Linear Algebra

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Linear Algebra for Machine Learning

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Preface

Linear algebra is one of the cornerstones of modern mathematics, with profound applications in computer science, engineering, physics, and the social sciences. It serves as the language of transformations, optimizations, and high-dimensional spaces, making it indispensable not only for pure mathematicians but also for practitioners in fields such as artificial intelligence, data science, and machine learning.

Traditionally, linear algebra is introduced at the sophomore level as the first rigorous encounter with vector spaces, linear transformations, and matrices. However, the way it is taught often varies. Some textbooks emphasize applications at the expense of mathematical depth, leaving students with a collection of computational techniques but little insight into the underlying structures. Others focus on the formal mathematical framework, avoiding computations and real-world connections. This book aims to strike a balance by presenting both the theoretical foundations and computational techniques of linear algebra while maintaining a strong geometric perspective.

One of the distinguishing features of this book is its integration of geometry throughout. The transformations of conic sections, for example, illustrate how diagonalizing a matrix corresponds to changing the basis of a vector space, revealing the deep connection between algebra and geometry. The notion of invariants—quantities that remain unchanged under transformations—appears repeatedly in discussions on eigenvalues, singular value decomposition, and other fundamental topics.

Beyond its traditional applications in physics and engineering, linear algebra has become an essential tool in artificial intelligence and data science. Machine learning algorithms, at their core, rely on linear algebraic structures. The representation of data as high-dimensional vectors, the optimization of loss functions, and the efficient computation of gradients in deep learning frameworks are all built on fundamental linear algebraic operations. Techniques such as Principal Component Analysis (PCA) for dimensionality reduction, singular value decomposition (SVD) for data compression, and gradient descent for optimization all rely on a solid understanding of linear algebra. Even neural networks, often perceived as highly nonlinear systems, can be analyzed as compositions of linear transformations with nonlinearity introduced via activation functions.

This book provides a comprehensive introduction to linear algebra while preparing students for more advanced applications in modern computational sciences. In later chapters, we explore optimization techniques, probability, statistics, and linear models—key topics for machine learning practitioners. The final chapters introduce neural networks from a linear algebraic perspective, offering insights into how matrices, vectors, and transformations underlie deep learning architectures.

A wide range of exercises is provided, from fundamental problems reinforcing key concepts to more challenging ones that connect to broader areas of mathematics. While this book assumes familiarity with calculus and basic discrete mathematics, no prior knowledge of linear algebra is required.

The material in this book has been shaped by years of teaching at the University of California, Irvine; the University of Idaho; the University of Vlora; and Oakland University. I am grateful to my students, whose engagement and curiosity have influenced this text. It is my hope that this book not only equips students with the mathematical tools necessary for their fields but also inspires an appreciation for the elegance and power of linear algebra.

Tony Shaska
Rochester, 2018

Chapter 1

Euclidean spaces, linear systems

We start this chapter with the familiar notion of Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ from previous lecture. Intuition from $\mathbb{R}^2$ and $\mathbb{R}^3$ will be used to generalize concepts for $\mathbb{R}^n$ including the norm, dot product of vectors, angles among vectors, and the geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$.

In Sec. 1.3, we introduce the matrices and their algebra. Using matrices to solve linear systems of equations involves computing the row-echelon form and the reduced row-echelon form of matrices. These are the so-called Gauss algorithm and Gauss - Jordan algorithm and are studied in Sec. 1.4. In Sec. 1.5 we study the inverses of matrices and algorithms of computing such matrices.

1.1 Vectors in Physics and Geometry

We will denote by $\mathbb{R}^2$ the xy-plane and by $\mathbb{R}^3$ the coordinate system in space. For any two given points P and Q, an **directed line segment** (P, Q) is the segment PQ. We call P the **initial point** and Q the **terminal point**. Two directed line segments (A, B) and (C, D) are called **equipollent** when the points A, B, D, C, in this order, form a parallelogram.

1.1.1 The plane $\mathbb{R}^2$

Every point in xy-plane is represented uniquely by an ordered pair (x, y). For any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ their distance is given by

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Let's denote the set of all directed line segments in $\mathbb{R}^2$ by S. In this set S we define the following relation: $(P_1, Q_1) \sim (P_2, Q_2)$ if the following hold

(i) lines P_1Q_1 and P_2Q_2 are parallel

(ii) $d(P_1, Q_1) = d(P_2, Q_2)$

(iii) directed line segments (P_1, Q_1) and (P_2, Q_2) have the same direction

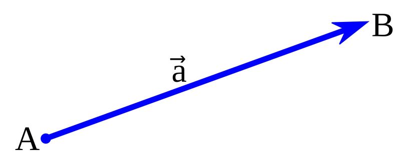


Figure 1.1: A Euclidean vector

Exercise 1. *Prove that ∼ is an equivalence relation.*

A **vector** is called an equivalence class from the above relation. Geometrically two directed line segments (A, B) and (C, D) are equivalent when they are equipollent. The equivalence class of the (A, B) will be denoted by $\overrightarrow{AB}$. The **magnitude** (or **length**) of the vector $\overrightarrow{AB}$ is simply the distance

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.1}$$

and from now on will be denoted by $\left\|\overrightarrow{AB}\right\|$.

Denote the set of all such equivalence classes by $V := S/\sim$. Hence, V the set of all vectors from the xy-plane. Moreover, the above three conditions are geometrically equivalent with moving the vector $\overrightarrow{P_1Q_1}$ in a parallel way over $\overrightarrow{OP}$, where O is the origin of the coordinate system. So we can assume that all vectors of V start at the origin O by picking for each equivalence class the representative that starts at the origin O. Elements of V will be denoted by bold letters throughout these lectures. Hence we have the following:

Lemma 1.1. *Thus, there is a one-to-one correspondence between the set of elements of V and points of the xy-plane, namely for any $P(x, y)$*

$$\mathbf{u} = \overrightarrow{OP} \longleftrightarrow P = (x, y)$$

Proof. Exercise □

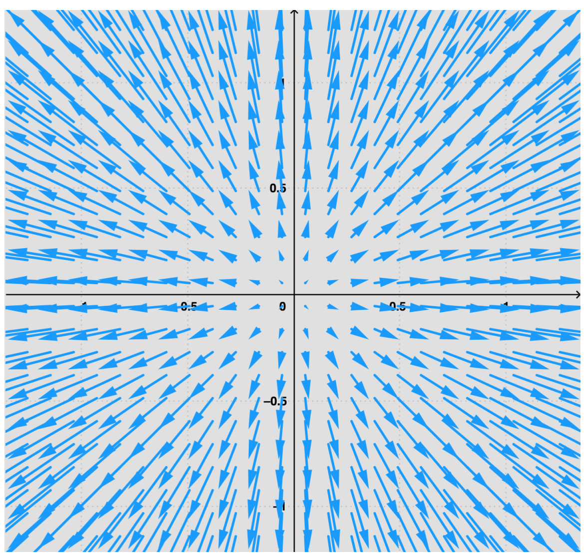


Figure 1.2: Vectors in $\mathbb{R}^2$

Hence, a vector $\mathbf{u} = \overrightarrow{OP}$ is identified with an ordered pair (x, y) and will be denoted by $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, in order to distinguish it from the point $P(x, y)$. Because of the above correspondence, from now on we will identify $V = \mathbb{R}^2$. We say that x and y are the **coordinates** of $\mathbf{u}$.

Example 1.1. *Let $P(1,2)$ and $Q(3,7)$ be given in $\mathbb{R}^2$. Find the coordinates of vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.*

Next we will see the addition and scalar multiplication of vectors. Most likely the reader is not new to such concepts since they are studied in a first course in elementary physics. We will focus on the algebraic and geometric point of view.

For any two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in V define the addition and scalar multiplication as

$$\mathbf{u} + \mathbf{v} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}, \qquad \text{and} \qquad r \cdot \mathbf{u} := \begin{bmatrix} ru_1 \\ ru_2 \end{bmatrix}, \tag{1.2}$$

where $r \in \mathbb{R}$. Geometrically scalar multiplication $r\mathbf{u}$ is described as in Fig. 1.3, where $r\mathbf{u}$ is a new vector with the same direction as $\mathbf{u}$ and length r-times the length of $\mathbf{u}$.

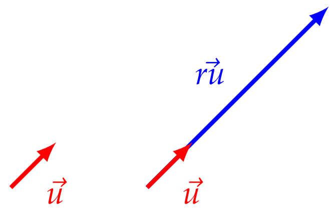


Figure 1.3: Multiplying by a scalar

Addition of two vectors $\mathbf{u}$ and $\mathbf{v}$ geometrically is described in Fig. 1.4.

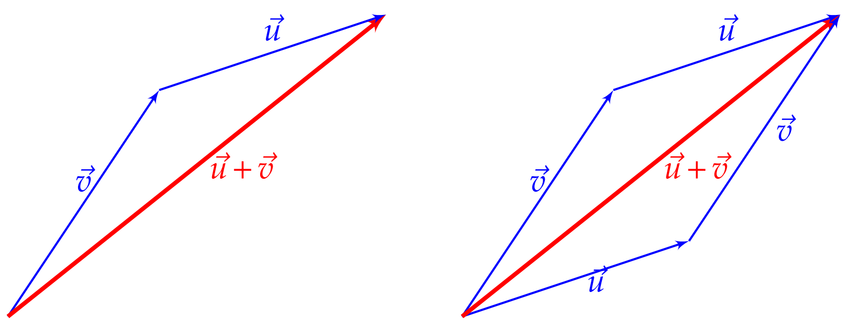


Figure 1.4: Addition of vectors

Exercise 2. *Prove that such definitions agree with addition and scalar multiplication defined in Eq. (1.2)*

The following exercise is elementary, but very interesting when we discuss determinants of matrices in coming lectures.

Exercise 3. *Given two vectors*

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

in $\mathbb{R}^2$ we can assume that both start at the origin.

(i) *Prove that the area of the parallelogram determined by these two vectors is* $A = |u_1 v_2 - u_2 v_1|$.
(ii) *Prove that the lines determined by* $\mathbf{u}$ *and* $\mathbf{v}$ *are perpendicular if and only if* $u_1 v_1 = u_2 v_2$.
(iii) *Determine the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

The following exercises explore further the geometry of the vectors in $\mathbb{R}^2$.

Exercises:

1. *If* A, B, C *are vertices of a triangle, find* $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

2. *Let* c *a positive real number and* O_1, O_2 *points on the xy-plane with coordinates* $(c,0)$ *and* $(-c,0)$ *respectively. Find the equation of all points* P *such that*

$$\left\|\overrightarrow{PO_1}\right\| + \left\|\overrightarrow{PO_2}\right\| = 2a,$$

for $a > c$.

3. *Let* $\triangle ABC$ *be a given triangle and* θ *the angle between* AB *and* AC. *Prove the Law of Cosines*

$$BC^2 = AB^2 + AC^2 - 2\,AB \cdot AC \cdot \cos\theta \qquad (1.3)$$

4. *Let* a *and* b *sides of a parallelogram and* d_1, d_2 *its diagonals. Prove that*

$$d_1^2 + d_2^2 = 2(a^2 + b^2).$$

5. *Prove that the diagonals of a parallelogram are perpendicular if and only if all sides are equal.*

6. *Prove that the distance* d *of a point* $P = (x_0, y_0)$ *from the line*

$$ax + by + c = 0$$

is given by

$$d = \frac{\left|ax_0 + by_0 + c\right|}{\sqrt{a^2 + b^2}}.$$

1.1.2 The space $\mathbb{R}^3$

Next we review briefly the geometry of the space and vectors in $\mathbb{R}^3$. Definition of vectors in $\mathbb{R}^3$ goes exactly the same with their definition in $\mathbb{R}^2$, by adding a third coordinate.
Recall that $\mathbb{R}^3$ is the Cartesian product

$$\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$$

and a point P in $\mathbb{R}^3$ is represented by an ordered triple (x_0, y_0, z_0) as shown in Fig. 1.5.

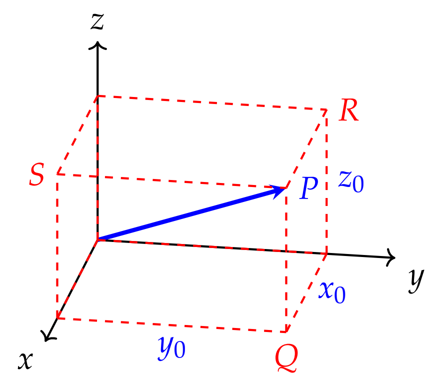


Figure 1.5: Coordinates of $P(x,y,z)$.

