ABELIAN VARIETIES WITH QUATERNION MULTIPLICATION

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Abstract. In this article we use a Prym construction to study low dimensional abelian varieties with an action of the quaternion group. In special cases we describe the Shimura variety parameterizing such abelian varieties, as well as a map to this Shimura variety from a natural parameter space of quaternionic abelian varieties. Our description is based on the moduli of cubic threefolds with nine nodes, a subject going back to C. Segre, which we study in some detail.

1. Introduction

In this article we study the moduli of low dimensional abelian varieties on which the quaternion group G acts. These abelian varieties with quaternion multiplication are realized as Prym varieties associated to G-Galois covers $\tilde{C}/C$ of curves. The Prym construction gives a map $\Phi : \mathcal{M} \to \mathrm{Shim}$ from an appropriate parameter space $\mathcal{M}$ to a Shimura variety Shim. Explicitly, $\mathcal{M}$ parameterizes triples $(C, \mathrm{Br}, \mathrm{type})$ consisting of a curve C, a finite subset Br over which $\tilde{C}$ is ramified, and the type of the ramification of $\tilde{C}$ above each point of Br. For dimension reasons, there are only five cases in which the resulting map Φ can be surjective. These are listed in (2). The two cases of unramified quaternion covers, which occur over a base curve C of genus 2 or 3, were considered independently by van Geemen and Verra [GV]. Their work concentrated mainly on the genus 3 case, while we study mainly the genus 2 case. The intersection of our results and theirs is therefore quite small, and is indicated in Remarks 6 and 14. In the unramified genus 2 case and two ramified cases, related to it through a degeneration, we go further and completely determine the relevant Shimura variety, which turns out to be the modular curve $Y_0(2)/w_2$ (see Corollary 13).

In a second part of our work we give a relationship between our Prym varieties and cubic threefolds with nine nodes, which extends Varley's treatment (see [Var, Don2]) of the ten nodal Segre cubic threefold. In particular, our study of nine nodal cubics gives that their moduli space is canonically the same modular curve. In Theorem 20 we

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describe several related moduli problems, a description which might be of independent interest (see e.g. [CLSS]). This allows us, in Theorem 24, to determine the fibers of our map when $\tilde{C}$ is an unramified cover of a curve C of genus 2. We also get the surjectivity of Φ in our case and in the two other related cases (Corollary 27). We thank I. Dolgachev for the reference to the classical work of C. Segre ([Seg]) on cubic threefolds with 9 nodes.

Our interest in the problem was raised by a question of W. Baily, connected with his attempt to find a moduli problem pertaining to the Plücker embedding of the Grassmanian $G(2,6)$ into $\mathbf{P}^{14}$ (the third exceptional domain in [Bai]). In email correspondence from 1997–1998 he suggested the following problem. Let $T^{**} \to T$ be a cyclic unramified four sheeted cover of a genus 7 trigonal curve T, and identify $\mathrm{Gal}\,(T^{**}/T)$ with $\langle \hat{\imath} \rangle$ (here $\hat{\imath} \in G$ is the standard quaternion of order 4). Let T^{*}/T be the intermediate 2-sheeted cover. The Prym variety $\mathrm{Prym}\,(T^{**}/T^{*})$ is principally polarized, it has dimension 12, and it admits an action by $\langle \hat{\imath} \rangle$. Baily remarked that the space of such covers T^{**}/T has dimension 12, as does the Shimura variety parameterizing 12-dimensional ppav's with a G-action. Based on this and other considerations, he asked whether $\mathrm{Prym}\,(T^{**}/T^{*})$ admits an action by G (extending the action by $\langle \hat{\imath} \rangle$). We sketch our (negative) solution in the Appendix. It illustrates the paucity of means for producing abelian varieties with quaternion action by geometric means, and thus indicates that the direct method in our paper is probably difficult to circumvent. It also proves a result which must be well-known (compare [Poo, KS]), namely that the endomorphism ring of a generic hyperelliptic jacobian is $\mathbf{Z}$.

2. The basic construction

The quaternion group G has order 8 and a center of order 2 generated by $\hat{\varepsilon}$. The standard generators $\hat{\imath}$, $\hat{\jmath}$ for G satisfy $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{\varepsilon}$ and $\hat{k} := \hat{\imath}\hat{\jmath} = \hat{\varepsilon}\hat{\jmath}\hat{\imath}$. It has a unique irreducible complex representation on which $\hat{\varepsilon}$ acts as -1. This representation St is 2-dimensional. It is not defined over $\mathbf{R}$, but its sum with itself has a model B over $\mathbf{Q}$, which is unique up to an isomorphism. In terms of the group ring $\mathbf{Q}G$ we may take $B = \mathbf{Q}G/(1+\hat{\varepsilon})\mathbf{Q}G$. The quotient of G by $\langle \hat{\varepsilon} \rangle$ is the Klein four-group $V_4 \simeq (\mathbf{Z}/2\mathbf{Z})^2$.

We shall study G-Galois covers of compact Riemann surfaces ("curves") $\tilde{C} \to C$. Putting $C_{\pm} = \tilde{C}/\langle \hat{\varepsilon} \rangle$, and $C_t = \tilde{C}/\langle t \rangle$ for $t = \hat{\imath}$, $\hat{\jmath}$, or $\hat{k}$, we obtain a tower

$$
\begin{array}{ccccc}
 & & \tilde{C} & & \\
 & & \downarrow \pi & & \\
 & & C_{\pm} & & \\
 & \swarrow & \downarrow & \searrow & \\
C_{\hat{\imath}} & & C_{\hat{k}} & & C_{\hat{\jmath}} \\
 & \searrow & \downarrow & \swarrow & \\
 & & C\,. & &
\end{array}
\tag{1}
$$

G acts on the Prym variety $P = \mathrm{Prym}\,\tilde{C}/C_{\pm}$, with $\hat{\varepsilon}$ acting as -1. Hence $H_1(P,\mathbf{Q})$ is a sum of copies of B and in particular P is even dimensional.

As we shall see in Section 3, P has a natural PEL structure in the sense of [Shi]. Hence we get a map $\Phi : \mathcal{M} \to \mathrm{Shim}$ from the parameter space $\mathcal{M}$ of quaternion covers $\tilde{C}/C$, into an appropriate moduli space of abelian varieties with PEL structure, which is a Shimura variety Shim.

Lemma 1. *Let $g = g(C)$ be the genus of C, and suppose there are a branch points of C over which $\tilde{C}$ is ramified. If our map Φ is surjective, with* $\dim \mathrm{Shim} > 0$*, then* (g, a) *must be* $(0, 4)$, $(1, 2)$, $(1, 3)$, $(2, 0)$, $(2, 1)$ *or* $(3, 0)$.

Proof. If the map is surjective, we must have $\dim \mathcal{M} \geq \dim \mathrm{Shim} > 0$ by our assumption. Clearly there are $3g-3+a$ parameters. On the other hand, suppose that $C_\pm$ is unramified over exactly a' of the branch points — we'll refer to such branch points as being of the first type, and to the others (over which $C_\pm$ is ramified) as being of the second type. The stabilizers in G of ramification points must be cyclic, necessarily ± 1 over points of the first type and of order 4 over points of the second type. By the Riemann-Hurwitz formula we get

$$g(C_\pm) = 4g - 3 + a - a' \qquad \text{and} \qquad g(\tilde{C}) = 8g - 7 + 3a - a',$$

and hence

$$\dim P = g(\tilde{C}) - g(C_\pm) = 4(g-1) + 2a.$$

In loc. cit. Shimura shows that $\dim \mathrm{Shim} = n(n-1)/2$ with $n = \dim P/2$. Our assumption $\dim \mathrm{Shim} > 0$ implies $2g - 2 + a = n \geq 2$. Moreover the surjectivity of Φ implies the second condition

$$3g - 3 + a \geq (2(g-1) + a)(2(g-1) + a - 1)/2,$$

or equivalently $4 \geq (2g+a-4)^2 + a = (n-2)^2 + a$, so $0 \leq a \leq 4$. We also have $n-2 \geq 0$, and these two conditions have the solutions indicated. □

Concerning the polarization, we have the following

Lemma 2. *The family of Pryms $\mathcal{P}$ is naturally principally polarized if $a = 0$. It is naturally isogenous to a principally polarized family $\mathcal{P}'$ if $a' = 0$.*

Proof. The first part is well-known. By [DL, Lemma 1], to get principally polarized varieties which are isogenous to the P's, we must make $C_\pm$ and $\tilde{C}$ singular by identifying the ramification points in pairs. To do so G-equivariantly is possible only if $a' = 0$, and only if we identify the two points above each branch point of C of the second type. □

For $t = \hat{\imath}$, $\hat{\jmath}$, or $\hat{k}$, suppose there are a_t points of the second type above which C_t is unramified. We get a disjoint partition of the branch locus Br corresponding to $a = |\mathrm{Br}| = a' + a_{\hat{\imath}} + a_{\hat{\jmath}} + a_{\hat{k}}$.

Lemma 3. *The a_t's all have the same parity. For each genus g and four non-negative integers a', a_t, with all the a_t's of the same parity, the quaternion towers with these invariants form a complex space $\mathcal{M} = \mathcal{M}(g; a', a_{\hat{\imath}}, a_{\hat{\jmath}}, a_{\hat{k}})$ which is quasi-projective.*

Proof. Most of the claims follow directly from the Riemann-Hurwitz formula. The parameter spaces are quasi-projective since they are finite covers of $\mathcal{M}_g$. $\square$

Lemma 3 shows that exactly five of the six cases allowed by Lemma 1 actually occur: we have found five principally polarized cases with quaternion action for which $\dim \mathcal{M} \geq \dim \mathrm{Shim} > 0$. In what follows we restrict to these cases. Since $a' = 0$, we change our notation from $\mathcal{M}(g; 0, a_{\hat{\imath}}, a_{\hat{\jmath}}, a_{\hat{k}})$ to $\mathcal{M}(g; a_{\hat{\imath}}, a_{\hat{\jmath}}, a_{\hat{k}})$. In the following table we give only the cases for which $a_{\hat{\imath}} \geq a_{\hat{\jmath}} \geq a_{\hat{k}}$.

(2)

	$g(C)$	$\dim P$	$\dim \mathcal{M}$	$\geq$	$\dim \mathrm{Shim}$
$\mathcal{M}(0,2,2,0)$	0	4	1		1
$\mathcal{M}(1,2,0,0)$	1	4	2		1
$\mathcal{M}(2,0,0,0)$	2	4	3		1
$\mathcal{M}(1,1,1,1)$	1	6	3		3
$\mathcal{M}(3,0,0,0)$	3	8	6		6

In [GV] van Geemen and Verra proved that the spaces $\mathcal{M}(g,0,0,0)$ were connected. We will generalize this to the general case. We have chosen to redo the unramified cases because we need the general set-up as well as the explicit forms of the corresponding homomorphisms of fundamental groups to prove the last part of the following Proposition:

Proposition 4. *The five spaces $\mathcal{M}(g; a_{\hat{\imath}}, a_{\hat{\jmath}}, a_{\hat{k}})$ above are connected and irreducible. The space $\mathcal{M}(1;2,0,0)$ can be viewed as a boundary component of $\mathcal{M}(2,0,0,0)$, so its points represent curves which are degenerations of curves represented by points of $\mathcal{M}(2,0,0,0)$. Similarly, $\mathcal{M}(0;2,2,0)$ can be viewed as a boundary component of $\mathcal{M}(1;2,0,0)$.*

Proof. For a given C and Br (with its partition), the tower (1) with the G-action is equivalent to a surjective homomorphism $\psi : \pi_1(C \setminus \mathrm{Br}, v) \to G$ up to an inner automorphism. (Since we are working up to inner automorphisms, the choice of the base point $v \in C$ is unimportant.) This ψ must send each loop γ_i around a branch point $P \in \mathrm{Br}$ to an element of order 4 in G: since $C_{\pm}/C$ is ramified at P, the composition ψ' of ψ with the quotient map to V_4 sends γ_i to an element $\neq \overline{1} \in V_4$. Hence each of our $\mathcal{M}$'s is a finite cover of the *connected, irreducible* moduli space of possible (C, Br). To prove $\mathcal{M}$ is connected and irreducible, it will therefore suffice to show that for a fixed (C, Br) the different ψ''s are contained in the image in $\mathcal{M}$ of an irreducible variety.

For this, let π be the group defined by generators α_i, β_i, for $i = 1, \dots, g$, and γ_j, $j = 1, \dots, a = |\mathrm{Br}|$, with one defining relation

$$[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] \gamma_1 \dots \gamma_a = 1.$$

We next present (C, Br) in the usual way: we view C as a $4g$-sided polygon with sides identified together in pairs. We moreover choose non-intersecting paths from the base point v, which we put at a corner of the polygon, to the "punctures", contained in the interior of the polygon. It is well-known that this gives a "standard" isomorphism μ of π with $\pi_1(C \setminus \mathrm{Br}, v)$. The data (C, Br, μ), where μ is such a standard isomorphism, given up to an inner automorphism, is the same as a point in the corresponding Teichmüller space $\mathcal{T} = \mathcal{T}(g, |\mathrm{Br}|)$. This is the usual construction of $\mathcal{T}$, which is known to be irreducible (hence connected) as a cover of $\mathcal{M}$. (In itself this is not enough to prove that $\mathcal{M}$ is connected, as several copies of $\mathcal{T}$ may be needed to cover it.) From each μ we get others belonging to $\mathcal{T}$ by elementary moves consisting of Dehn twists and braiding. These generate the mapping class group in $\mathrm{Aut}\,\mathcal{T}$ (on which inner automorphisms act trivially).

In the sequel it will be convenient to identify π with $\pi_1(C \setminus \mathrm{Br}, v)$ via some standard isomorphism. Then if $\psi, \psi' : \pi \to G$ as above differ by an element ν of the mapping class group (namely $\psi' = \psi\nu$), it follows that the points they represent lie in the same component of $\mathcal{M}$. To prove that $\mathcal{M}$ is connected and irreducible it thus suffices to show that any two ψ's differ by an element of the mapping class group. We will achieve this by a reduction to abelian subgroups and quotients of G.

The point of reducing to the abelian case is to be able to use the following well-known fact. The action of the mapping class group induced on the abelianization ($\simeq \mathbf{Z}^{2g+|\mathrm{Br}|}$) of $\pi_1(C \setminus \mathrm{Br}, v)$, surjects onto the group which permutes the γ_j's and which sends each α_i to $\sum_k n_{i,k}\alpha_k + n_{i,g+k}\beta_k + \sum_l m_{i,l}\gamma_l$ and each β_i to $\sum_k n_{g+i,k}\alpha_k + n_{g+i,g+k}\beta_k + \sum_l m_{g+i,l}\gamma_l$. Here the $2g \times 2g$ matrix $n_{i,j}$ is any integral matrix which is symplectic for the cup product, and $n_{i,j}$, $m_{i,j}$ are integers.

Using this fact we first bring to normal form the composition $\psi_\pm : \pi \to V_4$ of ψ with the quotient map $G \to V_4$. Since $\psi_\pm$ factors through the canonical quotient $H_1(C \setminus \mathrm{Br}, \mathbf{Z}/2\mathbf{Z})$ of $\pi = \pi_1(C \setminus \mathrm{Br}, v)$, we get by duality a copy $V_\pm$ of V_4 in $H^1(C \setminus \mathrm{Br}, \frac{1}{2}\mathbf{Z}/\mathbf{Z})$.

Consider the unramified cases first. Then this H^1 is the group $\mathrm{Jac}\,(C)[2]$ of points of order 2 of the jacobian

$$\mathrm{Jac}\,(C) = \mathrm{Pic}^{\,0}(C) = H^1(C, \mathcal{O}_C)/H^1(C, \mathbf{Z}).$$

Algebro-geometrically, $V_\pm$ is the kernel of the norm map $\mathrm{Nm} : \mathrm{Jac}\,(C_\pm) \to \mathrm{Jac}\,(C)$. We now have the following

Lemma 5. (1) *$V_\pm$ is isotropic for the Weil pairing w_2 on* $\mathrm{Jac}\,(C)[2]$.

(2) *Conversely, given a copy V of V_4 in* $\mathrm{Jac}\,(C)[2]$, *totally isotropic for the Weil pairing, let $\psi_\pm : \pi_1(C, v) \to V$ be the corresponding homomorphism. Then $\psi_\pm$ can be lifted to a homomorphism $\psi : \pi_1(C, v) \to G$. Two such lifts differ by multiplication by an arbitrary homomorphism $\chi : \pi_1(C, v) \to \pm 1$; in particular, there are 16 such ψ's.*

(3) *For each lift ψ let $\tilde{C}_\psi \to C_\pm$ be the corresponding cover. Then there are 4 inequivalent $\tilde{C}_\psi$'s, and for each of them there are 4 actions of G differing by inner automorphisms.*

Proof. 1. The symplectic group over $\mathbf{Z}$ surjects onto its mod 2 analog, and the Weil pairing on $\mathrm{Jac}\,(C)[2]$ "is" the mod 2 cup product. By Witt's theorem for symplectic forms we can bring $V_\pm$, and hence $\psi_\pm$, to normal form by a choice of some standard μ, so that we have either

1. $V_\pm$ is isotropic for the Weil pairing, and $\psi_\pm$ is given by $\psi_\pm(\alpha_1) = \bar{\imath}$, $\psi_\pm(\alpha_2) = \bar{\jmath}$, $\psi_\pm(\alpha_i) = 1$ for $i \geq 3$, and $\psi_\pm(\beta_i) = 1$ for all i, with $\bar{\imath}$, $\bar{\jmath}$ denoting the respective images of $\hat{\imath}$ and $\hat{\jmath}$ in V_4; or

2. The Weil pairing is non-degenerate on $V_\pm$, and $\psi_\pm$ is given by $\psi_\pm(\alpha_1) = \bar\imath$, $\psi_\pm(\beta_2) = \bar\jmath$, and $\psi_\pm(\alpha_i) = \psi_\pm(\beta_i) = 1$ for all $i \geq 2$.

