix}.$$

□

Exercises:

404. *Let the following data be given*

x	1	2	3	4	5
y	8	13	18	23	28

Find a linear function that best fits the data.

405. *Let the following data be given*

x	0	1	3	2	-2
y	2	1	-1	-5	4

Find a linear function that best fits the data.

406. *Find a degree 2 polynomial that best fits the following data:*

x	0	1	3	2	5
y	-2	-1	2	4	2

407. *Find a degree 3 polynomial that best fits the* *data:*

x	0	1	3	5	-2
y	2	3	5	0	-4

408. *Find the least squares solution to the system*

$$\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 3 \\ 3x_1 + 2x_2 = 1 \end{cases}$$

409. *Find the least squares solution to the system*

$$\begin{cases} x_1 - 11x_2 = 1 \\ 3x_1 + x_3 = 2 \\ x_1 + 2x_2 = 1 \\ 2x_1 + x_2 - x_3 = 31 \end{cases}$$

and the corresponding orthogonal projection.

410. *Find the least squares solution to the system*

$$\begin{cases} 5x_1 - 12x_2 = 4 \\ x_1 + 3x_2 = -2 \\ 6x_1 + 2x_2 = -1 \end{cases}$$

and the corresponding orthogonal projection.

411. *Find the least squares solution to the system*

$$\begin{cases} 3x_1 - x_2 + 3x_3 = 4 \\ 3x_1 + 7x_2 + x_3 = 3 \\ 3x_1 + 2x_2 - x_3 = 21 \\ 2x_1 + x_2 - x_3 = 4 \end{cases}$$

and the corresponding orthogonal projection.

10.2 Linear regression and its Connection to Least Squares

Suppose there are m data points in $\mathbb{R}^n$ and t targets.

Store the predictors in the matrix $X \in \mathbb{R}^{m\times n}$, with each row corresponding to one data point and store the targets in the matrix $Y \in \mathbb{R}^{m\times t}$, corresponding to X (i.e., the rows of X and Y are one-to-one mapped), say

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ \vdots & & & \\ x_{m,1} & x_{m,2} & \dots & x_{m,n} \end{bmatrix}, \quad Y = \begin{bmatrix} y_{1,1} & y_{1,2} & \dots & y_{1,t} \\ \vdots & & & \\ y_{m,1} & y_{m,2} & \dots & y_{m,t} \end{bmatrix},$$

In the lecture, we only consider a single target variable. Here Y consists of t target variables. Mathematically, linear regression solves this optimization problem:

Example 10.6. *Given X and Y, find the best $W \in \mathbb{R}^{n\times t}$ and $b \in \mathbb{R}^t$ such that*

$$Y \approx XW + \mathbf{1}b^T$$

Here, W is a $n\times t$ weight matrix, b is a bias vector of length t, and $\mathbf{1} \in \mathbb{R}^m$ is a column vector of ones.

Linear regression then solves this optimization problem: Find W^*, b^* such that

$$\min_{W,b} \|Y - (XW + \mathbf{1}b^T)\|_F^2$$

where $\|\cdot\|_F$ denotes the Frobenius norm. This is equivalent to minimizing the sum of the squared Euclidean norms of the columns of the residual matrix $Y - (XW + \mathbf{1}b^T)$.

Let $\mathbf{X} = [X, \mathbf{1}]$, i.e., append a column of 1 to the last column of X. Here $\mathbf{1} \in \mathbb{R}^m$ is a column vector of length m with every element being 1. Also, let $\mathcal{W} = \begin{bmatrix} W \\ b^T \end{bmatrix}$, where $\mathcal{W}$ is a $(n+1)\times t$ matrix, with the first n rows corresponding to W and the last row being the transpose of b. The linear regression model can then be written as $Y \approx \mathbf{X}\mathcal{W}$. The optimization problem becomes:

$$\min_{\mathcal{W}} \|Y - \mathbf{X}\mathcal{W}\|_F^2$$

This is a standard linear least squares problem. For each target variable (each column of Y), we are trying to find a linear combination of the columns of $\mathbf{X}$ that best approximates it in the least squares sense.

Consider a single target variable ($t = 1$), where $Y = \mathbf{y} \in \mathbb{R}^m$ and $\mathcal{W} = \mathbf{w} = \begin{bmatrix} W \\ b \end{bmatrix} \in \mathbb{R}^{n+1}$. The problem is to minimize $\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2$. From the theorem on least squares solutions, the optimal $\mathbf{w}^*$ satisfies the normal equations:

$$\mathbf{X}^T\mathbf{X}\mathbf{w}^* = \mathbf{X}^T\mathbf{y}$$

If $\mathbf{X}^T\mathbf{X}$ is invertible, the unique least squares solution is given by:

$$\mathbf{w}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

For multiple target variables ($t > 1$), we can apply this column-wise. Let $\mathbf{y}_j$ be the j-th column of Y and $\mathbf{w}_j$ be the j-th column of $\mathcal{W}$. Then, for each $j = 1, \dots, t$, the optimal $\mathbf{w}_j^*$ satisfies:

$$\mathbf{X}^T \mathbf{X} \mathbf{w}_j^* = \mathbf{X}^T \mathbf{y}_j$$

Combining these solutions into the matrix $\mathcal{W}^* = [\mathbf{w}_1^*, \dots, \mathbf{w}_t^*]$, we get the closed-form solution for linear regression:

$$\mathcal{W}^* = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T Y$$

You can then read W^* (the first n rows of $\mathcal{W}^*$) and b^{*T} (the last row of $\mathcal{W}^*$) directly from $\mathcal{W}^*$.

Let us see an example in $\mathbb{R}^3$ and some visual illustration of it.

Exercise 90. *Let a set of data in $\mathbb{R}^3$ be given by the points $P_1 = (1,2,3)$, $P_2 = (-1,1,1)$, $P_3 = (2,3,4)$, $P_4 = (-2,1,-3)$, $P_5 = (1,1,-1)$, $P_6 = (2,3,-2)$, $P_7 = (5,5,5)$, $P_8 = (-3,4,6)$.*

Find the regression plane for this data.

Solution: We want a plane with equation

$$ax + by + cz + d = 0$$

which best fits the data. So we want to determine the vector $\mathbf{v} = [a,b,c,d]^t$. We can consider the third coordinate as the target variable and the first two as predictors. Our model is

$$z \approx w_1 x + w_2 y + b,$$

which can be written as $w_1 x + w_2 y - z + b \approx 0$.

The matrices X and Y are given by

$$X = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 3 \\ -2 & 1 \\ 1 & 1 \\ 2 & 3 \\ 5 & 5 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -3 \\ -1 \\ -2 \\ 5 \\ 6 \end{bmatrix}$$

Then $\mathbf{X} = [X, \mathbf{1}] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 5 & 1 \\ -3 & 4 & 1 \end{bmatrix}$.

The augmented weight vector $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix}$ is given by $\mathbf{w} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T Y$. We computed earlier:

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 40 & 42 & 7 \\ 42 & 50 & 21 \\ 7 & 21 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{X}^T Y = \begin{bmatrix} 49 \\ 60 \\ 13 \end{bmatrix}$$

Solving the system $\begin{bmatrix} 40 & 42 & 7 \\ 42 & 50 & 21 \\ 7 & 21 & 8 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix} = \begin{bmatrix} 49 \\ 60 \\ 13 \end{bmatrix}$ will give the coefficients of the regression plane $z = w_1 x + w_2 y + b$.

Using a computational tool to solve this system, we find (approximately):

$$w_1 \approx 0.86, \quad w_2 \approx 0.51, \quad b \approx -0.82$$

So the regression plane is approximately given by $z = 0.86x + 0.51y - 0.82$, or $0.86x + 0.51y - z - 0.82 = 0$.

Then $\mathbf{v} = (X^t X)^{-1} X^t Y = \begin{bmatrix} 0.86 \\ 0.51 \\ -1 \\ -0.82 \end{bmatrix}$ (note the coefficient of z is -1 in our formulation). The data and the plane are given in the picture below. □

Exercise 91. *Consider a set of 6 data points in $\mathbb{R}^5$, where the first 4 components are the predictors and the 5th component is the target variable:*

$$\begin{aligned} P_1 &= (1,0,2,-1,3) \\ P_2 &= (0,1,-1,2,1) \\ P_3 &= (2,-1,1,0,4) \\ P_4 &= (-1,2,0,1,-2) \\ P_5 &= (1,1,1,1,2) \\ P_6 &= (0,0,-2,-2,-1) \end{aligned}$$

Find the linear regression model that predicts the 5th component (the target) based on the first 4 components (the predictors).