Let be given two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ in $\mathbb{R}^3$. We will show that the distance $|P_1P_2|$ between the two points is

$$||P_1P_2|| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

To verify this formula we construct a parallelepiped where the points P_1 and P_2 are vertices across from each other as in Fig. 1.6. If $A(x_2,y_1,z_1)$ and $B(x_2,y_2,z_1)$ are the other vertices as in Fig. 1.6, then

$$|P_1A| = |x_2-x_1|, \quad |AB| = |y_2-y_1|, \quad |BP_2| = |z_2-z_1|$$

Since the triangles $\triangle P_1BP_2$ and $\triangle P_1AB$ are right triangles, from the Pythagorean theorem we have

$$|P_1B|^2 = |P_1A|^2 + |AB|^2 \text{ and } |P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

Combining the two equations we have

$$\begin{aligned} ||P_1P_2||^2 &= ||P_1A||^2 + |||AB||^2 + ||BP_2||^2 \\ &= ||x_2-x_1||^2 + \left\|y_2-y_1\right\|^2 + ||z_2-z_1||^2 \\ &= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \end{aligned}$$

Thus,

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}} \tag{1.4}$$

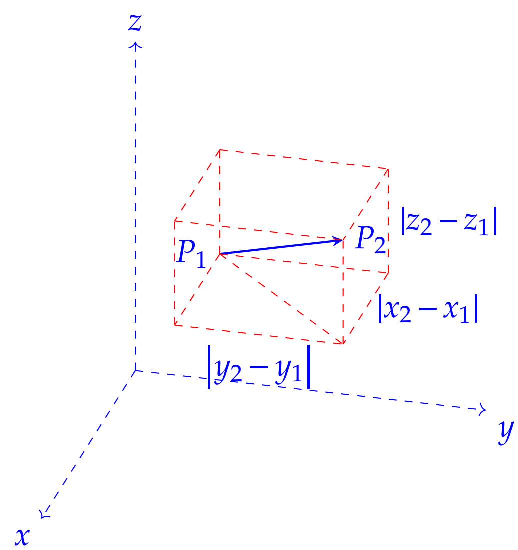
The distance between a point $P(x,y,z)$ and the origin is

$$\left\|\overrightarrow{OP}\right\| = \sqrt{x^2+y^2+z^2}.$$

Example 1.2. *Let $P(1,2,3)$ and $Q(4,2,1)$. Find the coordinates and the magnitude of the vector $\overrightarrow{PQ}$*

Solution: The coordinates of $\overrightarrow{PQ}$ are $\overrightarrow{PQ} = [3,0,-2]^t$ and its magnitude $\left\|\overrightarrow{PQ}\right\| = \sqrt{3^2+0^2+(-2)^2} = \sqrt{13}$. □

Figure 1.6: Distance between two points



Every point in $3d$-space is represented uniquely by an ordered triple (x, y, z). For any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ a **Euclidean vector** (or simply a **vector**) is frequently represented by a ray (a line segment with a definite direction), or graphically as an arrow connecting an **initial point** P_1 with a **terminal point** P_2, and denoted by $\overrightarrow{P_1P_2}$.

The **magnitude** (or **length**) of $\overrightarrow{P_1P_2}$ is simple the distance

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Let's denote the set of all 'vectors' in $\mathbb{R}^3$ by S. In this set S we define the following relation: $\overrightarrow{P_1Q_1} \sim \overrightarrow{P_2Q_2}$ if the following hold

(i) lines P_1Q_1 and P_2Q_2 are parallel

(ii) $\left\| \overrightarrow{P_1Q_1} \right\| = \left\| \overrightarrow{P_2Q_2} \right\|$

(iii) $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$ have the same direction

Exercise 4. *Prove that $\sim$ is an equivalence relation.*

Denote the set of all such equivalence classes by $V := S/\sim$. Hence, V the set of all equivalence classes of vectors from the xy-plane. Moreover, the above three conditions are geometrically equivalent with moving the vector $\overrightarrow{P_1Q_1}$ in a parallel way over $\overrightarrow{OP}$, where O is the origin of the coordinate system.

Then, a **vector** is called an equivalence class from the above relation. So we can assume that all vectors of V start at the origin O by picking for each equivalence class the representative that starts at the origin O. Elements of V will be denoted by bold letters throughout these lectures.

Lemma 1.2. *Thus, there is a one-to-one correspondence between the set of elements of V and points of the 3d-space, namely for any $P(x,y,x)$*

$$\mathbf{u} = \overrightarrow{OP} \longleftrightarrow P = (x,y,z)$$

Proof. Exercise ◻

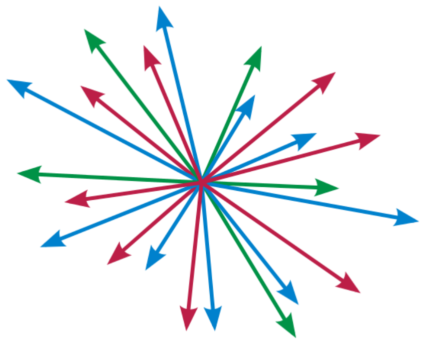


Figure 1.7: Representatives of the equivalence classes

Hence, a vector $u = \overrightarrow{OP}$ is identified with an ordered triple (x,y,z) and will be denoted by $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, in order to distinguish it from the point $P(x,y,z)$. Because of the above correspondence, from now on we will identify $V = \mathbb{R}^3$. We say that x, y and z are the **coordinates** of $\mathbf{u}$.

For any two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ we define the addition and scalar multiplication as in $\mathbb{R}^2$, namely

$$\mathbf{u} + \mathbf{v} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}, \quad r \cdot \mathbf{u} := \begin{bmatrix} ru_1 \\ ru_2 \\ ru_3 \end{bmatrix}.$$

where $r \in \mathbb{R}$. Since any two generic lines determine a plane, the geometric interpretation of addition and scalar multiplication of $\mathbb{R}^2$ is still valid in $\mathbb{R}^3$.

Sometimes it is more convenient to write vectors as **row vectors**. The **transpose** of the vector $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the row vector $u^t = [x,y,z]$ and the transpose of the row vector $[x,y,z]$ is the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. With these conventions the vector $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ will also be denoted by $\mathbf{u} = [x,y,z]^t$.

Exercise 5. *Find the coordinates of the vector $\overrightarrow{P_1P_2}$ when $P_1(1,1,2)$ and $P_2(2,4,6)$.*

Exercise 6. *Find* $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $\|\mathbf{v}\|$ *and* $\|\mathbf{v}-\mathbf{w}\|$, $\|\mathbf{v}+\mathbf{w}\|$, *and* $-2\mathbf{v}$, *if* $\mathbf{v}=[1,2,3]^t$ *and* $\mathbf{w}=[-1,2,-3]^t$.

Exercise 7. *Find* $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $\|\mathbf{v}\|$ *and* $\|\mathbf{v}-\mathbf{w}\|$, $\|\mathbf{v}+\mathbf{w}\|$, *and* $-2\mathbf{v}$, *if* $\mathbf{v}=[1,0,1]^t$ *and* $\mathbf{w}=[-1,-2,2]^t$

Properties of vector addition and multiplying by a scalar we can summarize below:

Theorem 1.1. *If* $\mathbf{u},\mathbf{v},\mathbf{w}$ *are three vectors in* $\mathbb{R}^3$ *and* $c,d\in\mathbb{R}$ *are scalars, then the following hold:*

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
(ii) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
(iii) $\mathbf{u}+\mathbf{0}=\mathbf{u}$
(iv) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
(vii) $(cd)\mathbf{u}=c(d\mathbf{u})$
(viii) $1\mathbf{u}=\mathbf{u}$

Proof. The proof is left as an exercise for the reader. □

Let's denote by V_3 the set of all vectors in the 3-dimensional space $\mathbb{R}^3$. Three vectors which play a special role in V_3 are

$$\mathbf{i}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{j}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

These vectors are called vectors of the **standard basis**. We will explain this terminology in more detail in the coming sections.

Exercise 8. *Prove that every vector in* $\mathbb{R}^3$ *is expressed in terms of vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. *In other words, if* $\mathbf{u}=\begin{bmatrix}a\\b\\c\end{bmatrix}$, *then we have*

$$\mathbf{u}==a\mathbf{i}+b\mathbf{j}+c\mathbf{k}.$$

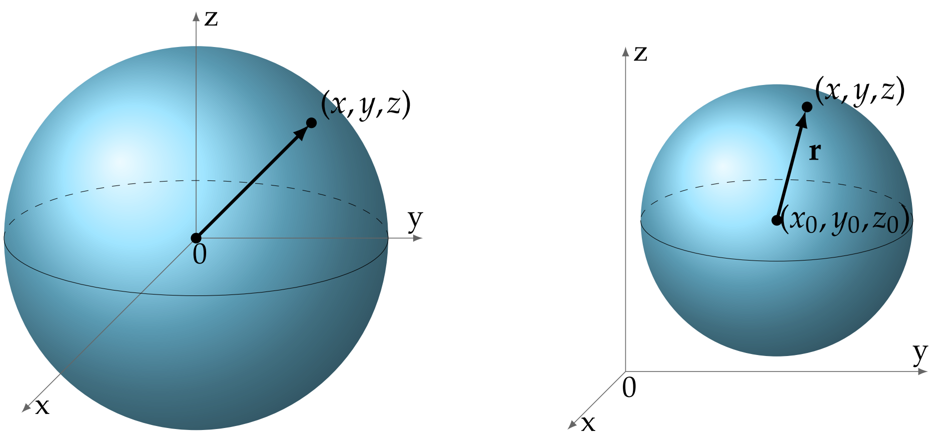
A vector u is called a **unit vector** if it has length 1. For example, vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors. A unit vector which has the same direction with a given vector $\mathbf{u}$ is a vector $\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{\mathbf{u}}{\|\mathbf{u}\|}$. In the next section we will formalize such definitions to the case of $\mathbb{R}^n$. The reader should make sure to fully understand the concepts from $\mathbb{R}^2$ and $\mathbb{R}^3$ before proceeding to $\mathbb{R}^n$.

Exercise 9. *Let* $\mathbf{v}=[x_0,y_0,z_0]^t$ *be a fixed vector in* $\mathbb{R}^3$. *Describe the set of all points* $P(x,y,z)$ *which satisfy* $\|\mathbf{u}-\mathbf{v}\|=1$, *where* $\mathbf{u}=[x,y,z]^t$.

Equation of the sphere

Using the distance formula above we can easily determine the equations of some simple geometric objects. The equation of the sphere with center at the point with coordinates (x_0,y_0,z_0) and radius r is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$



(a) Radius r, center $(0,0,0)$ (b) Radius r, center (x_0,y_0,z_0)

Figure 1.8: Spheres in $\mathbb{R}^3$

To prove this we just need the definition of the sphere, which is the set of all points $P(x,y,z)$ equidistant from the fixed point $Q(x_0,y_0,z_0)$ with a distance r from it. Thus, $\left\|\overrightarrow{QP}\right\| = r$. Squaring both sides we have $\left\|\overrightarrow{QP}\right\|^2 = r^2$ or

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2}$$

So the sphere with center at Q is the set of all terminal points of vectors with initial point at Q and magnitude r. When the center of the sphere is at the origin we have $x^2+y^2+z^2=r^2$ and in this case the sphere is the set of all terminal points of vectors with magnitude r and initial point at the origin.

However, not every sphere has an equation as above. Consider the following example:

Example 1.3. *Prove that the following equation represent a sphere and find its radius and its center*

$$4x^2+4y^2+4z^2-8x+16y=1.$$

Solution: Complete squares for $4x^2-8x$, $4y^2+16y$, and we have

$$(x-1)^2+(y+2)^2+z^2 = \frac{21}{4}$$

Thus the equation represents a sphere with center $(1,-2,0)$ and radius $\sqrt{\frac{21}{4}}$.

□

Remark 1.1. *Notice that the process of completing the square in each variable x,y,z gets complicated when the equation has cross terms xy, xz, and yz. We will learn how to handle such equations in later chapters.*

An equation in variables x and y represents a curve in $\mathbb{R}^2$ and a surface in $\mathbb{R}^3$. We illustrate with an easy example for which we construct the graph in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 1.4. *Construct the graph of $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$*

Solution: In $\mathbb{R}^2$ this equation represents a circle with radius 2 and center at the origin.