However in case 2 we get that $\psi([\alpha_i, \beta_i])$ is -1 for $i = 1$ and is 1 otherwise, which is incompatible with the defining relation of π_1. This shows that case 1 must hold, proving the first part of the lemma.

For the second part, suppose we are now in case 1 with $\psi_\pm$ in the above normal form. We can then lift it to the *normal form* $\psi_0 : \pi_1(C, v) \to G$ by setting

$$\psi_0(\alpha_1) = \hat\imath,\ \ \psi_0(\alpha_2) = \hat\jmath,\ \ \psi_0(\alpha_i) = 1 \quad \text{for } i \geq 3 \quad \text{and} \quad \psi_0(\beta_i) = 1 \text{ for all } i. \tag{3}$$

That ψ is well-defined and unique up to a homomorphism to ± 1 is clear. It is clear that there are 16 such homomorphism, concluding the proof of the second part.

3. Finally, notice that the image of $H^1(\tilde C, \mathbf{Z}/2\mathbf{Z}) \to H^1(C_\pm, \mathbf{Z}/2ZZ)$ is a copy of V, which has order 4. Hence there are exactly 4 possible $\tilde C$. Since conjugating a given G-action on a given $\tilde C$ by an element of G clearly gives 4 lifts of the same V-action of $C_\pm$, we get the last part of the Lemma.

□

We next show how to bring any lift ψ to normal form. We have $g = 2, 3$. Let F_1 be the free group on generators α_1, β_1, and let F_2 be the free group on α_i, β_i for $2 \leq i \leq g$. These groups come with obvious maps to π. (These maps are inclusions but we will not use it except to omit their maps to π from the notation.) Notice that F_1 is the fundamental group of a once punctured genus 1 surface, and that F_2 is the fundamental group of a once punctured genus $g - 1$ surface. In addition $\psi(F_1)$ is cyclic of order 4 on $\hat\imath$ and $\psi(F_2)$ is cyclic of order 4 on $\hat\jmath$. It is clear that each elementary transformation τ on F_1 or F_2 "is" an elementary transformation on π. Viewing the pair $(\psi(\alpha_1), \psi(\beta_1))$ and the $(2g-2)$-tuple $(\psi(\alpha_2), \dots, \psi(\beta_{2g}))$ as vectors in $(\mathbf{Z}/4\mathbf{Z})^2$, $(\mathbf{Z}/4\mathbf{Z})^{2g-2}$ respectively, we may use a symplectic transformation mod 4 to move them to the first unit vectors $e_1 = (1, 0, , \dots, 0)$ of the respective lengths 2, $2g - 2$. Lifting to the mapping class group as before, we may get the normal form $\psi(\alpha_1) = \hat\imath$, $\psi(\alpha_2) = \hat\jmath$, and the other generators map to 0. As was explained this implies that $\mathcal{M}$ is irreducible and connected. (The proof did not use the assumption $g = 2, 3$ so that $\mathcal{M}$ is irreducible and connected in general; however the map to the Shimura variety is not surjective for $g \geq 4$.)

We consider now the case of $\mathcal{M}(1; 1, 1, 1)$. To put $\psi_\pm$ in normal form we apply it to the defining relation of π to get $0 + \psi_\pm(\gamma_1) + \psi_\pm(\gamma_2) + \psi_\pm(\gamma_3) = 0 \in V_4$.

As none of the $\psi_\pm(\gamma_i)$'s may be 0, they must be a permutation of $\bar\imath$, $\bar\jmath$ and $\overline{k}$. An appropriate element h of the mapping class group then allows us to permute them so that $\psi_\pm(\gamma_1) = \bar\imath$, $\psi_\pm(\gamma_2) = \bar\jmath$, and $\psi_\pm(\gamma_3) = \overline{k}$. In addition we may assume that $\psi_\pm(\alpha_1) = \psi_\pm(\beta_1) = 0$ by taking an h inducing an appropriate translation modulo 2. Now $\psi_\pm$ is in (a unique) normal form. We next bring ψ to (a unique) normal form as well. First we use a translation modulo 4 to make $\psi(\alpha_1) = \psi(\beta_1) = 0$, which is possible as before since ψ maps the group generated by α_1, β_1 and γ_1 to a cyclic group of order 4. We have $\psi(\gamma_1) = \pm\hat\imath$ and $\psi(\gamma_2) = \pm\hat\jmath$, so using an inner automorphism of G we may assume both signs are 1, and then the defining relation forces $\psi(\gamma_3) = -\hat{k}$. This proves the uniqueness of a normal form and the irreducibility and connectedness of $\mathcal{M}(1;1,1,1)$ follows.

The remaining cases are similar. Straightforward computations, whose details we omit, give the normal forms $\psi(\alpha_1) = \hat{\jmath}$, $\psi(\beta_1) = 1$, $\psi(\gamma_1) = \psi(\gamma_2)^{-1} = \hat{\imath}$ for the component $\mathcal{M}(1;2,0,0)_{\hat{\imath}}$ and $\psi(\gamma_1) = \psi(\gamma_2)^{-1} = \hat{\imath}$, $\psi(\gamma_3) = \psi(\gamma_4)^{-1} = \hat{\jmath}$ for $\mathcal{M}(0;2,2,0)_{\hat{k}}$. To get the other possibilities we make a cyclic permutation on $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Notice that these are (outer) automorphisms of G; the other outer automorphisms — those of order 2, such as replacing $\psi(\alpha_1) = \hat{\imath}$ by $\hat{k}$ in the normal form for $\mathcal{M}(1;2,0,0)$, give equivalent forms.

The statements regarding degeneration follow by letting the curve acquire an ordinary double point. Let $S(g,n)$ be a curve of genus g with n punctures and with a base point $*$. There is a standard inclusion $S(1,2) \subset S(2,0)$ obtained by adding a handle connecting the two punctures. In our standard presentations for fundamental groups this corresponds to the map $\pi_1(S(1,2),*) \to \pi_1(S(2,0),*)$ given by $\gamma_1 \to \alpha_2$ and $\gamma_2 \to \beta_2\alpha_2^{-1}\beta_2$. Likewise, the map $\gamma_3 \to \alpha_1$, $\gamma_4 \to \beta_1\alpha_1^{-1}\beta_1$ gives the map on fundamental groups $\pi_1(S(0,4),*) \to \pi_1(S(1,2),*)$ corresponding to the standard inclusion $S(0,4) \subset S(1,2)$. Then our standard form above for the map $\psi : \pi_1(S(2,0),*) \to G$ induces $\psi(\gamma_1) = \hat{\imath}$, $\psi(\gamma_2) = \hat{\imath}^{-1} = -\hat{\imath}$, $\psi(\gamma_3) = \hat{\jmath}$, and $\psi(\gamma_4) = -\hat{\jmath}$, which are equivalent to the normal forms for the two degenerate cases. We omit the details. This completes the proof of Proposition 4. $\square$

Remark 6. *Van Geemen and Verra make a similar construction for the special case that the cover $\tilde{C}/C$ is unramified. They prove by a similar method the connectedness of the parameter spaces.*

3. The Shimura varieties

Let B be the Hamilton quaternion algebra over $\mathbf{Q}$. It is generated over $\mathbf{Q}$ by G with $\hat{\varepsilon}$ going to -1. Let $\mathrm{Tr}_{B/\mathbf{Q}} : B \to \mathbf{Q}$ be the reduced trace, defined by $\mathrm{Tr}(a_1+a_2\hat{\imath}+a_3\hat{\jmath}+a_4\hat{k} = 2a_1$. Let $b \mapsto \bar{b} : B \to B$ the main involution (or conjugation) $\bar{b} = \mathrm{Tr}_{B/\mathbf{Q}}b - b$. The order $\mathbf{M}' = \mathbf{Z}\langle 1, \hat{\imath}, \hat{\jmath}, \hat{k}\rangle$ is contained, with index 2, in a unique maximal order $\mathbf{M} = \mathbf{Z}\hat{u} + \mathbf{M}'$, where $\hat{u} = (1+\hat{\imath}+\hat{\jmath}+\hat{k})/2$. We have $(\mathbf{M}')^\times \simeq G$. (see [Vig]).

Set $P = \mathrm{Prym}\,(\tilde{C}/C_\pm)$ (see diagram (1)). From the exponential sheaf sequences $0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$ on $\tilde{C}$ and on $C_\pm$ we get an identification

$$H_1(P,\mathbf{Z}) \simeq \left(\mathrm{Ker}\, H^1(\tilde{C},\mathbf{Z}) \overset{\mathrm{Nm}}{\to} H^1(C_\pm,\mathbf{Z})\right) \simeq \left(\mathrm{Ker}\, H_1(\tilde{C},\mathbf{Z}) \overset{\pi_*}{\to} H_1(C_\pm,\mathbf{Z})\right) \tag{4}$$

The polarization pairing $\langle\,,\,\rangle : H_1(P,\mathbf{Z}) \times H_1(P,\mathbf{Z}) \to \mathbf{Z}$ is principal and $2\langle\,,\,\rangle$ is the restriction of the intersection pairing on $H_1(\tilde{C},\mathbf{Z})$.

Recall that a polarization on an abelian variety A determines a Rosati involution ρ on $\mathrm{End}\,(A) \otimes \mathbf{Q}$ (see [Mum]), characterized by the property

$$\langle mu,\, v\rangle = \langle u,\, \rho(m)v\rangle \tag{5}$$

for all u, v in $H_1, A, \mathbf{Q})$ and $m \in \mathrm{End}\,(A) \otimes \mathbf{Q}$.

We have the following

Lemma 7. *The action of G on $\tilde{C}$ induces an action of $\mathbf{M}'$ on P. The Rosati involution preserves $\mathbf{M}' \subset \operatorname{End} P$ and induces the main involution on $\mathbf{M}'$.*

Proof. From the definition of P we get a G action on P, with $\hat{\varepsilon}$ acting as $-\mathrm{Id}_P$. By linearity $\mathbf{M}' \simeq \mathbf{Z}G/(1+\hat{\varepsilon})\mathbf{Z}G$ acts on P. Since G preserves the orientation on $\tilde{C}$ it preserves the intersection pairing. This implies the relationship $\langle mu,\, v\rangle = \langle u,\, m^{-1}v\rangle$ for all $u, v \in H_1(P, \mathbf{Z})$ and $m \in G$. Since $m^{-1} = \overline{m}$ for any $m \in G$, we get (5) by linearity for all $m \in \mathbf{M}'$. $\square$

We now specialize to the case when $g = 2$. Then we have the following:

Theorem 8. *When $g = 2$ and the tower (1) is unramified we have the following:*

1. *The $\mathbf{M}'$ action on P extends (uniquely) to an $\mathbf{M}$ action.*
2. *Under this action $H_1(P, \mathbf{Z})$ is free of rank 2 over $\mathbf{M}$, and*
3. *with respect to an appropriate $\mathbf{M}$ basis λ_1, λ_2 of $H_1(P, \mathbf{Z})$ the polarization pairing is given by*

$$\langle \sum_i m_i \lambda_i,\, \sum_j n_j \lambda_j \rangle = \mathrm{Tr}_{B/\mathbf{Q}} \sum_i \sum_j v_{ij} \overline{m}_i n_j, \tag{6}$$

where $[v_{ij}] = \frac{1}{2} \begin{bmatrix} 2(\hat{\imath}+\hat{\jmath}) & -1-\hat{\imath} \\ 1-\hat{\imath} & 0 \end{bmatrix}$.

Proof. **1.** We must show that $\mathbf{M}H_1(P, \mathbf{Z})$, a priori contained in $H_1(P, \mathbf{Q}) \subset H^1(\tilde{C}, \mathbf{Q})$, is in fact contained in $H^1(\tilde{C}, \mathbf{Z})$. Since the space of towers $C_\pm/C$ is connected, it suffices to verify the inclusion $\mathbf{M}H_1 \subset H_1$ for one $C_\pm/C$ (and all compatible $\tilde{C}$'s). Let C be the smooth projective model of $y^2 = x^6 - 1$ and let P_ζ denote the Weierstrass point $(x, y) = (\zeta, 0)$ of C, where $\zeta^6 = 1$. Let $V \subset \operatorname{Jac} C[2]$ be the subgroup generated by divisor classes of $P_\zeta + P_{-\zeta} - K_C$ for $\zeta^3 = 1$. Then μ_6 acts on C via $\zeta : (x, y) \mapsto (\zeta x, y)$, preserves V, and induces on V a μ_3-action. It follows that G and μ_6 generate in $\operatorname{Aut} \tilde{C}$ a central extension $\tilde{A}_4$ of $A_4 = \mu_3 \ltimes V$ by $\{\pm 1\}$. This group is known to be the group of units of $\mathbf{M}$ ([Vig]), and it spans $\mathbf{M}$ additively. Similarly $\mathbf{M}'^\times = G$ spans $\mathbf{M}'$ additively. Hence $\mathbf{M}$ acts on $H_1(\tilde{C}, \mathbf{Z})$ extending the $\mathbf{M}'$ action.
2. Since $\mathbf{M}$ has class number 1 ([Vig]) and $H^1(P, \mathbf{Z})$ is torsion free, it follows that it is free, necessarily of rank 2 since $\dim P = 4$.
3. As was already remarked, $H_1(P, \mathbf{Q})$ is B-free since $-1 \in G$ acts on P as -1. As in the previous Lemma, the G-action shows that the polarization is B-skew-hermitian on $H_1(P, \mathbf{Q})$. Since the trace form $(x, y) \in B^2 \mapsto \mathrm{Tr}_{B/\mathbf{Q}} \overline{x} y$ is non-degenerate, there exist unique elements $v_{ij} \in B$ for which (6) holds. The skew-symmetry of the polarization implies that $v_{ji} = -\overline{v_{ji}}$. (This part holds in general, not just for the genus 2 case, except that the rank of $H_1(P, \mathbf{Q})$ over B is usually not 2).

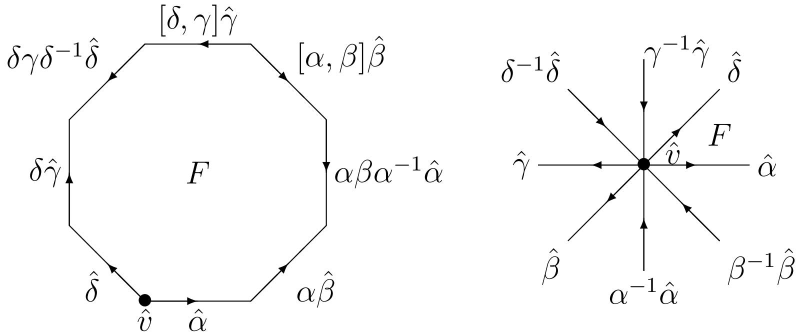
We will compute the pairing by an explicit (and lengthy) calculation, in the course of which we will in fact reprove parts (1) and (2).

Let C have genus 2 and let $\tilde{C} \to C$ be an unramified G-cover. Write

$$\pi_1(C, v) = \langle \alpha, \beta, \gamma, \delta \mid [\alpha, \beta][\gamma, \delta] = 1 \rangle$$

with v a base-point. As was shown we may (and do) let $\phi : \pi_1(C, v) \to G$ be the map characterized by $\phi(\alpha) = \hat{\imath}$, $\phi(\gamma) = \hat{\jmath}$, $\phi(\beta) = \phi(\delta) = 1$. In the universal cover C^{univ} of C choose a base-point $\hat{v}$ above v and lift α, β, γ, and δ to (not necessarily closed) paths $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$, starting at $\hat{v}$. Then C^{univ} is a copy of $\mathbf{R}^2$ subdivided into "octagons" by the paths $\{\tau.\xi \mid \tau \in \pi_1(C, v), \xi \in \{\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}\}\}$.

Figure 1



The left part of Figure 1 gives the sides of the (unique) octagon $F \subset \mathbf{R}^2$ whose boundary contains both $\hat{\alpha}$ and $\hat{\delta}$. The right part gives a *planar* neighborhood of $\hat{v}$.

Let R be the group ring $\mathbf{Z}\pi_1(C, v)$, and let $\mathcal{C}_{\bullet}$ be the chain complex

$$0 \to C_2 \to C_1 \to C_0 \to 0 \tag{7}$$

where C_i is the left R-free module on the basis set $\{F\}$, $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}\}$, and $\{\hat{v}\}$ for $i = 2$, 1, and 0 respectively.

The differentials ∂_1, ∂_2 are defined using the right and left sides of figure 1 respectively as follows:

$$\partial_2(rF) = r\left[(1 - \alpha\beta\alpha^{-1})\hat{\alpha} + (\alpha - [\alpha, \beta])\hat{\beta} + ([\delta, \gamma] - \delta)\hat{\gamma} + (\delta\gamma\delta^{-1} - 1)\hat{\delta}\right] \tag{8}$$

$$\partial_1(r\hat{\tau}) = r(\tau - 1)\hat{v} \tag{9}$$

for any $r \in R$, with τ denoting either of the symbols α, β, γ or δ.

For any right R-module M let $\mathcal{C}_{\bullet}(M)$ be the complex $M \otimes_R \mathcal{C}_{\bullet}$. For a basis element $\sigma = F, \ldots, \hat{v}$ as above, we will denote $1 \otimes_R \sigma \in \mathcal{C}_{\bullet}(M)$ respectively by F_M, $\hat{\alpha}_M, \ldots, \hat{\delta}_M$, $\hat{v}_M$. If M is an R-algebra, these elements form a free M-basis for $\mathcal{C}_{\bullet}(M)$, and we will write $mF_M, \ldots,$ for $m \otimes_R F, \ldots$ respectively. We will write $\mathcal{Z}_i(M)$ for the i-cycles of $\mathcal{C}_{\bullet}(M)$. The homology class of a cycle ξ will be denoted by $[\xi]$ or simply by ξ.