Solution: Let the predictor variables be x_1, x_2, x_3, x_4 and the target variable be y. We want to find a linear model of the form:

$$y \approx w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 + b$$

We can set up the matrices X and Y:

$$X = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \in \mathbb{R}^{6\times 4}, \quad Y = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -2 \\ 2 \\ -1 \end{bmatrix} \in \mathbb{R}^{6\times 1}$$

Now, we form the augmented predictor matrix $\mathbf{X}$ by adding a column of ones:

$$\mathbf{X} = [X, \mathbf{1}] = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 2 & -1 & 1 & 0 & 1 \\ -1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & 1 \end{bmatrix} \in \mathbb{R}^{6\times 5}$$

The augmented weight vector $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ b \end{bmatrix}$ is the least squares solution to $\mathbf{X}\mathbf{w} \approx Y$, given by $\mathbf{w} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T Y$, assuming $(\mathbf{X}^T\mathbf{X})$ is invertible.

Let's compute $\mathbf{X}^T\mathbf{X}$:

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 0 & 1 & -2 \\ -1 & 2 & 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 2 & -1 & 1 & 0 & 1 \\ -1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -1 & 3 & -4 & 2 \\ -1 & 6 & -3 & 5 & 3 \\ 3 & -3 & 11 & -3 & 3 \\ -4 & 5 & -3 & 11 & 3 \\ 2 & 3 & 3 & 3 & 6 \end{bmatrix}$$

Now, let's compute $\mathbf{X}^T Y$:

$$\mathbf{X}^T Y = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 0 & 1 & -2 \\ -1 & 2 & 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \\ -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3+0+8+2+2+0 \\ 0+1-4-4+2+0 \\ 6-1+4+0+2+2 \\ -3+2+0-2+2+2 \\ 3+1+4-2+2-1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \\ 13 \\ 1 \\ 7 \end{bmatrix}$$

To find $\mathbf{w}$, we need to solve the linear system $(\mathbf{X}^T\mathbf{X})\mathbf{w} = \mathbf{X}^T Y$:

$$\begin{bmatrix} 7 & -1 & 3 & -4 & 2 \\ -1 & 6 & -3 & 5 & 3 \\ 3 & -3 & 11 & -3 & 3 \\ -4 & 5 & -3 & 11 & 3 \\ 2 & 3 & 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \\ 13 \\ 1 \\ 7 \end{bmatrix}$$

Solving this system of linear equations (which would typically be done using numerical methods or computational software) will give us the values of w_1, w_2, w_3, w_4, and b. These coefficients define the linear regression model in $\mathbb{R}^5$.

Let's assume, after solving this system, we obtained a solution (for illustrative purposes):

$$w_1 \approx 1.0, \quad w_2 \approx -0.5, \quad w_3 \approx 0.2, \quad w_4 \approx 0.8, \quad b \approx 0.1$$

Then the linear regression model would be approximately:

$$y \approx 1.0x_1 - 0.5x_2 + 0.2x_3 + 0.8x_4 + 0.1$$

The process remains the same regardless of the number of dimensions. We form the augmented predictor matrix, compute $\mathbf{X}^T\mathbf{X}$ and $\mathbf{X}^T Y$, and then solve the resulting system of normal equations. The complexity of solving the system increases with the number of predictors.

$\square$

Charles Hermite (1822 - 1901)

Hermite was born in Dieuze, Moselle, on 24 December 1822, with a deformity in his right foot that would impair his gait throughout his life. He was the sixth of seven children of Ferdinand Hermite and his wife, Madeleine nee Lallemand. Ferdinand worked in the drapery business of Madeleine's family while also pursuing a career as an artist. The drapery business relocated to Nancy in 1828, and so did the family.

Hermite obtained his secondary education at College de Nancy and then, in Paris, at Coll?ge Henri IV and at the Lycee Louis-le-Grand. He read some of Joseph-Louis Lagrange's writings on the solution of numerical equations and Carl Friedrich Gauss's publications on number theory.

Hermite wanted to take his higher education at Ecole Polytechnique, a military academy renowned for excellence in mathematics, science, and engineering. Tutored by mathematician Eugene Charles Catalan, Hermite devoted a year to preparing for the notoriously difficult entrance examination. In 1842 he was admitted to the school. However, after one year the school would not allow Hermite to continue his studies there because of his deformed foot. He struggled to regain his admission to the school, but the administration imposed strict conditions. Hermite did not accept this, and he quit the Ecole Polytechnique without graduating.

In 1842, Nouvelles Annales de Mathematiques published Hermite's first original contribution to mathematics, a simple proof of Niels Abel's proposition of concerning the impossibility of an algebraic solution to equations of the fifth degree.

A correspondence with Carl Jacobi, begun in 1843 and continued the next year, resulted in the insertion, in the complete edition of Jacobi's works, of two articles by Hermite, one concerning the extension to Abelian functions of one of the theorems of Abel on elliptic functions, and the other concerning the transformation of elliptic functions.

After spending five years working privately towards his degree, in which he befriended eminent mathematicians Joseph Bertrand, Carl Gustav Jacob Jacobi, and Joseph Liouville, he took and passed the examinations for the baccalaureat, which he was awarded in 1847. He married Joseph Bertrand's sister, Louise Bertrand, in 1848.

In 1848, Hermite returned to the Ecole Polytechnique as repetiteur and examinateur d'admission. In 1856 he contracted smallpox. Through the influence of Augustin-Louis Cauchy and of a nun who nursed him, he resumed the practice of his Catholic faith. In July 1848, he was elected to the French Academy of Sciences. In 1869, he succeeded Jean-Marie Duhamel as professor of mathematics, both at the Ecole Polytechnique, where he remained until 1876, and at the University of Paris, where he remained until his death. From 1862 to 1873 he was lecturer at the Ecole Normale Superieure. Upon his 70th birthday, he was promoted to grand officer in the French Legion of Honour.

Hermite died in Paris on 14 January 1901, aged 78.

10.3 Polynomial Regression

Polynomial regression extends the concept of linear regression to model non-linear relationships between a scalar independent variable x and a scalar dependent variable y. Despite modeling a curve, it remains a linear model in terms of its parameters. The core idea is to augment the feature space by including polynomial powers of the original independent variable.

Given a dataset of m data points $\{(x_i, y_i)\}_{i=1}^m$, where $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$, polynomial regression of degree n aims to find a polynomial function that best fits the data:

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_n x^n + \epsilon, \tag{10.2}$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are the polynomial coefficients to be estimated, and ϵ represents the random error term, assumed to have a mean of zero.

10.3.1 Linear Algebra Formulation

To apply the principles of linear regression, we construct a design matrix $X \in \mathbb{R}^{m\times(n+1)}$ and a coefficient vector $\boldsymbol{\beta} \in \mathbb{R}^{n+1}$:

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \ldots & x_m^n \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}. \tag{10.3}$$

The vector of observed dependent variables is $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$, and the error vector is $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_m \end{bmatrix} \in \mathbb{R}^m$. The polynomial regression model in matrix form is:

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}. \tag{10.4}$$

10.3.2 Least Squares Solution

The coefficients $\boldsymbol{\beta}$ are typically estimated using the ordinary least squares (OLS) method, which minimizes the sum of the squared errors between the observed values $\mathbf{y}$ and the predicted values $\hat{\mathbf{y}} = X\boldsymbol{\beta}$:

$$\text{Minimize } L(\boldsymbol{\beta}) = \sum_{i=1}^{m} (y_i - \hat{y}_i)^2 = \|\mathbf{y} - X\boldsymbol{\beta}\|^2 = (\mathbf{y} - X\boldsymbol{\beta})^T(\mathbf{y} - X\boldsymbol{\beta}). \tag{10.5}$$

To find the $\boldsymbol{\beta}$ that minimizes $L(\boldsymbol{\beta})$, we take the gradient with respect to $\boldsymbol{\beta}$ and set it to zero:

$$\begin{aligned}\nabla_{\beta} L(\boldsymbol{\beta}) &= \nabla_{\beta}(\mathbf{y}^T\mathbf{y} - 2\boldsymbol{\beta}^T X^T\mathbf{y} + \boldsymbol{\beta}^T X^T X\boldsymbol{\beta}) \\ &= -2X^T\mathbf{y} + 2X^T X\boldsymbol{\beta} = \mathbf{0}.\end{aligned}$$

Solving for $\boldsymbol{\beta}$ yields the normal equations:

$$X^T X\boldsymbol{\beta} = X^T\mathbf{y}. \tag{10.6}$$

If the matrix $X^T X$ is invertible (which occurs when X has full column rank, typically when $m > n+1$ and the x_i values are distinct), the unique least squares solution for the coefficient vector is:

$$\boldsymbol{\beta}^* = (X^T X)^{-1} X^T\mathbf{y}. \tag{10.7}$$

This solution provides the polynomial coefficients that best fit the given data in the least squares sense.