In $\mathbb{R}^3$, the graph is all points $P(x, y, z)$, where $x^2 + y^2 = 4$ and the z-coordinate takes any value $z \in \mathbb{R}$. Hence, it is a right cylinder with radius $r = 2$ and exists the z-axis as in Fig. 1.9 □

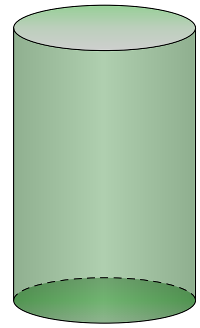


Figure 1.9: $x^2 + y^2 = 4$ in $\mathbb{R}^3$

1.1.3 Dot product

In $\mathbb{R}^2$, the **dot product** of two vectors $\mathbf{u} = [u_1, u_2]^t$ and $\mathbf{v} = [v_1, v_2]^t$ is defined as follows

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

For every two given vectors in $\mathbb{R}^3$, $\mathbf{u} = [u_1, u_2, u_3]^t$ and $\mathbf{v} = [v_1, v_2, v_3]^t$, **dot product** is called the real number $\mathbf{u} \cdot \mathbf{v}$ given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example 1.5. *Find the dot product in each case:*

(i) $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

(ii) $\mathbf{u} = [3, 0, -1]^t$, $\mathbf{v} = [2, 1, 7]^t$.

Solution: We have

i) $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 1 + 2 \cdot (-2) = 3 - 4 = -1$

ii) $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + 0 \cdot 1 + (-1) \cdot 7 = 6 + 0 - 7 = -1$.

□

The proof of the following is left as an exercise.

Theorem 1.2. *For every three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V_3 and $r \in \mathbb{R}$ we have*

(i) $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$

(ii) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,

(iii) $(r\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (r\mathbf{v})$

(iv) $\mathbf{u} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{u}$

(v) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

Definition 1.1. ***The angle*** *between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is called the smallest angle between them measured counterclockwise.*

Theorem 1.3. *If we denote by θ the angle between $\mathbf{u}$ and $\mathbf{v}$, then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \cos \theta$$

Proof. Using the cosine formula for the triangle OAB we have

$$||AB||^2 = ||OA||^2 + ||OB||^2 - 2\,||OA|| \cdot ||OB|| \cdot \cos\theta \qquad (1.5)$$

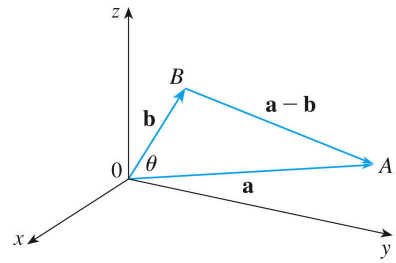


Figure 1.10

Since $||OA|| = ||\mathbf{u}||$, $||OB|| = ||\mathbf{v}||$, *and* $||BA|| = ||\mathbf{u}-\mathbf{v}||$, Eq. (1.5) becomes

$$||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\,||\mathbf{u}||\,||\mathbf{v}||\cos\theta \qquad (1.6)$$

The expression $||\mathbf{v}-\mathbf{w}||$, can be re-written as

$$\begin{aligned} ||\mathbf{u}-\mathbf{v}||^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = ||\mathbf{u}||^2 - 2\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2 \end{aligned}$$

Substituting in Eq. (1.6), we have

$$||\mathbf{v}||^2 - 2\mathbf{v}\cdot\mathbf{w} + ||\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 - 2\,||\mathbf{v}|| \cdot ||\mathbf{w}||\cos\theta$$

which implies $-2\mathbf{v}\cdot\mathbf{w} = -2\,||\mathbf{v}||\cdot||\mathbf{w}||\cos\theta$ and finally $\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||\cdot||\mathbf{w}||\cos\theta$. □

Corollary 1.1. *The angle θ between two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *is given by*

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||}.$$

Example 1.6. *Find the angle between the vectors* $\mathbf{v} = [1,-2,2]^t$ *and* $\mathbf{w} = [2,-2,-1]^t$.

Solution: First $||\mathbf{v}|| = \sqrt{1+4+4} = \sqrt{9} = 3$ and $||\mathbf{w}|| = \sqrt{4+4+1} = \sqrt{9} = 3$. Also, $\mathbf{v}\cdot\mathbf{w} = 1(2) + (-2)(-2) + 2(-1) = 4$. Then $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \frac{4}{3\cdot 3} = \frac{4}{9}$ and the angle between two vectors is $\theta = \cos^{-1}\left(\frac{4}{9}\right)$.

□

Dy vectors are called **orthogonal** if the angle between them is $\theta = \pi/2$. Thus, we have a corollary of Thm. 1.3, which gives an if and only if condition to determine if two vectors are orthogonal.

Corollary 1.2. *Two nonzero vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are orthogonal if and only if* $\mathbf{v}\cdot\mathbf{w} = 0$.

For orthogonal vectors we use the notation $\mathbf{v} \perp \mathbf{w}$.

Example 1.7. *Determine if vectors* $\mathbf{v} = [1,-5,2]^t$ *and* $\mathbf{w} = [3,1,1]^t$ *are orthogonal.*

Solution: We have $\mathbf{v}\cdot\mathbf{w} = 1\cdot 3 + (-5)\cdot 1 + 2\cdot 1 = 0$, so vectors $\mathbf{w},\mathbf{w}$ are orthogonal. □

Since $\cos\theta > 0$, for $0 \leq \theta \leq \pi/2$ and $\cos\theta < 0$ for $\pi/2 \leq \theta \leq \pi$, we have another corollary of Thm. 1.3

Corollary 1.3. *If θ is the angle between two vectors* $\mathbf{v}$ *and* $\mathbf{w}$, *then*

$$\mathbf{v}\cdot\mathbf{w} = \begin{cases} > 0 & \text{for} \quad 0 \leq \theta < \pi/2 \\ 0 & \text{for} \quad \theta = \pi/2 \\ < 0 & \text{for} \quad \pi/2 < \theta \leq \pi \end{cases} \qquad (1.7)$$

Directional angles of a nonzero vector $\mathbf{u}$ are the angles α, β, γ of this vector with the major axes of the coordinate system as in Fig. 1.11. The cosine functions of these angles, $\cos\alpha, \cos\beta, \cos\gamma$, are called **directional cosines** of the vector $\mathbf{u}$.

Using Cor 1.1 we have

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{i}}{\|\mathbf{u}\|\cdot\|\mathbf{i}\|} = \frac{u_1}{\|\mathbf{u}\|} \tag{1.8}$$

and similarly for the other two angles

$$\cos\beta = \frac{\mathbf{u}\cdot\mathbf{j}}{\|\mathbf{u}\|\cdot\|\mathbf{j}\|} = \frac{u_2}{\|\mathbf{u}\|} \qquad \cos\gamma = \frac{\mathbf{u}\cdot\mathbf{k}}{\|\mathbf{u}\|\cdot\|\mathbf{k}\|} = \frac{u_3}{\|\mathbf{u}\|} \tag{1.9}$$

Using equations Eq. (1.8) and Eq. (1.9) , we square them and get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1.10}$$

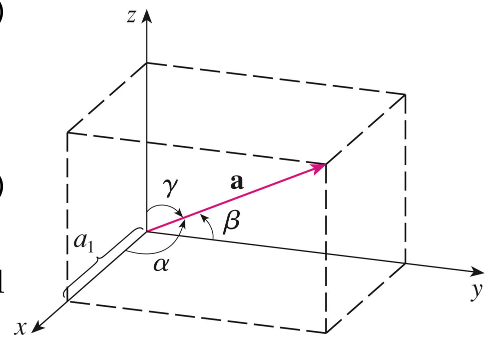


Figure 1.11: Directional angles

For $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ we have $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \|\mathbf{u}\|\cos\alpha \\ \|\mathbf{u}\|\cos\beta \\ \|\mathbf{u}\|\cos\gamma \end{bmatrix} = \|\mathbf{u}\| \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$. Then,

$$\boxed{\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}} \tag{1.11}$$

So, the directional cosines of the vector u are the components of a unit vector with the same direction as $\mathbf{u}$.

Example 1.8. *Determine directional cosines and directional angles for the vector* $\mathbf{u} = [2,1,-4]^t$

Solution: First $\|\mathbf{u}\| = \sqrt{4+1+16} = \sqrt{21}$ then from Eq. (1.8) and Eq. (1.9), we have $\cos\alpha = \frac{2}{\sqrt{21}}$, $\cos\beta = \frac{1}{\sqrt{21}}$, $\cos\gamma = \frac{-4}{\sqrt{21}}$ and respectively $\alpha = 1.119$, $\beta = 1.351$, $\gamma = 2.632$. □

1.1.4 Cross product

Given vectors $\mathbf{u} = [u_1, u_2, u_3]^t$ and $\mathbf{v} = [v_1, v_2, v_3]^t$, then their **cross product** is defined as

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Another way to remember this formula is as the determinant of the 3 by 3 matrix

$$\begin{aligned} \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\,\mathbf{i} + (u_3v_1 - u_1v_3)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k} \end{aligned} \tag{1.12}$$

Let us see an example.

Example 1.9. *For vectors* $\mathbf{u} = [2,1,-1]^t$ *and* $\mathbf{v} = [-3,4,1]^t$, *find* $\mathbf{u}\times\mathbf{v}$ *and* $\mathbf{v}\times\mathbf{u}$.

Solution: From the definition we have

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ -3 & 4 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix}\mathbf{k} = (1+4)\cdot\mathbf{i} - (2-3)\cdot\mathbf{j} + (8+3)\cdot\mathbf{k} = 5\cdot\mathbf{i} + \mathbf{j} + 11\cdot\mathbf{k}\end{aligned}$$

Also, $\mathbf{v}\times\mathbf{u}$

$$\begin{aligned}\mathbf{v}\times\mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 2 & 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 4 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 4 \\ 2 & 1 \end{vmatrix}\mathbf{k} = (-4-1)\,\mathbf{i} - (3-2)\,\mathbf{j} + (-3-8)\,\mathbf{k} = -5\,\mathbf{i} - \mathbf{j} - 11\,\mathbf{k}\end{aligned}$$

□

Theorem 1.4. *If* θ *is the angle between the vectors* $\mathbf{u}$ *and* $\mathbf{v}$, $(0 \le \theta \le \pi)$, *then*

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta. \tag{1.13}$$

Proof. From the definition we have:

$$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 \quad + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta\end{aligned}$$

taking square roots of both sides and keeping in mind that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \ge 0$ when $0 \le \theta \le \pi$, we have

$$\boxed{\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta}$$

□

Corollary 1.4. *Two nonzero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel if and only if* $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

The geometric interpretation of Thm. 1.4 is the area of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$, have the same initial point then they define a parallelogram with base $\|\mathbf{u}\|$ and height $\|\mathbf{v}\|\sin\theta$. Its area is

$$\boxed{S = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta = \|\mathbf{u}\times\mathbf{v}\|} \tag{1.14}$$

Thus, geometrically the magnitude of the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.

Example 1.10. *Find the area of the parallelogram determined by the points* $P=(1,4,6)$, $Q=(-2,5,-1)$, *and* $R=(1,-1,1)$.

Solution: From the discussion above in Thm. 1.4, we have $\overrightarrow{PQ}=\begin{bmatrix}-1-2\\5-4\\-1-6\end{bmatrix}=\begin{bmatrix}-3\\1\\-7\end{bmatrix}$ and $\overrightarrow{PR}=\begin{bmatrix}1-1\\-1-4\\1-6\end{bmatrix}=\begin{bmatrix}0\\-5\\-5\end{bmatrix}$. Their cross product is

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&1&-7\\0&-5&-5\end{vmatrix}=\begin{vmatrix}1&-7\\-5&-5\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3&-7\\0&-5\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3&1\\0&-5\end{vmatrix}\mathbf{k}=-40\,\mathbf{i}-15\,\mathbf{j}+15\,\mathbf{k}$$

and its magnitude $\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\sqrt{(-40)^2+(-15)^2+(15^2)}=5\sqrt{82}$. □

Theorem 1.5. *The cross product of two nonzero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is orthogonal with the vectors* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof. To show that $\mathbf{u}\times\mathbf{v}$ is orthogonal with $\mathbf{u}$, it is enough to show that their dot product is zero. So

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}=\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=u_2v_3u_1-u_3v_2u_1+u_3v_1u_2-u_1v_3u_2+u_1v_2u_3-u_2v_1u_3=0$$

Similarly $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=0$. Thus the cross product is orthogonal with vectors $\mathbf{u}$ and $\mathbf{v}$. □

In the picture it is illustrated the **right hand rule** of determining the direction of the cross product.