In the special case $M = R$ we get back $\mathcal{C}_{\bullet}(R) = \mathcal{C}_{\bullet}$, the cellular (or CW) chain complex for C^{univ}. Since this description is clearly $\pi_1(C, v)$-equivariant, we get the chain complex for $\Gamma\backslash C^{\mathrm{univ}}$, for any subgroup $\Gamma \subset \pi_1(C, v)$, by taking $M = \mathbf{Z}\left(\Gamma\backslash\pi_1(C, v)\right)$. If Γ is normal in $\pi_1(C, v)$ then this description is $\pi_1(C, v)/\Gamma$-equivariant. If now $\Gamma = \operatorname{Ker}\phi$ then

$\pi_1(C,v)/\Gamma \simeq G$ and the formulas (8), (9), with $M = \mathbf{Z}G$ simplify to

$$\partial_2(F_{\mathbf{Z}G}) = (\hat{\imath}-1)\hat{\beta}_{\mathbf{Z}G} + (\hat{\jmath}-1)\hat{\delta}_{\mathbf{Z}G} \tag{10}$$

$$\partial_1(\hat{\alpha}_{\mathbf{Z}G}) = (\hat{\imath}-1)v_{\mathbf{Z}G}, \quad \partial_1(\hat{\beta}_{\mathbf{Z}G}) = \partial_1(\hat{\delta}_{\mathbf{Z}G}) = 0, \quad \text{and} \quad \partial_1(\hat{\gamma}_{\mathbf{Z}G}) = (\hat{\jmath}-1)\hat{v}_{\mathbf{Z}G}. \tag{11}$$

Similarly, (7) with $M = \mathbf{Z}V_4$ and (10), (11) also describe the chain complex for $C_\pm$ if we replace $F_{\mathbf{Z}G}, \ldots, \hat{v}_{\mathbf{Z}G}$ by $F_{\mathbf{Z}V_4}, \ldots, \hat{v}_{\mathbf{Z}V_4}$, and $\hat{\imath}, \ldots$ by their images in $\mathbf{Z}V_4$.

The order $\mathbf{M}' \simeq \mathbf{Z}G/(1+\hat{\varepsilon})\mathbf{Z}G$ is identified with $\mathrm{Ker}\,(\mathbf{Z}G \to \mathbf{Z}V_4) = (1-\hat{\varepsilon})\mathbf{Z}G$, by sending the image of $m \in \mathbf{Z}G$ in $\mathbf{M}'$ to $(1-\hat{\varepsilon})m$. Since $\mathcal{C}_{\textstyle\cdot}$ is free, we see that the projection $\pi : \tilde{C} \to C_\pm$ induces a G-equivariant exact sequence

$$0 \to \mathcal{C}_{\textstyle\cdot}(\mathcal{M}') \to \mathcal{C}_{\textstyle\cdot}(\mathbf{Z}G) \overset{\pi_*}{\to} \mathcal{C}_{\textstyle\cdot}(\mathbf{Z}V_4) \to 0. \tag{12}$$

We can (and will) therefore identify $\mathcal{C}_{\textstyle\cdot}(\mathcal{M}')$ with $\mathrm{Ker}\,(\mathcal{C}_{\textstyle\cdot}(\mathbf{Z}G) \overset{\pi_*}{\to} \mathcal{C}_{\textstyle\cdot}(\mathbf{Z}V_4)) = (1-\hat{\varepsilon})\mathcal{C}_{\textstyle\cdot}(\mathbf{Z}G)$. As above, the identification is between the image of $m \in \mathcal{C}_{\textstyle\cdot}(\mathbf{Z}G)$ in $\mathcal{C}_{\textstyle\cdot}(\mathbf{Z}V_4)$ and $(1-\hat{\varepsilon})m \in \mathcal{C}_{\textstyle\cdot}(\mathcal{M}')$. In particular, for the basis elements we have $F_{\mathbf{M}'} = (1-\hat{\varepsilon})F_{\mathbf{Z}G}, \ldots, \hat{v}_{\mathbf{M}'} = (1-\hat{\varepsilon})\hat{v}_{\mathbf{Z}G}$. We now have the following

Proposition 9. (1) *The torsion subgroup of* $H_1(\mathcal{C}_{\textstyle\cdot}(\mathcal{M}'))$ *has order* 2 *and the quotient* $L = H_1(\mathcal{C}_{\textstyle\cdot}(\mathcal{M}')/\text{torsion}$ *is naturally* $H_1(P,\mathbf{Z})$.

(2) *The natural* $\mathbf{M}'$ *action on* L *extends (uniquely) to an* $\mathbf{M}$ *action, and* L *is* $\mathbf{M}$*-free on the classes of the cycles* $\lambda_1 = (\hat{\imath}+1)\hat{\alpha}_{\mathbf{M}'} - (\hat{\jmath}+1)\hat{\gamma}_{\mathbf{M}'}$ *and* $\lambda_2 = \hat{\beta}_{\mathbf{M}'}$.

Proof. (1) The homology sequence of 12 gives the exact sequence

$$0 \to \mathbf{Z}/2\mathbf{Z} \to H_1(\mathcal{C}_{\textstyle\cdot}(\mathcal{M}')) \to H_1(\tilde{C},\mathbf{Z}) \to H_1(C_\pm,\mathbf{Z}),$$

since $H_2(\tilde{C},\mathbf{Z}) \overset{\pi_*}{\to} H_2(C_\pm,\mathbf{Z})$ is identified with $\mathbf{Z} \overset{\deg\pi}{\to} \mathbf{Z}$, and $\deg\pi = 2$. As $H_1(\tilde{C},\mathbf{Z})$ is torsion-free, the result follows from (4).

(2) To analyze $H_1(\mathcal{C}_{\textstyle\cdot}(\mathcal{M}'))$ write $C_1(\mathbf{M}') = C' \oplus C''$, where $C' = \mathrm{span}\{\hat{\alpha}_{\mathbf{M}'}, \hat{\gamma}_{\mathbf{M}'}\}$ and $C'' = \mathrm{span}\{\hat{\beta}_{\mathbf{M}'}, \hat{\delta}_{\mathbf{M}'}\}$. By (10) and (11) we have

$$H_1(\mathcal{C}_{\textstyle\cdot}(\mathcal{M}')) = \mathrm{Ker}\,(\partial_1 \mid C') \oplus (C''/\partial_2(\mathbf{M}'F))$$

We will show that

(a) $\mathrm{Ker}\,(\partial_1|C')$ is preserved by multiplication by $\mathbf{M}$ and is $\mathbf{M}$-free on λ_1.

(b) $\partial_2(\mathbf{M}F) \subset C''$ and the $\mathbf{M}'$-module structure on $C''/\partial_2(\mathbf{M}F)$ extends uniquely to an $\mathbf{M}$-module structure, for which C'' is $\mathbf{M}$-free on λ_2.

□

These two assertions imply Proposition 9.2. In turn they follow from the following:

Lemma 10. *Let* $\chi : B^2 \to B$ *and* $\psi : B \to B^2$ *be the maps of left* B*-modules given by*

$$\chi(x,y) = x(\hat{\imath}-1) + y(\hat{\jmath}-1) \quad \text{and} \quad \psi(x) = x(\hat{\imath}-1, \hat{\jmath}-1).$$

Then

(1) $(\mathbf{M}' \oplus \mathbf{M}') \cap \mathrm{Ker}\,\chi = \mathbf{M}(\hat{\imath}+1, -\hat{\jmath}-1)$.

(2) *We have* $(\mathbf{M}' \oplus \mathbf{M}') \cap \psi(B) = \psi(\mathbf{M})$ *and* $(\mathbf{M}' \oplus \mathbf{M}') + \psi(B) = \mathbf{M}(1,0) + \psi(B)$*; the last sum is direct.*

Proof. Observe the isomorphism of (commutative) rings

$$\mathbf{M}'/2\mathbf{M}' \simeq \mathbf{F}_2[\varepsilon, \varepsilon']/\varepsilon^2 = (\varepsilon')^2 = 0$$

given by $1 + \hat{\imath} \mapsto \varepsilon$, $1 + \hat{\jmath} \mapsto \varepsilon'$. Then $2\hat{u} = 1 + \hat{\imath} + \hat{\jmath} + \hat{k} \in \mathbf{M}'$ maps to $1 + (1+\varepsilon)(1+\varepsilon') + (1+\varepsilon)(1+\varepsilon') = \varepsilon\varepsilon'$. We now prove the two assertions of the lemma.

(1) Set $z = (\hat{\imath}+1, -\hat{\jmath}-1) \in B^2$. Then $\chi(z) = 0$, so $Bz \subset \operatorname{Ker}\chi$. In addition uz maps to $\varepsilon\varepsilon'(\varepsilon, \varepsilon') = (0,0)$ in $(\mathbf{M}'/2\mathbf{M}')^2$, so $(u/2) \cdot z$ is in $\mathbf{M}' \oplus \mathbf{M}'$. Hence RHS $\subset$ LHS. Conversely, suppose $\chi(x,y) = 0$. Then for $t = x(\hat{\imath} - 1) = y(1 - \hat{\jmath})$ we have $t(\hat{\imath}+1, -\hat{\jmath}-1) = -2(x,y)$, so that $\operatorname{Ker}\chi \subset Bz$. If in addition $(x,y) \in \mathbf{M}' \oplus \mathbf{M}'$ then $t \in \mathbf{M}'$ has reduction $\bar{t}$ to $\mathbf{M}'/2\mathbf{M}'$ which is divisible both by ε and by ε', hence $\bar{t}$ is a multiple of $\varepsilon\varepsilon'$ which is the reduction of $2\hat{u}$. It follows that $t \in 2\hat{u}\mathbf{Z} + 2\mathbf{M}'$, so that $t/2 \in \hat{u}\mathbf{Z} + \mathbf{M}' = \mathbf{M}$ as asserted.

(2) Notice that the sum on the RHS is direct. Since

$$\psi(1+j) \equiv (2\hat{u}, 0) \pmod{2\mathbf{M}' \oplus 2\mathbf{M}'}$$

we see that $(\hat{u}, 0) \in$ LHS, so that RHS $\subset$ LHS. Conversely, $(\hat{\jmath}-1)^{-1}(\hat{\imath}-1) \in \mathbf{M}$, so that $(0,1) = \psi((\hat{\jmath}-1)^{-1}) - ((\hat{\jmath}-1)^{-1}(\hat{\imath}-1), 0)$ belongs to the RHS, giving LHS $\subset$ RHS. This completes the proof of the Lemma and hence of the Proposition.

□

We shall now compute the polarization form of P in terms of the $\mathbf{M}$-basis λ_1, λ_2 of L. For this we shall use the embedding of L into $H_1(\mathcal{C}_{\bullet}(\mathbf{Z}G))$. By (6) we have that

$$\langle \sum_i m_i\lambda_i,\, \sum_j n_j\lambda_j \rangle = \sum_i \sum_j \operatorname{Tr} v_{ij}\overline{m_i}n_j$$

for any $m_i, n_i \in B$ and appropriate v_{ij}'s. To determine them, we need only determine $\langle m\lambda_i,\, \lambda_j \rangle$ for any $1 \le i \le j \le 2$, and $m \in \{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$. Denote the intersection pairing on $H_1(\mathcal{C}_{\bullet}(\mathbf{Z}G)) \simeq H_1(\tilde{C}, \mathbf{Z})$ by $\langle\,,\,\rangle_{\tilde{C}}$. Then for any $x, y \in \mathcal{Z}_1(\mathbf{Z}G)$ mapping to $\overline{x}, \overline{y} \in \mathcal{Z}_1(\mathbf{M}')$ we have the basic formula

$$\langle \overline{x},\, \overline{y} \rangle = \frac{1}{2}\langle (1-\hat{\varepsilon})x,\, (1-\hat{\varepsilon})y \rangle_{\tilde{C}} = \frac{1}{2}\langle x,\, (1-\hat{\varepsilon})^2 y \rangle_{\tilde{C}} = \langle x,\, (1-\hat{\varepsilon})y \rangle_{\tilde{C}}. \tag{13}$$

Indeed, the first equality is the definition of the polarization pairing on P, the second follows from the general formula $\langle gx,\, gy \rangle = \langle x,\, y \rangle$ for all $g \in G$, and the third holds since $(1-\hat{\varepsilon})^2 = 2(1-\hat{\varepsilon})$.

We will let $\tilde{v}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$ denote the images of $\hat{v}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ in $\tilde{C}$.

Computation of v_{22}: We are assuming that $\phi : \pi_1(C, v) \to G$ is in normal form (3). Thus $\tilde{\beta}$ is a closed loop. We have $\langle \hat{\beta}_{\mathbf{Z}G},\, \hat{\beta}_{\mathbf{Z}G} \rangle_{\tilde{C}} = 0$, and since $g\tilde{\beta}$ and $\tilde{\beta}$ are clearly disjoint for $g \in G - 1$, we get $\langle \hat{\beta}_{\mathbf{Z}G},\, g\hat{\beta}_{\mathbf{Z}G} \rangle_{\tilde{C}} = 0$ for all $g \in G$. Therefore $\langle \lambda_2,\, \lambda_2 \rangle = \operatorname{Tr}_{B/\mathbf{Q}} v_{22} m = \langle \hat{\beta}_{\mathbf{Z}G},\, (1-\hat{\varepsilon})m\hat{\beta}_{\mathbf{Z}G} \rangle_{\tilde{C}} = 0$ for all $m \in \mathbf{M}'$, whence $v_{22} = 0$.

Computation of v_{11}: Lift λ_1 to the cycle $\sigma = (1+\hat\imath)\hat\alpha_{\mathbf{Z}G} - (1+\hat\jmath)\hat\gamma_{\mathbf{Z}G} \in \mathcal{C}_{\cdot}(\mathbf{Z}G)$, represented by the following:

Figure 2

$$\begin{tikzcd}
& \hat\imath\tilde v \arrow[dr, "\hat\imath\tilde\alpha"] & \\
\tilde v \arrow[ur, "\tilde\alpha"] \arrow[dr, "\tilde\gamma"'] & & \hat\varepsilon\tilde v\ , \\
& \hat\jmath\tilde v \arrow[ur, "\hat\jmath\tilde\gamma"'] &
\end{tikzcd}$$

By (13) we need to compute $\langle\sigma,\,(1-\hat\varepsilon)l\sigma\rangle_{\tilde C}$ for $l = 1,\ \hat\imath,\ \hat\jmath$, and $\hat k$. Now for any $x \in \mathcal{Z}_1(\mathbf{Z}G)$ and $g \in G$, $g \neq \pm 1$ we have $\langle x,\,\hat\varepsilon x\rangle_{\tilde C} = \langle \hat\varepsilon x,\,\hat\varepsilon^2 x\rangle_{\tilde C} = -\langle x,\,\hat\varepsilon x\rangle_{\tilde C}$. Hence $\langle x,\,\hat\varepsilon x\rangle_{\tilde C} = 0$. Since $\hat\varepsilon g = g^{-1}$, we likewise get $\langle x,\,\hat\varepsilon g x\rangle_{\tilde C} = \langle gx,\,x\rangle_{\tilde C} = -\langle x,\,gx\rangle_{\tilde C}$. In particular

$$\langle\lambda_1,\,\lambda_1\rangle = \langle\sigma,\,(1-\hat\varepsilon)\sigma\rangle_{\tilde C} = \langle\sigma,\,\sigma\rangle_{\tilde C} - \langle\sigma,\,\hat\varepsilon\sigma\rangle_{\tilde C} = 0 - 0 = 0.$$

Next, for $g = \hat\imath,\ \hat\jmath$, or $\hat k$ we have $\langle\lambda_1,\,g\lambda_1\rangle = \langle\sigma,\,(1-\hat\varepsilon)g\sigma\rangle_{\tilde C} = 2\langle\sigma,\,g\sigma\rangle_{\tilde C}$; we will compute each case separately. For convenience, we will simplify Figure 2 to $\begin{pmatrix} 1 & \begin{smallmatrix}\hat\imath\\ \hat\jmath\end{smallmatrix} & \hat\varepsilon\end{pmatrix}$, and we will likewise represent $\hat k\sigma$ by the diagram $\begin{pmatrix} \hat k & \begin{smallmatrix}\hat\jmath\\ \hat\varepsilon\hat\imath\end{smallmatrix} & \hat\varepsilon\hat k\end{pmatrix}$ etc. An element $g \in G$ appears in the diagram for a translate $g'\sigma$ of σ if and only if $g'\sigma$ passes through $g\tilde v$. Likewise one can reconstruct the 1-cells participating in $g'\sigma$ (with their signs).

The case $g = \hat k$: The four 1-cells $\tilde\alpha$, $\hat\imath\tilde\alpha$, $\tilde\gamma$, and $\hat\jmath\tilde\gamma$ in the support of σ are distinct from their translates by $\hat k$. Hence σ and $\hat k\sigma$ can intersect only at points of $\tilde C$ over v. As is clear from the diagrams for σ and for $\hat k\sigma$, these points of intersection are the translates of $\tilde v$ by $\{1, \hat\imath, \hat\jmath, \hat\varepsilon\} \cap \{\hat k, \hat\jmath, \hat\varepsilon\hat\imath, \hat\varepsilon\hat k\} = \{\hat\jmath\}$. By Figure 1 the local picture at $\hat\jmath\tilde v$, when lifted to C^{univ} and translated to $\hat v$, is the left part of Figure 3:

The right hand side of Figure 3 shows that after a homotopy σ and $\hat k\sigma$ do not meet. Hence $\langle\lambda_1,\,\hat k\lambda_1\rangle = 2\langle\sigma,\,\hat k\sigma\rangle_{\tilde C} = 0$.