10.3.3 Examples

Let's illustrate polynomial regression with a couple of examples.

Example 10.7 (Quadratic Regression). *Suppose we have the following dataset:* $\{(1,2),(2,5),(3,10),(4,17)\}$. *We want to fit a quadratic polynomial ($n = 2$) of the form $y = \beta_0 + \beta_1 x + \beta_2 x^2$.*

The design matrix X and the target vector $\mathbf{y}$ are:

$$X = \begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ 5 \\ 10 \\ 17 \end{bmatrix}.$$

First, we compute $X^T X$:

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}.$$

Next, we compute $X^T\mathbf{y}$:

$$X^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 10 \\ 17 \end{bmatrix} = \begin{bmatrix} 2+5+10+17 \\ 2+10+30+68 \\ 2+20+90+272 \end{bmatrix} = \begin{bmatrix} 34 \\ 110 \\ 384 \end{bmatrix}.$$

Now, we need to solve the system $(X^T X)\boldsymbol{\beta} = X^T \mathbf{y}$*:*

$$\begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 34 \\ 110 \\ 384 \end{bmatrix}.$$

Solving this system of linear equations (e.g., using Gaussian elimination or by computing $(X^T X)^{-1}$*), we find the coefficients:*

$$\beta_0 = 1, \quad \beta_1 = 0, \quad \beta_2 = 1.$$

Thus, the best-fit quadratic polynomial is $y = 1 + 0x + 1x^2 = 1 + x^2$.

Example 10.8 (Cubic Regression)**.** *Consider a dataset where we suspect a cubic relationship:* $\{(0,1),(1,6),(2,17),(3,34)\}$. *We want to fit a cubic polynomial* ($n = 3$) *of the form* $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$.

The design matrix X *and the target vector* $\mathbf{y}$ *are:*

$$X = \begin{bmatrix} 1 & 0 & 0^2 & 0^3 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 6 \\ 17 \\ 34 \end{bmatrix}.$$

We would then compute $X^T X$ *and* $X^T \mathbf{y}$*:*

$$X^T X = \begin{bmatrix} 4 & 6 & 14 & 36 \\ 6 & 14 & 36 & 98 \\ 14 & 36 & 98 & 276 \\ 36 & 98 & 276 & 794 \end{bmatrix}, \quad X^T \mathbf{y} = \begin{bmatrix} 58 \\ 150 \\ 404 \\ 1158 \end{bmatrix}.$$

Solving the system $(X^T X)\boldsymbol{\beta} = X^T \mathbf{y}$ *yields the coefficients:*

$$\beta_0 = 1, \quad \beta_1 = 2, \quad \beta_2 = 3, \quad \beta_3 = 0.$$

Thus, the best-fit cubic polynomial is $y = 1 + 2x + 3x^2 + 0x^3 = 1 + 2x + 3x^2$.

These examples demonstrate how to set up the design matrix and the normal equations for polynomial regression. Solving the resulting linear system provides the coefficients of the best-fit polynomial.

10.3.4 Model Evaluation, Applications, Advantages, Disadvantages, and Regularization

The subsequent sections on model evaluation, applications, advantages, disadvantages, and regularization remain relevant and well-described in your original text. They provide important context for understanding the practical aspects of polynomial regression.

By understanding the linear algebra formulation and the least squares solution, along with the practical considerations discussed in the other sections, you have a comprehensive overview of polynomial regression.

10.4 Logistic Regression

Logistic regression is a powerful and widely used statistical model for binary classification problems. Despite its name containing "regression," it is fundamentally a classification algorithm. It models the probability of a binary outcome (e.g., success/failure, yes/no, 0/1) based on a set of independent variables (predictors). Logistic regression achieves this by using a linear combination of the input features and applying a non-linear function, the sigmoid function (also known as the logistic function), to the result. From a linear algebra perspective, it combines a linear transformation of the feature space with this non-linear mapping, making it a bridge between linear models and advanced machine learning techniques like neural networks.

10.4.1 The Sigmoid Function

The core of logistic regression is the sigmoid function, denoted by $\sigma(z)$, which maps any real-valued number z to a value between 0 and 1. The sigmoid function is defined as:

$$\sigma(z) = \frac{1}{1+e^{-z}} \tag{10.8}$$

The shape of the sigmoid function is an "S" curve. It has the following important properties:

- $\sigma(z) \in (0,1)$ for all $z \in \mathbb{R}$.
- $\lim_{z\to\infty} \sigma(z) = 1$.
- $\lim_{z\to-\infty} \sigma(z) = 0$.
- $\sigma(0) = 0.5$.
- $\sigma'(z) = \sigma(z)(1-\sigma(z))$.

The output of the sigmoid function can be interpreted as the probability of the positive class (typically labeled as 1). Its smooth "S" shape ensures a gradual transition, which is key to modeling probabilities rather than hard boundaries.

10.4.2 The Logistic Regression Model

In logistic regression, we model the probability of the positive class given a set of input features $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ using a linear combination of these features, similar to linear regression. However, instead of directly predicting the outcome, we pass this linear combination through the sigmoid function:

$$P(y=1|\mathbf{x};\mathbf{w},b) = \sigma(\mathbf{w}^T\mathbf{x}+b) = \frac{1}{1+e^{-(\mathbf{w}^T\mathbf{x}+b)}} \tag{10.9}$$

Here:

- $\mathbf{x}$ is the vector of input features (independent variables).
- $\mathbf{w} = [w_1, w_2, \ldots, w_n]^T$ is the vector of weights (coefficients) associated with each feature.
- b is the bias term (intercept).
- $\mathbf{w}^T\mathbf{x} + b$ is the linear combination of the features, just like in linear regression.
- $\sigma(\mathbf{w}^T\mathbf{x} + b)$ is the predicted probability that the output variable y is 1 given the input features $\mathbf{x}$.

The probability of the negative class ($y = 0$) is then given by:

$$P(y=0|\mathbf{x};\mathbf{w},b) = 1 - P(y=1|\mathbf{x};\mathbf{w},b) = 1-\sigma(\mathbf{w}^T\mathbf{x}+b) = \sigma(-(\mathbf{w}^T\mathbf{x}+b)) = \frac{e^{-(\mathbf{w}^T\mathbf{x}+b)}}{1+e^{-(\mathbf{w}^T\mathbf{x}+b)}} \tag{10.10}$$

To emphasize the linear algebra foundation, we can augment $\mathbf{x}$ with a 1 (i.e., $\mathbf{x}_{aug} = [x_1, x_2, \ldots, x_n, 1]^T \in \mathbb{R}^{n+1}$) and define $\mathbf{w}_{aug} = [w_1, w_2, \ldots, w_n, b]^T$, so we have

$$P(y=1|\mathbf{x};\mathbf{w}_{aug}) = \sigma(\mathbf{w}_{aug}^T\mathbf{x}_{aug}) \tag{10.11}$$

This matrix notation highlights the linear transformation from features to probabilities.

10.4.3 Decision Boundary

To make a classification decision, we typically set a threshold on the predicted probability. A common threshold is 0.5. If $P(y=1|\mathbf{x};\mathbf{w},b) \geq 0.5$, we predict $y = 1$; otherwise, we predict $y = 0$.

The decision boundary is the set of points $\mathbf{x}$ for which $P(y=1|\mathbf{x};\mathbf{w},b) = 0.5$. This occurs when the argument of the sigmoid function is zero:

$$\mathbf{w}^T\mathbf{x} + b = 0 \tag{10.12}$$

This equation represents a hyperplane in the feature space, which separates the regions where the model predicts one class versus the other. This is why logistic regression is considered a linear classifier – its decision boundary is linear. Geometrically, $\mathbf{w}$ determines the hyperplane's orientation (as its normal vector), while b shifts its position from the origin. The sigmoid function softens this boundary, assigning probabilities that decrease exponentially with distance from the hyperplane, as illustrated in Figure 10.5.