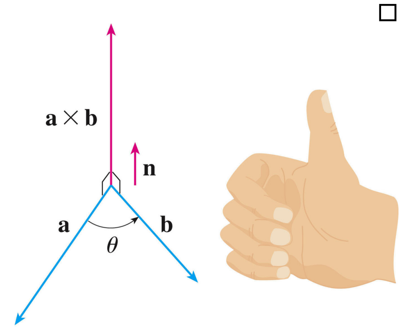


Figure 1.12: Cross product

Example 1.11. *If a plan is defined by the points* $A(1,0,0)$, $B(2,-1,3)$ *and* $C=(1,1,1)$, *find a vector orthogonal with it.*

Solution: We take

$$\overrightarrow{AB}=\begin{bmatrix}2-1\\-1-0\\3-0\end{bmatrix}=\begin{bmatrix}1\\-1\\3\end{bmatrix}\quad\text{and}\quad\overrightarrow{AC}=\begin{bmatrix}1-1\\1-0\\1-0\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}.$$

The cross product is

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&3\\0&1&1\end{vmatrix}=\begin{vmatrix}-1&3\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&3\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\0&1\end{vmatrix}\mathbf{k}=-4\,\mathbf{i}-\mathbf{j}+\mathbf{k}$$

Thus, the vector $-4\,\mathbf{i}-\mathbf{j}+2\,\mathbf{k}$ is orthogonal to the plane passing through A, B, and C □

Theorem 1.6. *For any vectors* $\mathbf{u},\mathbf{v},\mathbf{w}$ *in* V^3, *and* $r\in\mathbb{R}$, *the following are true:*

(i) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$
(ii) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$
(iii) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$
(iv) $(r\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(r\mathbf{v}) = r(\mathbf{u}\times\mathbf{v})$
(v) $\mathbf{u}\times\boldsymbol{0} = \boldsymbol{0}\times\mathbf{u} = \boldsymbol{0}$
(vi) $\mathbf{u}\times\mathbf{u} = \boldsymbol{0}$
(vii) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$
(viii) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

Proof. We will only prove vii), since the rest are easy exxercises. If $\mathbf{u} = [u_1,u_2,u_3]^t$, $\mathbf{v} = [v_1,v_2,v_3]^t$ and $\mathbf{w} = [w_1,w_2,w_3]^t$, then

$$
\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= u_1(v_2w_3-v_3w_2)+u_2(v_3w_1-v_1w_3)+u_3(v_1w_2-v_2w_1)\\
&= u_1v_2w_3-u_1v_3w_2+u_2v_3w_1-u_2v_1w_3+u_3v_1w_2-u_3v_2w_1 \qquad (1.15)\\
&= (u_2v_3-u_3v_2)w_1+(u_3v_1-u_1v_3)w_2+(u_1v_2-u_2v_1)w_3 = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}
\end{aligned}
$$

This completes the proof. □

1.1.5 Mixed product

Given vectors $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3$ with coordinates $\mathbf{u} = [u_1,u_2,u_3]^t$, $\mathbf{v} = [v_1,v_2,v_3]^t$, and $\mathbf{w} = [w_1,w_2,w_3]^t$. The **mixed product** of $\mathbf{u},\mathbf{v},\mathbf{w}$ is called expression $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$. From Eq. (1.15) we notice that

$$
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} \qquad (1.16)
$$

The geometric interpretation is that the absolute value of the mixed product is: the volume of the parallelepiped defined by vectors $\mathbf{u},\mathbf{v},\mathbf{w}$. Thus

$$
\boxed{V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|} \qquad (1.17)
$$

Example 1.12. *Find the volume of the parallelepiped defined by vectors* $\mathbf{u} = [2,1,3]^t$, $\mathbf{v} = [-1,3,2]^t$ *and* $\mathbf{w} = [1,1,-2]^t$.

Solution: We have

$$
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 2 & 1 & 3\\ -1 & 3 & 2\\ 1 & 1 & -2\end{vmatrix} = 2\begin{vmatrix} 3 & 2\\ 1 & -2\end{vmatrix} - 1\begin{vmatrix} -1 & 2\\ 1 & -2\end{vmatrix} + 3\begin{vmatrix} -1 & 3\\ 1 & 1\end{vmatrix} = 2(-8)-1(0)+3(-4) = -28
$$

So $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |-28| = 28$. □

Example 1.13. *Prove that vectors* $\mathbf{u} = [1,4,-7]^t$, $\mathbf{v} = [2,-1,4]^t$ *and* $\mathbf{w} = [0,-9,18]^t$ *lie on the same plane.*

Solution: We have

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 0$$

Since the volume is zero, then the vectors lie on the same plane. □

Exercises:

7. *For given vectors find their cross product and verify if the vectors are orthogonal.*
(i) $\mathbf{u} = [5,2,-1]^t$, $\mathbf{v} = [7,2,-10]^t$
(ii) $\mathbf{u} = [4,4,-3]^t$, $\mathbf{v} = [2,6,4]^t$
(iii) $\mathbf{u} = [1,2,0]^t$, $\mathbf{v} = [1,0,3]^t$
(iv) $\mathbf{u} = [5,1,-1]^t$, $\mathbf{v} = [-1,0,2]^t$
(v) $\mathbf{u} = 3\,\mathbf{i}+2\,\mathbf{j}+4\,\mathbf{k}$, $\mathbf{v} = \mathbf{i}-2\,\mathbf{j}-3\,\mathbf{k}$
(vi) $\mathbf{u} = -\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}$, $\mathbf{v} = -3\,\mathbf{i}+6\,\mathbf{j}+3\,\mathbf{k}$

8. *Find* $\mathbf{u}\times\mathbf{v}$ *and* $\mathbf{v}\times\mathbf{u}$, *for vectors* $\mathbf{u} = [0,1,3]^t$ *and* $\mathbf{v} = [1,1,2]^t$

9. *For vectors* $\mathbf{u} = [3,1,2]^t$, $\mathbf{v} = [-1,1,0]^t$, *and* $\mathbf{w} = [0,0,-4]^t$, *prove that* $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

10. *Find the area of the triangle determined by*
(i) $P = (5,1,-2)$, $Q = (4,-4,3)$, $R = (2,4,0)$
(ii) $P = (4,0,2)$, $Q = (2,1,5)$, $R = (-1,0,-1)$.

11. *Find a unit vector which is orthogonal with vectors* $\mathbf{u} = [1,0,1]^t$ *and* $\mathbf{v} = [1,3,5]^t$.

12. *Prove that* $\mathbf{0}\times\mathbf{u} = \mathbf{u}\times\mathbf{0}$ *for every vector* $\mathbf{u}$ *in* V_3.

13. *Prove that* $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$ *for all vectors in* V_3.

14. *Find the area of the parallelogram with vertices:*
(i) $A(2,1,3)$, $B(1,4,5)$, $C(2,5,3)$, $D(3,2,1)$.
(ii) $A(-2,2)$, $B(1,4)$, $C(6,6)$, *and* $D(3,0)$.
(iii) $A(1,2,3)$, $B(1,3,6)$, $C(3,7,3)$, $D(3,8,6)$.

15. *Find* $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$ *and* $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})$.

16. *Prove that* $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.

17. *The angle between two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is* $\pi/6$ *and* $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$. *Find* $\|\mathbf{u}\times\mathbf{v}\|$.

18. *Find a vector which is orthogonal to the plane passing through* P,Q,R, *and find the area of the triangle* PQR.
(i) $P(3,0,6)$, $Q(2,1,5)$, $R(-1,3,4)$.
(ii) $P(1,2,3)$, $Q(1,0,1)$, $R(-1,3,1)$.
(iii) $P(2,0,-3)$, $Q(5,2,2)$, $R(3,1,0)$.

19. *Find the volume of the parallelepiped determined by the vectors*
(i) $\mathbf{u} = [1,1,3]^t$, $\mathbf{v} = [2,1,4]^t$, $\mathbf{w} = [5,1,-2]^t$
(ii) $\mathbf{u} = [1,3,2]^t$, $\mathbf{v} = [7,2,-10]^t$, $\mathbf{w} = [1,0,1]^t$.

20. *For the given vectors compute* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ *and* $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$.
(i) $\mathbf{u} = [1,1,1]^t$, $\mathbf{v} = [3,0,2]^t$, $\mathbf{w} = [2,2,2]^t$.
(ii) $\mathbf{u} = [1,0,2]^t$, $\mathbf{v} = [-1,0,3]^t$, $\mathbf{w} = [2,0,-2]^t$.

21. *Show that vectors* $\mathbf{u} = 2\,\mathbf{i}+3\,\mathbf{j}+\mathbf{k}$, $\mathbf{v} = \mathbf{i}-\mathbf{j}$, *and* $\mathbf{w} = 7\,\mathbf{i} = 3\,\mathbf{j}+2\,\mathbf{k}$ *are coplanar.*

22. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are unit vectors in* V_3, *when is the vector* $\mathbf{v}\times\mathbf{w}$ *also a unit vector?*

23. *Prove that if* $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ *for all* $\mathbf{v}$ *in* V_3, *then* $\mathbf{v} = \mathbf{0}$.

24. *Prove that for all vectors* $\mathbf{v},\mathbf{w}$ *in* V_3:

$$\|\mathbf{v}\times\mathbf{w}\| + |\mathbf{v}\cdot\mathbf{w}| = \|\mathbf{v}\|^2\cdot\|\mathbf{w}\|^2.$$

25. *Given* $\mathbf{u},\mathbf{v},\mathbf{x}\in\mathbb{R}^3$ *such that* $\mathbf{u}\times\mathbf{x} = \mathbf{v}$, *where* $\mathbf{u}\neq\mathbf{0}$. *Prove that*
(i) $\mathbf{u}\cdot\mathbf{v} = 0$
(ii) $\mathbf{x} = \frac{\mathbf{v}\times\mathbf{u}}{\|\mathbf{u}\|^2} + \lambda\mathbf{u}$ *is a solution of the equation* $\mathbf{u}\times\mathbf{x} = \mathbf{v}$ *for every scalar* $\lambda\in\mathbb{R}$.

26. *Prove the Jacobi identity*

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}.$$

27. *For all vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *in* V_3*, prove that*

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b}))\mathbf{c} - (\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}))\mathbf{d}$$

1.1.6 Equation of lines

A line in $\mathbb{R}^3$ is uniquely determined when it passes through a point P and has a given direction. Let $P = (x_0, y_0, z_0)$ a point in $\mathbb{R}^3$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a nonzero vector. Denote by L the line passing through P and is parallel to a vector $\mathbf{v}$; Fig. 1.13. Denote by $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ the vector $\vec{OP}$. The vector

$$\mathbf{r}(t) := \mathbf{r}_0 + t\cdot\mathbf{v}, \quad \text{for} \quad t \in \mathbb{R} \tag{1.18}$$

determines every point of the line L. Hence,

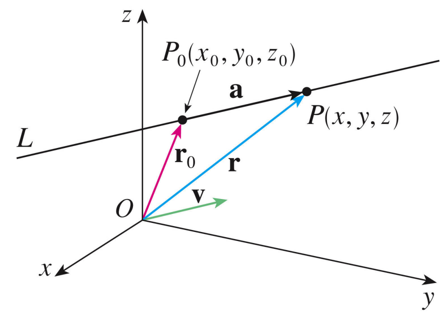


Figure 1.13: Equation of lines

Lemma 1.3. *For a given point* $P = (x_0, y_0, z_0)$ *and a nonzero vector* $\mathbf{v} \in \mathbb{R}^3$*, the line* L *which passes through* P *and is parallel with the vector* $\mathbf{v}$ *has equation*

$$\boxed{r(t) = r_0 + t\cdot\mathbf{v}, \quad \text{for} \quad r_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad \text{and} \quad t \in (-\infty, \infty)} \tag{1.19}$$

Notice the correspondence between a the vector and its endpoint. Since $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then its endpoint $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ is the point

$$(x_0 + at, y_0 + bt, z_0 + ct).$$

Hence, we have a parametric representation of the line L in terms of the parameter t:

For a point $P(x_0, y_0, z_0)$ and a nonzero vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the line L, passing through P and parallel $\mathbf{v}$, consists in all points (x, y, z) such that

$$\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty} \tag{1.20}$$

Notice that in the above two interpretations, the point P is obtained when $t = 0$. Coordinates a, b, c are called **directional numbers** and the vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is called the **directional vector** of the line L.