The case $g = \hat\imath$: Here the supports of σ and of $\hat\imath\sigma \leftrightarrow \begin{pmatrix} \hat\imath & \begin{smallmatrix}\hat\varepsilon\\ \hat k\end{smallmatrix} & \hat\varepsilon\hat\imath\end{pmatrix}$ intersect along $\hat\imath\tilde\alpha$. The left part of Figure 4 shows a neighborhood of $\hat\imath\tilde\alpha$ lifted to $\tilde C$: the part of σ represented in it is $\tilde\alpha + \hat\imath\tilde\alpha - \hat\jmath\tilde\gamma$ and that of $\hat\imath\sigma$ is $-\hat\imath\tilde\gamma + \hat\imath\tilde\alpha + \hat\varepsilon\tilde\alpha$. To obtain this picture we combine the local pictures offered by Figure 1 at both endpoints $\hat\imath\tilde v$ and $\hat\varepsilon\tilde v$ of $\hat\imath\tilde\alpha$. The right hand part of Figure 4 represents homologous paths, and it follows that $\langle\lambda_1,\,\hat\imath\lambda_1\rangle = 2\langle\sigma,\,\hat\imath\sigma\rangle_{\tilde C} = -2$.

The case $g = \hat\jmath$: Here $\hat\jmath\sigma$ corresponds to $\begin{pmatrix} \hat\jmath & \begin{smallmatrix}\hat\varepsilon\hat k\\ \hat\varepsilon\end{smallmatrix} & \hat\varepsilon\hat\jmath\end{pmatrix}$, so σ and $\hat\jmath\sigma$ intersect along $\hat\jmath\tilde\gamma$ as in Figure 5. We handle it exactly as we handled Figure 4, to obtain $\langle\lambda_1,\,\hat\jmath\lambda_1\rangle = 2\langle\sigma,\,\hat\jmath\sigma\rangle_{\tilde C} = -2$.

Figure 3

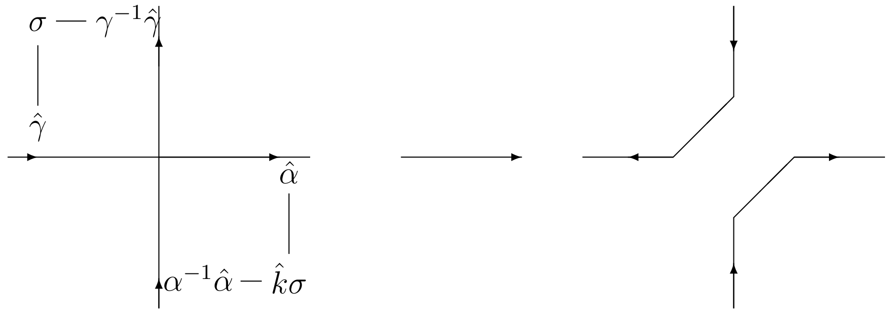


Figure 4

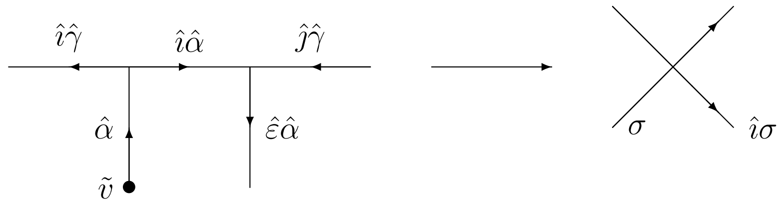
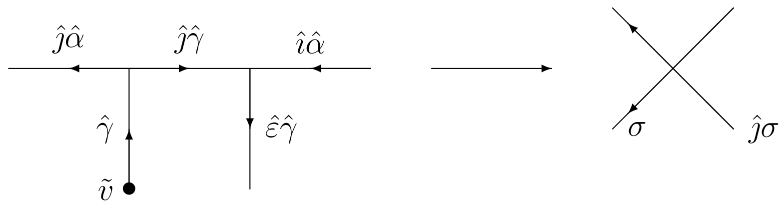


Figure 5

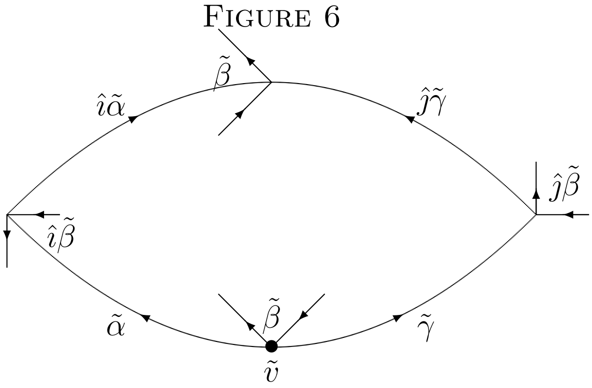


Now set $v_{11} = t_1 + t_2\hat{\imath} + t_3\hat{\jmath} + t_4\hat{k}$, with $t_i \in \mathbf{R}$. We get $2t_1 = \mathrm{Tr} v_{11} = 0$, $-2t_4 = \mathrm{Tr} v_{11}\hat{k} = 0$, $-2t_2 = \mathrm{Tr} v_{11}\hat{\imath} = -2$ and $-2t_3 = \mathrm{Tr} v_{11}\hat{\jmath} = -2$, so $v_{11} = \hat{\imath} + \hat{\jmath}$.

Computation of v_{12}: We shall evaluate $\langle \sigma,\, g\hat{\beta}_{\mathbf{Z}G}\rangle_{\tilde{C}}$ for all $g \in G$. The loops σ and $g\tilde{\beta}$ can only intersect at points of $\tilde{C}$ above $\tilde{v}$. These intersection points are the translates of $\tilde{v}$ by the elements of $\{1, \hat{\imath}, \hat{\jmath}, \hat{\varepsilon}\} \cap \{g\}$. Figure 6 shows how σ intersects the four translates $g\tilde{\beta}$, for $g \in \{1, \hat{\imath}, \hat{\jmath}, \hat{\varepsilon}\}$. To verify it one translates the basic Figure 1 to the four points above v in the support of σ.

This implies that $\langle \sigma,\, \hat{\beta}_{\mathbf{Z}G}\rangle_{\tilde{C}} = 0$, $\langle \sigma,\, \hat{\imath}\hat{\beta}_{\mathbf{Z}G}\rangle_{\tilde{C}} = 1$, $\langle \sigma,\, \hat{\jmath}\hat{\beta}_{\mathbf{Z}G}\rangle_{\tilde{C}} = 0$, and $\langle \sigma,\, \hat{\varepsilon}\hat{\beta}_{\mathbf{Z}G}\rangle_{\tilde{C}} = 1$. The other intersections $\langle \sigma,\, g\hat{\beta}_{\mathbf{Z}G}\rangle_{\tilde{C}}$ are trivially 0. Hence for $g = 1$, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ we have $\langle \lambda_1,\, g\lambda_2\rangle = \langle \sigma,\, g\beta\rangle - \langle \sigma,\, \hat{\varepsilon} g\beta\rangle = -1$, 1, 0, and 0 respectively. Writing $v_{12} =$

FIGURE 6



$t_1 + t_2\hat{\imath} + t_3\hat{\jmath} + t_4\hat{k}$ we see as before that $v_{12} = (-1-i)/2$. Hence $v_{21} = (1-i)/2$. This concludes the proof of Theorem 8. $\square$

Remark 11. *Set $\lambda_1' = -(1+\hat{\imath})\lambda_1 + (\hat{\imath}+\hat{k})\lambda_2$ and $\lambda_2' = \lambda_2$. Then λ_1' and λ_2' constitute a B-basis for $H_1(P, \mathbf{Q})$. The pairing is given in this basis as in* (6) *but with the matrix $[v_{ij}'] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.*

We omit the routine verification.

The ideal $\mathbf{P} = (1+\hat{\imath})\mathbf{M}$ is two-sided, of index 4 in $\mathbf{M}$, and contained in $\mathbf{M}'$ with index 2. Let $\langle\ ,\ \rangle_{\mathbf{M}'} : \mathbf{M}' \times \mathbf{M}' \to \mathbf{Z}$ map (m_1, m_2) to $\frac{1}{2}\mathrm{Tr}_{B/\mathbf{Q}}(\overline{m_1}m_2)$. We will prove the following:

Theorem 12. *For an elliptic curve $E/\mathbf{C}$ let A_E be the 4-dimensional abelian variety $\mathbf{M}' \otimes E$ with the polarization*

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_{\mathbf{M}'} \otimes \langle\ ,\ \rangle_E,$$

where $\langle\ ,\ \rangle_E$ is the standard polarization on E. Then

1. *The polarization $\langle\ ,\ \rangle$ is principal. It is preserved by $\mathbf{M}'$ up to similitudes. As a polarized variety A_E is isomorphic to E^4 with the product polarization.*
2. *Let $\alpha : E \to E'$ be an isogeny of degree 2 of elliptic curves, so that $H_1(E', \mathbf{Z})$ contains $H_1(E, \mathbf{Z})$ with index 2. In the isogeny class of A_E, let B_α be the polarized abelian variety characterized by*

$$H_1(B_\alpha, \mathbf{Z}) = 2\mathbf{M} \otimes H_1(E', \mathbf{Z}) + \mathbf{P} \otimes H_1(E, \mathbf{Z}) \subset H_1(A, \mathbf{Q}). \tag{14}$$

   Then the induced pairing on $H_1(B_\alpha, \mathbf{Z})$ makes B_α into a principally polarized abelian variety with an $\mathbf{M}$-action.
3. *Each Prym variety $\mathrm{Prym}\,(\tilde{C}/C_\pm)$ in case $\mathcal{M}(2,0,0,0)$ is isomorphic to some B_α as a polarized abelian variety together with the $\mathbf{M}$-action.*

Proof. The first part is clear when using the standard basis 1, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ of $\mathbf{M}'$, since $\langle\ ,\ \rangle_{\mathbf{M}'}$ is then the standard Euclidean pairing (and $\mathbf{M}'$ acts via similitudes). For the second part, let e, f be a symplectic basis for $H_1(E,\mathbf{Z})$ so that $\frac{1}{2}e$ and f are a basis for $H_1(E',\mathbf{Z})$. Then

$$H_1(B_\alpha,\mathbf{Z}) = \mathbf{M}\otimes e \oplus \mathbf{P}\otimes f \subset H_1(A,\mathbf{Q})\otimes B.$$

Observe that $\mathrm{Tr}_{B/\mathbf{Q}}$ is even on $\mathbf{P}$. It follows that $\langle\ ,\ \rangle$ is integral on $H_1(B_\alpha,\mathbf{Z})$, hence it defines a polarization on B_α. Since the lattices $H_1(B_\alpha,\mathbf{Z})$ and $H_1(A_E,\mathbf{Z})$ have the same volume, it follows that this polarization is principal as asserted. The $\mathbf{M}$-action preserves $H_1(B_\alpha,\mathbf{Z})$ since $\mathbf{P}$ is an $\mathbf{M}$-ideal, and the second part follows.

To prove the third part, we use Remark 11 to write $H_1(P,\mathbf{Q}) = B\otimes L$, where $L = \mathbf{Z}\lambda_1'\oplus\mathbf{Z}\lambda_2'$. By a standard exceptional isomorphism of Lie groups, the group J of $B_{\mathbf{R}}$-linear similitudes of $H_1(A_E,\mathbf{R})$ is then $\mathrm{GL}\,(L\otimes\mathbf{R})\times B^*/\sim$, where we identify $(t,1)\sim(1,t)$ for any scalar t (see e. g. [Hel, Chap. IX.4.B.xi]). Let $h_P:\mathbf{C}^*\to J\subset \mathrm{GSp}\,(H_1(P,\mathbf{R}),\langle\ ,\ \rangle)$ be the Hodge type of P, in the sense of [Del, Section 4]. Since $h_P(\sqrt{-1}\,)$ is a Cartan involution of J, the image of h_P must centralize the compact factor subgroup of J consisting of the norm 1 elements in B^*. Therefore h_P factorizes through a Hodge type $h_0 = h_P:\mathbf{C}^*\to\mathrm{GL}\,(L\otimes\mathbf{R})$. The lattice $L\subset L\otimes\mathbf{R}$ then determines an elliptic curve, characterized by the properties that its Hodge type is h_0, and that $H_1(E,\mathbf{Z}) = L$. The over-lattice $L' = \mathbf{Z}\lambda_1\oplus\mathbf{Z}\lambda_2$ defines similarly an elliptic curve E' with an isogeny $\alpha: E'\to E$ of degree 2. From the definitions, the resulting equality $H_1(\mathrm{Prym}\,,\mathbf{Q}) = B\otimes L$ is compatible with the B-action, with the Hodge structure, and with the (B-hermitian) polarization. Now define B_α by (14) above. Since λ_1' and λ_2' are a symplectic basis for L, it follows that we have an equality of lattices in this space

$$H_1(\mathrm{Prym}\,,\mathbf{Z}) = \mathbf{M}\lambda_1\oplus\mathbf{M}\lambda_2 = \mathbf{P}\lambda_1'\oplus\lambda_2'\mathbf{M} = H_1(B_\alpha,\mathbf{Z}),$$

so that $\mathrm{Prym}\simeq B_\alpha$ as asserted. $\square$

Corollary 13. *Let $Y_0(2)$ be the modular curve parameterizing elliptic curves with an isogeny $\alpha: E\to E'$ of degree 2, and let w_2 be the modular (Atkin-Lehner) involution of $Y_0(2)$, sending α to its dual isogeny. Then the quotient curve $Y_0(2)/w_2$ is isomorphic to the Shimura curve* Shim *parameterizing the PEL data of Theorem 12.2 (see* [Del, Shi]*) via the assignment $\phi:\alpha\mapsto B_\alpha$.*

Proof. By Theorem 12(2), $\phi: Y_0(2)\to\mathrm{Shim}$ is a morphism. Moreover by Theorem 12(3) ϕ is surjective. Since we work over $\mathbf{C}$ we know that analytically $Y_0(2) = \Gamma_0(2)\backslash\mathcal{H}$, where $\Gamma_0(2)$ is the subgroup of $\mathrm{SL}\,(2,\mathbf{Z})$ consisting of the matrices $\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\mathrm{SL}\,(2,\mathbf{Z})$ for which $2|c$. Since ϕ is modular it induces an isomorphism $\Gamma\backslash\mathcal{H}\xrightarrow{\sim}\mathrm{Shim}$ for some congruence subgroup Γ of $\mathrm{PSL}\,(2,\mathbf{R})$ containing $\Gamma_0(2)$. Let e, f be the symplectic basis for L as in the proof of Theorem 12(2). In terms of this basis Let $W_2: B\otimes L\to B\otimes L$ be the involution $R_{(1+\hat\imath)/2}\otimes\begin{bmatrix} 0 & -1\\ 2 & 0\end{bmatrix}$, where $R_{(1+\hat\imath)/2}$ acts as right multiplication by $(1+\hat\imath)/2$. Then W_2 is clearly (left) B-linear. Moreover, it preserves $H_1(B_\alpha,\mathbf{Z})$ and the polarization, and its L component generates the normalizer $N_0(2)$ of $\Gamma_0(2)\subset\mathrm{SL}\,(L)$. To see this, we compute

$$W_2(\mathbf{M}e \oplus \mathbf{P}f) = \mathbf{P}(1+\hat{\imath})/2e \oplus 2\mathbf{M}(1+\hat{\imath})f = \mathbf{M}e \oplus \mathbf{P}f; \quad \text{likewise,}$$

$$\begin{aligned}\langle W_2(m_1e+m_2f),\, W_2(n_1e+n_2f)\rangle &= \langle m_2\tfrac{1+\hat{\imath}}{2}e + m_1(1+\hat{\imath})f,\, -n_2\tfrac{1+\hat{\imath}}{2}e + n_1(1+\hat{\imath})f\rangle \\ &= \mathrm{Tr}_{B/\mathbf{Q}}(-\tfrac{1-\hat{\imath}}{2}\overline{m_2}n_1(1+\hat{\imath}) + (1-\hat{\imath})\overline{m_1}n_2\tfrac{1+\hat{\imath}}{2}) \\ &= \mathrm{Tr}_{B/\mathbf{Q}}(\overline{m_1}n_2 - \overline{m_2}n_1) \\ &= \langle m_1e+m_2f,\, n_1e+n_2f\rangle.\end{aligned}$$

The last part is well-known. Hence our Γ contains $N_0(2)$. Since $N_0(2)/\pm 1$ is known to be maximal as a fuchsian group, we get $\Gamma = N_0(2)$, proving our assertion. $\square$

Remark 14. *Parts 1 and 2 of Theorem 12 are stated in* [GV, Proposition 2.6]*, and their approach is a geometric version of our explicit argument for Theorem 12.3. However, the lengthy analysis which we needed to determine the pairing and deduce its properties, which in their terminology would have amounted to the analysis of the contraction map, is not done in their paper.*

4. Cubics with nine nodes

In this section we will study cubic threefolds X with nine nodes (i. e. ordinary double points). In the next section these will be related to the quaternionic abelian varieties through some Prym-theoretic constructions.

The maximal number of nodes that a cubic threefold X can have is 10, and this happens if and only if X is (projectively) the Segre cubic (see [Seg, Var, Don2] and Lemma 17 below). We thank Igor Dolgachev for telling us about the beautiful work [Seg]. In it, C. Segre studies cubics with n nodes, $6 \leq n \leq 10$. He starts with the subvarieties $S \subset D \subset \mathbf{P}^8$, where $\mathbf{P}^8$ is the projectivization of the vector space of 3x3 matrices, D is the locus det $= 0$ of singular matrices, and S is the locus of rank-1 matrices which we would nowadays call the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^2 \hookrightarrow \mathbf{P}^8$. The intersection of D with a generic subspace $\mathbf{P}^4 \subset \mathbf{P}^8$ is a cubic threefold with 6 nodes at the points of $\mathbf{P}^4 \cap S$. By moving the $\mathbf{P}^4$ subspace into special position, the cubic threefold $\mathbf{P}^4 \cap D$ can be made to have n nodes, $6 \leq n \leq 10$. Segre states that conversely, any cubic with $n \geq 6$ isolated nodes can be obtained this way, and proceeds to a detailed case by case analysis. In the case of interest to us, he points out that 9-nodal cubics appear in the pencil generated by two completely reducible cubics, i.e. $x_1x_2x_3 + \alpha x_4x_5x_6 = 0$, where the x_i are six general linear coordinates on $\mathbf{P}^4$ satisfying a single linear relation which we can write as $\sum_i x_i = 0$. He states that all 9-nodal cubics arise this way. The properties of such cubics are then straightforward to determine.