10.4.4 Learning the Parameters: Maximum Likelihood Estimation

The goal of training a logistic regression model is to find the optimal values for the weights $\mathbf{w}$ and the bias b that best fit the training data. This is typically done using the principle of maximum likelihood estimation (MLE).

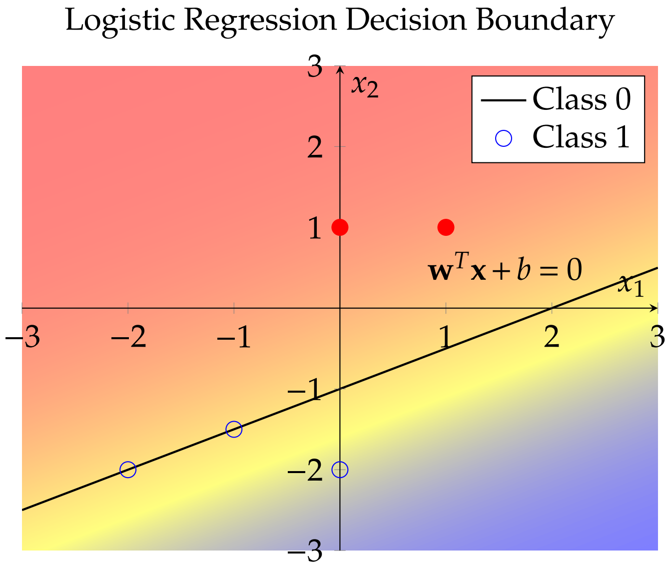


Figure 10.5: A 2D decision boundary $w_1x_1 + w_2x_2 + b = 0$ with points labeled 0 (blue circles) and 1 (red stars). The gradient represents $\sigma(\mathbf{w}^T\mathbf{x}+b)$, transitioning from 0 (black) through red and yellow to 1 (white).

Given a training dataset of m labeled examples $\{(\mathbf{x}_i, y_i)\}_{i=1}^m$, where $y_i \in \{0,1\}$, the likelihood of observing this dataset given the parameters $\mathbf{w}$ and b is:

$$L(\mathbf{w},b) = \prod_{i=1}^{m} P(y_i|\mathbf{x}_i;\mathbf{w},b) \tag{10.13}$$

We can write $P(y_i|\mathbf{x}_i;\mathbf{w},b)$ as:

$$P(y_i|\mathbf{x}_i;\mathbf{w},b) = [P(y=1|\mathbf{x}_i;\mathbf{w},b)]^{y_i}[P(y=0|\mathbf{x}_i;\mathbf{w},b)]^{1-y_i} \tag{10.14}$$

To make the optimization easier, we often work with the log-likelihood:

$$\begin{aligned}
\ell(\mathbf{w},b) &= \log L(\mathbf{w},b) && (10.15)\\
&= \sum_{i=1}^{m}[y_i \log P(y=1|\mathbf{x}_i;\mathbf{w},b) + (1-y_i)\log P(y=0|\mathbf{x}_i;\mathbf{w},b)] && (10.16)\\
&= \sum_{i=1}^{m}[y_i \log \sigma(\mathbf{w}^T\mathbf{x}_i+b) + (1-y_i)\log(1-\sigma(\mathbf{w}^T\mathbf{x}_i+b))] && (10.17)\\
&= \sum_{i=1}^{m}[y_i(\mathbf{w}^T\mathbf{x}_i+b) - \log(1+e^{(\mathbf{w}^T\mathbf{x}_i+b)})] && (10.18)
\end{aligned}$$

The goal is to find the values of $\mathbf{w}$ and b that maximize this log-likelihood function. This is typically done using gradient-based optimization algorithms like gradient ascent (since we want to maximize) or gradient descent (if we minimize the negative log-likelihood, which is a common loss function called the binary cross-entropy loss). The negative log-likelihood is:

$$J(\mathbf{w},b) = -\ell(\mathbf{w},b) = -\sum_{i=1}^{m} [y_i \log \sigma(\mathbf{w}^T \mathbf{x}_i + b) + (1-y_i)\log(1-\sigma(\mathbf{w}^T \mathbf{x}_i + b))] \tag{10.19}$$

The gradient of the log-likelihood with respect to the weights w_j and the bias b can be calculated as

$$\frac{\partial \ell(\mathbf{w},b)}{\partial w_j} = \sum_{i=1}^{m} (y_i - \sigma(\mathbf{w}^T \mathbf{x}_i + b)) x_{ij} \tag{10.20}$$

$$\frac{\partial \ell(\mathbf{w},b)}{\partial b} = \sum_{i=1}^{m} (y_i - \sigma(\mathbf{w}^T \mathbf{x}_i + b)) \tag{10.21}$$

For minimization, the gradient of J is:

$$\frac{\partial J}{\partial w_j} = \sum_{i=1}^{m} [\sigma(\mathbf{w}^T \mathbf{x}_i + b) - y_i] x_{ij} \tag{10.22}$$

$$\frac{\partial J}{\partial b} = \sum_{i=1}^{m} [\sigma(\mathbf{w}^T \mathbf{x}_i + b) - y_i] \tag{10.23}$$

10.4.5 Linear Algebra Formulation for Training

We can express the training process in a more compact linear algebra form. Let X be the design matrix where each row is a feature vector $\mathbf{x}_i^T$, and let $\mathbf{y}$ be the vector of labels y_i. We can also include the bias term by augmenting the feature matrix with a column of ones and including the bias b in the weight vector $\mathbf{w}_{aug} = [b, w_1, \ldots, w_n]^T$. The augmented feature vector is $\mathbf{x}_{i,aug} = [1, x_{i1}, \ldots, x_{in}]^T$.

Then the linear combination becomes $\mathbf{w}_{aug}^T \mathbf{x}_{i,aug}$. The predicted probability for all data points can be written as a vector $\hat{\mathbf{p}}$ where $\hat{p}_i = \sigma(\mathbf{w}_{aug}^T \mathbf{x}_{i,aug})$. Define $X_{aug} \in \mathbb{R}^{m\times(n+1)}$ as the augmented design matrix. The gradient of the negative log-likelihood (the loss function we want to minimize) with respect to $\mathbf{w}_{aug}$ can be expressed as:

$$\nabla_{\mathbf{w}_{aug}} J(\mathbf{w}_{aug}) = \frac{1}{m} X_{aug}^T (\hat{\mathbf{p}} - \mathbf{y}) \tag{10.24}$$

Gradient descent updates the weights as:

$$\mathbf{w}_{aug}^{(t+1)} = \mathbf{w}_{aug}^{(t)} - \alpha \nabla_{\mathbf{w}_{aug}} J(\mathbf{w}_{aug}^{(t)}) \tag{10.25}$$

where α is the learning rate. This matrix formulation leverages linear algebra operations (e.g., inner products from Section 6.1), enhancing computational efficiency.

Example 10.9. *Consider a simple binary classification problem with one feature. Suppose we have the following data points:*

- *Feature:* $x = -1$, *Class:* $y = 0$
- *Feature:* $x = 0$, *Class:* $y = 0$
- *Feature:* $x = 1$, *Class:* $y = 1$
- *Feature:* $x = 2$, *Class:* $y = 1$

We want to train a logistic regression model to classify these points. Our model is $P(y = 1|x; w, b) = \sigma(wx + b)$. *We would initialize* w *and* b *(e.g., to 0). Then, we would iteratively update* w *and* b *using gradient descent based on the gradients of the log-likelihood (or the binary cross-entropy loss) calculated on these data points.*

For instance, for the first data point $(x = -1, y = 0)$, *the predicted probability is* $\hat{p}_1 = \sigma(-w + b)$. *The contribution to the gradient would involve* $(0 - \hat{p}_1) \cdot (-1)$ *for* w *and* $(0 - \hat{p}_1) \cdot 1$ *for* b. *We would do this for all data points and sum the contributions to get the overall gradient, which we then use to update* w *and* b. *This process is repeated until convergence.*

After training, we would have the optimal values for w *and* b, *and we could use the decision boundary* $wx + b = 0$ *to classify new data points.*

Now, let's compute one step explicitly. The augmented design matrix and labels are:

$$X_{aug} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Initialize $\mathbf{w}_{aug} = [w, b]^T = [0, 0]^T$. *Then* $X_{aug}\mathbf{w}_{aug} = [1, 1, 1, 1]^T$, *so* $\hat{\mathbf{p}} = \sigma([1, 1, 1, 1]^T) \approx [0.731, 0.731, 0.731, 0.731]^T$ *(since* $\sigma(1) \approx 0.731$*). The gradient is:*

$$\nabla J = \frac{1}{4} X_{aug}^T(\hat{\mathbf{p}} - \mathbf{y}) = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.731 \\ 0.731 \\ -0.269 \\ -0.269 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -0.538 \\ 0.924 \end{bmatrix} = \begin{bmatrix} -0.1345 \\ 0.231 \end{bmatrix}$$

With $\alpha = 0.1$, *update:*

$$\mathbf{w}_{aug}^{(1)} = [0, 0]^T - 0.1 \cdot [-0.1345, 0.231]^T = [0.01345, -0.0231]^T$$

This step shifts the decision boundary toward better separating the classes.