If in Eq. (1.20), we have that $a \neq 0$, then solving for t, we have: $t = \frac{x-x_0}{a}$. We can also solve for y or z with the condition that b or c are nonzero. So we have $t = \frac{y-y_0}{b}$ or $t = \frac{z-z_0}{c}$. Hence,

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \tag{1.21}$$

If $a = 0$ then $x = x_0 + at$, hence $x = x_0 + 0 \cdot t = x_0$. Then, we have

$$x = x_0 \qquad \frac{y-y_0}{b} = \frac{z-z_0}{c} \tag{1.22}$$

Hence, the line L is on the plane $x = x_0$. Similarly for $b = 0$, or $c = 0$.

Example 1.14. *Find the equation of the line L passing through $P(2,3,5)$ and parallel to the vector $\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$, in all three forms. Find two points of L different from P.*

Solution: Denote by $\mathbf{r}_0 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and from Eq. (1.19), the line L has equation

$$\mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}, \quad \textit{for } -\infty < t < \infty.$$

For the parametric form , L consists of all points (x, y, z) such that

$$x = 2 + 4t, \quad y = 3 - t, \quad z = 5 + 6t, \quad \textit{for } -\infty < t < \infty$$

The symmetric equation of L is all points (x, y, z) such that

$$\frac{x-2}{4} = \frac{y-3}{-1} = \frac{z-5}{6}$$

Taking $t = 2$ and $t = 3$ we get $(10, 1, 17)$ and $(14, 0, 23)$ in L. □

The line going through two points

Given $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ two distinct points in $\mathbb{R}^3$ and L the line going through them. Denote by

$$\mathbf{r}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

two vectors with endpoints P_1 and P_2. Then, from Fig. 1.14, $\mathbf{r}_2 - \mathbf{r}_1$ is the vector from P_1 to P_2. Thus, $\overrightarrow{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1$. If we multiply $\mathbf{r}_2 - \mathbf{r}_1$ with a scalar t, and add that to the vector $\mathbf{r}_1$, we will have the line L for all values of t in $\mathbb{R}$. Thus points of the line are given by

$$\mathbf{r}(t) = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1),$$

for $t \in \mathbb{R}$. Then, the vector, parametric, or symmetric, equation of the line passing through P_1 and P_2 are.

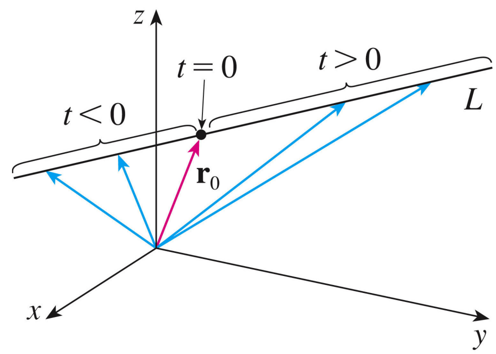


Figure 1.14: Equation of the line

Vector equation:

$$\mathbf{r}(t) = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1), \quad \textit{for } -\infty < t < \infty \tag{1.23}$$

Parametric equation:

$$x = x_1 + (x_2 - x_1) \cdot t, \quad y = y_1 + (y_2 - y_1) \cdot t, \quad z = z_1 + (z_2 - z_1) \cdot t, \quad \textit{for} -\infty < t < \infty \tag{1.24}$$

Symmetric equation:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \textit{for } x_1 \neq x_2,\ y_1 \neq y_2, \text{ and } z_1 \neq z_2 \tag{1.25}$$

1.1.7 Planes

Let $\mathbf{n} = [a, b, c]^t$ be a nonzero vector which is orthogonal to the plane P. Such vector is called **normal vector** of the plane. Let (x, y, z) be a point of P. Then, the vector $\mathbf{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$ is on the plane P; see Fig. 1.15.

Thus if $\mathbf{r} \neq \mathbf{0}$, then $\mathbf{r} \perp \mathbf{n}$, and so $\mathbf{n} \cdot \mathbf{r} = 0$. If $\mathbf{r} = \mathbf{0}$ then we have $\mathbf{r} \cdot \mathbf{n} = 0$. Conversely, if (x, y, z) is a point in $\mathbb{R}^3$ such that $\mathbf{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \neq \mathbf{0}$ and $\mathbf{n} \cdot \mathbf{r} = 0$, then $\mathbf{r} \perp \mathbf{n}$ and (x, y, z) is in P. Thus, we have:

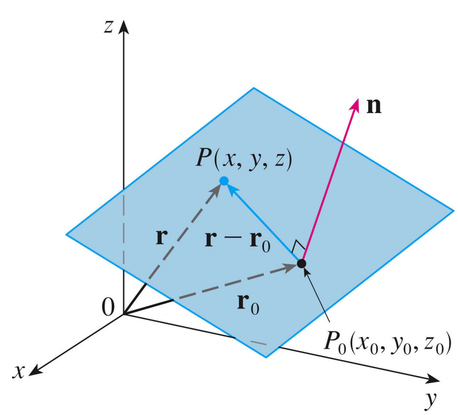


Figure 1.15: A normal vector with the plane

Lemma 1.4. *Let P be a plane and (x_0, y_0, z_0) a point in P. Let $\boldsymbol{n} = [a, b, c]^t$ be a nonzero vector orthogonal to the plane P. Then, the plane P consists of all points (x, y, z) such that*

$$\boldsymbol{n} \cdot \boldsymbol{r} = 0 \tag{1.26}$$

where $\boldsymbol{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$; *or*

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \tag{1.27}$$

Eq. (1.26) is called **vector equation** of the plane and Eq. (1.27) is called **scalar equation** of the plane. Expanding Eq. (1.27) we get

$$\boxed{ax + by + cz + d = 0} \tag{1.28}$$

where $d = -(ax_0 + by_0 + cz_0)$. Eq. (1.28) is called **linear equation of the plane**.

Example 1.15. *Find the equation of the plane passing through $Q(1,3,2)$, $R(3,-1,6)$ and $S(5,2,0)$.*

Solution: Vectors $\overrightarrow{QR}$ and $\overrightarrow{QS}$ are given by $\overrightarrow{QR} = [2, -4, 4]^t$ and $\overrightarrow{QS} = [4, -1, -2]^t$. Since these vectors are on the plane, their cross product is orthogonal to the plane and it is a normal vector of the plane. Thus

$$\mathbf{n} = \overrightarrow{QR} \times \overrightarrow{QS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{i} + 14\,\mathbf{k}$$

With the point Q and normal vector $\mathbf{n}$, the equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

Thus $6x + 10y + 7z = 50$. □

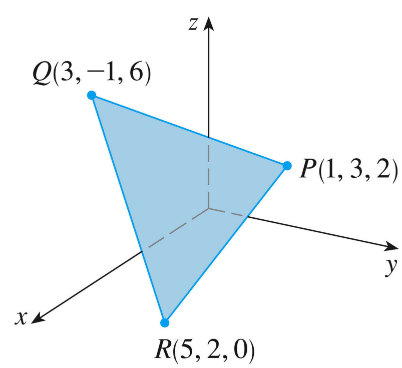


Figure 1.16

Two planes are **parallel** if their normal vectors are parallel . If planes are not parallel , then they intersect along a line. The **angle between two planes** is called the angle between their normal vectors.

Example 1.16. (i) *Find the angle between two planes $x + y + z = 1$ and $x - 2y + 3z = 1$.*

(ii) *Find the equation of the line of intersection between these two planes.*

Solution: i) Normal vectors are $\mathbf{n}_1 = \begin{bmatrix} 1\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ Then the angle is

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|} = \frac{2}{\sqrt{42}}.$$

Thus, $\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$. Part ii) is left as an exercise. □

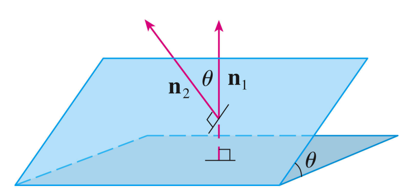


Figure 1.17

1.1.8 The distance between a point and a plane

The distance between a point and a plane is the length of the orthogonal line from the given point to the point of intersection with the plane.

Lemma 1.5. *Let $P_1(x_1, y_1, z_1)$ be a point and P a plan with equation l $ax+by+cz+d=0$, which does not contain P_1. Then, the distance of P_1 from P is:*

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \tag{1.29}$$

Proof. Let $P_0(x_0, y_0, z_0)$ be a point of the plane P, and denote by $\mathbf{v}$ the corresponding vector $\overrightarrow{P_0P_1}$. Then,

$$\mathbf{v} = \begin{bmatrix} x_1-x_0 \\ y_1-y_0 \\ z_1-z_0 \end{bmatrix}$$

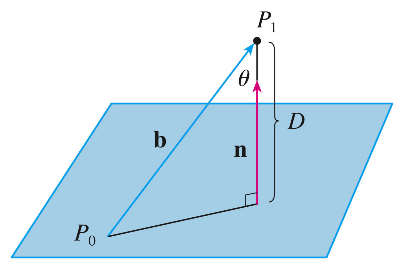


Figure 1.18: Distance of the point to the plane

From Fig. 1.18 we can see that the distance D from P_1 to the plane P, is the magnitude of the projection of $\mathbf{v}$ over the normal vector $\mathbf{n} = [a,b,c]^t$.

Thus

$$d = \left\|\text{proj}_{\mathbf{v}}(\mathbf{n})\right\| = \frac{\mathbf{n}\cdot\mathbf{v}}{\|\mathbf{n}\|} = \frac{\left|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)\right|}{\sqrt{a^2+b^2+c^2}} = \frac{\left|(ax_1+by_1+cz_1)-(ax_0+by_0+cz_0)\right|}{\sqrt{a^2+b^2+c^2}}$$

Since P_0 is in the plane, then it satisfies the equation of the plane. Hence, $ax_0+by_0+cz_0+d=0$, from which we have $(ax_0+by_0+cz_0) = -d$. Therefore, the distance D is

$$\boxed{d = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

□

Example 1.17. *Find the distance of the point $(2,4,-5)$ to the plane $5x-3y+z-10=0$.*

Solution: Using the above formula we have

$$D = \frac{|5(2)-3(4)+1(-5)-10|}{\sqrt{5^2+(-3)^2+1^2}} = \frac{|-17|}{\sqrt{35}} = \frac{17}{\sqrt{35}} \approx 2.87$$

□

Example 1.18. *Find the distance between the two planes* $10x+2y-2z=5$ *and* $5x+y-z=1$

Solution: Normal vectors of these two planes are $\begin{bmatrix}10\\2\\-2\end{bmatrix}$ and $\begin{bmatrix}5\\1\\-1\end{bmatrix}$. They are parallel, and therefore planes are parallel . To find the distance, it is enough to take a point in one of the planes and find its distance to the other plane using the formula (1.29).

Take the point $\left(\frac{1}{2},0,0\right)$ in the first plane. Then we have

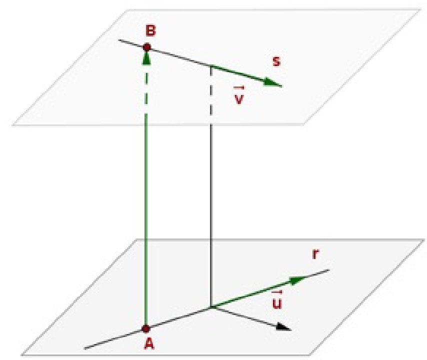
$$D = \frac{\left|5\cdot\frac{1}{2}+1\cdot 0-1\right|}{\sqrt{5^2+1^2+(-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

□

Two lines which are not in the same plane and do not intersect ate called **skew lines**

Example 1.19. *Given two lines with parametric equations as follows*

$$L_1: \quad x = 1+t, \quad y = -2+3t, \quad z = 4-t$$
$$L_2: \quad x = 2s, \quad y = 3 = s, \quad z = -3+4s$$



Prove that these are skew lines. Find the distance between them.

Solution: The lines are not parallel because their directional vectors

$$\mathbf{u} = \begin{bmatrix}1\\3\\-1\end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix}2\\1\\4\end{bmatrix},$$

are not parallel . They also do not intersect because the system

$$\begin{cases} 1+t = 2s \\ -2+3t = 3+s \\ 4-t = -3+4s \end{cases}$$

has no solutions. Thus, these are skew lines.

Since they do not intersect we can consider them in two parallel planes, say P_1 and P_2. The distance between L_1 and L_2 is is the same as the distance between P_1 and P_2, which can be found as follows.

A normal vector with these two planes must be orthogonal with vectors **u** and **v**. Thus a normal vector could be their cross product. Thus,

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & -1\\ 2 & 1 & 4\end{vmatrix} = 13\,\mathbf{i} - 6\,\mathbf{j} - 5\,\mathbf{k}$$

Now we can find the equation of each plane, say P_2.