In this section we give a modern treatment of these results and explain their modular interpretation. The cubics with 9 nodes turn out to form an irreducible family with many nice properties. In fact we have the following:

Theorem 15. (1) *Let X be a cubic threefold with at least* 9 isolated *singularities over an algebraically closed field. Then the singular locus X_{sing} of X consists of* 9 *or* 10 *nodes.*

(2) *Through every node of an X as in (1) pass 4 planes contained in X. Each plane P' contained in a cubic threefold X' with at most isolated singularities along P' contains at most 4 singularities of X. It contains exactly 4 singularities if and only if they are all nodes for X.*

Proof. Fix an isolated singularity O of X. The lines through O contained in X form an algebraic set $C_{O,X}$ in the projectivized tangent space $\mathbf{P} = \mathbf{P}(T_O(\mathbf{P}^4)) \simeq \mathbf{P}^3$. Choose homogeneous coordinates $[x;y;z;w;u]$ of $\mathbf{P}^4$, where $O = [0;0;0;0;1]$. Then X is given by an equation $f = uq + c = 0$, and $C_{O,X}$ by $q = c = 0$, where q and c are a quadric and a cubic in (x, y, z, w) respectively. To relate the singularities of X and of $C_{O,X}$, let ∇ denote the gradient in the x, y, z, w variables. We will need the following facts:

Lemma 16. *Assume that X is given by $f = uq + c$ as above, with $O = [0;0;0;0;1]$ an isolated singularity as before. Then*
(1) *The singularities of X are given by $q = c = 0$ and $u\nabla q + \nabla c = 0$. The singularities of $C_{O,X}$ are the points of $\mathbf{P}$ where ∇q and ∇c are linearly dependent.*
(2) *Let $P \neq O$ be any singularity of X, and let $p \in \mathbf{P}$ denote the point corresponding to the line $\overline{OP}$. Then q is nonsingular at p, i. e. $\nabla q(p) \neq 0$.*

Proof. The first part is immediate. For the second part, put p at $[0, 0, 0, 1, 0]$. If q were singular at p then $q = q(x, y, z)$ would not depend on w. Moreover we would have $c = c'(x, y, z) + wq'(x, y, z)$. But then X would be singular along the entire line $\overline{OP}$, contradicting our assumption. □

Note that $C_{O,X}$ is a curve: otherwise q and c would have a component in common, yielding $f = Lq'$ for a linear form L and a quadric q'. But then X would have at most one isolated singularity (the vertex of the cone $q' = 0$). Observe also that a line ℓ through two singularities of any cubic threefold Y given by $\{g = 0\}$ is contained in Y, and that a line through three singularities of Y is contained in Y_{sing}: indeed, a cubic polynomial g defining Y vanishes in the first case to order 2 at the 2 singularities so it vanishes on ℓ, while in the second case each partial derivative of g vanishes at three points of ℓ and has degree 2, so is $\equiv 0$.

Returning to our X we see from this and from Lemma 16 that a line on X through O corresponds to an (isolated) singularity of $C_{O,X}$ if and only if it contains an (isolated) singularity — necessarily unique — of X, which is different from O. In particular X has $m + 1$ isolated singularities if $C_{O,X}$ has $m > 0$ isolated singularities.

To prove part 1 it now suffices to prove that q has maximal rank $r = 4$, so that O is a node, and that $C_{O,X}$ has at most one additional singularity to the 8 we know.

Notice first that $r \geq 2$ by Lemma 16(2). Suppose next that $r = 2$. Then we may write $q = wz$. The singularities of X are on the union of the two hyperplanes $z = 0$ and $w = 0$ but not on their intersection (by Lemma 16(2)). Thus $C_{O,X}$ is contained in the union of the corresponding planes in $\mathbf{P}$. On each plane $C_{O,X}$ is defined by the cubic equation $c = 0$, so it can have at most 3 isolated singularities. Since $C_{O,X}$ cannot be singular along the intersection $z = w = 0$, we see that altogether $C_{O,X}$ has at most 6 isolated singularities, contradicting our assumption.

Suppose now that $r = 3$. Then $q = 0$ defines a quadric cone S_0 in $\mathbf{P}$, whose vertex e cannot be in $(C_{O,X})_{\text{sing}}$ by Lemma 16(2). We shall obtain a contradiction by showing that the intersection C' of a cubic surface in $\mathbf{P}$ with S_0 can have at most 6 (isolated) singularities away from e. For this let S be the blowup of S_0 at e. It is well-known that $\operatorname{Pic} S$ is freely generated by the exceptional divisor E and the proper transform F of a line on S through e. Let H be the pullback to S of the hyperplane class $\mathcal{O}_{\mathbf{P}}(1)_{|S}$. Then $2F+E \equiv H$ in $\operatorname{Pic} S$. A divisor class $aE+bF$ contains a reduced and irreducible curve B if and only if either $(a, b) = (0, 1)$ (and then B is a fiber F), or $(a, b) = (1, 0)$ (and then B is E), or if $b \geq 2a > 0$ (and then B is in $|aH+cF|$ with $c \geq 0$). Indeed intersecting B with E and with F shows these conditions are necessary, and their sufficiency follows from Bertini's theorem, since the general member in $aH + cF$ is smooth, hence irreducible. The canonical class is $K_S = -2E - 4F = -2H$ and hence $K_S^2 = 8$. By adjunction, the arithmetic genus of an irreducible curve in $|aE + bF|$ as above is 0 in the first two cases $(a, b) = (1, 0), (0, 1)$, is 0 if $a = 1$ and $c \geq 0$, is $1 + c$ if $a = 2$ and $c \geq 0$, and is 4 if $(a, b) = (3, 6)$. Since an irreducible curve of arithmetic genus g has at most g singularities, a member of $H + cF$ is smooth.

We will now show that the proper transform C'' of C' has at most 6 nodes not on E by examining the types of irreducible components that $C'' \in |3H|$ can have. Each component is E, or some F, or of type $aH + cF$ with $3 \geq a > 0$. A component with $a = 3$ can have at most 4 singularities, a component with $a = 2$ has at most 1 singularity, and a component with $a = 0$ is nonsingular. In particular, since a component with $a = 3$ must be all C'' and $4 < 7$, this case cannot occur. Similarly, a component D with $a = 2$ cannot occur, since the other components are either H — then $D \in |2H|$, and there are at most one singularity on D, none on H and four points of intersection, making a total of at most $5 < 7$ nodes. Otherwise, there are $2 - c$ components of type F and one component equal to E, and there are at most $1 + 2(2 - c) < 7$ nodes. In conclusion, only $a = 1$ occurs. Next, the number k of components of type E clearly cannot be more than 3. It cannot be 3 since the other components will be only F's and E, without any nodes not on E; k cannot be 2, since then we will have one component of type $H + cF$ and $4 - k$ fibers, giving at most $4 - k < 7$ nodes; if $k = 1$ we have components D_1, D_2 of types $aH + c_iF$, $i = 1, 2$, and $2 - c_1 - c_2$ fibers: this gives at most $5 < 7$ nodes. Finally, when $k = 0$ we get 3 components of type H and there are at most 6 nodes.

We now know that $r = 4$, so the locus of $q = 0$ is a nonsingular quadric S. Let F, F' be the two standard rulings of S by lines. As before we want to find the maximal number of isolated singularities p_i that a member C' of $3(F + F')$ can have. Applying the same type of analysis as before we find that when this number is eight or more, C' breaks into two fibers of F, two fibers of F', and a member of the hyperplane class $H \simeq F + F'$, all intersecting transversely. Moreover H is reducible if and only if C' has 9 nodes. Part 1. of the Theorem follows.

The explicit description of such a curve C' in our case $C' = C_{O,X}$ shows that on each of the 4 line components l of $C_{O,X}$ there are 3 singularities. Hence each of the planes $\overline{Ol}$ contains the three corresponding nodes of X in addition to the node O. This plane intersects X in at least the 6 lines joining any two of these 4 nodes of X, hence is contained in X. This gives 4 planes through O contained in X.

Finally, let Π' be a plane contained in a cubic threefold $Y : \{g = 0\}$. If the plane is given by $w = z = 0$, then the singularities of Y along Π' are the intersection of the two conics $\partial g/\partial z = \partial g/\partial w = 0$. A point in the intersection is a node for Y if and only if the intersection is transverse there, and all the intersections are transverse if and only if there are precisely 4 of them, proving part 2 of the Theorem. □

Using the Theorem we can describe the cubic threefolds having at least 9 nodes:

Lemma 17. *A cubic threefold over a scheme S containing nine given nodes is projectively equivalent to one given by*

$$X(\alpha) : \qquad x_1x_2x_3 + \alpha x_4x_5x_6 = x_1 + \cdots + x_6 = 0 \tag{15}$$

in $\mathbf{P}^5/S$, where α is in $\mathbf{G}_m(S)$. Under the evident $S_3 \times S_3$ symmetry, the nine nodes are the orbit of $O_{3,6} = (0, 0, 1, 0, 0, -1)$. Over an algebraically closed field there are 10 *nodes precisely in the Segre case $a = 1$, and then the* 10*th node is* $(1, 1, 1, -1, -1, -1)$.

Proof. Take affine coordinates x_1', x_2', x_4', x_5' for $\mathbf{A}^4$ so that the origin O is one of the given nodes of X. Let T/S be the locus in the grassmanian $\mathrm{Gr}\,(2, T_O(\mathbf{A}^4))/S$ of planes through O which are contained in X and whose intersection with the singular locus X_{sing} of X is supported on the given nodes. The explicit description of $C_{O,X}$ obtained in the proof of Theorem 15 gives that T is étale of degree 4 over S. We also know that T corresponds to 4 lines on $C_{O,X}$, which intersect mutually according to the graph of the sides of a square. These intersections represent 4 of the given nodes of X, so that monodromy acts trivially on the square, and it follows that T is a trivial (product) covering of S. The projectivized tangent cone to X at O, which is a nonsingular quadric Q, contains these two lines in each of its rulings which are marked, i.e.. the étale cover of S which these lines define is a trivial (product) cover. Hence we may choose the coordinates so that these lines are $x_i' = x_j' = 0$ for $1 \le i, j \le 2$. The tangent cone is then $x_1'x_2' + tx_4'x_5' = 0$ for some $t \in \mathcal{O}_S^\times$, and replacing x_4' by tx_4' we may assume $t = 1$. Then $C_{O,X}$ is the intersection of the tangent cone above with $x_1'x_2'\ell_1 = 0$, where ℓ_i denotes an $\mathcal{O}_S$-linear function of x_1', x_2', x_4', and x_5' for any i. Homogenizing, we see that an equation of X in $\mathbf{P}^4$ is given by

$$(x_3' + \ell_2)(x_1'x_2' + x_4'x_5') + x_1'x_2'\ell_1 = 0.$$

Taking $y_3 = x_3' + \ell_2 + \ell_1$, $y_i = x_i'$ for $i = 1, 2, 4$, and 5, and $m = x_3' + \ell_2$ gives the equation $y_1y_2y_3 + y_4y_5\sum_{i=1}^5 \alpha_i y_i = 0$. The coefficients $\alpha_i \in \mathcal{O}_S$ are invertible on S: if $\alpha_1(s) = 0$ for a geometric point s of S, then the tangent cone to the singularity $O_3 = [0; 0; 1; 0; 0]$ is reducible, and similarly for $i = 2$ or 3; if $\alpha_4 = 0$ (respectively $\alpha_5 = 0$ then $O_4 = [0; 0; 0; 1; 0]$ (respectively $O_5 = [0; 0; 0; 0; 1]$) is a singularity whose tangent cone is reducible. In each case we get a non-nodal singularity on X, contradicting our assumption. Thus α_i is invertible, so we may replace each y_i by $x_i = \alpha_i y_i$. Setting $x_6 == -\sum_{i=1}^5 x_i$ we get the desired form.

To determine the singularities we must find the points when the gradients of the two equations in (15) defining $X(\alpha)$ are dependent. If any x_i is 0 we get that two of x_1, x_2,

x_3, and two of x_4, x_5, and x_6 are 0. This leads to the nine nodes of type $O_{3,6}$. Else we find the 10th node as indicated with $\alpha = 1$. We omit the details. $\square$

Note that our formulas are characteristic free. In characteristic > 3 the Segre cubic threefold is usually given by the S_6-symmetric equations

$$\sum_{i=1}^{6} y_i = \sum_{i=1}^{6} y_i^3 = 0.$$

The coordinate change $y_i = x_j + x_k - x_i$, for $\{i, j, k\} = \{1, 2, 3\}$ or $\{4, 5, 6\}$, transforms our form into the other in an $(S_3 \times S_3) \rtimes S_2$-equivariant way.

We will show that the function α in Lemma 17 is unique and that the family (15) is universal; this will require (a part of) the following

Proposition 18. *Let X be a cubic threefold with nine given nodes (over any base, with some fibers possibly having a 10th node). Then we have the following:*

(1) *X contains 9 planes in bijection with the nodes, with a node p corresponding to a plane Π if for every other plane Π' we have $p \in \Pi' \Leftrightarrow \Pi'$ is transversal to Π. In particular, if X has exactly 9 nodes, then it contains exactly (these) 9 planes.*

(2) *There are exactly six* non-transversal plane systems N_k, $k \in \{1, \ldots, 6\}$, *namely sets of three pairwise non-transverse planes among these nine on X, and also six* transversal plane systems, *namely sets T_c of three pairwise transverse planes.*

(3) *Two different transverse plane systems T_c are either disjoint or have one plane in common. Likewise any two different non-transverse plane systems N_k are either disjoint or have one plane in common.*

(4) *If we define two plane systems to be equivalent whenever they are equal or disjoint, then this is indeed an equivalence relation on each type of system, so there are two equivalence classes A, B consisting of three transverse plane systems each, as well as two equivalence classes I, J consisting of three non-transverse plane systems each. In this way the planes in X are put in bijection with both $A \times B$ and $I \times J$: each plane is in a unique system of each type A, B, I, J.*

Proof. In the coordinates of Lemma 17, the node $O_{i,j}$ having 1 at the i'th coordinate, -1 at the j'th coordinate, and 0 elsewhere corresponds to the plane $\Pi_{i,j} = \{x_i = x_j = 0\}$ for any $i \in I := \{1, 2, 3\}$ and $j \in J := \{4, 5, 6\}$. Two planes $\Pi_{i,j}, \Pi_{i',j'}$ are transversal iff the index sets $\{i, j\}, \{i', j'\}$ are disjoint. So the 6 non-transversal plane systems

$$\{N_k |\ k \in I \cup J\}$$

are

$$N_i = \{\Pi_{i,j} |\ j \in J\}$$

and

$$N_j = \{\Pi_{i,j} |\ i \in I\},$$

where i, j, k run over $I, J,\ I \cup J$ respectively. Similarly, the 6 transversal plane systems T_c are:

$$\Pi_{1,i}, \Pi_{2,j}, \Pi_{3,k}, \quad \{i, j, k\} = \{4, 5, 6\} = J.$$

In other words, the index c runs over the 6 bijections of I with J. The symmetric group S_3 permutes the three indices $\{4,5,6\} = J$ and hence also the six transversal plane systems T_c. The plane systems of class A then consist of one orbit of the alternating group $A_3 \subset S_3$, while B consists of the other. For example, we could take A to consist of the systems:

$$\{1,4\},\{2,5\},\{3,6\},$$

$$\{1,5\},\{2,6\},\{3,4\},$$

and

$$\{1,6\},\{2,4\},\{3,5\},$$

while B consists of the remaining systems:

$$\{1,4\},\{2,6\},\{3,5\},$$

$$\{1,5\},\{2,4\},\{3,6\},$$

and

$$\{1,6\},\{2,5\},\{3,4\}.$$

All the assertions of our Proposition are now straightforward. □

Definition 19. *An* allowable marking *of a cubic threefold with nine given nodes is a bijection of the given nodes with $A \times B$ for which there exist the nodes-planes configuration indexed as in Proposition 18. For $a \in A$ and $b \in B$ we will mark the corresponding node by O_{ab}. (Note that there are* ***no*** *natural identifications among the 4 sets A, B, I, J, yet the product $A \times B$* ***is*** *naturally identified with $I \times J$: The plane systems T_a, T_b from distinct equivalence classes have a unique plane in common, say $\Pi_{i,j}$, and the bijection then takes $a \times b$ to $i \times j$. The marking is such that $O_{a,b} = O_{i,j}$ and correspondingly $\Pi_{a,b} = \Pi_{i,j}$.)*

We can now strengthen Lemma 17:

Theorem 20. **(1)** *The moduli problem of classifying nine-nodal cubic threefolds with an allowable marking is represented by $\mathbf{G}_m$. Let α be the usual coordinate of $\mathbf{G}_m$. Then a universal family is given by* (15)*. The universal family is (allowably) marked by letting each marked node O_{ab} of X be the point of $\mathbf{P}^4$ having* 1 *at the ith coordinate,* -1 *at the jth coordinate, and* 0 *elsewhere.*
(2) *The moduli problem of classifying cubic threefolds with nine unmarked nodes but with marked plane systems A, B is coarsely represented by the same (under the forgetting functor) $\mathbf{G}_m$.*
(3) *Over an algebraically closed field, $X(\alpha)$ and $X(\beta)$ of* (15) *are isomorphic if and only if $\beta = \alpha^{\pm 1}$. The involution interchanging A and B on $\mathbf{G}_m$ above is given by $\alpha \mapsto \alpha^{-1}$.*
(4) *The moduli problem of classifying cubic threefolds with unmarked nine nodes is coarsely represented by $\mathbf{G}_m/(\alpha \sim \alpha^{-1}) = \mathbf{A}^1$, with coordinate $b = \alpha + \alpha^{-1}$.*