10.4.6 Advantages and Disadvantages

Advantages:

- Simple and easy to implement.
- Provides probabilistic interpretation of the output.
- Can be regularized to prevent overfitting (see Section 8.4).

- Performs well on linearly separable data.

Disadvantages:

- Assumes a linear relationship between the features and the log-odds of the outcome.
- Can underperform when the decision boundary is highly non-linear.
- Sensitive to multicollinearity in the features (mitigable via PCA, Section 10.1).

10.4.7 Extension to Multiclass Classification

Logistic regression can be extended to handle multiclass classification problems (where the outcome variable has more than two categories) using techniques like:

- **One-vs-Rest (OvR):** Train a separate binary logistic regression classifier for each class, where one class is treated as the positive class and all other classes are treated as the negative class.
- **Multinomial Logistic Regression (Softmax Regression):** Directly models the probabilities of each class using a softmax function, which generalizes the sigmoid function to multiple classes:

$$P(y = k|\mathbf{x}) = \frac{e^{\mathbf{w}_k^T \mathbf{x}_{aug}}}{\sum_{j=1}^{K} e^{\mathbf{w}_j^T \mathbf{x}_{aug}}}$$

  where K is the number of classes.

This section provides a foundational understanding of logistic regression as a linear model for binary classification, covering its mathematical formulation, training process, and key characteristics. It also connects to machine learning: $\sigma(\mathbf{w}^T\mathbf{x} + b)$ resembles a neuron's output in neural networks (Chapter 14), and gradient descent aligns with backpropagation (Section 14.4).

412. *For the dataset* $X_{aug} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{y} = [0, 1, 1]^T$*, compute one gradient descent step with* $\mathbf{w}_{aug} = [0, 0]^T$, $\alpha = 0.1$.

Chapter 11

Tensors

This chapter introduces the concept of tensors, which are a generalization of scalars, vectors, and matrices. Tensors are essential for understanding many areas of mathematics, physics, and computer science, particularly machine learning.

11.1 Tensor Product

Let U and V be vector spaces over a field $\mathbb{F}$. A *tensor product* of U and V is a pair $(W, \otimes)$, where W is a vector space over $\mathbb{F}$ and

$$\otimes : U \times V \to W$$

is a bilinear map, satisfying the following *universal property*:

For any vector space Z over $\mathbb{F}$ and any bilinear map $\phi : U \times V \to Z$, there exists a *unique* linear map $\tilde{\phi} : W \to Z$ such that the diagram commutes:

$$\begin{tikzcd} U \times V \arrow[r, "\otimes"] & W \arrow[d, "\tilde{\phi}"] \\ & Z \end{tikzcd}$$

11.1.1 The Bilinear Map $\otimes$

The map $\otimes : U \times V \to W$ takes pairs of vectors $(u, v) \in U \times V$ and maps them to elements $u \otimes v \in W$. It is bilinear, meaning:

$$\begin{aligned} (\alpha u) \otimes v &= \alpha(u \otimes v) \\ u \otimes (\alpha v) &= \alpha(u \otimes v) \\ (u_1 + u_2) \otimes v &= u_1 \otimes v + u_2 \otimes v \\ u \otimes (v_1 + v_2) &= u \otimes v_1 + u \otimes v_2 \end{aligned}$$

for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, and $\alpha \in \mathbb{F}$. Elements of W of the form $u \otimes v$ are called *simple tensors* or *elementary tensors*.

Lemma 11.1. *The bilinear map $\otimes$ is distributive over addition in both arguments, as shown by the properties above.*

Proof. The properties follow directly from the definition of bilinearity in the construction of W. For instance, $(u_1+u_2)\otimes v-(u_1\otimes v+u_2\otimes v)$ is an element of the subspace S, hence zero in the quotient $W=F(U\times V)/S$. $\square$

413. *Verify that* $(2u_1-u_2)\otimes(v_1+v_2)=2u_1\otimes v_1+2u_1\otimes v_2-u_2\otimes v_1-u_2\otimes v_2$ *using the bilinearity properties.*

414. *If u_1,u_2 are linearly dependent in U, show that $u_1\otimes v$ and $u_2\otimes v$ are linearly dependent in $U\otimes V$ for any $v\in V$.*

11.1.2 The Universal Property

The universal property states that $\otimes$ is the "most general" bilinear map. Specifically, for any bilinear map $\phi: U\times V\to Z$ (where Z is any vector space), there exists a unique linear map $\tilde{\phi}: W\to Z$ such that $\phi=\tilde{\phi}\circ\otimes$. This means the following diagram commutes:

$$\begin{array}{ccc} U\times V & \xrightarrow{\otimes} & W \\ & & \downarrow{\scriptstyle\tilde{\phi}} \\ & & Z \end{array}$$

This property completely characterizes the tensor product.

415. *Suppose $\phi: U\times V\to Z$ is bilinear and satisfies $\phi(u,v)=0$ for all $u\in U$, $v\in V$. Prove that the corresponding $\tilde{\phi}: W\to Z$ is the zero map.*

416. *Construct a bilinear map $\phi:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ defined by $\phi\left(\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\right)=ac+bd$. Find the corresponding linear map $\tilde{\phi}:\mathbb{R}^2\otimes\mathbb{R}^2\to\mathbb{R}$.*

11.1.3 Existence and Construction

We construct the tensor product as follows:

1. **Free Vector Space:** Let $F(U\times V)$ be the free vector space generated by the set $U\times V$. Elements of $F(U\times V)$ are formal linear combinations of pairs (u,v), i.e., elements of the form $\sum_{i=1}^n \alpha_i(u_i,v_i)$ where $\alpha_i\in\mathbb{F}$, $u_i\in U$, and $v_i\in V$.

2. **Subspace of Relations:** Let S be the subspace of $F(U\times V)$ generated by the elements:

$$\begin{aligned}(\alpha u,v)-\alpha(u,v)\\(u,\alpha v)-\alpha(u,v)\\(u_1+u_2,v)-(u_1,v)-(u_2,v)\\(u,v_1+v_2)-(u,v_1)-(u,v_2)\end{aligned}$$

   These elements enforce the bilinearity relations.

3. **Tensor Product Space:** Define the tensor product space W as the quotient space $W = F(U \times V)/S$.

4. **Bilinear Map:** Define the bilinear map $\otimes : U \times V \to W$ by $u \otimes v = (u,v) + S$, where $(u,v)+S$ denotes the coset of (u,v) in W.

It can be verified that $(W, \otimes)$ constructed in this way satisfies the universal property.

417. *Show that the elements generating S ensure that $\otimes$ is bilinear by verifying one of the relations in the quotient space.*

418. *If $U = \mathbb{R}$ and $V = \mathbb{R}^2$, describe the elements of $F(U \times V)$ and identify one element in S.*

11.1.4 Uniqueness (Up to Isomorphism)

Suppose $(W, \otimes)$ and $(W', \otimes')$ are two tensor products of U and V. By the universal property of $(W, \otimes)$, there exists a unique linear map $F : W \to W'$ such that $F(u \otimes v) = u \otimes' v$ for all $u \in U$ and $v \in V$. Similarly, there exists a unique linear map $G : W' \to W$ such that $G(u \otimes' v) = u \otimes v$.

Now, consider the map $G \circ F : W \to W$. For any simple tensor $u \otimes v \in W$,

$$(G \circ F)(u \otimes v) = G(F(u \otimes v)) = G(u \otimes' v) = u \otimes v.$$

Since simple tensors span W, $G \circ F$ must be the identity map on W. A similar argument shows that $F \circ G$ is the identity on W'. Therefore, F and G are isomorphisms, inverses of each other, demonstrating that the tensor product is unique up to isomorphism.