Take a point in L_2 by choosing $s = 0$. Then the point $(0,3,-3)$ is in L_2 and therefore in P_2. Thus, the equation for P_2 is

$$13(x-0) - 6(y-3) - 5(z+3) = 0$$

or $13x-6y-5z+3=0$. Taking $t=0$ in the equation for L_1 we find the point $(1,-2,4)$ in P_1. Thus the distance between the lines L_1 and L_2 is the same as the distance from the point $(1,-2,4)$ to the plane $13-6y-5z+3=0$. From above we have

$$D = \frac{|13\cdot 1 - 6(-2) - 5\cdot 4 + 3|}{13^2 + (-6)^2 + (-5)^2} = \frac{8}{\sqrt{230}}.$$

□

Exercises:

28. *Determine if the lines L_1 and L_2 are parallel, intersect, or are skew lines.*

(i) $L_1 : \mathbf{u}(t) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $L_2 : \mathbf{v}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

(ii) $L_1 : \mathbf{u}(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, $L_2 : \mathbf{v}(t) = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$

29. *Is the line passing through points $P_1(-4,-6,1)$ and $P_2(-2,0,-3)$ parallel to the line passing through the points $Q_1(10,18,4)$ and $Q_2(5,3,14)$?*

30. *Find a and c such that the point $(a,1,c)$ is on the line passing through the points $P(0,2,3)$ and $Q(2,7,5)$.*

31. *Find the equation of the plane which contains the point $(-1,2,-3)$ and is orthogonal to the vector* $[4,5,-1]^t$.

32. *Find the equation of the plane which contains the point $(6,3,2)$ and is orthogonal to the vector* $[-2,1,5]^t$.

33. *Find the equation of the plane which contains the point $(4,0,-3)$ and has normal vector* $\mathbf{j}+2\mathbf{k}$.

34. *Find the equation of the plane which contains the point $(5,1,-2)$ and has normal vector* $[4,-4,3]^t$.

35. *Find the equation of the plane which passes through the point $(-2,8,10)$ and is orthogonal with the line $x=1=t, y=2t, z=4-3t$.*

36. *Find the equation of the plane which passes through the point $(4,-2,3)$ and is parallel with the plane $3x-7z=12$.*

37. *Find the equation of the plane which passes through points $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$.*

38. *Find the equation of the plane which passes through points $(1,0,3)$, $(2,01)$, and $(3,3,1)$.*

39. *Find the equation of the plane which contains the point $(1,0,0)$ and the line* $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

40. *Find the equation of the plane which passes through the origin and is orthogonal to the plane $x+y-z=2$.*

41. *Find the equation of the plane which passes through the point $-1,2,1$ and contains the intersection line of the two planes $x=y-z=2$ and $2x-y+3z=1$.*

42. *Find the intersection line of the two planes:*

(i) $x+3y-3z-6=0$ *and* $2x-y+z+2=0$.

(ii) $3x+y-5z=0$ *and* $x+2y+z+4=0$.

43. *Find point of intersection of the line $\frac{x-6}{4} = y+3 = z$ with the plane $x+3y+2z-6=0$.*

44. *Find point of intersection of the line $x=y-1=2z$ with the plane $4x-y+3z=8$.*

45. *Find point of intersection of the line* $x = 1 + 2t, y = 4t, z = 2 - 3t$ *with the plane* $x + 2y - z + 1 = 0$.

46. *How can we find the the angle between two planes? Find the angle between the two planes* $x + y + z = 2$ *and* $x + 2y + 3z = 8$.

47. *Find cosine of the angle between two planes* $x + y + z = 0$ *and* $x + 2y + 3z = 1$.

48. *Find the lengths of the sides of the triangle with vertices* $A(3,-2,1)$, $B(1,2,-3)$, $C(3,4,-2)$. *Determine if this triangle is regular.*

49. *Finds the distance of the point* $(-5,3,4)$ *from each coordinate plane.*

50. *Find the magnitude of the force which has its projections on the coordinate axis as* $x = -6$, $y = -2$, *and* $z = 9$.

51. *Prove that the triangle with vertices* $A(1,-2,1)$ $B(3,-3,1)$ *and* $C(4,0,3)$ *is a right triangle.*

52. *Find the equation of the sphere with center at the point* $(4,-2,3)$ *and radius* $r = \sqrt{3}$.

53. *Find the equation of the sphere with center at the point* $(-1,3,2)$ *and radius* $r = \sqrt{3}$.

54. *Find the equation of the sphere with center at the point* $(2,3,4)$ *and radius 5. Where does the sphere intersect the coordinate planes?*

55. *Find the equation of the sphere which passes through the point* $(4,3,-1)$ *and has the center at* $(3,8,1)$.

56. *Prove that the following equations represent a sphere, find its center and its radius.*

(i) $x^2 + y^2 + z^2 - 6x + 4y + 2z = -17$
(ii) $x^2 + y^2 + z^2 = 4x - 2y$
(iii) $x^2 + y^2 + z^2 = x + y + z$
(iv) $x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$
(v) $16x^2 + 16y^2 + 16z^2 - 96x + 32y = 5$

57. *(a) Prove that the middle of the segment which is determined by the points* $A(a_1, b_1, c_1)$ *and* $B(a_2, b_2, c_2)$ *is the point with coordinates*

$$\left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}, \frac{c_1 + c_2}{2} \right)$$

(b) Find the lengths of the three medians of the triangle with vertices $A(4,1,5)$, $B(1,2,3)$, $C(-2,0,5)$.

Determine the inequalities which determine the following regions.

58. *The region between the plane* xy *and* $z = 5$.

59. *The region which consists of all points between spheres of radii* r *and* R *with center at the origin, where* $r < R$.

60. *Find the equation of the sphere with has the same center with* $x^2 + y^2 + z^2 - 6x + 4z - 36 = 0$ *and passes through the point* $(2,5,-7)$.

61. *Prove that the set of all points whose distance from* $A(-1,5,3)$ *is twice the distance from* $B(6,2,-2)$, *is a sphere.*

62. *Determine an equation for the set of points equidistant from* $A(-1,5,3)$ *and* $B(6,2,-2)$.

63. *Draw the vector* $\overrightarrow{AB}$, *when* A *and* B *are given as below and find its equivalent with the initial point at the origin.*

(i) $A = (0,3,1), \quad B = (2,3,-1)$
(ii) $A = (4,0,-2), \quad B = (4,2,1)$
(iii) $A = (2,0,3), \quad B = (3,4,5)$
(iv) $A = (0,3,-2), \quad B = (2,4,-1)$

64. *Find* $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} - 3\mathbf{b}$, $\|\mathbf{a}\|$ *and* $\|\mathbf{a} - \mathbf{b}\|$, *if* $\mathbf{a} = [5,-12]^t$ *and* $\mathbf{b} = [3,6]^t$.

65. *Find* $\mathbf{a} - \mathbf{b}$, $\mathbf{a} + 2\mathbf{b}$, $\|\mathbf{a}\|$ *and* $\|\mathbf{a} - \mathbf{b}\|$, *if* $\mathbf{a} = [1,2,-3]^t$ *and* $\mathbf{b} =]-2,-1,5]^t$.

66. *Find* $\mathbf{a} + \mathbf{b}$, $3\mathbf{a} - 2\mathbf{b}$, $\|\mathbf{a}\|$ *and* $\|\mathbf{a} - \mathbf{b}\|$, *if* $\mathbf{a} = [2,-4,4]^t$ *and* $\mathbf{b} = [0,2,-1]^t$.

67. *Find* $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$, $\|\mathbf{v}\|$ *and* $\|\mathbf{v} - \mathbf{w}\|$, $\|\mathbf{v} + \mathbf{w}\|$, *and* $-2\mathbf{v}$, *if* $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$.

68. *Find the unit vector which has the same direction with the vector* $-3\mathbf{i}+7\mathbf{j}$.

69. *Find the unit vector which has the same direction with the vector* $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

70. *Find the unit vector which has the same direction with the vector* $\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$.

71. *Find a vector which has the same direction with the vector* $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, *but has length 3*

72. *Find a vector which has the same direction with the vector* $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$, *but has length 6.*

73. *Let be given the vectors* $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(i) *Find the vector* $\mathbf{u}$ *such that* $\mathbf{u}+\mathbf{v}+\mathbf{w} = \mathbf{i}$.

(ii) *Find the vector* $\mathbf{u}$ *such that* $\mathbf{u}+\mathbf{v}+\mathbf{w} = 2\mathbf{j}+\mathbf{k}$.

74. *If* A, B, C *are vertices of a triangle, find* $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$.

75. *Draw the vectors* $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. *Determine graphically if there exist the scalars s and t such that* $\mathbf{w} = s\mathbf{u}+t\mathbf{v}$. *Find the values for s and t.*

76. *Let be given* $\mathbf{u}$ *and* $\mathbf{v}$ *two nonzero vectors not parallel in* $\mathbb{R}^2$. *Prove that if* $\mathbf{w}$ *is any vector in* $\mathbb{R}^2$, *then there exist two scalars s and t such that* $\mathbf{w} = s\mathbf{u}+t\mathbf{v}$.

77. *Is the property from the previous problem true for* $\mathbb{R}^3$? *Explain.*

78. *Let* $\mathbf{a} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{a}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, *and* $\mathbf{a}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. *Describe the set of all points (x,y,z) which satisfy*

$$\|\mathbf{a}-\mathbf{a}_1\| + \|\mathbf{a}-\mathbf{a}_2\| = \lambda,$$

where $\lambda > \|\mathbf{a}_1 - \mathbf{a}_2\|$.

1.2 Euclidean n- space $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the following Cartesian product

$$\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

A vector $\mathbf{u}$ in $\mathbb{R}^n$ will be defined as an ordered tuple $(u_1, \dots, u_n)$ for $u_i \in \mathbb{R}$, $i = 1, \dots, n$ and denoted by $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$.

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such as $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ we define the **vector addition** and **scalar multiplication** as follows:

$$\mathbf{u}+\mathbf{v} := \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}, \qquad r\,\mathbf{v} := \begin{bmatrix} rv_1 \\ \vdots \\ rv_n \end{bmatrix}. \tag{1.30}$$

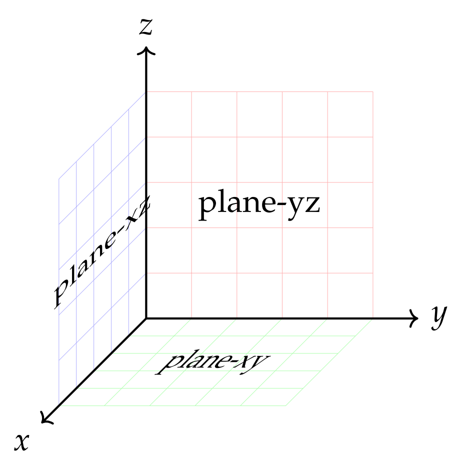


Figure 1.19: Euclidean space $\mathbb{R}^3$.

A **Euclidean** n**-space** is the set of vectors together with **vector addition** and **scalar multiplication** defined as above. Elements of $\mathbb{R}^n$ are called **vectors** and all $r \in \mathbb{R}$ are called **scalars**. The vector $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is called the **zero vector**. By a vector $\mathbf{u}$ we usually mean a **column vector** unless otherwise stated. The row vector $[u_1, \ldots, u_n]$ is called the **transpose** of $\mathbf{u}$ and denoted by

$$\mathbf{u}^t = [u_1, \ldots, u_n]$$

For the addition and scalar multiplication we have the following properties.

Theorem 1.7. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and* r, s *scalars in* $\mathbb{R}$. *The following are satisfied:*

(i) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,
(ii) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,
(iii) $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$,
(iv) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$,
(v) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$,
(vi) $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$,
(vii) $(rs)\mathbf{u} = r(s\mathbf{u})$,
(viii) $1\mathbf{u} = \mathbf{u}$.

Proof. Exercise. □

Two vectors $\mathbf{v}$ and $\mathbf{u}$ are called **parallel** if there exists an $r \in \mathbb{R}$ such that $\mathbf{v} = r\mathbf{u}$.

Definition 1.2. *Given vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \mathbb{R}^n$ *and* $r_1, \ldots, r_s \in \mathbb{R}$, *the vector*

$$r_1\mathbf{v}_1 + \cdots + r_s\mathbf{v}_s$$

is called a ***linear combination of vectors*** $\mathbf{v}_1, \ldots, \mathbf{v}_s$.