Proof. **(1)** Let X/S be a family of cubic threefolds with 9 nodes, marked $\{O_{a,b} = O_{i,j}\}$ as above, over a base scheme S. We view the ambient $\mathbf{P}^4/S$ as the hyperplane $x_1+\cdots+x_6 = 0$ in $\mathbf{P}^5/S$. We will show that there are unique coordinates on $\mathbf{P}^4$ so that each node O_{ij} and each plane Π_{ij} on X goes to its namesake in $\mathbf{P}^4$: indeed, the proof of Lemma 17 started by doing this for $O_{3,6}$. Then, perhaps after permuting x_1 with x_2 and/or x_4 with x_5, we got it also for the Π_{ij}'s with $i = 1, 2$ and $j = 4, 5$. As there is no pair of permutations of I and J fixing these, the rigidity of the configuration of nodes and planes on X of Proposition 18 now forces each node and plane of X to go to its namesake in $\mathbf{P}^4$ as asserted. The 9 nodes are in general position in $\mathbf{P}^4$, in the sense that a linear automorphism of $\mathbf{P}^4$ fixing them (pointwise) is the identity. Hence the coordinates are indeed unique. In other words, X/S is the pull-back of the family $X(\alpha)/\mathbf{G}_m$ via a unique morphism $S \to \mathbf{G}_m$ compatible with the markings. This is what we had to show.
(2) The $S_3 \times S_3$ action on $I \times J$ acts trivially on $\alpha \in \mathbf{G}_m$ (from part (1)), and dividing this $\mathbf{G}_m$ by the trivial action(!) gives the claim.
(3) Notice that the permutations of $I \times J$ of the form $\sigma_I \times \sigma_J$ are realized by linear automorphisms of $\mathbf{P}^4$ preserving each $X(\alpha)$. In addition, exchanging x_i with x_{3+i}, for $1 \le i \le 3$ gives an isomorphism θ of $X(\alpha)$ with $X(\alpha^{-1})$. This isomorphism interchanges the classes I and J. Now let $\phi : X(\alpha) \to X(\beta)$ be an isomorphism. Since the singularities of these threefolds are in codimension 3, the weak Lefschetz theorem tells us that their Picard groups are those of the ambient projective space, namely $\mathbf{Z}$, with the hyperplane class as canonical generator. Hence ϕ must preserve it, and so is induced by a linear automorphism of $\mathbf{P}^4$. The rigidity of the plane systems of Proposition 18 shows that after composition with $\sigma_I \times \sigma_J$ and possibly with θ, our ϕ must map each node O_{ab} of $X(\alpha)$ to its namesake in $X(\beta)$. As was already remarked, this forces ϕ to be the identity, and in particular $\beta = \alpha^{\pm 1}$ as asserted.
(4) This follows again by dividing $\mathbf{G}_m$ by $\alpha \sim \alpha^{-1}$. □

Recall that the lines on a cubic threefold X having at most isolated singularities form a surface $F(X)$, called the Fano surface of X. Let O be a node of X and as before, let $C_{O,X}$ be the curve of lines in X through O as before. If X is generic (among cubics with O as a node) then $F(X)$ is identified with $\mathrm{Sym}^2 C_{O,X}$, where a line ℓ on X not passing through O is mapped to the two lines through O which the plane $\overline{O\ell}$ cuts on X. For cubic threefolds with 9 or 10 nodes $F(X)$ is reducible and is described as follows:

Proposition 21. *Let X be any cubic threefold with nine nodes and let X' be the Segre cubic threefold. Then we have the following*

(1) *$F(X')$ consists of fifteen dual planes $\Pi^{\prime,*}_{ij}$, for $1 \le i < j \le 6$, and of six* rulings *R'_i, $1 \le i \le 6$, namely the set of lines on X meeting each plane in the ith plane system.*
(2) *For any plane system on X let R_i, $1 \le i \le 6$ be the corresponding ruling. Then R_i is a surface.*

(3) *$F(X)$ consists of the nine dual planes Π^*_{ab}, for $(a,b) \in A \times B$, and the six rulings R_i, for $i \in A \cup B = \{1,\dots,6\}$. If we view $F(X)$ as a cycle on the grassmanian $G(2,\mathbf{P}^4)$, then each component counts with multiplicity 1.*

(4) *Under the degeneration $a \to 1$ of X to X' given in Lemma 17, the plane Π^*_{ij} goes to the plane $\Pi'^{,*}_{ij}$ for $1 \le i \le 3$ and $4 \le j \le 6$, and the ruling R_i degenerates to $R'_i + \Pi'^{,*}_{jk}$ whenever $\{i,j,k\} = \{1,2,3\}$ or $\{4,5,6\}$.*

Proof. The first part is well-known (see e.g. [Don2]). It is convenient to fix the node $O = O_{3,6}$ and to let $C_{O,X'}$ be the curve of lines on X' through O as before. Write $C_{O,X'}$ as a union of lines $C_{O,X'} = \cup_{i=1}^6 L_i$, where the first three are of type $(1,0)$ and the last three of type $(0,1)$ on the quadric. Then the components of $\mathrm{Sym}^2 C_{O,X'}$ correspond to those of $F(X')$ by $\{L_i, L_j\} \leftrightarrow \Pi^*_{ij}$ and $\mathrm{Sym}^2 L_i \leftrightarrow R'_i$.

The second part follows easily from equation (15). For the remaining parts, consider a degeneration of X to X' in which the node $O = O_{36}$ is fixed, and where the curve $C_{O,X}$ acquires in the limit $C_{O,X'}$ another node. We can write $C_{O,X} = D \cup L_{16} \cup L_{26} \cup L_{35} \cup L_{3,4}$, where the L's are fixed lines of types $(1,0),(1,0),(0,1),(0,1)$ on the quadric and D is the conic (of type $(1,1)$) degenerating to two lines. Then the components of $\mathrm{Sym}^2 C_{O,X}$ correspond to those of $F(X)$ as follows.

$$
\begin{aligned}
&\mathrm{Sym}^2 L_{ij} \leftrightarrow \Pi^*_{ij} \text{ for } \{i,j\} = \{1,6\},\{2,6\},\{34\},\{3,5\};\\
&\{L_{i6}, L_{3j}\} \leftrightarrow \Pi^*_{k\ell} \text{ for } \{i,k\} = \{1,2\} \text{ and } \{j,\ell\} = \{4,5\};\\
&\{L_{34}, L_{35}\} \leftrightarrow R_3 \text{ and } \{L_{16}, L_{26}\} \leftrightarrow R_6;\\
&\{L_{i6}, D\} \leftrightarrow R_i \text{ and } \{L_{3j}, D\} \leftrightarrow R_j \text{ for } i \in \{1,2\} \text{ and } j \in \{4,5\};\\
&\mathrm{Sym}^2 D \leftrightarrow \Pi^*_{36}.
\end{aligned}
\tag{16}
$$

Since we know how $C_{O,X}$ degenerates to $C_{O,X'}$, we know how $F(X) = \mathrm{Sym}^2 C_{O,X}$ degenerates to $F(X') = \mathrm{Sym}^2 C_{O,X'}$: Π^*_{ij} is constant for $1 \le i \le 3$ and $4 \le j \le 6$, and R_i degenerates to $R'_i + \Pi^*_{jk}$ whenever $\{i,j,k\} = \{1,2,3\}$ or $\{4,5,6\}$. Since we know (see [Don2]) that each component of $F(X')$ is simple when $F(X')$ is viewed as a cycle on the grassmanian $G(2,\mathbf{P}^4)$, it follows that our 9 planes and 6 rulings account for all of the components of $F(X)$ (and each is simple on $F(X)$ when viewed as a cycle on $G(2,\mathbf{P}^4)$). □

For future use we summarize in the following proposition the intersection pattern of the components of $F(X)$:

Proposition 22. 1. $\Pi^*_{14} \cap \Pi^*_{15} = \emptyset$.
2. $\Pi^*_{14} \cap \Pi^*_{25}$ *consists of the one point corresponding to the line* $\Pi_{14} \cap \Pi_{25}$.
3. $\Pi^*_{14} \cap R_1$ *consists of the one point corresponding to the line* $\overline{O_{15}O_{16}}$.
4. $\Pi^*_{14} \cap R_2$ *is the line of lines in* Π_{14} *through* O_{34}.
5. $R_1 \cap R_2$ *consists of five points.*
6. $R_1 \cap R_4$ *consists of the conic D of lines in X passing through O_{36} and meeting Π_{36}, and the two points corresponding to the lines $\overline{O_{25}O_{14}}$ and $\overline{O_{15}O_{24}}$.*

The other intersections are obtained by applying an automorphism and a monodromy. We omit the routine proof.

5. The genus two case

5.1. **More symmetry.** When the base curve C has genus 2, there is a group $(\mathbf{Z}/2\mathbf{Z})^3$ extending the symmetry group $(\mathbf{Z}/2\mathbf{Z})^2$ which acts on $C_\pm$ in other genera. To see this, we use the following construction.

Start with three pairs of points

$$a_{i,\epsilon},\ i = 1,2,3,\ \epsilon = 0,1$$

in $\mathbf{P}^1$. Let $\mathbf{P}^1_i$ be the double cover of $\mathbf{P}^1$ branched at the two points $a_{i,\epsilon},\ \epsilon = 0,1$. The fiber product $\mathbf{P}^1_1 \times_{\mathbf{P}^1} \mathbf{P}^1_2 \times_{\mathbf{P}^1} \mathbf{P}^1_3$ is a curve ${}_5C$ of genus 5. It admits a $(\mathbf{Z}/2\mathbf{Z})^3$ action. The seven level-1 quotients, or quotients by subgroups $(\mathbf{Z}/2\mathbf{Z})^2$, are the three $\mathbf{P}^1_i$ plus the three elliptic curves E_i branched at the four points $a_{j,\epsilon},\ j \neq i,\ \epsilon = 0,1$, and the genus 2 hyperelliptic curve ${}_2C$ branched at all six points. The seven level-2 quotients, or quotients by subgroups $\mathbf{Z}/2\mathbf{Z}$, consist of the three elliptic curves $\widetilde{E_i} := \mathbf{P}^1_j \times_{\mathbf{P}^1} \mathbf{P}^1_k$ (where $\{i,j,k\} = \{1,2,3\}$), plus the three genus 3 curves ${}_3C_i := \mathbf{P}^1_i \times_{\mathbf{P}^1} E_i$, and one more genus 3 curve ${}_3C$ whose quotients are the three E_i.

We can recover an instance of diagram (1)) by relabelling:

$${}_5C = C_\pm,\ {}_3C_1 = C_{\hat{\imath}},\ {}_3C_2 = C_{\hat{\jmath}},\ {}_3C_3 = C_{\hat{k}},\ {}_2C = C,$$

and choosing a double cover $\widetilde{C} \to C_\pm$ such that $\widetilde{C}$ is Galois over each ${}_3C_i$ with group $\mathbf{Z}/4\mathbf{Z}$. But in fact, any diagram (1) with $g = 2$ arises this way, and uniquely. The point is that the base curve ${}_2C$ is hyperelliptic, and the hyperelliptic involution acts on $J({}_2C)$ as -1, so it preserves all points of order 2 and all double covers. In fact, any double cover such as $C_{\hat{\imath}} \to {}_2C$ is Galois over $\mathbf{P}^1$ with group $(\mathbf{Z}/2\mathbf{Z})^2$ and quotients ${}_2C, E_{\hat{\imath}}, \mathbf{P}^1_{\hat{\imath}}$ of genera $2,1,0$ respectively. In particular, this gives three double covers $\mathbf{P}^1_{\hat{\imath}}$, $\mathbf{P}^1_{\hat{\jmath}}$, $\mathbf{P}^1_{\hat{k}}$ of $\mathbf{P}^1$. If we relabel them $\mathbf{P}^1_i,\ i = 1,2,3$, we are back in the situation of the previous paragraph.

5.2. **Even more symmetry.** Now start with a set S of five points in $\mathbf{P}^1$. We label the five distinct elements, in any order, as i,j,k,l,m. There are $15 = 2^{(5-1)} - 1$ non-empty even subsets of S, giving 15 branched double covers of $\mathbf{P}^1$. These are the 15 level-1 quotients of their common Galois closure, a curve ${}_5C$ which is Galois over $\mathbf{P}^1$ with group $(\mathbf{Z}/2\mathbf{Z})^4$. We enumerate all the quotients of ${}_5C$:

Level 1

10 rational curves	$\mathbf{P}^1_{i,j}$ is branched at 2 points $i, j \in S$
5 elliptic curves	E_i is branched at 4 points $S \setminus i$

Level 2

10 rational curves	$\mathbf{P}^1_{i,j,k}$ has quotients $\mathbf{P}^1_{i,j}, \mathbf{P}^1_{i,k}, \mathbf{P}^1_{j,k}$
15 elliptic curves	$E_{ij,kl} := \mathbf{P}^1_{i,j} \times_{\mathbf{P}}^1 \mathbf{P}^1_{k,l}$
10 genus 2 curves	${}_2C_{i,j}$ has quotients $\mathbf{P}^1_{i,j}, E_i, E_j$

Level 3

5 elliptic curves	$\widetilde{E}_i$ has level-1 quotients: $E_i, \mathbf{P}^1_{j,k}, \mathbf{P}^1_{j,l}, \mathbf{P}^1_{j,m}, \mathbf{P}^1_{k,l}, \mathbf{P}^1_{k,m}, \mathbf{P}^1_{l,m}$, and level-2 quotients: $\mathbf{P}^1_{j,k,l}, \mathbf{P}^1_{j,k,l}, \mathbf{P}^1_{j,k,l}, \mathbf{P}^1_{j,k,l}, E_{jk,lm}, E_{jl,km}, E_{jm,kl}$
10 genus 3 curves	${}_3C_{i,j}$ has level-1 quotients: $\mathbf{P}^1_{i,j}, \mathbf{P}^1_{k,l}, \mathbf{P}^1_{k,m}, \mathbf{P}^1_{l,m}, E_k, E_l, E_m$ and level-2 quotients: $\mathbf{P}^1_{k,l,m}, E_{ij,kl}, E_{ij,km}, E_{ij,lm}, {}_2C_{k,l}, {}_2C_{k,m}, {}_2C_{l,m}$.

Level 4

1 genus 5 curve	${}_5C$

Note that $\widetilde{E}_i$ is Galois over E_i with group $(\mathbf{Z}/2\mathbf{Z})^2$ and intermediate covers $E_{jk,lm}$, $E_{jl,km}$, and $E_{jm,kl}$. These are all the double covers of E_i; so $\widetilde{E}_i$ is isomorphic to E_i, and the degree 4 map $\widetilde{E}_i \to E_i$ is multiplication by 2.

The Galois group of ${}_5C$ over $\mathbf{P}^1_{ij}$ is $(\mathbf{Z}/2\mathbf{Z})^3$. In $\mathbf{P}^1_{ij}$ we have six branch points in three pairs, namely the inverse images of $S \setminus \{i, j\}$. So the curve ${}_5C$ can be viewed as our previous $C_\pm$ in ten distinct ways, over the ten rational curves $\mathbf{P}^1_{ij}$ and the corresponding genus-2 base curves ${}_2C_{i,j}$.

In this special case we can also describe quaternion covers $\widetilde{C} \to C_\pm$ quite explicitly. Let q be either of the two points of E_m above $m \in S \subset \mathbf{P}^1$. Its inverse image in $\widetilde{E}_m$ is the set of four points q_a, $a = 1, 2, 3, 4$ satisfying $2q_a = q$. Now on $\widetilde{E}_m$ we have a natural line bundle $\mathcal{L}_m \in \mathrm{Pic}^2(\widetilde{E}_m)$ such that $\mathcal{L}_m^{\otimes 2}$ has a section s vanishing at the four points q_a. Namely, $\mathcal{L}_m$ is isomorphic to $\mathcal{O}_{\widetilde{E}_m}(2q_a)$, for any a. The inverse image in $\mathcal{L}_m$ of the section s, under the squaring map, gives a double cover ${}_3C_m \to \widetilde{E}_m$ branched at the four points q_a. Explicitly, if we write the equation of $\widetilde{E}_m$ as a double cover of $\mathbf{P}^1_{i,j,k}$ as:

$$y^2 = \Pi_{a=1}^4 (x - \lambda_a),$$

with q_a the point with coordinates $(x = \lambda_a, y = 0)$, then ${}_3C_m$ has equation

$$y^4 = \Pi_{a=1}^4 (x - \lambda_a).$$

In particular, ${}_3C_m$ is $\mathbf{Z}/4\mathbf{Z}$-Galois over $\mathbf{P}^1_{i,j,k}$. It follows that the fiber product:

$$\widetilde{C} := {}_5C \times_{\widetilde{E}_m} {}_3C_m = {}_3C_{l,m} \times_{\mathbf{P}^1_{ijk}} {}_3C_m$$

is a $\mathbf{Z}/4\mathbf{Z}$-Galois cover of ${}_3C_{l,m}$. Similarly, this same $\widetilde{C}$ is also a $\mathbf{Z}/4\mathbf{Z}$-Galois cover of ${}_3C_{i,m}$, ${}_3C_{j,m}$, and ${}_3C_{k,m}$. In particular, $\widetilde{C}$ is quaternionic over $\mathbf{P}^1_{ijk}$ (and also over $\mathbf{P}^1_{ijl}, \mathbf{P}^1_{ikl}$, and $\mathbf{P}^1_{jkl}$).

We note that what we get this way is a special case of the general genus 2 quaternionic towers (1): the general case depends on 3 parameters, while this special case depends on only two parameters. The curves ${}_5C$ in this two dimensional family are known as Humbert curves, cf. [Don2, Var]. Varley shows [Var] that the covers $\widetilde{C} \to {}_5C$ all have the same Prym, a certain 4-dimensional non-hyperelliptic ppav with 10 vanishing theta nulls.