Theorem 11.1. *The tensor product $U \otimes V$ is unique up to isomorphism.*

Proof. The existence of isomorphisms F and G with $G \circ F = \mathrm{id}_W$ and $F \circ G = \mathrm{id}_{W'}$ establishes that $W \cong W'$. ◻

419. *Prove that if $F : W \to W'$ is an isomorphism satisfying $F(u \otimes v) = u \otimes' v$, then F^{-1} satisfies $F^{-1}(u \otimes' v) = u \otimes v$.*

420. *If $U = V = \mathbb{R}$, show that $U \otimes V \cong \mathbb{R}$ by constructing an explicit isomorphism.*

11.1.5 Spanning Set and Dimension

The set $\{u \otimes v \mid u \in U, v \in V\}$ spans W, but it is not generally a basis. If $\dim(U) = m$ and $\dim(V) = n$, then $\dim(U \otimes V) = mn$.

Theorem 11.2. *If U and V are finite-dimensional vector spaces with $\dim(U) = m$ and $\dim(V) = n$, then $\dim(U \otimes V) = mn$.*

Proof. Let $\{e_1, \dots, e_m\}$ be a basis for U and $\{f_1, \dots, f_n\}$ be a basis for V. The set $\{e_i \otimes f_j \mid 1 \le i \le m, 1 \le j \le n\}$ spans $U \otimes V$ and has mn elements. To show it is a basis, suppose $\sum_{i,j} a_{ij}(e_i \otimes f_j) = 0$. Define a bilinear map $\phi : U \times V \to \mathbb{F}$ by $\phi(e_i, f_j) = a_{ij}$ and extend linearly. By the universal property, there exists $\tilde{\phi} : U \otimes V \to \mathbb{F}$ with $\tilde{\phi}(e_i \otimes f_j) = a_{ij}$. Since $\sum a_{ij}(e_i \otimes f_j) = 0$, $\tilde{\phi} = 0$, implying $a_{ij} = 0$ for all i, j. Thus, the set is linearly independent and a basis. ◻

421. *If $U = \mathbb{R}^2$ and $V = \mathbb{R}^3$, compute* $\dim(U \otimes V)$ *and write down a basis for $U \otimes V$.*

422. *Prove that if U or V is infinite-dimensional, then $U \otimes V$ is infinite-dimensional.*

11.1.6 Operations on Tensor Products

- **Scalar Multiplication:** $\alpha(u \otimes v) = (\alpha u) \otimes v = u \otimes (\alpha v)$
- **Addition:** Defined by linearity. If $w_1 = \sum_i a_i(u_i \otimes v_i)$ and $w_2 = \sum_j b_j(u'_j \otimes v'_j)$, then

$$w_1 + w_2 = \sum_i a_i(u_i \otimes v_i) + \sum_j b_j(u'_j \otimes v'_j).$$

Remark: Tensor products are fundamental in machine learning, particularly in deep learning frameworks like TensorFlow, where multi-dimensional arrays (tensors) represent data and weights. Operations like addition and scalar multiplication on tensor products enable efficient computation of gradients and updates in neural networks.

423. *Compute* $3\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}2\\1\end{pmatrix}\right) + 2\left(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\3\end{pmatrix}\right)$ *in* $\mathbb{R}^2 \otimes \mathbb{R}^2$.

424. *Show that the set of all simple tensors $\{u \otimes v \mid u \in U, v \in V\}$ is not a basis if* $\dim(U), \dim(V) > 1$.

11.1.7 Basis Representation and Elementary Matrices

Let $\{e_1, e_2, \dots, e_m\}$ be a basis for U and $\{f_1, f_2, \dots, f_n\}$ be a basis for V. Then the set $\{e_i \otimes f_j \mid 1 \le i \le m, 1 \le j \le n\}$ forms a basis for $U \otimes V$, and any element $w \in U \otimes V$ can be written as:

$$w = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(e_i \otimes f_j),$$

where $a_{ij} \in \mathbb{F}$. The dimension of $U \otimes V$ is thus mn, as previously stated.

To represent the tensor product in matrix form, consider vectors $u = \sum_{i=1}^m u_i e_i \in U$ and $v = \sum_{j=1}^n v_j f_j \in V$. The tensor $u \otimes v$ can be expressed as:

$$u \otimes v = \sum_{i=1}^{m} \sum_{j=1}^{n} u_i v_j (e_i \otimes f_j).$$

If we order the basis $\{e_i \otimes f_j\}$ (e.g., lexicographically: $e_1 \otimes f_1, e_1 \otimes f_2, \dots, e_m \otimes f_n$), then $u \otimes v$ corresponds to a vector in $\mathbb{F}^{mn}$ with components $u_i v_j$.

For example, let $U = V = \mathbb{R}^2$ with basis $\{e_1, e_2\}$ and $\{f_1, f_2\}$, respectively. For $u = \begin{pmatrix}u_1\\u_2\end{pmatrix}$ and $v = \begin{pmatrix}v_1\\v_2\end{pmatrix}$, the tensor $u \otimes v$ in the basis $\{e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2\}$ is:

$$u \otimes v = u_1 v_1 (e_1 \otimes f_1) + u_1 v_2 (e_1 \otimes f_2) + u_2 v_1 (e_2 \otimes f_1) + u_2 v_2 (e_2 \otimes f_2),$$

represented as the vector:

$$\begin{pmatrix} u_1v_1 \\ u_1v_2 \\ u_2v_1 \\ u_2v_2 \end{pmatrix}.$$

This is equivalent to the Kronecker product of the coordinate vectors $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

425. *Express* $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ *as a linear combination of the basis* $\{e_i \otimes f_j\}$ *for* $\mathbb{R}^2 \otimes \mathbb{R}^2$.

426. *If* $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* $v = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, *compute the coordinates of* $u \otimes v$ *in the basis* $\{e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2\}$.

11.1.8 Elementary Tensor Matrices and the Kronecker Product

For finite-dimensional spaces, we can represent simple tensors as matrices via the Kronecker product, which provides a concrete realization of the tensor product for matrices. For matrices $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{p \times q}$, their Kronecker product $A \otimes B$ is an $mp \times nq$ matrix defined as:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix},$$

where $a_{ij}B$ is the matrix B scaled by the scalar a_{ij}. This operation is bilinear and aligns with the tensor product when matrices are viewed as elements of vector spaces (e.g., $\mathbb{F}^{m \times n} \cong \mathbb{F}^{mn}$).

Consider $U = \mathbb{R}^m$ and $V = \mathbb{R}^n$. The tensor $e_i \otimes f_j$, where e_i is the i-th standard basis vector in $\mathbb{R}^m$ and f_j is the j-th standard basis vector in $\mathbb{R}^n$, corresponds to the $m \times n$ matrix with a 1 in the (i, j)-th position and 0s elsewhere. For example, in $\mathbb{R}^2 \otimes \mathbb{R}^2$:

- $e_1 \otimes f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (as a flattened 4×1 vector: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$),
- $e_1 \otimes f_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (as a flattened vector: $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$),
- $e_2 \otimes f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (as a flattened vector: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$),

- $e_2 \otimes f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (as a flattened vector: $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$).

These elementary tensors form a basis for $\mathbb{R}^2 \otimes \mathbb{R}^2$, and any tensor can be written as a linear combination of them.

Here are additional examples of the Kronecker product:

- **Vectors:** Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Then:

$$u \otimes v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 \\ 1 \cdot 4 \\ 2 \cdot 3 \\ 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix},$$

  a 4×1 vector.

- **Matrices:** Let $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \end{pmatrix}$. Then:

$$A \otimes B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 3 & 4 \end{pmatrix} \\ 2 \cdot \begin{pmatrix} 3 & 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix},$$

  a 2×2 matrix.

For a larger example, consider the tensor product of a 2×3 matrix and a 3×4 matrix. Let:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

The Kronecker product $A \otimes B$ is a 6×12 matrix:

$$A \otimes B = \begin{pmatrix} 1 \cdot B & 2 \cdot B & 3 \cdot B \\ 4 \cdot B & 5 \cdot B & 6 \cdot B \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 2 & 4 & 6 & 8 & 3 & 6 & 9 & 12 \\ 5 & 6 & 7 & 8 & 10 & 12 & 14 & 16 & 15 & 18 & 21 & 24 \\ 9 & 10 & 11 & 12 & 18 & 20 & 22 & 24 & 27 & 30 & 33 & 36 \\ 4 & 8 & 12 & 16 & 5 & 10 & 15 & 20 & 6 & 12 & 18 & 24 \\ 20 & 24 & 28 & 32 & 25 & 30 & 35 & 40 & 30 & 36 & 42 & 48 \\ 36 & 40 & 44 & 48 & 45 & 50 & 55 & 60 & 54 & 60 & 66 & 72 \end{pmatrix}$$

Remark: The Kronecker product is widely used in applications such as quantum mechanics (to describe composite systems via tensor products of state spaces) and machine learning (e.g., in tensor decomposition methods like CP decomposition for multi-dimensional data analysis).