Definition 1.3. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_s$ *be vectors in* $\mathbb{R}^n$. *The* ***span*** *of these vectors, denoted by* $Span\,(\mathbf{v}_1, \ldots, \mathbf{v}_s)$, *is the set in* $\mathbb{R}^n$ *of all linear combinations of* $\mathbf{v}_1, \ldots, \mathbf{v}_s$.

$$Span\,(\mathbf{v}_1, \ldots, \mathbf{v}_s) = \left\{ r_1\mathbf{v}_1 + \cdots + r_s\mathbf{v}_s \mid r_i \in \mathbb{R}, i = 1, \ldots, s \right\}$$

Exercise 10. *Let* $V = \mathbb{R}^3$ *be the 3-dimensional Euclidean space and*

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

vectors in V. *Determine* $Span\,(\mathbf{i}, \mathbf{j})$. *What about* $Span\,(\mathbf{i}, \mathbf{j}, \mathbf{k})$*?*

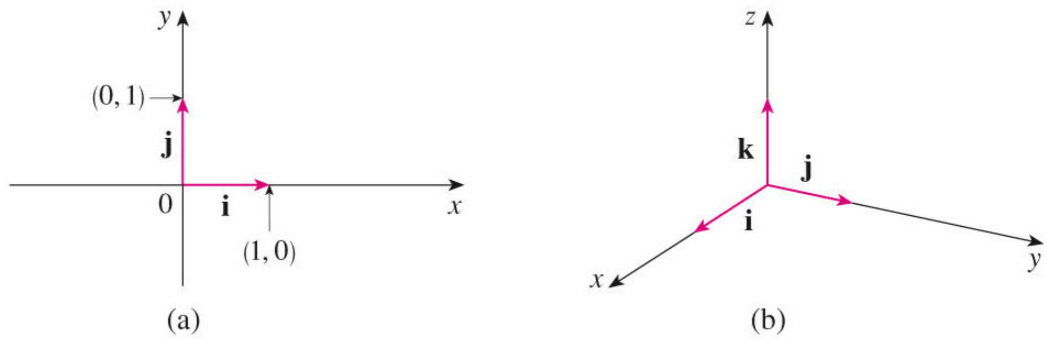


Figure 1.20: Standard basis for $\mathbb{R}^2$ and $\mathbb{R}^3$

Proof. If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then

$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$$

Hence, every vector in $\mathbb{R}^3$ can be expressed as a linear combination of vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Therefore, Span $(\mathbf{i}, \mathbf{j}, \mathbf{k}) = \mathbb{R}^3$. □

Definition 1.4. *Vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are called **linearly independent** if*

$$r_1 \mathbf{u}_1 + \cdots + r_n \mathbf{u}_n = 0$$

implies that

$$r_1 = \cdots = r_n = 0,$$

*otherwise, we say that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are **linearly dependent**.*

Exercise 11. *Prove that $\mathbf{i}, \mathbf{j}, \mathbf{k}$, given above, are linearly independent.*

In the coming sections we will see that the concept of linear independence is one of the most important concepts of linear algebra. Our strategy will be to try to generalize all concepts of $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$. Of course the geometric interpretation in $\mathbb{R}^n$ doesn't make sense, but this will not deter us to assign the same names to abstract concepts in $\mathbb{R}^n$ as we had for $\mathbb{R}^2$ and $\mathbb{R}^3$.

1.2.1 Subspaces of $\mathbb{R}^n$

A subset $U \subset \mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if the following hold:

(i) $\mathbf{0} \in U$
(ii) $\forall \mathbf{u}, \mathbf{v} \in U$, $\mathbf{u} + \mathbf{v} \in U$
(iii) $\forall \lambda \in \mathbb{R}$, $\forall \mathbf{u} \in U$, we have that $\lambda \mathbf{u} \in U$.

Property ii) is usually referred to as U is **closed under addition** and property iii) as U is **closed under scalar multiplication**. A subspace U of $\mathbb{R}^n$ is called **proper** if $U \neq \{0\}$ and $U \neq \mathbb{R}^n$. The concept of a subspace is very important and we will study it in detail in the next chapter.

Exercise 12. *Prove that every line and every plane in $\mathbb{R}^3$ which passes through the point $O(0,0,0)$ is a subspace.*

Lemma 1.6. *Let $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{R}^n$. Prove that Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is a subspace of $\mathbb{R}^n$.*

Proof. The zero vector $\mathbf{0}$ is in Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ since it can be written as

$$\mathbf{0} = 0\,\mathbf{u}_1 + \cdots + 0, u_n.$$

Let $\mathbf{v}_1, \mathbf{v}_2 \in$ Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$. Then exist scalar $r_1, \dots r_n$ and $s_1 \dots s_n$ such that

$$\mathbf{v}_1 = r_1\mathbf{u}_1 + \cdots r_n\mathbf{u}_n \quad \text{and} \quad \mathbf{v}_2 = s_1\mathbf{u}_1 + \cdots s_n\mathbf{u}_n$$

Thus

$$\mathbf{v}_1 + \mathbf{v}_2 = (r_1 + s_1)\mathbf{u}_1 + \cdots + (r_n + s_n)\mathbf{u}_n$$

is also a vector in Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$. Hence, Span $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is closed under addition. Similarly we show that it is also closed under scalar multiplication.

□

Exercise 13. *Let P be a plane in $\mathbb{R}^3$ with equation*

$$ax + by + cz = d.$$

Determine the values of a, b, c, d such that the set of points of P forms a subspace of $\mathbb{R}^3$.

Solution: For P to be a subspace the vector $\mathbf{0}$ must be in P. Hence, point $O(0,0,0)$ must be in P. This implies that $d = 0$. The plane P is closed under addition and scalar multiplication since the sum of any two vectors is on the same plane determined by the two vectors (similarly for the multiplication by a scalar).

□

Exercise 14. *From Exe. 10 we know that every vector $\mathbf{u} \in \mathbb{R}^2$, such that $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, can be written as $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$. Using this fact, can you determine all subspaces of $\mathbb{R}^2$?*

1.2.2 Norm and dot product

In this section we study two very important concepts of Euclidean spaces; that of the dot product and the norm. The concept of the dot product will be generalized later to that of inner product for any vector space.

Definition 1.5. *Let* $\mathbf{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$. *The **norm** of $\mathbf{u}$, denoted by $\|\mathbf{u}\|$, is defined as*

$$\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

The norm has the following properties:

Theorem 1.8. *For any vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and any scalar* $r \in \mathbb{R}$ *the following are true:*

(i) $\|\mathbf{u}\| \geq 0$ *and* $\|\mathbf{u}\| = 0$ *if and only if* $\mathbf{u} = 0$

(ii) $\|r\mathbf{u}\| = |r| \, \|\mathbf{u}\|$

(iii) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Proof. The proof of i) and ii) are easy and left as exercises. The proof of iii) is completed in Lem. 1.9

□

A **unit vector** is a vector with norm 1. Notice that for any nonzero vector $\mathbf{u}$ the vector $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ is a unit vector. Let

$$\mathbf{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} := \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

be vectors in $\mathbb{R}^n$. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ (sometimes called the **inner product**) is defined as follows:

$$\boxed{\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n,} \tag{1.31}$$

or sometimes denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$. Notice the identity $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$, which is very useful.

Lemma 1.7. *The dot product has the following properties:*

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(ii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(iii) $r(\mathbf{u} \cdot \mathbf{v}) = (r\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (r\mathbf{v})$

(iv) $\mathbf{u} \cdot \mathbf{u} \geq 0$, *and* $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = 0$

Proof. Use the definition of the dot product to check all i) through iv).

□

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are called **perpendicular** if $\mathbf{u} \cdot \mathbf{v} = 0$.

Lemma 1.8 (Cauchy-Schwartz inequality)**.** *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any vectors in* $\mathbb{R}^n$*. Then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$$

Proof. If one of the vectors is the zero vector, then the inequality is obvious. So we assume that $\mathbf{u}, \mathbf{v}$ are nonzero. For any $r, s \in \mathbb{R}^n$ we have $\|r\mathbf{v} + s\mathbf{u}\| \geq 0$. Then,

$$\|r\mathbf{v} + s\mathbf{u}\|^2 = (r\mathbf{v} + s\mathbf{u}) \cdot (r\mathbf{v} + s\mathbf{u}) = r^2 (\mathbf{v} \cdot \mathbf{v}) + 2rs (\mathbf{v} \cdot \mathbf{u}) + s^2 (\mathbf{u} \cdot \mathbf{u}) \geq 0$$

Take $r = \mathbf{u} \cdot \mathbf{u}$ and $s = -\mathbf{v} \cdot \mathbf{u}$. Substituting in the above we have:

$$\begin{aligned} \|r\mathbf{v} + s\mathbf{u}\|^2 &= (\mathbf{u} \cdot \mathbf{u})^2 (\mathbf{v} \cdot \mathbf{v}) - 2(\mathbf{u} \cdot \mathbf{u}) (\mathbf{v} \cdot \mathbf{u})^2 + (\mathbf{v} \cdot \mathbf{u})^2 (\mathbf{u} \cdot \mathbf{u}) \\ &= (\mathbf{u} \cdot \mathbf{u}) \left[(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{u})^2 \right] \geq 0 \end{aligned}$$

Since $(\mathbf{u} \cdot \mathbf{u}) = \|\mathbf{u}\|^2 > 0$ then $\left[(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{u})^2 \right] \geq 0$. Hence,

$$(\mathbf{v} \cdot \mathbf{u})^2 \leq (\mathbf{u} \cdot \mathbf{u}) (\mathbf{v} \cdot \mathbf{v}) = \|\mathbf{u}\|^2 \cdot \|\mathbf{v}\|^2$$

and $|\mathbf{u}\cdot\mathbf{v}| \leq ||\mathbf{u}||\cdot||\mathbf{v}||$.

□

Lemma 1.9 (Triangle inequality). *For any two vectors* $\mathbf{u}, \mathbf{v}$ *in* $\mathbb{R}^n$ *the following hold*

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}||+||\mathbf{v}||$$

Proof. We have

$$\begin{aligned}
||\mathbf{u}+\mathbf{v}||^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= (\mathbf{u}\cdot\mathbf{u})+2(\mathbf{u}\cdot\mathbf{v})+(\mathbf{v}\cdot\mathbf{v}) = ||\mathbf{u}||^2+2(\mathbf{u}\cdot\mathbf{v})+||\mathbf{v}||^2 \leq ||\mathbf{u}||^2+2\,|\mathbf{u}\cdot\mathbf{v}|+||\mathbf{v}||^2 \\
&\leq ||\mathbf{u}||^2+2\cdot||\mathbf{u}||\cdot||\mathbf{v}||+||\mathbf{v}||^2 = (||\mathbf{u}||+||\mathbf{v}||)^2
\end{aligned}$$

Hence, $||\mathbf{v}+\mathbf{u}|| \leq ||\mathbf{v}||+||\mathbf{u}||$.

□

Example 1.20. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two given vectors and* θ *the angle between them. Prove that*

$$\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||\cdot||\mathbf{v}||\cos\theta$$

Hence, we have the following definition. The **angle** between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined to be

$$\theta := cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\cdot||\mathbf{v}||}\right)$$

From Lem. 1.8 we have that

$$-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\cdot||\mathbf{v}||} \leq 1$$

Hence, the angle between two vectors is well defined.

Example 1.21. *Find the angle between* $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Solution: *Using the above formula we have* $\theta = \cos^{-1}\left(\frac{(2,-1,2)\cdot(-1,-1,1)}{\sqrt{9}\cdot\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{9}\right)$. *Then* $\theta \approx 1.377$ *radians or* $\theta \approx 78.90°$. □

Let $P(x_1,\ldots,x_n)$ and $Q(y_1,\ldots,y_n)$ be points in $\mathbb{R}^n$. The **Euclidean distance** between P and Q is defined as

$$d(P,Q) := \left\|\overrightarrow{PQ}\right\| = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

The **distance between two vectors** $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is defined as the distance between P and Q.

Exercise 15. *Prove that the distance* $d(\mathbf{u},\mathbf{v})$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *is* $d(\mathbf{u},\mathbf{v}) = ||\mathbf{u}-\mathbf{v}||$.