5.3. **Abelian fourfolds and cubic threefolds.** We need to recall some features of the Prym map in genus 5. Our references in this subsection are [Don1, Don2]. Let $\mathcal{A}_g$ be the moduli space of g-dimensional ppav's, and $\mathcal{RA}_g$ the moduli space of g-dimensional ppav's with a marked point of order 2. Let $\mathcal{M}_g$ be the moduli space of curves of genus g, and $\mathcal{R}_g$ the moduli space of curves with a marked point of order 2 in their Jacobian. Let $\mathcal{C}$ be the moduli space of cubic threefolds whose only singularities are some ordinary double points. There is a corresponding moduli space $\mathcal{RC}$ of cubic threefolds together with a point of order 2 in their intermediate Jacobian. In fact, this space splits into even and odd components: $\mathcal{RC} = \mathcal{RC}^+ \cup \mathcal{RC}^-$, distinguished by an appropriate $\mathbf{Z}/2\mathbf{Z}$-valued function. Similarly, let $\mathcal{Q}$ be the moduli space of plane quintic curves Q whose only singularities are some ordinary double points. There is a corresponding moduli space $\mathcal{RQ}$ of plane quintic curves together with a point of order 2 in their compactified Jacobian. Again, this space splits into even and odd components: $\mathcal{RQ} = \mathcal{RQ}^+ \cup \mathcal{RQ}^-$, distinguished by an appropriate $\mathbf{Z}/2\mathbf{Z}$-valued function.

One of the basic results about the Prym map:

$$\mathcal{P} : \mathcal{R}_5 \to \mathcal{A}_4$$

is that it factors through a rational map:

$$\kappa : \mathcal{R}_5 \to \mathcal{RC}^+$$

followed by a birational isomorphism:

$$\chi : \mathcal{RC}^+ \to \mathcal{A}_4.$$

These are constructed as follows. For more details, see [Don2].

- A pair (X, l), where $X \subset \mathbf{P}^4$ is a cubic threefold (with nodes at worst) and $l \subset \mathbf{P}^4$ is a line contained in X, determines a plane quintic curve Q and an (odd) double cover $\widetilde{Q}_\sigma \to Q$. Explicitly, $\widetilde{Q}_\sigma$ parameterizes the lines in X meeting l, while Q parameterizes the planes through l which intersect X residually in two additional lines.
- Conversely, given Q and an odd σ, we can recover the pair (X, l). When Q is non-singular, X is characterized as the unique cubic threefold whose intermediate Jacobian is isomorphic to $\mathrm{Prym}\,(Q, \sigma)$. In case Q is singular, we describe an

explicit construction below, in the proof of the implication (2) $\Rightarrow$ (1) in Theorem 25.

- A non-hyperelliptic curve $C \in \mathcal{M}_5$ determines a plane quintic curve Q and an (even) double cover $\widetilde{Q}_\nu \to Q$ such that $\mathcal{P}(Q,\nu) \cong \mathrm{Jac}\,(C)$. Explicitly, $\widetilde{Q}_\nu$ is the singular locus of the theta divisor of $\mathrm{Jac}\,(C)$, so it parameterizes linear systems g^1_4 on C, while Q is the quotient of $\widetilde{Q}_\nu$ by the involution -1 of $\mathrm{Jac}\,(C)$, and it parameterizes quadrics of rank 4 in $\mathbf{P}^4$ through the canonical image of C.
- In the above situation, a point of order 2: $\mu \in \mathrm{Jac}\,(C)[2]$, determines via Mumford's isomorphism (cf. [Mum2] or [Don2, Theorem 1.4.2]), a pair of points of order two: $\sigma, \nu\sigma \in \mathrm{Jac}\,(Q)[2]$. One of these, say $\nu\sigma$, is even, while the other, σ, is odd.
- So, given $(C,\mu) \in \mathcal{R}_5$, we set $\kappa(C,\mu) = (X,\delta)$, where X is the cubic threefold corresponding to (Q,σ), and δ is the image of ν under Mumford's isomorphism. It is automatically even.
- Finally, given $(X,\delta) \in \mathcal{RC}^+$, choose a line l in X. The pair (X,l) determines the quintic Q and its odd double cover $\widetilde{Q}_\sigma \to Q$, while δ determines a second cover $\widetilde{Q}_\nu \to Q$. Then $\mathrm{Prym}\,(Q,\nu)$ is the Jacobian of a curve C which can be described explicitly, and σ descends to a point $\mu \in \mathrm{Jac}\,(C)[2]$. We then set $\chi(X,\delta) := \mathrm{Prym}\,(C,\mu)$. The result turns out to be independent of the choice of l, by the tetragonal construction.

5.4. **The main results.** Our main result is that under the correspondence χ, the four dimensional quaternionic abelian varieties correspond to the nine-nodal cubic threefolds. Each of these nine-nodal cubic threefolds comes equipped with a natural, "allowable" point of order two. Before giving the precise statement of the theorem, we need to explain this lift.

Let X be a nine-nodal cubic threefold. We use the notation of section 4. Let $R = R_6$ be a ruling consisting of all lines in X meeting three transversal planes $\Pi_{16}, \Pi_{26}, \Pi_{36}$, and let $l \in R$ be one such line. This determines a plane quintic $Q = Q_l$ and its double cover $\widetilde{Q}_\sigma$. We note that $\widetilde{Q}_\sigma$ contains the three pencils L^0_a of lines in the plane Π_{a6} which pass through the point $\Pi_{a6} \cap l$, for $a \in A = \{1,2,3\}$. Therefore the plane quintic Q contains three lines L_a, and residually a conic D. It follows that the cover $\widetilde{Q}_\sigma \to Q$ is étale: $\widetilde{Q}_\sigma = (\cup_{a=1}^3 \cup_{\epsilon=0}^1 L^\epsilon_a) \cup D^0 \cup D^1$, where each D^δ meets each L^ϵ_a in one point.

Our X represents a point of the moduli space $\mathcal{C}$. A lift of X to $\mathcal{RC}^+$ is determined by a second double cover $\widetilde{Q}_\nu$ of Q, with class ν which is orthogonal to σ and even. There is a natural choice for such a double cover, and hence for the lift, namely the unique allowable one: $\widetilde{Q}_\nu$ is the unique cover which is branched over all nine nodes of Q. Explicitly, $\widetilde{Q}_\nu = E_1 \cup E_2 \cup E_3 \cup {}_2\widetilde{D}$, where $E_a \to L_a$ is a double cover branched at the four points where L_a meets the other components, and ${}_2\widetilde{D} \to D$ is a double cover branched at the six intersection points of the conic with the lines.

Lemma 23. *The allowable cover $\widetilde{Q}_\nu \to Q$ constructed from a pair (X,l) is even and orthogonal to σ.*

We can now state our main results.

Theorem 24. *The correspondence χ takes the nine-nodal cubic threefolds (with their unique allowable lift to $\mathcal{RC}^+$) to the four dimensional quaternionic abelian varieties.*

This follows immediately from the following more detailed version:

Theorem 25. *The following data are equivalent:*

(1) *Pairs (X, l) where X is a nine-nodal cubic threefold and l is a line in a ruling R on X.*
(2) *Pairs $(Q, \widetilde{Q}_\sigma)$ where $Q = L_1 \cup L_2 \cup L_3 \cup \Delta$ is a reducible quintic consisting of three lines and a conic, and $\widetilde{Q}_\sigma \to Q$ is an étale double cover, $\widetilde{Q}_\sigma = (\cup_{a=1}^3 \cup_{\epsilon=0}^1 L_a^\epsilon) \cup \Delta^0 \cup \Delta^1$, where each Δ^δ meets each L_a^ϵ in one point, and L_a^ϵ meets $L_{a'}^{\epsilon'}$ if and only if $a \neq a'$ and $\epsilon \neq \epsilon'$.*
(3) *A curve $\widetilde{C} \in \mathcal{M}_9$ with a fixed-point free action of the quaternion group G.*

Proof. (of Theorem 25 and Lemma 23)

(1) $\Rightarrow$ (2):
We saw above how to go from (X, l) to a reducible plane quintic $Q = L_1 \cup L_2 \cup L_3 \cup \Delta$ and an étale double cover $\widetilde{Q}_\sigma = (\cup_{a=1}^3 \cup_{\epsilon=0}^1 L_a^\epsilon) \cup \Delta^0 \cup \Delta^1$. The intersection properties of the components of $\widetilde{Q}_\sigma$ can be determined directly from the explicit formula (15). An alternative is to return to the degeneration used in Lemma 21, in which X goes to the Segre cubic and $l \in R_6$ goes to $l' \in R_6'$. The cover $\widetilde{Q'}_\sigma \to Q'$ corresponding to (X', l') is easy to determine, because of the larger symmetry present in this case. It was described, for example, in [Don2], formula (5.17.4):

$$Q' = (\cup_{i=1}^5 L_i^\epsilon)/(p_{i,j} \sim p_{j,i},\ i \neq j),$$

$$\widetilde{Q'}_\sigma = (\cup_{i=1}^5 \cup_{\epsilon=0}^1 L_i^\epsilon)/(p_{i,j}^0 \sim p_{j,i}^1,\ i \neq j). \tag{17}$$

Under our degeneration, the conic Δ splits into $L_4 \cup L_5$. The cover $\widetilde{Q}_\sigma \to Q$ given in the theorem is the only one which specializes correctly.

(2) $\Rightarrow$ (1):
To go in the opposite direction, consider first the more general situation, where we start with a pair $(Q, \widetilde{Q}_\sigma)$, where Q is any quintic with at least one node o over which $\widetilde{Q}$ is étale. We can explicitly exhibit the corresponding cubic threefold X and line l as follows. Projection from o shows that the partial normalization T of Q at o is a trigonal curve of arithmetic genus 5, with a double cover $\widetilde{T}$ obtained by normalizing $\widetilde{Q}$ above o. The trigonal construction takes the pair $T, \widetilde{T}$ to a curve B of genus 4 which comes equipped with a g_4^1 linear system. The canonical map sends B to $\mathbf{P}^3$, and the homogeneous ideal of the image is generated by a quadric f_2 and a cubic f_3. The inhomogeneous equation $f_2 + f_3 = 0$ then determines a Zariski open piece of our cubic X as a hypersurface

in affine 4-space, and X is recovered as the closure in $\mathbf{P}^4$. Hence B can be naturally identified with the curve $C_{O,X}$ of lines on X through O (introduced in Theorem 15). By Proposition 21, the Fano surface $F(X)$ parameterizing lines in X can be described as the symmetric product S^2B modulo certain identifications. The g^1_4 linear system on B is necessarily of the form $\omega_B(-p-q)$ for two points p, q in (the smooth part of) B, where ω_B is the canonical bundle. We then recover l as the line corresponding to the point of $F(X)$ given by the image of $p+q \in S^2B$. From section (5.11.2) of [Don2] it follows that this construction is indeed inverse to our construction of $(Q, \widetilde{Q}_\sigma)$ from (X, l).

Returning to our special case, we now see that it merely remains to check that this line l lies indeed on a ruling (and not on a dual plane). Assume (as we may) that, in the previous notation, $O = O_{3,6}$. Then we will show a more precise result:

Claim 26. *The line l is on the ruling R_3 or R_6 if and only if the node o is on the intersection of two lines L_i, L_j of Q; on the other hand, l is on one of the rulings R_1, R_2, R_4 or R_5 if and only if the node o is on the intersection of a line of Q and the conic D. (The monodromy action permutes all the cases of a given type.)*

To see this we use the analysis in (16): this tells us which components of B must contain the points p, q in order for the line l to be on a given ruling R_i:

$$\begin{array}{llll} l \in R_3 & & & \Leftrightarrow p \in L_{34}, \;\; q \in L_{35} \\ l \in R_6 & & & \Leftrightarrow p \in L_{16}, \;\; q \in L_{26} \\ l \in R_i & \text{for} & i = 1,2 & \Leftrightarrow p \in L_{i6}, \;\; q \in D \\ l \in R_j & \text{for} & j = 4,5 & \Leftrightarrow p \in L_{3j}, \;\; q \in D. \end{array} \tag{18}$$

As in the general case, the degree 4 map $\pi : B \to \mathbf{P}^1$ is given by projecting from the line $\overline{pq} \subset \mathbf{P}^3$. Here this projection has degree 0 (i.e. it is constant) on the line components of B through p, q; on the remaining line components of B the degree of π is 1, and on D it is 1 when q lies on D ("the second case") and 2 otherwise ("the first case"). To prove the claim, we must show that the degree 4 map $\pi : B \to \mathbf{P}^1$ arises from the double cover $\widetilde{Q}_\sigma \to Q$ of the trigonal curve $\pi' : Q \to \mathbf{P}^1$ by the trigonal construction. There are two cases to examine, namely when π' is projection from the intersection of two lines ("the first case"), and when it is projection from a point of intersection of Δ and a line "the second case"). As before, π' has degree 0 on the lines through the center of projection, and it is straightforward to see from the definition of the trigonal construction, that the two cases we distinguished for π' yield the respective cases we distinguished for π.

(3) $\Rightarrow$ (2)**:** Recall first that by part (3) of Lemma 5 there are 4 possible double covers $\tilde{C}$'s covering a given $C_\pm$ in (3). Similarly, there are 4 possible double covers $\widetilde{Q}_\sigma$ in (2): for each $a = 1, 2, 3$ we must choose which of the two points of Δ^0 which lie above the points where Δ and L_a intersect is on L^0_a. Of the resulting $2^3 = 8$ possibilities each choice is isomorphic with the "opposite" one, obtained by interchanging Δ^0 with Δ^1 and each chosen point of $\Delta \cap L_a$ with the other one. We now claim that the parameter spaces, $\mathcal{R}_3$ for the coverings $\tilde{C}/C_\pm$ and $\mathcal{R}_2$ for the $\widetilde{Q}_\sigma/Q$'s, form irreducible spaces. For $\mathcal{R}_3$ this is Proposition 4. For $\mathcal{R}_2$ notice first that the space of Q's (conics and three lines) is

manifestly irreducible. The same is then true for the allowable covers $\widetilde{Q}_\nu$ of Q by their uniqueness. Moreover, monodromy allows us to "turn around" individually each of the lines so its points of intersection with the conic are interchanged. This shows that the 4 covers $\widetilde{Q}_\sigma$'s of a given Q are in the same component, proving the claim.

Now suppose that we are given the curve $\tilde{C}$ with an action of G, hence the quotient $C_\pm$ and the entire tower (1). Let $\widetilde{Q}_\nu \to Q$ be the quintic double cover corresponding to the genus-5 curve $C_\pm$, and let $\widetilde{Q}'_\sigma \to Q$ be the double cover inducing $\widetilde{C} \to C_\pm$ via Mumford's isomorphism, cf. [Mum2] or [Don2, Theorem 1.4.2]. As we saw in subsection 5.1, $C_\pm$ has three elliptic quotient curves $\widetilde{E_i},\ i = 1,2,3$. It follows that $\widetilde{Q}_\nu$, which parameterizes linear systems g^1_4 on $C_\pm$, contains three elliptic curves, which can be canonically identified with the Picard varieties $\mathrm{Pic}^2(\widetilde{E_i})$. Therefore, Q contains three lines $L_i = \mathbf{P}^1_i$, and residually a conic Δ. We claim that the double cover $\widetilde{Q}'_\sigma \to Q$ is one of the double covers $\widetilde{Q}_\sigma \to Q$ described in part (2) of the theorem.

This is known to be true after we specialize the general curves $C_\pm$ of subsection 5.1 to the Humbert curves of 5.2: the conic Δ breaks further to two lines L_4, L_5, so that each of Δ^0 and Δ^1 breaks into two lines $L^0_4 \cup L^1_5$ and $L^1_4 \cup L^0_5$ respectively. The double cover obtained from the Segre cubic is specified in (17) and agrees with the double cover $\widetilde{Q}_\sigma \to Q$ described in part (2) of the theorem.

We now claim that the same must hold in general, namely that for every $\tilde{C}$ in (3) the covering $\tilde{C}_\sigma$ we obtained is one of the 4 covers given in (2). Indeed, let $\mathcal{Q}$ be the irreducible variety parameterizing the reducible quintics as in part (2) of the theorem, and let $\mathcal{RQ} \to \mathcal{Q}$ be the ètale cover parameterizing all ètale double covers of such quintics. We are given two irreducible subcovers $\mathcal{R}_2 \to \mathcal{Q}$ and $\mathcal{R}_3 \to \mathcal{Q}$ of $\mathcal{RQ} \to \mathcal{Q}$, parametrizing the ètale double covers coming from (2) and (3) respectively. (We noted that each is a four-sheeted cover.) Now two irreducible subcovers of an ètale cover must either be disjoint or coincide. But our $\mathcal{R}_2$ and $\mathcal{R}_3$ intersect at the Segre point $\widetilde{Q}_{\sigma,0}/Q_0$, at which $Q = Q_0$ consists of five lines. They therefore coincide as claimed.

(2) ⇒ (3):

As we saw above, the allowable double cover $\widetilde{Q}_\nu \to Q$ is uniquely determined by Q. We recover $C_\pm$ as the unique curve whose Jacobian is isomorphic (as a ppav) to $\mathrm{Prym}\,(\widetilde{Q}_\nu/Q)$, and $\widetilde{C} \to C_\pm$ is the double cover induced via Mumford's isomorphism from $\widetilde{Q}'_\sigma \to Q$. This is clearly the inverse of the previous construction, so we are done with the theorem.

If we start with data (3), the construction of $\widetilde{Q}_\sigma \to Q$ involves Mumford's isomorphism, so the cover $\widetilde{Q}_\sigma \to Q$ is automatically orthogonal to $\widetilde{Q}_\nu \to Q$. Moreover, this cover $\widetilde{Q}_\nu \to Q$ is even, since its Prym is the Jacobian of a curve $C_\pm$ (rather than the intermediate Jacobian of a cubic threefold). By the theorem, it follows that the same holds if we start with data (1), proving the lemma.

□

Now that we have a fairly complete description of the Prym map on the parameter space $\mathcal{M}(2;0,0,0)$ of proposition 4, it is easy to specialize further and find its behavior on the boundary strata $\mathcal{M}(1;2,0,0)$ and $\mathcal{M}(0;2,2,0)$. We work out the latter in some detail.