427. *Compute the Kronecker product* $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and express the result as a matrix.*

428. *If A is a 2×2 matrix and B is a 3×3 matrix, what is the size of $A \otimes B$? Verify your answer with an example.*

11.2 Tensor Operations

11.2.1 Tensor Addition and Scalar Multiplication

Tensors of the same order can be added element-wise:

$$(\mathbf{A}+\mathbf{B})_{i_1 i_2 \dots i_n} = A_{i_1 i_2 \dots i_n} + B_{i_1 i_2 \dots i_n}$$

Tensors can be multiplied by a scalar:

$$(\alpha \mathbf{A})_{i_1 i_2 \dots i_n} = \alpha A_{i_1 i_2 \dots i_n}$$

429. *If* $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$*, compute* $2\mathbf{A}+3\mathbf{B}$*.*

430. *Show that tensor addition is commutative:* $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$*.*

11.2.2 Tensor Product (Outer Product)

The tensor product (also known as the outer product) of two tensors $\mathbf{A}$ (of order m) and $\mathbf{B}$ (of order n) results in a tensor $\mathbf{C}$ of order $m+n$:

$$(\mathbf{A}\otimes\mathbf{B})_{i_1 i_2 \dots i_m j_1 j_2 \dots j_n} = A_{i_1 i_2 \dots i_m} B_{j_1 j_2 \dots j_n}$$

Remark 11.1. *machine learning, the outer product is used in algorithms like singular value decomposition (SVD) and in constructing covariance matrices, which are critical for dimensionality reduction techniques such as PCA.*

431. *Compute the outer product of* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *and* $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ *and verify it matches the Kronecker product result.*

432. *If* $\mathbf{A}$ *is a* 2×3 *matrix, what is the order of* $\mathbf{A}\otimes\mathbf{A}$*? Describe its dimensions.*

11.2.3 Contraction

Contraction is an operation that reduces the order of a tensor. It involves summing over a pair of indices. For example, contracting a 2-order tensor (matrix) over its two indices gives the trace:

$$Tr(\mathbf{A}) = \sum_i A_{ii}$$

For higher-order tensors, contraction can be performed over any pair of indices.

Lemma 11.2. *The trace of a rank-1 matrix* $\mathbf{A} = u \otimes v^T$ *(where* $u, v \in \mathbb{R}^n$*) is the dot product* $u^T v$*.*

Proof. Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. Then $\mathbf{A}_{ij} = u_i v_j$, and $Tr(\mathbf{A}) = \sum_i A_{ii} = \sum_i u_i v_i = u^T v$. □

Remark 11.2. *Contraction is key in physics (e.g., Einstein notation for tensor operations in relativity) and machine learning (e.g., reducing tensor dimensions in neural network optimization).*

433. *Compute the trace of* $\begin{pmatrix}1\\2\end{pmatrix}\otimes\begin{pmatrix}3 & 4\end{pmatrix}$ *and verify it equals the dot product of the vectors.*

434. *If* $\mathbf{A}$ *is a* 3×3 *matrix with* $Tr(\mathbf{A}) = 5$, *what is* $Tr(2\mathbf{A})$*?*

11.2.4 Examples of Tensor Operations

Consider $U = V = \mathbb{R}^2$. Let $u = \begin{pmatrix}1\\2\end{pmatrix}$ and $v = \begin{pmatrix}3\\4\end{pmatrix}$ in $\mathbb{R}^2$. Their tensor product $u\otimes v$ as a 2×2 matrix is:

$$u\otimes v = \begin{pmatrix}1\\2\end{pmatrix}\otimes\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1\cdot 3 & 1\cdot 4\\ 2\cdot 3 & 2\cdot 4\end{pmatrix} = \begin{pmatrix}3 & 4\\ 6 & 8\end{pmatrix}.$$

Now, let $w = \begin{pmatrix}5\\6\end{pmatrix}$. Then $w\otimes v = \begin{pmatrix}15 & 20\\ 18 & 24\end{pmatrix}$. Adding these tensors:

$$(u\otimes v) + (w\otimes v) = \begin{pmatrix}3 & 4\\ 6 & 8\end{pmatrix} + \begin{pmatrix}15 & 20\\ 18 & 24\end{pmatrix} = \begin{pmatrix}18 & 24\\ 24 & 32\end{pmatrix}.$$

Scalar multiplication: $2(u\otimes v) = 2\begin{pmatrix}3 & 4\\ 6 & 8\end{pmatrix} = \begin{pmatrix}6 & 8\\ 12 & 16\end{pmatrix}$.

For contraction, the trace of $u\otimes v$ is:

$$Tr(u\otimes v) = 3 + 8 = 11.$$

435. *Using* $u = \begin{pmatrix}1\\2\end{pmatrix}$ *and* $v = \begin{pmatrix}3\\4\end{pmatrix}$, *compute* $u\otimes v + v\otimes u$ *and find its trace.*

436. *Verify that* $Tr(3(u\otimes v)) = 3Tr(u\otimes v)$ *for the example above.*

Chapter 12

Introduction to Machine Learning

This chapter provides a brief overview of key machine learning concepts to tie together the linear algebra principles discussed in the previous chapters. We will explore the fundamental differences between supervised and unsupervised learning, the crucial bias-variance tradeoff, model selection and evaluation techniques like cross-validation, and a high-level introduction to some other common machine learning algorithms, highlighting their connections to linear algebra.

12.1 Supervised vs. Unsupervised Learning

Machine learning can be broadly categorized into two main types: supervised learning and unsupervised learning.

Supervised learning involves learning a mapping from inputs to outputs based on labeled training data. The training data consists of pairs $(\mathbf{x}_i, \mathbf{y}_i)$, where $\mathbf{x}_i$ is the input and $\mathbf{y}_i$ is the corresponding desired output (or label). The goal is to learn a function f such that $f(\mathbf{x}_i) \approx \mathbf{y}_i$ for unseen inputs. Examples of supervised learning tasks include:

* **Classification:** Predicting a categorical label (e.g., classifying images as cats or dogs).

* **Regression:** Predicting a continuous value (e.g., predicting house prices).

Linear algebra plays a crucial role in many supervised learning algorithms. For instance, linear regression, logistic regression, and support vector machines (SVMs) all heavily rely on matrix operations, vector spaces, and other linear algebra concepts.

Unsupervised learning, on the other hand, involves learning patterns and structures in data without explicit labels. The training data consists only of inputs $\mathbf{x}_i$, and the goal is to discover relationships or representations within the data itself. Examples of unsupervised learning tasks include:

* **Clustering:** Grouping similar data points together (e.g., grouping customers based on their purchasing behavior).

* **Dimensionality Reduction:** Reducing the number of features while preserving important information (e.g., Principal Component Analysis - PCA).

Unsupervised learning algorithms often utilize linear algebra techniques. For example, PCA relies heavily on eigenvalue decomposition and singular value decomposition (SVD) to find the principal components of the data. Clustering algorithms like k-means often involve calculating distances between data points, which can be expressed using vector norms.

12.2 The Bias-Variance Tradeoff

A fundamental challenge in machine learning is the **bias-variance tradeoff**. This tradeoff describes the balance between a model's ability to fit the training data (low bias) and its ability to generalize to unseen data (low variance).

* **Bias:** Bias refers to the error introduced by simplifying assumptions made by the model to make the target function easier to learn. A high-bias model underfits the training data and may fail to capture complex relationships. * **Variance:** Variance refers to the model's sensitivity to fluctuations in the training data. A high-variance model overfits the training data, memorizing noise and performing poorly on unseen data.

Ideally, we want a model with both low bias and low variance. However, decreasing bias often increases variance, and vice versa. Finding the right balance is crucial for good generalization performance. Linear algebra techniques, such as regularization (L1 and L2), can be used to control model complexity and manage the bias-variance tradeoff. Regularization adds penalties to the loss function, which in turn affects the weight matrices and other linear algebra components of the model.

12.3 Model Selection and Evaluation (Cross-Validation)

Model selection is the process of choosing the best model from a set of candidate models. **Model evaluation** is the process of assessing the performance of a trained model. A key challenge is that we want to estimate how well a model will perform on unseen data, but we only have access to the training data.