Consider now a subspace V in $\mathbb{R}^n$. The distance between P and V is defined as

$$d(P,V) := min \left\{ d(\overrightarrow{OP}, \mathbf{v}) \mid \mathbf{v} \in V \right\}$$

The concept of the distance on Euclidean spaces is widely used in communication theory and more specifically coding theory. A **linear code** C is a subspace of a vector space $\mathbb{F}^n$, where $\mathbb{F}$ is a finite field. Its **minimum distance** is

$$d(C) := min_{\mathbf{v} \neq \mathbf{0}} \left\{ d(\mathbf{0}, \mathbf{v}) \mid \mathbf{v} \in C \right\}$$

Then we say that this is an $[n,d]$ code. One of the classical results of coding theory is that we can detect up to $(d-1)$ errors and can correct up to $\left[\frac{d-1}{2}\right]$ of them.

1.2.3 Projections

Consider vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ having the same initial point. The **projection vector** of $\mathbf{v}$ onto $\mathbf{u}$, denoted by $\text{proj}_{\mathbf{u}}(\mathbf{v})$ is the vector with initial point the same as that of $\mathbf{v}$ and terminal point obtained by dropping a perpendicular from the terminal point of $\mathbf{v}$ on the line determined by $\mathbf{u}$. Thus,

$$\left\| \text{proj}_{\mathbf{u}}(\mathbf{v}) \right\| := \left\| \overrightarrow{AO} \right\| = \|\mathbf{v}\| \cdot \cos(C\hat{A}B) = \|\mathbf{v}\| \cdot \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}.$$

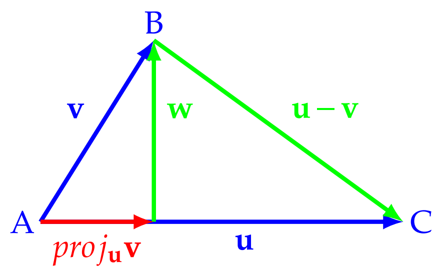


Figure 1.21: The projection of $\mathbf{v}$ onto $\mathbf{u}$

We can multiply by the unit vector $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ to get

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u}^2} \mathbf{u} \qquad (1.32)$$

If we want a vector perpendicular to $\mathbf{u}$ we have

$$\boxed{\mathbf{w} = \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u}^2} \mathbf{u}.} \qquad (1.33)$$

We will see later in the course how this idea is generalized in $\mathbb{R}^n$ to the process of orthogonalization.

Exercise 16. *The above discussion provides a method that for any two given vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *we can determine a vector* $\mathbf{w}$ *which is perpendicular to* $\mathbf{u}$. *Can you devise a similar argument for three vectors* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$*? In other words, determine* $\mathbf{v}$ *and* $\mathbf{w}$ *from* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *such that the set of vectors* $\{\mathbf{u}_1, \mathbf{v}, \mathbf{w}\}$ *are pairwise perpendicular.*

Exercise 17. *Show that the distance from a point* $P = (x_0, y_0)$ *to a line* $L : ax + by + c = 0$ *is given by* $d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$.

Solution: The line L intersect the y-axis at $A\left(0,-\frac{c}{b}\right)$ and the x-axis at $B\left(-\frac{c}{a},0\right)$. Let $\mathbf{u} = \overrightarrow{AB} = c\begin{bmatrix} -\frac{1}{a} \\ \frac{1}{b} \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AP} = \begin{bmatrix} x_0 \\ y_0 + \frac{c}{b} \end{bmatrix}$. Then the distance d from the point P to the line L is $d = \left\| \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) \right\|$. Use the formula from Eq. (1.33) to prove the result. □

Exercise 18. *Let* $\mathbf{u},\mathbf{v} \in \mathbb{R}^3$*. Prove that the formulas Eq. (1.32) and Eq. (1.33) still hold.*

Next we consider the problem of finding the projection of a vector $\mathbf{w}$ on the plane P determined by two vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^3$. Denote by $\mathbf{n} = \frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}$ the unit normal vector to the plane P, say $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then the plane has equation $ax + by + cz = 0$. The projection of $\mathbf{w}$ onto the plane P is

$$\text{proj}_P(\mathbf{w}) = \mathbf{w} - \text{proj}_{\mathbf{n}}(\mathbf{w}) = \mathbf{w} - \frac{\mathbf{n}\cdot\mathbf{w}}{\mathbf{n}^2}\mathbf{n} = \mathbf{w} - (\mathbf{n}\cdot\mathbf{w})\,\mathbf{n}, \quad (1.34)$$

since $\mathbf{n}$ is a unit vector. Summarizing, we have:

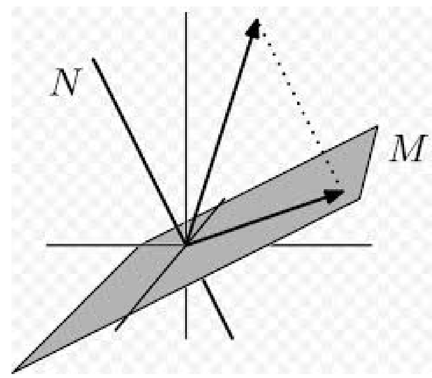


Figure 1.22: Projection on a plane

Lemma 1.10. *Let* $\mathbf{u},\mathbf{v} \in \mathbb{R}^3$ *and* $U = $ *Span* $(\mathbf{u},\mathbf{v})$*. If* $\mathbf{u}$ *and* $\mathbf{v}$ *are not co-linear then the projection of any vector* $\mathbf{w} \in \mathbb{R}^3$ *onto the space* U *is given by the formula*

$$\boxed{proj_U(\mathbf{w}) = \mathbf{w} - (\boldsymbol{n}\cdot\mathbf{w})\,\boldsymbol{n},} \quad (1.35)$$

where $\boldsymbol{n}$ *is a unit vector perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Before we generalize this result to $\mathbb{R}^n$ let us see a computational example.

Example 1.22. *Let* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ *be vectors in* $\mathbb{R}^3$*. Find the projection of* $\mathbf{u}$ *onto the* $\mathbf{vw}$*-plane.*

Solution: The normal vector for the $\mathbf{vw}$-plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -3 \\ -1 & -1 & -1 \end{vmatrix} = -5\mathbf{i} + 5\mathbf{j} + 0\mathbf{k} = -5\mathbf{i} + 5\mathbf{j}$$

We normalize this vector as $\mathbf{n} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. From the above formula we have

$$\text{proj}_P(\mathbf{u}) = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\,\mathbf{n} = \mathbf{u} - \frac{1}{\sqrt{2}}\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

□

In Lem. 1.11 we give another formula for $\text{proj}_{\mathbb{P}}(\mathbf{x})$, which does not include the normal vector $\mathbf{n}$. While the Lemma has a very simple proof, its generalization to $\mathbb{R}^n$ is quite important as we will see in Lem. 1.10 and Lem. 5.1.

Lemma 1.11. *Let V be a subspace of $\mathbb{R}^3$ such that $V = \text{Span}\,(\mathbf{v}_1, \mathbf{v}_2)$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors and perpendicular to each other. Prove that*

$$proj_V(\mathbf{x}) = (\mathbf{v}_1 \cdot \mathbf{x}) \cdot \mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{x}) \cdot \mathbf{v}_2. \tag{1.36}$$

Solution: This is a simple geometry problem. Let P denote the endpoint of the vector $\mathbf{x}$ and Q the endpoint of the vector $\text{proj}_V(\mathbf{x})$. Denote by $\mathbf{a}$ and $\mathbf{b}$ the projections of Q on $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Then

$$\text{proj}_V(\mathbf{x}) = \|\mathbf{a}\| \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \|\mathbf{b}\| \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \|a\|\,\mathbf{v}_1 + \|b\|\,\mathbf{v}_2,$$

since $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$ However, since $\mathbf{a} = \text{proj}_{\mathbf{v}_1}(\mathbf{x})$ and $\mathbf{b} = \text{proj}_{\mathbf{v}_2}(\mathbf{x})$. we have

$$\|\mathbf{a}\| = \frac{\mathbf{v}_1 \cdot \mathbf{x}}{\|\mathbf{v}_1\| \cdot \|\mathbf{x}\|} \cdot \mathbf{x} = \frac{\mathbf{v}_1 \cdot \mathbf{x}}{\|\mathbf{v}_1\|} = \mathbf{v}_1 \cdot \mathbf{x}.$$

Similarly $\|\mathbf{b}\| = \mathbf{v}_2 \cdot \mathbf{x}$. This completes the proof. □

Exercise 19. *Let V be a subspace in $\mathbb{R}^3$ and P a point in $\mathbb{R}^3$. The **distance** $d(P,V)$ between P and the subspace V is called the shortest distance between P and all points of V. In other words,*

$$d(P,V) = \min \{d(P,Q) \mid \overrightarrow{OP} \in V\}$$

Prove that

$$d(P,V) = \left\| \overrightarrow{OP} - proj_V(\overrightarrow{OP}) \right\|$$

We will generalize the concept of the projection to a subspace of $\mathbb{R}^n$ in coming lectures when we study projections; see Lem. 5.1. Projection formulas will be used in the so called Gram-Schmidt algorithm and in the QR-factorization of matrices and will be generalized to any positive definite inner product; see Chap. 5.

Exercises:

79. *Show that the formal definitions of the addition and scalar multiplication in $\mathbb{R}^2$ agree with the geometric interpretations.*

80. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ given as* $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$, *and* $\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. *Compute* $2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$.

81. *Let* $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ -6 \end{bmatrix}$. *Compute* $2\mathbf{u} + 3\mathbf{v}$.

82. *Let* $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ *and* $\mathbf{u} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. *Find scalars r, s such that*

$$r\,\mathbf{v} + s\,\mathbf{u} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

83. *What does it mean for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ to* *be linearly dependent?*

84. *What is the span of* $\begin{bmatrix}0\\1\end{bmatrix}$ *and* $\begin{bmatrix}1\\0\end{bmatrix}$ *in* $\mathbb{R}^2$*?*

85. *Let* $\mathbf{u}, \mathbf{v}$*, and* $\mathbf{w}$ *be given vectors as below*

$$\mathbf{u} = \begin{bmatrix}1\\2\\0\end{bmatrix}, \ \mathbf{v} = \begin{bmatrix}3\\4\\0\end{bmatrix}, \ \mathbf{w} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Can $\mathbf{w}$ *be a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$*? What is geometrically the span of* $\mathbf{u}$ *and* $\mathbf{v}$*?*

86. *Find the area of the triangle determined by the vectors*

$$\mathbf{u} = \begin{bmatrix}1\\2\\2\end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix}2\\2\\-3\end{bmatrix}.$$

87. *Use vectors to decide whether the triangle with vertices* $A = (1,-3,-2)$*,* $B = (2,0,-4)$*, and* $C = (6,-2,-5)$ *is right angled.*

88. *Prove that the triangle with vertices* $A(-2,4,0)$*,* $B(1,2,-1)$ *and* $C(-1,1,2)$ *is regular.*

89. *In the third octant find the point* P *the distances of which from the three coordinate axis are* $d_x = \sqrt{10}$*,* $d_y = \sqrt{5}$*,* $d_z = \sqrt{13}$*.*

90. *Show that for any two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *the following is true*

$$(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = 0 \iff ||\mathbf{v}|| = ||\mathbf{w}||$$

91. *Find the angle between the vectors* $\mathbf{u} = \begin{bmatrix}1\\2\\2\end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix}2\\2\\-3\end{bmatrix}$ *and the area of the triangle determined by them.*

92. *Let* $\mathbf{u}$ *be the unit vector tangent to the graph of* $y = x^2 + 1$ *at the point* $(2,5)$*. Find a vector* $\mathbf{v}$ *perpendicular to* $\mathbf{u}$*.*

93. *For what values of* t *are the vectors* $\mathbf{u} = \begin{bmatrix}1\\0\\t\end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix}t\\-t\\t^2\end{bmatrix}$ *perpendicular?*

94. *Let the vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and coordinates*

$$\mathbf{u} = \begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}2\\2\\-3\end{bmatrix}, \quad \textit{and} \quad \mathbf{w} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

Compute the volume of the parallelepiped determined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$*.*

95. *Let the vectors* $\mathbf{u}, \mathbf{v} \in\in \mathbb{R}^3$ *be given as* $[\mathbf{u} = [1,2,2]^t$ *and* $\mathbf{v} = [1,2,-3]^t$*. Find the projection of* $\mathbf{u}$ *on* $\mathbf{v}$*.*

96. *Let* $\mathbf{u} = [1,2,2]^t$*,* $\mathbf{v} = [2,2,-3]^t$*, and* $\mathbf{w} = [-1,-1,-1]^t$*. Find the projection of* $\mathbf{u}$ *onto the* $\mathbf{vw}$*-plane.*