We want the genus 2 curve C at the bottom of tower (1) to degenerate, within a fixed fiber of the Prym map, to a rational curve with two nodes, and we want to trace what happens to $C_\pm, \widetilde{C}$ in this limit. This can be easily arranged, in the language of subsection 5.1, for example by letting the branch points $a_{i,0}, a_{i,1}$ coincide for $i = 2,3$. We relabel the surviving points $a_{1,0}, a_{1,1}, a_2, a_3$. At level 1 of the $(\mathbf{Z}/2\mathbf{Z})^3$-diagram we find that $\mathbf{P}^1_1$ remains a smooth rational curve, doubly covering $\mathbf{P}^1$ with branch points $a_{1,0}, a_{1,1}$; but $\mathbf{P}^1_i$ for $i = 2,3$ degenerates to a reducible curve, consisting of two copies $\mathbf{P}^1_{i,\epsilon_i}$ of $\mathbf{P}^1$, $\epsilon_i \in (\mathbf{Z}/2\mathbf{Z})$, intersecting each other above a_i. It follows immediately that ${}_5C = C_\pm$ has four components $C_{\epsilon_2,\epsilon_3}$, $\epsilon_2, \epsilon_3 \in \mathbf{Z}/2\mathbf{Z}$. Each component $C_{\epsilon_2,\epsilon_3}$ is isomorphic, as a double cover of $\mathbf{P}^1$, to $\mathbf{P}^1_1$, i.e. it is branched at $a_{1,0}, a_{1,1}$. Component $C_{\epsilon_2,\epsilon_3}$ meets component $C_{1+\epsilon_2,\epsilon_3}$ in the two points above a_2; it meets component $C_{\epsilon_2,1+\epsilon_3}$ in the two points above a_3; and it does not meet component $C_{1+\epsilon_2,1+\epsilon_3}$. Finally, up to isomorphism there is only one quaternionic cover $\widetilde{C} \to C_\pm$, namely the unique allowable cover of $C_\pm$. Let $E(a_{1,0}, a_{1,1}, a_2, a_3)$ be the elliptic curve which is the double cover of $\mathbf{P}^1_1$ branched at the four points above $a_2, a_3 \in \mathbf{P}^1$. Then $\widetilde{C}$ consists of four copies of the same elliptic curve $E(a_{1,0}, a_{1,1}, a_2, a_3)$ glued at their ramification points. The Prym is then isogenous to $(E(a_{1,0}, a_{1,1}, a_2, a_3))^{\times 4}$.

In a special case, this degeneration picture was obtained in Remark 3 on the last page of [Var]; in fact, Varley's situation is precisely the case of our subsection 5.2, where two pairs among the five points $S \subset \mathbf{P}^1$ coalesce, say i and l go to 0 while j and m go to ∞ and k is at 1. The six points on $\mathbf{P}^1_{i,j}$ then coincide as in the previous paragraph: $a_{1,0} = 1, a_{1,1} = -1, a_2 = 0, a_3 = \infty$, showing that the Prym of an Humbert curve is isogenous to the fourth power of the harmonic elliptic curve. In our more general setting, the elliptic curve $E(a_{1,0}, a_{1,1}, a_2, a_3)$ is arbitrary: we can take for instance $a_{1,0} = 0, a_{1,1} = \infty$. We then see that $E(0,\infty,a_2,a_3)$ is the double cover of $\mathbf{P}^1_1$ branched at $\pm\sqrt{a_2}, \pm\sqrt{a_3}$, and this has variable modulus. We conclude:

Corollary 27. *The Prym map Φ sends $\mathcal{M}(2;0,0,0)$ and each of the spaces $\mathcal{M}(1;2,0,0)$ and $\mathcal{M}(0;2,2,0)$ onto the Shimura curve* Shim *parameterizing 4-dimensional ppav's A with quaternionic multiplication. Each 4-dimensional ppav A with quaternionic multiplication is isogenous to the fourth power $E^{\times 4}$ of some elliptic curve E, and every E occurs for some A. (The precise isogeny is given in Corollary 13.)*

6. Appendix

In this appendix we will sketch the proof of the following result, mentioned in the Introduction.

Theorem 28. *Let T^{**}/T be a cyclic, 4-sheeted Galois unramified cover of a (smooth projective complex irreducible) general trigonal curve T of genus $g > 1$. Identify* $\mathrm{Gal}\,(T^{**}/T)$

with $\langle\hat{\imath}\rangle$, and let T^*/T be the intermediate 2-sheeted cover. Then the $\langle\hat{\imath}\rangle$-action on $P = \mathrm{Prym}\,(T^{**}/T^*)$ does not extend to a G-action.

Proof. The locus of hyperelliptic curves in $\mathcal{M}_g$ is in the closure of the trigonal locus. Hence existence of a G-action in the trigonal case implies the same for each hyperelliptic curve H. In the trigonal case there is only one type of cyclic covers (under monodromy), but over the hyperelliptic locus there are several types of unramified double covers H^*/H and a-fortiori of 4-sheeted cyclic covers H^{**}/H. A double cover H^*/H is determined by the choice of a subset D of even cardinality of the Weierstrass points of H, up to replacing this subset by its complement. The irreducibility of the space of T^{**}/T over the trigonal locus implies that the G-action must then exist for each such (hyperelliptic) type. Thus it suffices to consider the easiest type, when the set D consists of two Weierstrass points, which we take as the points of H over $0, \infty$ in $\mathbf{P}^1$. In terms of a coordinate x on $\mathbf{P}^1$, the curve H is given by an affine equation $y^2 = xf(x)$, where f has degree $2g$ and simple roots. Then H^* is given by $(y/u)^2 = f(u^2)$ (with $x = u^2$). In particular, H^* is hyperelliptic. We will need the following two lemmas:

Lemma 29. *In this case P is the square of a general hyperelliptic jacobian $J(C)$ (up to isogeny).*

Lemma 30. *For a general hyperelliptic curve C we have* $\mathrm{End}\,(\mathrm{Jac}\,(C)) = \mathbf{Z}$.

Assuming Lemmas 29, 30 it is easy to conclude the proof of the Theorem. Indeed, the $\mathbf{Q}$-endomorphism ring of P is then $\mathrm{Mat}_{2\times 2}(\mathbf{Q})$. The group G must then embed into $\mathrm{GL}_2(\mathbf{Q})$ (the action of G on P must be faithful since $-1 \in G$ acts as -1 on P). This is a contradiction, because G does not embed even into $\mathrm{GL}_2(\mathbf{R})$. ☐

In the proof of Lemma 29 we will need the following third Lemma:

Lemma 31. *Let F be a curve, let F^* be an unramified double cover of F corresponding to the $\sigma \in \mathrm{Jac}\,F[2]$, and let $F{**}$ be an unramified double cover of F^* corresponding to the class $\sigma^* \in \mathrm{Jac}\,F^*[2]$. Then F^{**} is cyclic Galois over F if and only if the norm map $\mathrm{Nm} : \mathrm{Jac}\,F^* \to \mathrm{Jac}\,F$ maps σ^* to σ.*

Proof. We will prove this presumably well-known Lemma 31 since we do not know a reference for it. Assume first that F^{**}/F is cyclic (of order 4). As in the proof of Proposition 4 we present the fundamental group $\pi = \pi_1(F, \dot{})$ of F as generated by a standard (symplectic) basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$. We assume, as we may, that F^{**} is the cover corresponding to the kernel of the homomorphism $\pi \to \mathbf{Z}/4\mathbf{Z}$ given by sending α_1 to 1 (in $\mathbf{Z}/4\mathbf{Z}$) and the other generators to 0. Then we may think of F as a genus one curve E, with fundamental group generated by α_1 and β_1, to which a "tail" T of genus $g-1$ is attached. In this model F^{**} can be viewed as a genus one curve E^{**}, with fundamental group generated by $\alpha^4\alpha_1$ and β_1, with four copies of T attached. The deck transformations of F^{**}/F rotate E cyclically by quarter-turns along $4\alpha_1$ and permute cyclically the four T-tails. The intermediate cover F^* is then similarly a genus one curve E^*, whose fundamental group is generated by $\alpha^* = 2\alpha_1$ and β_1, to which two copies of T are attached, with the deck transformation rotating by half turns along $2\alpha_1$ while permuting these two copies of T. This is summarized by the following diagram:

$$
\begin{array}{lccccccc}
 & \circlearrowleft & T & & \curvearrowleft & & & \\
 & & | & & & & & \\
(19) & T- & E^{**}-T & \to & T-E^{*}-T & \to & E-T \\
 & & | & & & & & \\
 & & T & & & & &
\end{array}
$$

Under the duality between $\mathrm{Jac}\,(F)[2]$ and $H_1(F,\mathbf{Z}/2\mathbf{Z})$ we readily see that σ is dual to α_1. Indeed $H_1(F,\mathbf{Z}) = H_1(E,\mathbf{Z}) \oplus H_1(T,\mathbf{Z})$, with $H_1(E,\mathbf{Z}) = \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\beta_1$. Likewise $H_1(F^*,\mathbf{Z}) = H_1(E^*,\mathbf{Z}) \oplus H_1(T,\mathbf{Z})^2$, with $H_1(E^*,\mathbf{Z}) = \mathbf{Z}\alpha^* \oplus \mathbf{Z}\beta_1$. The projection from F^* to F visibly maps $H_1(F^*,\mathbf{Z})$ onto the subgroup $2\mathbf{Z}\alpha_1 \oplus \mathbf{Z}\beta_1 \oplus H_1(T,\mathbf{Z})$ of $H_1(F^*,\mathbf{Z})$. Reducing the coefficients modulo 2 we see that the annihilator of this image modulo 2 in $H_1(F,\mathbf{Z}/2\mathbf{Z})$, under the intersection pairing, is indeed (the class modulo 2 of) α_1, which is therefore dual to σ. Likewise α^* is dual to σ^*. But the norm map above is dual to the pull-back from $H_1(F,\mathbf{Z}/2\mathbf{Z})$ to $H_1(H^*,\mathbf{Z}/2\mathbf{Z})$, which indeed maps the class of α_1 to the class of α^*. This proves the"only if" part of our claim.

To prove the "if" part (which is our main concern), observe that the space $\mathcal{R}_g^{(n)}$ of cyclic unramified n-sheeted covers of curves of genus g is connected by the same argument as in the proof of Proposition 4. It forms an étale cover of $\mathcal{M}_g$, whose degree is the cardinality of $\mathbf{P}^{2g}(\mathbf{Z}/n\mathbf{Z})$. The space $\mathcal{R}^{(2)}\mathcal{R}_g^{(2)}$ of unramified 2-sheeted covers of unramified 2-sheeted covers of curves of genus g is therefore étale over $\mathcal{R}_g^{(2)}$. It contains two subcovers of $\mathcal{R}_g^{(2)}$, namely $\mathcal{R}_g^{(4)}$ and the subspace $\mathcal{R}_\sigma$ of $\mathcal{R}^{(2)}\mathcal{R}_g^{(2)}$ determined by the condition that $\mathrm{Nm}(\sigma^*) = \sigma$. The Lemma asserts that these subcovers are equal, and the "only if" direction just proved shows the inclusion of $\mathcal{R}_g^{(4)}$ into $\mathcal{R}_\sigma$. To prove equality it suffices to show equality of the degrees of these two subcovers over $\mathcal{R}_g^{(2)}$. The degree of $\mathcal{R}_g^{(4)}/\mathcal{R}_g^{(2)}$ is $|\mathbf{P}^{2g}(\mathbf{Z}/4\mathbf{Z})/\mathbf{P}^{2g}(\mathbf{Z}/2\mathbf{Z})| = 2^{2g-1}$. The degree of $\mathcal{R}_\sigma/\mathcal{R}_g^{(2)}$ is the cardinality of the kernel of the norm map Nm. This cardinality is the quotient of the order of the source $|H_1(F^*,\mathbf{Z}/2\mathbf{Z})| = 2^{2(2g-1)}$ of Nm by the order of its image. As explained above, this image is dual to the pull-back π^* on homology. Hence this image has order $|H_1(F,\mathbf{Z}/2\mathbf{Z})|/|\mathrm{Ker}\,(\pi^*)| = 2^{2g}/2$, so that the degree of $\mathcal{R}_\sigma/\mathcal{R}_g^{(2)}$ is 2^{2g-1}, which is the same as the degree of $\mathcal{R}_g^{(4)}/\mathcal{R}_g^{(2)}$, concluding the proof of the Lemma. $\square$

Proof. (of Lemma 29) Observe first that the even sets of Weierstrass points of a hyperelliptic curve V modulo complementation are in natural bijection with the points $\mathrm{Jac}\,(V)[2]$ of order 2 of its Jacobian. by Lemma 31 the cover H^{**}/H is cyclic if (and only if) the norm map

$$\mathrm{Nm} : \mathrm{Jac}\,(H^*) \to \mathrm{Jac}\,(H)$$

maps σ^* to σ, where σ^* and σ are the points of order 2 corresponding to H^{**}/H^* and H^*/H respectively. Let $q_1, \ldots, q_{2g} \in \mathbf{P}^1$ be the branch points of $H/\mathbf{P}^1$ other than $\infty, 0$. Let H' be the double cover of $\mathbf{P}^1$ branched over $q_1, \ldots, q_{2g}$, and let $\mathbf{P}'$ be the u-line, namely the double cover of $\mathbf{P}^1$ branched above $\infty, 0$. Let q_i^+, q_i^- be the inverse images of each q_i in $\mathbf{P}'$. These are the branch points of the hyperelliptic cover $H^*/\mathbf{P}'$. Let T be the $\mathbf{Z}/2\mathbf{Z}$-module freely generated by the q_i's, Let T' be the $\mathbf{Z}/2\mathbf{Z}$-module freely generated by the $q_i^{\pm}$'s, and let $t \in T$ and $t' \in T'$ be the sums of the respective free generators. Let T_0' be the kernel of the natural degree map $\deg : T' \to \mathbf{Z}/2\mathbf{Z}$, and let T^* be the quotient $T_0'/\langle t' \rangle$. Then we have natural isomorphisms

$$\mathrm{Jac}\,(H)[2] \simeq T \qquad \text{and} \qquad \mathrm{Jac}\,(H^*) \simeq T^*.$$

Under these identifications σ goes to t, while $\mathrm{Nm}(q_i^+) = \mathrm{Nm}(q_i^-) = q_i$ for each i. It follows that the covers H^{**}'s which are cyclic over H correspond to those σ^*'s which are a section of the natural projection $\{q_i^+, q_i^-\} \to \{q_i\}$, namely to a choice of either q_i^+ or q_i^- for each i. Let H_1 be the double cover of $\mathbf{P}'$ branched above the points of σ^*, and let H_2 be the double cover of $\mathbf{P}'$ branched along the complementary section. The following diagram shows the various curves:

$$\begin{tikzcd}
 & & H^{**} \arrow[dl] \arrow[d] \arrow[dr] & \\
 & H^* \arrow[dl] \arrow[d] \arrow[dr] & H_1 \arrow[d] & H_2 \arrow[dl] \\
H' \arrow[dr] & H \arrow[d] & \mathbf{P}' \arrow[dl] & \\
 & \mathbf{P}^1 & &
\end{tikzcd}$$

It follows that H^{**} is the fibred product $H_1 \times_{\mathbf{P}'} H_2$. At the same time it is clear that the involution $u \mapsto -u$ of $\mathbf{P}'/\mathbf{P}^1$ exchanges the branch loci of H_1 and H_2, so that these curves are isomorphic to each other (and are otherwise arbitrary for their fixed genus). Therefore

$$\mathrm{Prym}\,(H^{**}/H^*) \simeq \mathrm{Jac}\,(H_1) \times \mathrm{Jac}\,(H_2) \simeq \mathrm{Jac}\,(H_1)^2,$$

concluding the proof of the Lemma.

Remark 32. *One can show that* $\mathrm{Prym}\,(H^{**}/H^*)$ *is in fact* isomorphic *to* $\mathrm{Jac}\,(H_1) \times \mathrm{Jac}\,(H_2)$ *using the bigonal construction (see* [Don2]*). We omit the proof since we do not need this.*

$\square$

Proof. (of Lemma 30) The statement is well known when the genus $g(C)$ of C is 1, so we assume by induction that we know the result up to genus $g-1$ and prove it for $g(C) = g > 1$. We have a canonical embedding

$$\mathrm{End}\,\mathrm{Jac}\,C \hookrightarrow \mathrm{End}\,H^1(\mathrm{Jac}\,C, \mathbf{Z}) = \mathrm{End}\,H^1(C, \mathbf{Z}).$$

Consider a degeneration of C to a one point union $C_0 = C_1 \cup C_2$, where each C_i is a general hyperelliptic curve of genus $g_i > 0$ (if $g_1 = g_2$ we also assume that the Jacobians of C_1 and of C_2 are not isogenous). Then $\mathrm{Jac}\,C_0 = \mathrm{Jac}\,C_1 \times \mathrm{Jac}\,C_2$. The family of cohomology groups $H^1(C, \mathbf{Z})$ and the Hodge structures on $H^1(C)$ are continuous at C_0. Hence $\mathrm{End}\,\mathrm{Jac}\,C$ embeds into $\mathrm{End}\,\mathrm{Jac}\,C_0$. By our assumptions and by induction, we get a decomposition $\mathrm{End}\,\mathrm{Jac}\,C_0 \simeq \mathbf{Z} \oplus \mathbf{Z}$ corresponding to the decomposition $H^1(C_0, \mathbf{Z}) =$

$H^1(C_1, \mathbf{Z}) \oplus H^1(C_2, \mathbf{Z})$. Since it is possible to make the degeneration of C to C_0 so as to give different such decompositions, and since $\operatorname{End} \operatorname{Jac} C$ must respect them all, it follows that $\operatorname{End} \operatorname{Jac} C = \mathbf{Z}$ as asserted. $\square$

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