Cross-validation is a technique used to address this challenge. The basic idea is to divide the training data into k subsets (or folds). For each fold, we train the model on the remaining $k-1$ folds and evaluate it on the held-out fold. This process is repeated k times, and the performance estimates are averaged to obtain a more robust evaluation of the model. Common cross-validation techniques include k-fold cross-validation and leave-one-out cross-validation.

Linear algebra isn't directly involved in the cross-validation process itself, but it's essential for the underlying models being evaluated. The models being compared in cross-validation are often based on linear algebra.

12.4 A Very High-Level Introduction to Other ML Algorithms

Many other machine learning algorithms rely heavily on linear algebra. Here are a few examples:

* **Support Vector Machines (SVMs):** SVMs use linear algebra to find the optimal hyperplane that separates data points of different classes. The kernel trick, used in SVMs, implicitly

maps data to a high-dimensional feature space, which is often represented using linear algebra.

* **Decision Trees:** While decision trees are not directly based on linear algebra, some variations, like oblique decision trees, use linear combinations of features at each node, which can be represented using vectors and matrices.

* **k-Nearest Neighbors (k-NN):** k-NN is a simple algorithm that classifies a data point based on the labels of its k nearest neighbors. Calculating distances between data points, a key step in k-NN, involves vector norms.

This chapter has provided a brief introduction to key machine learning concepts. While the focus has been on providing a high-level overview, it's important to remember that linear algebra forms the mathematical backbone of many machine learning algorithms. A solid understanding of linear algebra is crucial for developing, understanding, and applying these powerful techniques.

Chapter 13

Neural Networks: A Linear Algebra Perspective

Neural networks are powerful computational models inspired by the structure of the human brain. They are widely used in machine learning for tasks like image recognition, natural language processing, and many others. From a linear algebra perspective, neural networks are essentially compositions of linear transformations and non-linear activation functions. This chapter focuses on the linear algebra aspects of neural networks, emphasizing how matrix multiplications represent layers, how backpropagation leverages the chain rule, and how optimization algorithms are employed. We will not delve deeply into specific neural network architectures, as that is a broader topic.

13.1 Neural Network Layers as Linear Transformations

At the heart of a neural network lies the concept of a layer. A layer performs a transformation on its input, and this transformation can be largely expressed using linear algebra. Consider a single layer with m neurons, receiving an input vector $\mathbf{x} \in \mathbb{R}^n$. Each neuron computes a weighted sum of the inputs and adds a bias term:

$$z_i = \sum_{j=1}^{n} w_{ij} x_j + b_i, \quad i = 1, 2, \ldots, m, \tag{13.1}$$

where w_{ij} are the weights connecting input x_j to neuron i, and b_i is the bias for neuron i. This can be compactly written in matrix form:

$$\mathbf{z} = W\mathbf{x} + \mathbf{b}, \tag{13.2}$$

where $\mathbf{z} \in \mathbb{R}^m$ is the vector of weighted sums, $W \in \mathbb{R}^{m \times n}$ is the weight matrix, $\mathbf{x} \in \mathbb{R}^n$ is the input vector, and $\mathbf{b} \in \mathbb{R}^m$ is the bias vector. This equation represents a linear transformation of the input vector $\mathbf{x}$, followed by a translation by the bias vector $\mathbf{b}$. The weight matrix W can be viewed as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

13.2 Activation Functions: Introducing Non-linearity

The linear transformation in a layer is followed by a non-linear **activation function** σ. This function is applied element-wise to the vector $\mathbf{z}$:

$$\mathbf{y} = \sigma(\mathbf{z}), \tag{13.3}$$

where $\mathbf{y} \in \mathbb{R}^m$ is the output of the layer. Common activation functions include:

* **Sigmoid:** $\sigma(z) = \frac{1}{1+e^{-z}}$. The sigmoid function squashes the input to a range between 0 and 1. It was historically popular but is less used now due to vanishing gradients. * **Tanh:** $\sigma(z) = \tanh(z)$. The tanh function squashes the input to a range between -1 and 1. It is similar to sigmoid but centered at 0, which often leads to faster convergence. * **ReLU (Rectified Linear Unit):** $\sigma(z) = \max(0, z)$. ReLU is a piecewise linear function that outputs 0 if the input is negative and the input itself if it is positive. It is very popular due to its simplicity and effectiveness in mitigating the vanishing gradient problem.

Activation functions introduce non-linearity, which is crucial for neural networks to learn complex patterns in data. Without non-linearity, a neural network would simply be a composition of linear transformations, which could be represented by a single linear transformation. Specifically, if we had two linear layers: $\mathbf{z}_1 = W_1\mathbf{x} + \mathbf{b}_1$ and $\mathbf{z}_2 = W_2\mathbf{z}_1 + \mathbf{b}_2$, then $\mathbf{z}_2 = W_2(W_1\mathbf{x} + \mathbf{b}_1) + \mathbf{b}_2 = (W_2W_1)\mathbf{x} + (W_2\mathbf{b}_1 + \mathbf{b}_2)$, which is just another linear transformation with weight matrix W_2W_1 and bias vector $W_2\mathbf{b}_1 + \mathbf{b}_2$.

13.3 Multi-Layer Networks: Composing Linear Transformations

A neural network consists of multiple layers stacked together. The output of one layer becomes the input to the next. For a network with L layers, the output $\mathbf{y}^{(L)}$ can be expressed as:

$$\mathbf{y}^{(L)} = \sigma_L(W^{(L)}\mathbf{x}^{(L-1)} + \mathbf{b}^{(L)}), \tag{13.4}$$

where $\mathbf{x}^{(L-1)} = \mathbf{y}^{(L-1)}$ is the output of the $(L-1)$-th layer. This shows how matrix multiplications and activation functions are composed to form the overall transformation performed by the network. The composition of linear transformations and non-linear activations allows the network to learn highly non-linear and complex mappings from input to output.

13.4 Backpropagation: The Chain Rule in Action

Training a neural network involves adjusting the weights and biases to minimize a **loss function** L that measures the difference between the network's output $\hat{\mathbf{y}}$ and the true target values $\mathbf{y}$. This is typically done using **gradient descent** or related optimization algorithms. A crucial step is computing the gradient of the loss function with respect to each weight and bias. This is where **backpropagation** comes in.

Backpropagation is an efficient algorithm for computing gradients by leveraging the **chain rule** of calculus. Because the network is a composition of functions, the gradient of the loss function with respect to a weight in an earlier layer can be computed by propagating the gradient backwards through the network. Specifically, if we have two layers, the gradient of the loss with respect to the weights W_1 in the first layer can be computed using the chain rule:

$$\frac{\partial L}{\partial W_1} = \frac{\partial L}{\partial \mathbf{y}_2}\frac{\partial \mathbf{y}_2}{\partial \mathbf{z}_2}\frac{\partial \mathbf{z}_2}{\partial \mathbf{y}_1}\frac{\partial \mathbf{y}_1}{\partial \mathbf{z}_1}\frac{\partial \mathbf{z}_1}{\partial W_1}, \tag{13.5}$$

where $\mathbf{y}_1$ and $\mathbf{z}_1$ are the output and weighted sum of the first layer, and $\mathbf{y}_2$ and $\mathbf{z}_2$ are the output and weighted sum of the second layer. This process is repeated for all layers, moving backwards from the output layer to the input layer.

13.5 Optimization Algorithms: Minimizing the Loss

Once the gradients are computed using backpropagation, optimization algorithms are used to update the weights and biases. A basic algorithm is **gradient descent**:

$$\theta_{t+1} = \theta_t - \alpha \nabla L(\theta_t), \tag{13.6}$$

where θ represents the network's parameters (weights and biases), α is the **learning rate**, and $\nabla L(\theta)$ is the gradient of the loss function.

More sophisticated optimization algorithms, such as **Adam**, **RMSprop**, and others, use adaptive learning rates and other techniques to improve convergence. These algorithms often involve matrix operations and vector manipulations. For example, Adam updates the parameters using exponentially decaying averages of the past gradients and the squared gradients.

13.6 Conclusion

This chapter has highlighted the crucial role of linear algebra in neural networks. Matrix multiplications are used to represent the linear transformations within each layer, and the chain rule is essential for backpropagation. Optimization algorithms, which often rely on linear algebra operations, are then used to update the network's parameters. While neural networks involve many other concepts, understanding the underlying linear algebra is fundamental to grasping how these powerful models work.

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Linear Algebra
ISBN: 987-0-97545-414-5

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